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Multiplicity results for a boundary value problem with integral boundary conditions

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Abstract
The main purpose of this paper is to establish existence and multiplicity of positive solutions for a system of fourth-order boundary value problem with multi-point and integral conditions. To prove our results, we used Leggett–Williams fixed point theorem. An example is presented to illustrate our main results.

Keywords Fourth-order differential equation · Multi-point and integral boundary conditions · Leggett–Williams fixed point theorem · Positive solution

Mathematics Subject Classification 34B15 · 34B25 · 34B18

1 Introduction

Boundary value problems for ordinary differential equations play a very important role in theory and application see for example [8,14,16,17]. They describe a large number of nonlinear problems in physics, biology and chemistry. For example, the deformations of an elastic beam are described by a fourth-order differential equation, often referred to as the beam equation, which has been studied under a variety of boundary conditions [1,8,10]. This kind

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of problem was studied by many authors via various methods, such as the Leray–Schauder continuation method, the topological degree theory, the shooting method, fixed point theorems on cones, the critical point theory, and the lower and upper solution method, we refere the readers to [2–5,8,9,12] and the references therein.

Recently, Sun et al. [15] investigated the existence of positive solutions for the following fourth-order boundary value problem:

$$u^{(4)}(t) + f(t, u(t)) = 0, \; 0 < t < 1 \tag{1.1}$$

$$u(0) = u'(0) = u''(0) = 0, \tag{1.2}$$

$$u_i''(1) - \alpha u_i''(\eta) = \lambda, \tag{1.3}$$

where $\alpha \in [0, \frac{1}{\eta})$, $0 < \eta < 1$ are constants, $\lambda \in [0, +\infty)$ is a parameter, $f(t, u(t))$ singular at $t = 0$ and $t = 1$. Using Guo–Krasnosel'skii fixed point theorem the authors prove that (1.1)–(1.3) has at least one positive solution. In this paper, we generalize the results in [15] to a multi-point boundary value problem of the form.

$$u_i^{(4)}(t) + f_i(t, \mathbf{u}(t), \mathbf{u}'(t), \mathbf{u}''(t)) = 0, \; 0 < t < 1, \tag{1.4}$$

subject to multi-points and integral boundary conditions

$$\begin{cases} u_i(0) = h_{1,i}(\psi_1[u_1], \ldots, \psi_1[u_n]), \\ u_i'(0) = h_{2,i}(\psi_2[u_1], \ldots, \psi_2[u_n]), \\ u_i''(0) = 0, \\ u_i''(1) = \sum_{j=1}^{p} \beta_{j,i} u_i''(\eta_{j,i}) + h_{3,i}(\psi_3[u_1], \ldots, \psi_3[u_n]), \end{cases} \tag{1.5}$$

where $\mathbf{u}(s) = (u_1(s), \ldots, u_n(s))$, for $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, p\}$, $k \in \{1, 2, 3\}$, $\beta_{j,i} \geq 0$, $\eta_{j,i} > 0$ such that $0 \leq \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i} < 1$, $f_i : (0, 1) \times \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+$ are continuous functions and may be singular at $t = 0, 1$, $h_{k,i} : \mathbb{R}^n \to \mathbb{R}_+$ are continuous functions, $\psi_k : C([0, 1]) \to \mathbb{R}$ is a linear functional defined in the Lebesgue–Stieltjes sense by $\psi_k[w] := \int_0^1 w(s)d\phi_k(s)$, where ϕ_k is a function of bounded variation. Note that if $h_{k,i}(\psi_1[u_1], \ldots, \psi_1[u_n]) = \sum_{k=0}^{n} |a_{k,i}| u_i(\xi_{k,i})$, then, we have multi-point boundary conditions. The particularity of problem (1.4)–(1.5) is that the boundary condition involves multi-points and nonlinear integral conditions, which leads to extra difficulties.

In the special case, our problem reduces to the following classical boundary value problems coupled with the cantilever beam boundary conditions:

$$\begin{cases} u^{(4)}(t) + f(t, u(s)) = 0, \; 0 < t < 1, \\ u(0) = u'(0) = u''(0) = u''(1) = 0. \end{cases} \tag{1.6}$$

Note that problem (1.4)–(1.5) is a generalization of system (1.1)–(1.3). However, to the best of the authors knowledge, there are no results for triple positive solutions of the nonlinear differential equation (1.4) jointly with conditions (1.5) by using the Leggett–Williams fixed-point theorem. The aim of this paper is to fill the gap in the relevant literature. This paper is structured as follows: in next section, we give some properties of the Green's function associated to the problem (1.4)–(1.5) and transform problem (1.4)–(1.5) into Hammerstein integral equations. Moreover, we show some preliminary results which are used along the paper. In Sect. 3, we state the main theorems and give the proofs. Indeed, we firstly apply the well known Leggett–Williams fixed point theorem to prove the existence of at least three positive solutions, and after, by induction method we show the existence of countably many positive solutions for the problem (1.4)–(1.5). An example is presented in Sect. 4 to illustrate our main results.

2 Preliminaries

In this section we present some preliminary results which are useful in the proofs of the main results. First let us give the definition and some properties of the Green's function. Unless otherwise specified, the letters i and k in the remainder of this work always denote arbitrary integers in $\{1, 2, \ldots, n\}$ and in $\{1, 2, 3\}$ respectively.

Lemma 2.1 *Let $h_i \in C([0, 1]; \mathbb{R})$ and $g_{k,i} \in \mathbb{R}$, then the problem*

$$\begin{cases} u_i^{(4)}(t) + h_i(t) = 0, \;\; 0 < t < 1, \\ u_i(0) = g_{1,i}, \\ u_i'(0) = g_{2,i}, \\ u_i''(0) = 0, \\ u_i''(1) = \sum_{j=1}^{p} \beta_{j,i} u_i''(\eta_{j,i}) + g_{3,i} \end{cases} \tag{2.1}$$

is equivalent to

$$u_i(t) = \int_0^1 \left(G(t,s) + \frac{t^3}{6} K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2} \right) h_i(s)\, ds + \varphi_i(t),$$

where

$$G(t,s) = \frac{1}{6} \begin{cases} (1-s)t^3, & \text{if } 0 \le t \le s, \\ (1-s)t^3 - (t-s)^3, & \text{if } s \le t \le 1, \end{cases} \tag{2.2}$$

$$\varphi_i(t) = \frac{K_i g_{3,i} t^3}{6} + t g_{2,i} + g_{1,i}, \tag{2.3}$$

and K_i be such that

$$K_i \left(1 - \sum_{j=1}^{p} \beta_{j,i} \eta_{j,i} \right) = 1. \tag{2.4}$$

Proof We can integrate equation (2.1) to obtain

$$u_i(t) = -\frac{1}{6} \int_0^t (t-s)^3 h_i(s)\, ds + \frac{1}{6} C_{3,i} t^3 + \frac{1}{2} C_{2,i} t^2 + C_{1,i} t + C_{0,i}.$$

By the boundary conditions $u_i(0) = g_{1,i}$, $u_i'(0) = g_{2,i}$ and $u_i''(0) = 0$ we have $C_{0,i} = g_{1,i}$, $C_{1,i} = g_{2,i}$ and $C_{2,i} = 0$.

On the other hand, from the condition $u_i''(1) = \sum_{j=1}^{p} \beta_{j,i} u_i''(\eta_{j,i}) + g_{3,i}$, we obtain

$$C_{3,i} = K_i \int_0^1 (1-s) h_i(s)\, ds - K_i \sum_{j=1}^{p} \beta_{j,i} \int_0^{\eta_{j,i}} (\eta_{j,i} - s) h_i(s)\, ds + K_i g_{3,i},$$

where K_i is given by (2.4). It follows from the above informations that

$$u_i(t) = \frac{t^3}{6}\left(K_i \int_0^1 (1-s)h_i(s)\,ds \; - K_i \sum_{j=1}^{p} \beta_{j,i} \int_0^{\eta_{j,i}} (\eta_{j,i}-s)h_i(s)\,ds + K_i g_{3,i}\right)$$
$$+ t\, g_{2,i} + g_{1,i} - \frac{1}{6}\int_0^t (t-s)^3 h_i(s)\,ds$$
$$= t^3 \sum_{j=1}^{p} \beta_{j,i} \times \left(\eta_{j,i}\int_0^1 \frac{(1-s)}{6} h_i(s)\,ds - \int_0^{\eta_{j,i}} \frac{(\eta_{j,i}-s)}{6} h_i(s)\,ds\right)$$
$$+ \frac{t^3}{6}\int_0^1 (1-s)h_i(s)\,ds + \frac{t^3}{6} K_i\, g_{3,i} + t\, g_{2,i} + g_{1,i} - \frac{1}{6}\int_0^t (t-s)^3 h_i(s)\,ds$$
$$= \int_0^1 \left(G(t,s) + \frac{t^3}{6} K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2}\right) h_i(s)\,ds + \varphi_i(t),$$

where $G(t, s)$ and $\varphi_i(t)$ are given by (2.8) and (2.3) respectively. The proof of Lemma 2.1 is now completed. □

Now, we need some properties of the Green function $G(t, s)$ for more details, we refer the interested reader to [7,11,15].

Lemma 2.2 *The Green function has the following property:*

Let $\varphi(s) = \dfrac{(1-s)s}{2}$*, we have:*

1. - *For all* $(t, s) \in [0, 1] \times [0, 1]$, $0 \le G(t, s) \le 2\varphi(s)$.
   - *For all* $(t, s) \in [0, 1] \times [0, 1]$, $0 \le \dfrac{\partial G(t, s)}{\partial t} \le \varphi(s)$.
   - *For all* $(t, s) \in [0, 1] \times [0, 1]$, $0 \le \dfrac{\partial^2 G(t, s)}{\partial t^2} \le 2\varphi(s)$.
2. *Let* $\theta \in \left(0, \frac{1}{2}\right)$*, then*
   - *For all* $(t, s) \in [\theta, 1-\theta] \times [0, 1]$, $G(t, s) \ge \dfrac{\theta^3}{3}\varphi(s)$.
   - *For all* $(t, s) \in [\theta, 1-\theta] \times [0, 1]$, $\dfrac{\partial G(t, s)}{\partial t} \ge \theta^2\varphi(s)$.
   - *For all* $(t, s) \in [\theta, 1-\theta] \times [0, 1]$, $\dfrac{\partial^2 G(t, s)}{\partial t^2} \ge \theta\varphi(s)$.

The Leggett–Williams fixed point theorem is the main tools for proving the multiplicity results. For the convenience of the reader, we present here the Leggett–Williams fixed point theorem [13].

Let P be a cone in a real Banach space E, $0 < a < b$ and let β be a nonnegative continuous concave functional on K. Define the convex sets P_r and $P(\beta, a, b)$ by

$$P_r = \{x \in K \mid \|x\| \le r\}$$

and

$$P(\beta, a, b) = \{x \in K \mid a \le \beta(x), \|x\| \le b\}.$$

Theorem 2.3 (Leggett–Williams fixed point theorem) (see [13]) *Let* $A : \overline{P}_c \to \overline{P}_c$ *be completely continuous operator and* β *be a nonnegative continuous concave functional on* P *such that* $\beta(x) \le \|x\|$ *for* $x \in \overline{P}_c$. *Suppose there exist* $0 < a < b < d \le c$ *such that*

(A_1) $\{x \in P(\beta, b, d); \beta(x) > b\} \neq \emptyset$ *and* $\beta(Ax) > b$ *for* $x \in P(\beta, b, d)$
(A_2) $\|Ax\| < a$ *for* $\|x\| \leq a$,
(A_3) $\beta(Ax) > b$ *for* $x \in P(\beta, b, c)$ *with* $\|Ax\| > d$.

Then A has at least three fixed points x_1, x_2, x_3 *in* $\overline{P}_c$ *such that*
$\|x_1\| < a$, $\beta(x_2) > b$ *and* $\|x_3\| > a$ *with* $\beta(x_3) < b$.

For convenience, we introduce the following notations. Define

$$M_i = \max_{d\in\{0,1,2\}} \max_{t\in[0,1]} \int_0^1 \frac{\partial^d H_i(t,s)}{\partial t^d} ds$$

where

$$H_i(t,s) = G(t,s) + \frac{t^3}{6} K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2}.$$

Put

$$m_d(\theta) = \min_{t\in[\theta,1-\theta]} \int_\theta^{1-\theta} \frac{\partial^d G(t,s)}{\partial t^d} ds, \quad d \in \{0, 1, 2\}$$

and

$$L_{1,i} = \frac{1}{\psi_1[1]}, \quad L_{2,i} = \frac{1}{\psi_2[1]}, \quad L_{3,i} = \frac{1}{K_i \psi_3[1]}.$$

The basic space used in this paper is a real Banach space $E = (C^2([0,1];\mathbb{R}))^n$ equipped with the norm

$$\|\mathbf{u}\| = \sum_{i=1}^{n} \sum_{d=0}^{2} \|u_i^{(d)}\|_\infty.$$

Let

$$E^+ = \{\mathbf{u} = (u_1, \ldots, u_n) \in E, u_i(t) \geq 0, u_i'(t) \geq 0, u_i''(t) \geq 0, t \in [0,1], i \in \{1, \ldots, n\}\}.$$

Then the set

$$K(\theta) = \left\{ \mathbf{u} \in E^+, \min_{t\in[\theta,1-\theta]} \sum_{i=1}^{n} \sum_{d=0}^{2} u_i^{(d)}(t) \geq \gamma(\theta)\|\mathbf{u}\| \right\}$$

is a cone of E, where $\theta \in (0, \frac{1}{2})$ and $\gamma(\theta) = \frac{\theta^3}{6}$. The following result follows immediately from Lemma 2.1

Corollary 2.4 *Assume that* $h_{k,i} \in C([0,1] \times \mathbb{R}^n, \mathbb{R}_+)$ *and* $f_i \in C((0,1) \times \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n, \mathbb{R}_+)$. *Then,* $\boldsymbol{u} \in E$ *is a solution of* (1.4)–(1.5) *if and only if*

$$\boldsymbol{T}(\boldsymbol{u}) = \boldsymbol{u},$$

where $\boldsymbol{T}$ *is the operator defined on E by*

$$\boldsymbol{T}(\boldsymbol{u}) = (T_1(\boldsymbol{u}), \ldots, T_n(\boldsymbol{u})),$$

and for all $t \in [0,1]$.

$$T_i(\boldsymbol{u})(t) = \int_0^1 \left(G(t,s) + \frac{t^3}{6} K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2} \right) f_i(s, \boldsymbol{u}(s), \boldsymbol{u}'(s), \boldsymbol{u}''(s))\, ds + P_i(t), \tag{2.5}$$

with

$$P_i(t) = h_{1,i}\left(\psi_1[u_1], \ldots, \psi_1[u_n]\right) + t\, h_{2,i}\left(\psi_2[u_1], \ldots, \psi_2[u_n]\right) + \frac{K_i t^3}{6} h_{3,i}\left(\psi_3[u_1], \ldots, \psi_3[u_n]\right). \tag{2.6}$$

Definition 2.5 A function $\mathbf{u} = (u_1, \ldots, u_n)$ is called a nonnegative solution of (1.4)–(1.5) if $\mathbf{u}$ satisfies (1.4)–(1.5) and $u_i \geq 0$ in $[0, 1]$. If in addition, $u_i(t) > 0$ in $[0, 1]$, then, u is called a positive solution.

Lemma 2.6 *Let $\theta \in \left(0, \frac{1}{2}\right)$ and assume that $\int_0^1 f_i(s, x, y, z)\, ds < +\infty$, for any $x, y, z \in [0, +\infty)$ then, the operator $\boldsymbol{T}$ given by (2.5) maps $K(\theta)$ into itself, i.e., $\boldsymbol{T} : K(\theta) \to K(\theta)$. Moreover, $\boldsymbol{T}$ is completely continuous that is $\boldsymbol{T}$ is continuous and maps bounded sets into precompact sets.*

Proof Let $\mathbf{u} \in K(\theta)$, then, from the positivity of the Green function, it is easy to see that for all $t \in [0, 1]$

$$T_i(\mathbf{u})(t) \geq 0, \quad T_i(\mathbf{u})'(t) \geq 0 \quad \text{and} \quad T_i(\mathbf{u})''(t) \geq 0.$$

Thus, to prove that $\mathbf{T}(K(\theta)) \subset K(\theta)$, it suffices to prove that

$$\min_{t \in [\theta, 1-\theta]} \sum_{i=1}^{n} \sum_{d=0}^{2} T_i(\mathbf{u})^{(d)}(t) \geq \gamma(\theta) \|\mathbf{T}(\mathbf{u})\|.$$

Indeed, for all $t \in [0, 1]$,

$$\begin{aligned}
|T_i(\mathbf{u})(t)| &\leq \int_0^1 \left(G(t,s) + \frac{t^3}{6} K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2} \right) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds + P_i(t) \\
&\leq \int_0^1 \left(\frac{1}{6} K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2} \right) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds + P_i(1) \\
&\quad + 2 \int_0^1 \varphi(s) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds.
\end{aligned}$$

Then

$$\begin{aligned}
\|T_i(\mathbf{u})\|_\infty \leq& \int_0^1 \left(\frac{1}{6} K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2} \right) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds + P_i(1) \\
&+ 2 \int_0^1 \varphi(s) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds.
\end{aligned}$$

On the other hand, it follows from Lemma 2.2 that, for all $t \in [\theta, 1-\theta]$,

$$T_i(\mathbf{u})(t) \geq \int_0^1 \left(\frac{\theta^3}{6} K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2} \right) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds$$

$$
\begin{aligned}
&+\frac{\theta^3}{3}\int_0^1 \varphi(s) f_i(t,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds+\frac{\theta^3}{6}K_i h_{3,i}(\psi_2[u_1],\ldots,\psi_2[u_n])\\
&+\theta h_{2,i}(\psi_2[u_1],\ldots,\psi_2[u_n])+h_{1,i}(\psi_2[u_1],\ldots,\psi_2[u_n])\\
\geq{}& \frac{\theta^3}{6}\left[\int_0^1\left(K_i\sum_{j=1}^{p}\beta_{j,i}\frac{\partial^2 G(\eta_{j,i},s)}{\partial t^2}\right) f_i(t,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds\right.\\
&+2\int_0^1 \varphi(s) f_i(t,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds+K_i h_{3,i}(\psi_2[u_1],\ldots,\psi_2[u_n])\\
&\left.+\frac{6}{\theta^2}h_{2,i}(\psi_2[u_1],\ldots,\psi_2[u_n])+\frac{6}{\theta^3}h_{1,i}(\psi_2[u_1],\ldots,\psi_2[u_n])\right]\\
\geq{}& \frac{\theta^3}{6}\left[\int_0^1\left(K_i\sum_{j=1}^{p}\beta_{j,i}\frac{\partial^2 G(\eta_{j,i},s)}{\partial t^2}\right) f_i(t,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds\right.\\
&\left.+2\int_0^1 \varphi(s) f_i(t,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds+P_i(1)\right]\\
\geq{}& \frac{\theta^3}{6}\|T_i(\mathbf{u})\|_\infty.
\end{aligned}
$$

Then, we obtain

$$\min_{t\in[\theta,1-\theta]} T_i(\mathbf{u})(t)\geq \gamma(\theta)\|T_i(\mathbf{u})\|_\infty. \tag{2.7}$$

In addition, we have

$$
\begin{aligned}
|T_i(\mathbf{u})'(t)| &\leq \int_0^1\left(\frac{\partial G(t,s)}{\partial t}+\frac{t^2}{2}K_i\sum_{j=1}^{p}\beta_{j,i}\frac{\partial^2 G(\eta_{j,i},s)}{\partial t^2}\right) f_i(t,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds+P_i'(t)\\
&\leq \int_0^1\left(\frac{1}{2}K_i\sum_{j=1}^{p}\beta_{j,i}\frac{\partial^2 G(\eta_{j,i},s)}{\partial t^2}\right) f_i(t,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds+P_i'(1)\\
&\quad+\int_0^1 \varphi(s) f_i(t,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|T_i(\mathbf{u})'\|_\infty \leq{}& \int_0^1\left(\frac{1}{2}K_i\sum_{j=1}^{p}\beta_{j,i}\frac{\partial^2 G(\eta_{j,i},s)}{\partial t^2}\right) f_i(t,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds+P_i'(1)\\
&+\int_0^1 \varphi(s) f_i(t,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds.
\end{aligned}
$$

It follows from Lemma 2.2 that, for all $t\in[\theta,1-\theta]$,

$$
\begin{aligned}
T_i(\mathbf{u})'(t) \geq{}& \int_0^1\left(\frac{\theta^2}{2}K_i\sum_{j=1}^{p}\beta_{j,i}\frac{\partial^2 G(\eta_{j,i},s)}{\partial t^2}\right) f_i(t,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds\\
&+\theta^2\int_0^1 \varphi(s) f_i(t,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds+\frac{\theta^2}{2}K_i\left(h_{3,i}(\psi_3[u_1],\ldots,\psi_3[u_n])\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ h_{2,i}(\psi_2[u_1], \ldots, \psi_2[u_n]) \\
&\geq \theta^2 \left(\int_0^1 \left(\frac{1}{2} K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2} \right) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds \right. \\
&+ \int_0^1 \varphi(s) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds + \frac{1}{2} K_i h_{3,i}(\psi_2[u_1], \ldots, \psi_2[u_n]) \\
&\left. + \frac{1}{\theta^2} h_{2,i}(\psi_2[u_1], \ldots, \psi_2[u_n]) \right) \\
&\geq \theta^2 \left(\int_0^1 \left(\frac{1}{2} K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2} \right) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds \right. \\
&\left. + \int_0^1 \varphi(s) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds + P_i'(1) \right) \\
&\geq \theta^2 \| T_i(\mathbf{u})' \|_\infty.
\end{aligned}
$$

Thus

$$
\min_{t \in [\theta, 1-\theta]} T_i(\mathbf{u})'(t) \geq \gamma(\theta) \| T_i(\mathbf{u})' \|_\infty.
$$

Besides,

$$
\begin{aligned}
|T_i(\mathbf{u})''(t)| &\leq \int_0^1 \left(\frac{\partial^2 G(t, s)}{\partial t^2} + t K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2} \right) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds + P_i''(t) \\
&\leq \int_0^1 \left(K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2} \right) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds \\
&\quad + 2 \int_0^1 \varphi(s) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds + P_i''(1).
\end{aligned}
$$

Moreover, it follows from Lemma 2.2 that for each $t \in [\theta, 1-\theta]$

$$
\begin{aligned}
T_i(\mathbf{u})''(t) &\geq \theta \int_0^1 \left(K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2} \right) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds \\
&\quad + \theta \int_0^1 \varphi(s) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds + \theta K_i h_{3,i}(\psi_2[u_1], \ldots, \psi_2[u_n]) \\
&\geq \frac{\theta}{2} \left(\int_0^1 \left(2K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2} \right) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds \right. \\
&\quad \left. + \int_0^1 2\varphi(s) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds + 2K_i h_{3,i}(\psi_2[u_1], \ldots, \psi_2[u_n]) \right) \\
&\geq \frac{\theta}{2} \left(\int_0^1 \left(K_i \sum_{j=1}^{p} \beta_{j,i} \frac{\partial^2 G(\eta_{j,i}, s)}{\partial t^2} \right) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds \right.
\end{aligned}
$$

$$+\int_0^1 2\varphi(s) f_i(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds + P_i''(1)\Bigg)$$
$$\geq \frac{\theta^3}{6}\|T_i(\mathbf{u})''\|_\infty.$$

We deduce that $\mathbf{T}(K(\theta)) \subset K(\theta)$.

Now we prove the operator $\mathbf{T}$ is completely continuous. For any natural number m ($m \geq 2$), we set, for all $u, v, w \in [0, +\infty)$

$$f_{i,m}(t, u, v, w) = \begin{cases} \inf_{t<s\leq \frac{1}{m}} f_i(s, u, v, w), & \text{if } 0 \leq t \leq \frac{1}{m}, \\ f_i(t, u, v, w), & \text{if } \frac{1}{m} \leq t \leq 1 - \frac{1}{m}, \\ \inf_{1-\frac{1}{m}\leq s<t} f_i(s, u, v, w), & \text{if } 1 - \frac{1}{m} \leq t \leq 1. \end{cases} \tag{2.8}$$

Then $f_{i,m} : [0, 1] \times \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n \to [0, +\infty)$ is continuous and $0 \leq f_{i,m}(t, u, v, w) \leq f_i(t, u, v, w)$ for all $t \in (0, 1)$.

Let $T_{i,m}(\mathbf{u})(t) = \int_0^1 H_i(t, s) f_{i,m}(s, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds + P_i(t)$ and $\mathbf{T}_m(\mathbf{u}) = (T_{1,m}(\mathbf{u}), \ldots, T_{n,m}(\mathbf{u}))$.

Since $[0, 1]$ is compact, $f_{i,m}$ and H_i are continuous, it is easy to show by using of Arzel–Ascoli theorem [6] that $\mathbf{T}_m$ is completely continuous. Furthermore, for any $R > 0$, set $B_R = \{\mathbf{u} \in K(\theta) : \|\mathbf{u}\| \leq R\}$, then $\mathbf{T}_m$ converges uniformly to $\mathbf{T}$ as $m \to \infty$. In fact, for all $d \in \{0, 1, 2\}$, we denote by $J_d = \max_{(t,s)\in[0,1]\times[0,1]} \frac{\partial^d H_i(t,s)}{\partial t^d}$. For $R > 0$ and $\mathbf{u} \in B_R$, we have

$$\begin{aligned}
&|T_{i,m}(\mathbf{u})^{(d)}(t) - T_i(\mathbf{u})^{(d)}(t)| \\
&\quad\leq J_d \int_0^1 |f_{i,m}(s, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s)) - f_i(s, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))|\, ds \\
&\quad\leq J_d \int_0^{\frac{1}{m}} |f_{i,m}(s, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s)) - f_i(s, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))|\, ds \\
&\qquad + J_d \int_{1-\frac{1}{m}}^1 |f_{i,m}(s, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s)) - f_i(s, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))|\, ds \\
&\quad\leq J_d \left(\int_0^{\frac{1}{m}} f_i(s, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds + \int_{1-\frac{1}{m}}^1 f_i(s, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds\right) \\
&\qquad \to 0 \text{ as } (m \to \infty)
\end{aligned}$$

So we conclude that $\mathbf{T}_m$ converges uniformly to $\mathbf{T}$ as $m \to \infty$. Thus, $\mathbf{T}$ is completely continuous. The proof is completed. □

3 Main results and proofs

Let $\beta : K(\theta) \to [0, +\infty)$ be a functional defined by:

$$\beta(\mathbf{u}) = \min_{t\in[\theta,1-\theta]} \sum_{i=1}^n \sum_{d=0}^2 u_i^{(d)}(t).$$

Then, it is easy to see that β is a nonnegative continuous and concave functional on $K(\theta)$, moreover, for each $\mathbf{u} = (u_1, \ldots, u_n) \in K(\theta)$, one has

$$\beta(\mathbf{u}) \leq \|\mathbf{u}\|.$$

Let $p_i, q_{k,i}$ be positive numbers such that $\sum_{i=1}^{n} \frac{1}{p_i} + \frac{5}{3q_{3,i}} + \frac{2}{q_{2,i}} + \frac{1}{q_{1,i}} \leq 1$.

Our first existence result is the following:

Theorem 3.1 *Let a, b, c in $\mathbb{R}$ such that $0 < a < b < \frac{b}{\gamma(\theta)} \leq c$. Assume that*

(H_1) *For all $u \in \mathbb{R}^n$ such that $\sum_{i=1}^{n} u_i \in [0, c]$, we have*

$$h_{k,i}(u) \leq \frac{L_{k,i}}{q_{k,i}} \sum_{i=1}^{n} u_i \;\; \text{for all } k \in \{1, 2, 3\}.$$

(H_2) *For all $u_k = (u_{k,1}, \ldots, u_{k,n})$ such that $\sum_{k=1}^{3} \sum_{i=1}^{n} u_{k,i} \in [0, c]$, we have*

$$f_i(t, u_1, u_2, u_3) \leq \frac{c}{3p_i M_i}, \;\; t \in [0, 1].$$

(H_3) *For all $u_k = (u_{k,1}, \ldots, u_{k,n})$ such that $\sum_{k=1}^{3} \sum_{i=1}^{n} u_{k,i} \in [0, a]$, we have*

$$f_i(t, u_1, u_2, u_3) \leq \frac{a}{3p_i M_i}, \;\; t \in [0, 1].$$

(H_4) *For all $u_k = (u_{k,1}, \ldots, u_{k,n})$ such that $\sum_{k=1}^{3} \sum_{i=1}^{n} u_{k,i} \in \left[b, \frac{b}{\gamma(\theta)}\right]$ we have*

$$f_i(t, u_1, u_2, u_3) \geq \frac{b}{n \sum_{d=0}^{2} m_d(\theta)}, \;\; t \in [\theta, 1 - \theta].$$

Then the boundary value problem (1.4)–(1.5) has at least three nonnegative solutions $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ in $\overline{P}_c$ such that $\|\boldsymbol{u}_1\| < a$, $\beta(\boldsymbol{u}_2) > b$ and $\|\boldsymbol{u}_3\| > a$ with $\beta(\boldsymbol{u}_3) < b$.

Proof First, let us prove that the operator $\mathbf{T}$ maps $\overline{P}_c$ into itself. Indeed, if $\mathbf{u} = (u_1, \ldots, u_n) \in \overline{P}_c$, then $\|\mathbf{u}\| \leq c$. Moreover, by hypothesis (H_1), we get

$$h_{1,i}(\psi_1[u_1], \ldots, \psi_1[u_n]) \leq \frac{L_{1,i}}{q_{1,i}}(\psi_1[u_1 + \cdots + u_n]) \leq \frac{L_{1,i}}{q_{1,i}}\psi_1[1]\|\mathbf{u}\| \leq \frac{c}{q_{1,i}},$$
$$h_{2,i}(\psi_2[u_1], \ldots, \psi_2[u_n]) \leq \frac{L_{2,i}}{q_{2,i}}(\psi_2[u_1 + \cdots + u_n]) \leq \frac{L_{2,i}}{q_{2,i}}\psi_2[1]\|\mathbf{u}\| \leq \frac{c}{q_{2,i}}$$

and

$$h_{3,i}(\psi_3[u_1], \ldots, \psi_3[u_n]) \leq \frac{L_{3,i}}{q_{3,i}}(\psi_3[u_1 + \cdots + u_n]) \leq \frac{L_{3,i}}{q_{3,i}}\psi_3[1]\|\mathbf{u}\| \leq \frac{c}{K_i q_{3,i}}.$$

Thus, from hypothesis (H_2), we have

$$\begin{aligned}
\|T_i(\mathbf{u})\|_\infty &= \max_{t\in[0,1]} \int_0^1 H_i(t, s) f_i(s, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s))\, ds + \max_{t\in[0,1]} P_i(t) \\
&\leq \max_{t\in[0,1]} \int_0^1 H_i(t, s)\, ds \frac{c}{3p_i M_i} + \frac{c}{6q_{3,i}} + \frac{c}{q_{2,i}} + \frac{c}{q_{1,i}} \\
&\leq \frac{c}{3p_i} + \frac{c}{6q_{3,i}} + \frac{c}{q_{2,i}} + \frac{c}{q_{1,i}},
\end{aligned}$$

$$\begin{aligned}\|T_i(\mathbf{u})'\|_\infty &= \max_{t\in[0,1]}\int_0^1 \frac{\partial H_i(t,s)}{\partial t} f_i(s,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds + \max_{t\in[0,1]}\frac{\partial P_i(t)}{\partial t}\\ &\le \max_{t\in[0,1]}\int_0^1 \frac{\partial H_i(t,s)\,ds}{\partial t}\frac{c}{3p_iM_i} + \frac{c}{2q_{3,i}} + \frac{c}{q_{2,i}}\\ &\le \frac{c}{3p_i} + \frac{c}{2q_{3,i}} + \frac{c}{q_{2,i}} + \frac{c}{q_{1,i}}\end{aligned}$$

and

$$\begin{aligned}\|T_i(\mathbf{u})''\|_\infty &= \max_{t\in[0,1]}\int_0^1 \frac{\partial^2 H_i(t,s)}{\partial t^2} f_i(s,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds + \max_{t\in[0,1]}\frac{\partial^2 P_i(t)}{\partial t^2}\\ &\le \max_{t\in[0,1]}\int_0^1 \frac{\partial^2 H_i(t,s)\,ds}{\partial t^2}\frac{c}{3p_iM_i} + \frac{c}{2q_{3,i}} + \frac{c}{q_{2,i}}\\ &\le \frac{c}{3p_i} + \frac{c}{q_{3,i}} + \frac{c}{q_{2,i}},\end{aligned}$$

which yields to

$$\begin{aligned}\|\mathbf{T}(\mathbf{u})\| &= \sum_{i=1}^n\sum_{d=0}^2 \|T_i(\mathbf{u})^{(d)}\|_\infty\\ &\le \sum_{i=1}^n \frac{c}{3p_i} + \frac{c}{6q_{3,i}} + \frac{c}{q_{2,i}} + \frac{c}{q_{1,i}}\\ &\quad + \sum_{i=1}^n \frac{c}{3p_i} + \frac{c}{2q_{3,i}} + \frac{c}{q_{2,i}} + \frac{c}{q_{1,i}}\\ &\quad + \sum_{i=1}^n \frac{c}{3p_i} + \frac{c}{q_{3,i}} + \frac{c}{q_{2,i}}\\ &= \sum_{i=1}^n \frac{c}{p_i} + \frac{5c}{3q_{3,i}} + \frac{2c}{q_{2,i}} + \frac{c}{q_{1,i}} \le c.\end{aligned}$$

Hence, $\|\mathbf{T}(\mathbf{u})\| \le c$, that is, $\mathbf{T} : \overline{P}_c \to \overline{P}_c$. It is easy to prove by Arzel–Ascoli [6] that the operator $\mathbf{T}$ is completely continuous. In the same way, the condition (H_3) implies that the condition (A_2) of Theorem 2.3 is satisfied.

We now show that condition (A_1) of Theorem 2.3 is satisfied. Clearly, if

$$\mathbf{u}(t) = \left(\frac{b}{2\times 3n} + \frac{b}{2\times\gamma(\theta)3n}\right)(1,\ldots,1),$$

then, $\beta(\mathbf{u}) > b$ and $\|\mathbf{u}\| \le \frac{b}{\gamma(\theta)}$, that is

$$\left\{\mathbf{u}\in P\left(\beta, b, \frac{b}{\gamma(\theta)}\right); \beta(\mathbf{u}) > b\right\} \ne \emptyset.$$

Let $\mathbf{u} = (u_1,\ldots,u_n) \in P\left(\beta, b, \frac{b}{\gamma(\theta)}\right)$, then, from (H_4) we have

$$b \le \sum_{i=1}^n\sum_{d=0}^2 u_i^{(d)}(t) \le \frac{b}{\gamma(\theta)}, \qquad t\in[\theta, 1-\theta].$$

Moreover

$$\begin{aligned}\beta(\mathbf{T}(\mathbf{u})) &= \min_{t\in[\theta,1-\theta]}\sum_{i=1}^{n}\sum_{d=0}^{2}\int_0^1 \frac{\partial^d G(t,s)}{\partial t^d} f_i(s,\mathbf{u}(s),\mathbf{u}'(s),\mathbf{u}''(s))\,ds + \min_{t\in[\theta,1-\theta]}\frac{\partial^d P_i(t)}{\partial t^d}\\ &\geq \sum_{i=1}^{n}\sum_{d=0}^{2}\min_{t\in[\theta,1-\theta]}\int_\theta^{1-\theta}\frac{\partial^d G(t,s)}{\partial t^d}\,ds\frac{b}{n\sum_{d=0}^{2} m_d(\theta)}\\ &\geq b.\end{aligned}$$

Therefore, condition (A_1) of Theorem 2.3 is satisfied.

Finally, if

$$\mathbf{u} = (u_1,\ldots,u_n) \in P(\beta,b,c) \quad \text{and} \quad \|\mathbf{T}(\mathbf{u})\| > \frac{b}{\gamma(\theta)},$$

then

$$\beta(\mathbf{T}(\mathbf{u})) = \min_{t\in[\theta,1-\theta]}\sum_{i=1}^{n}\sum_{d=0}^{2} T_i(\mathbf{u})^{(d)}(t) \geq \gamma(\theta)\|\mathbf{T}(\mathbf{u})\| \geq \gamma(\theta)\frac{b}{\gamma(\theta)} = b.$$

Therefore, the condition (A_3) of Theorem 2.3 is also satisfied. By Theorem 2.3, there exist three positive solutions $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ such that $\|\mathbf{u}_1\| < a$, $\beta(\mathbf{u}_2) > b$ and $\|\mathbf{u}_3\| > a$ with $\beta(\mathbf{u}_3) < b$. The proof of Theorem 3.1 is now completed. □

From the proof of Theorem 3.1, it is easy to see that, if the conditions like (H_1)-(H_4) are appropriately combined, we can obtain an arbitrary number of positive solutions of problem (1.4)–(1.5). More precisely, let m be an arbitrary positive integer with $m \geq 1$. Assume that there exist numbers b_j $(1 \leq j \leq m-1)$ and c_l $(1 \leq l \leq m)$ such that

$$0 < c_1 < b_1 < \frac{b_1}{\gamma(\theta)} \leq c_2 < b_2 < \frac{b_2}{\gamma(\theta)} \leq c_3 < \cdots \leq c_{m-1} < b_{m-1} < \frac{b_{m-1}}{\gamma(\theta)} \leq c_m,$$

then, if we replace the hypothesis (H_1)–(H_4) of Theorem 3.1 by the following hypothesis:

$(H_{m,1})$ For all $1 \leq l \leq m$ and $u \in \mathbb{R}^n$ such that $\sum_{i=1}^n u_i \in [0, c_l]$, we have

$$h_{k,i}(u) \leq \frac{L_{k,i}}{q_{k,i}}\sum_{j=1}^{n} u_j, \text{ for all } k \in \{1,2,3\}.$$

$(H_{m,2})$ For all $1 \leq l \leq m$ and $(u_1,u_2,u_3) \in \mathbb{R}^{3n}$ such that $\sum_{k=1}^3\sum_{i=1}^n u_{k,i} \in [0, c_l]$, we have

$$f_i(t,u_1,u_2,u_3) \leq \frac{c_l}{3p_iM_i}, \quad t \in [0,1].$$

$(H_{m,3})$ For all $1 \leq j \leq m-1$ and $(u_1,u_2,u_3) \in \mathbb{R}^{3n}$ such that $\sum_{k=1}^3\sum_{i=1}^n u_{k,i} \in \left[b_j, \frac{b_j}{\gamma(\theta)}\right]$, we have

$$f_i(t,u_1,u_2,u_3) \geq \frac{b_j}{n\sum_{d=0}^{2} m_d(\theta)}, \quad t \in [\theta, 1-\theta].$$

we obtain the following result:

Theorem 3.2 *Under hypothesis* $(H_{m,1}) - (H_{m,3})$, *problem* (1.4)–(1.5) *has at least* $2m-1$ *nonnegative solutions in* $\overline{P_{c_m}}$.

Proof In order to prove Theorem 3.2, observe that for $m = 1$, we know from (H_3) that $\mathbf{T} : \overline{P_{c_1}} \to P_{c_1}$. Then it follows from Schauder fixed point theorem that (1.4)–(1.5) has at least one positive solution in $\overline{P_{c_1}}$. Moreover, for $m = 2$, it is clear that Theorem 3.1 holds (with $a = c_1$, $b = b_1$ and $c = c_2$). Then, we can obtain three positive solutions x_2, x_3, and x_4.

Along this way, we can finish the proof by the induction method. To this aim, we suppose that there exist numbers b_j $(1 \le j \le m)$ and c_l $(1 \le l \le m+1)$ such that

$$0 < c_1 < b_1 < \frac{b_1}{\gamma(\theta)} \le c_2 < b_2 < \frac{b_2}{\gamma(\theta)} \le \cdots \le c_m < b_m < \frac{b_m}{\gamma(\theta)} \le c_{m+1},$$

and $(H_{m+1,1})$, $(H_{m+1,2})$ and $(H_{m+1,3})$ hold true. We know by the inductive hypothesis that (1.4)–(1.5) has at least $2m-1$ positive solutions u_i $(i = 1, 2, \ldots, 2m-1)$ in $\overline{P_{c_m}}$. At the same time, it follows from Theorem 3.1, $(H_{m+1,1})$, $(H_{m+1,2})$ and $(H_{m+1,3})$ that (1.4)–(1.5) has at least three positive solutions $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $P_{c_{m+1}}$ such that $\|\mathbf{u}\| < c_m$, $\beta(\mathbf{v}) > b_m$ and $\|\mathbf{w}\| > c_m$ with $\beta(\mathbf{w}) < b_m$. Obviously, $\mathbf{v}$ and $\mathbf{w}$ are not in $\overline{P_{c_m}}$. Therefore, (1.4)–(1.5) has at least $2m+1$ nonnegative solutions in $P_{c_{m+1}}$. This completes the proof. □

We can generalize the above result and present the following result which is especially important and useful in applications.

Theorem 3.3 *Under the assumptions of Theorem 3.2. If the following additional assumption:*

$$\text{there exists } t_{0,i} \in (0,1) \text{ such that } f_i(t_{0,i}, x, y, z) > 0, \quad \forall x, y, z \in \mathbb{R}^n_+, \tag{3.1}$$

holds true. Then (1.4)–(1.5) has at least $2m-1$ positive solutions in $\overline{P_{c_m}}$.

Proof Let $u_{i,l}$, for $l \in \{1, \ldots, 2m-1\}$ be the $2m-1$ nonnegative solutions of problem (1.4)–(1.5) whose existence is guaranteed by Theorem 3.2. Then, $u_{i,l}$ satisfy the following integral equation

$$u_{i,l}(t) = \int_0^1 H_i(t,s) f_i(s, \mathbf{u}_l(s), \mathbf{u}_l'(s), \mathbf{u}_l''(s))ds + P_i(t).$$

Indeed, on the contrary case we can find $t^* \in (0,1)$ such that $u_{i,l}(t^*) = 0$. Since $u_{i,l}(t) \ge 0$, $u'_{i,l}(t) \ge 0$ and $u''_{i,l}(t) \ge 0$ for all $t \in [0,1]$, we have

$$u_{i,l}(t^*) = 0 = \int_0^1 H_i(t^*,s) f_i(s, \mathbf{u}_l(s), \mathbf{u}_l'(s), \mathbf{u}_l''(s))ds + P_i(t^*) \ge 0.$$

Since the functions H_i and f_i are nonnegative and continuous, we obtain

$$H_i(t^*,s) f_i(s, \mathbf{u}_l(s), \mathbf{u}_l'(s), \mathbf{u}_l''(s)) = 0 \quad \text{almost everywhere s.}$$

Since $f_i(s, \mathbf{u}_l(s), \mathbf{u}_l'(s), \mathbf{u}_l''(s)) \ge 0$ and H_i is positive on $(0,1)$, we deduce that

$$f_i(s, \mathbf{u}_l(s), \mathbf{u}_l'(s), \mathbf{u}_l''(s)) = 0 \quad \text{almost everywhere s.}$$

Now, by the condition (3.1) and the continuity of the functions f_i, we deduce that there exists a subset $\Omega \subset (0,1)$ with $\mu(\Omega) > 0$ where μ is the Lebesgue measure on $[0,1]$ such that $f_i(s, \mathbf{u}_l(s), \mathbf{u}_l'(s), \mathbf{u}_l''(s)) > 0$ on Ω and this is a contradiction. This ends the proof. □

Remark 3.4 It is clear that the conclusion of Theorem 3.2 remains valid if we replace condition (3.1) by: There exist $k_0 \in \{1,2,3\}$ for all $i \in \{1, \ldots, n\}$, there exists, $t_{0,i} \in (0,1)$ such that $h_{k_0,i}(t_{0,i}, x) > 0$ for all $x \in \mathbb{R}^n_+$.

Remark 3.5 In the special case when the functions f_i are nondecreasing with respect to the second, third and the fourth variable on (0, 1), the condition (3.1) can be replaced by

$$\text{For all } i \in \{1,\ldots,n\}, \text{ there exists } t_{0,i} \in (0,1) \text{ such that } f_i(t_{0,i}, \mathbf{0}, \mathbf{0}, \mathbf{0}) > 0, \tag{3.2}$$

where $\mathbf{0} = (0,\ldots,0) \in \mathbb{R}^n$.

4 Example

In this section, we present an example to illustrate our main theorems. Let f_1 and f_2 be two functions defined by:

$$f_1(t,u_1,u_2,u_3) = \begin{cases} \dfrac{\sin^2(21\pi u)}{5} + \dfrac{2}{10} + \dfrac{1}{100}e^{-\frac{1}{1-t}}, & \text{if } 0 \le u \le \frac{1}{2}, \\ \left(u - \dfrac{1}{2}\right)^2 + \dfrac{4}{10} + \dfrac{1}{100}e^{-\frac{2u}{1-t}}, & \text{if } \frac{1}{2} \le u \le 1, \\ \dfrac{2935}{100}u - \dfrac{287}{10} + \dfrac{1}{100}e^{-\frac{2u}{1-t}}, & \text{if } 1 \le u \le 2, \\ \dfrac{\left|\cos\left(\frac{\pi u}{4}\right)\right|}{1000} + 30 + \dfrac{1}{100}e^{-\frac{2u}{1-t}}, & \text{if } 2 \le u \le 768, \\ \dfrac{30001}{1000}e^{768-u} + \dfrac{2u}{5}\left|\sin\left(\dfrac{\pi u}{768}\right)\right| + \dfrac{1}{100}e^{-\frac{2u}{1-t}}, & \text{if } 768 \le u, \end{cases}$$

$$f_2(t,u_1,u_2,u_3) =$$

$$\begin{cases} \dfrac{\sqrt{\frac{3+3u}{2}}}{|\sin(\pi u)| + 14} + \dfrac{1}{1000}\left|\cos\left(\dfrac{1}{\sqrt{t-t^2}}\right)\right|, & \text{if } 0 \le u \le 1, \\ \ln\left(\dfrac{1+u}{2}\right) + \dfrac{\sqrt{3}}{14} + 30\left|\cos\left(\dfrac{\pi}{2}u\right)\right| + \dfrac{1}{1000}\left|\cos\left(\dfrac{u}{\sqrt{t-t^2}}\right),\right| & \text{if } 1 \le u \le 2, \\ \left(\ln\left(\dfrac{3}{2}\right) + \dfrac{\sqrt{3}}{14}\right)\left|\cos\left(\dfrac{\pi}{2}u\right)\right| + 30 + \dfrac{1}{1000}\left|\cos\left(\dfrac{2}{\sqrt{t-t^2}}\right),\right| & \text{if } 2 \le u \le 768, \\ \left(\ln\left(\dfrac{3}{2}\right) + 30 + \dfrac{\sqrt{3}}{14}\right) + \dfrac{u}{10}|\sin(\pi u)| + \dfrac{1}{1000}\left|\cos\left(\dfrac{2}{\sqrt{t-t^2}}\right),\right| & \text{if } 768 \le u, \end{cases}$$

where $u_k = (u_{1,k}, u_{2,k})$ and $u = \sum_{i=1}^{2}\sum_{k=1}^{3} u_{i,k}$.

For $i = 1, 2$ and $v = \sum_{i=1}^{2} u_{i,1}$, we define the functions $h_{k,i}$ as follows:

$$h_{1,i}(u_1) = \begin{cases} \dfrac{\ln(1+v)}{100\,i\,\ln(i+2)\,(v+1)}, & \text{if } t \in [0,1], 0 \le v \le 2, \\ \dfrac{\ln(3)}{300\,i\,\ln(i+2)}, & \text{if } t \in [0,1], 2 \le v, \end{cases}$$

$$h_{2,i}(u_1) = \begin{cases} \dfrac{v\,e^{v}}{40\,i\,e^{2i}(v+1)}, & \text{if } t \in [0,1], 0 \le v \le 2, \\ \dfrac{v\,e^{2}}{120\,i\,e^{2i}}, & \text{if } t \in [0,1], 2 \le v, \end{cases}$$

$$h_{3,i}(u_1) = \begin{cases} \dfrac{7}{2250\ |\sin i|} \times \dfrac{v\ln(2)}{2(1+v)\left(\sqrt{iv}+1\right)}, & \text{if } t \in [0,1], 0 \le v \le 2, \\ \dfrac{7}{2250\,|\sin i|} \times \dfrac{2\ln(2)}{6\left(\sqrt{2i}+1\right)}, & \text{if } t \in [0,1], 2 \le v, \end{cases}$$

and we consider the following boundary value problem:

$$\begin{cases} u_1^{(4)}(t) + f_1(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s)) = 0,\ 0 < t < 1, \\ u_2^{(4)}(t) + f_2(t, \mathbf{u}(s), \mathbf{u}'(s), \mathbf{u}''(s)) = 0,\ 0 < t < 1, \\ u_1(0) = h_{1,1}(\psi_1[u_1], \psi_1[u_2]), \\ u_2(0) = h_{1,2}(\psi_1[u_1], \psi_1[u_2]), \\ u_1'(0) = h_{2,1}(\psi_2[u_1], \psi_1[u_2]), \\ u_2'(0) = h_{2,2}(\psi_2[u_1], \psi_1[u_2]), \\ u_1''(0) = u_2''(0) = 0, \\ u_1''(1) = \sum_{j=1}^{2} \beta_{j,i} u_1''(\eta_{j,1}) + h_{3,1}(\psi_3[u_1], \psi_3[u_2]), \\ u_2''(1) = \sum_{j=1}^{2} \beta_{j,i} u_2''(\eta_{j,2}) + h_{3,2}(\psi_3[u_1], \psi_3[u_2]). \end{cases} \tag{4.1}$$

We shall apply Theorem 3.3 in the following special cases

$a = 1$, $b = 2$, $c = 844.8$, $\theta = \frac{1}{4}$, $\gamma(\theta) = \frac{\theta^3}{6}$, $\frac{b}{\gamma(\theta)} = 768$, $m_0(\theta) = \frac{1}{1536}$, $m_1(\theta) = \frac{1}{128}$, $m_2(\theta) = \frac{1}{16}$, $\psi_1[1] = 1$, $\psi_2[1] = 2$, $\psi_3[1] = 3$, $\beta_{j,i} = \frac{i}{5}$, $\eta_{j,i} = \frac{i}{6}$, $K_1 = \frac{15}{14}$, $K_2 = \frac{15}{11}$, $K_3 = \frac{5}{2}$, $M_{0,1} = \frac{17}{336}$, $M_{1,1} = \frac{37}{336}$, $M_{2,1} = \frac{5}{28}$, $M_1 = \frac{5}{28}$, $M_{0,2} = \frac{17}{264}$, $M_{1,2} = \frac{5}{33}$, $M_{2,2} = \frac{23}{88}$, $M_2 = \frac{23}{88}$, $L_{1,1} = 1$, $L_{2,2} = 1$, $L_{3,1} = \frac{1}{2}$, $L_{1,2} = \frac{1}{2}$, $L_{2,1} = \frac{14}{45}$, $L_{3,2} = \frac{14}{45}$, $q_{1,i} = 100\ \ln(i+2)$, $q_{2,i} = 20i$, $q_{3,i} = 100\,|\sin i|$ and $p_i = e^i$.

We can easily know that the following statements hold:

1. By calculating we have

$$\sum_{i=1}^{2} \frac{1}{p_i} + \frac{5}{3q_{3,i}} + \frac{2}{q_{2,i}} + \frac{1}{q_{1,i}} = 0.707666 \le 1,$$

and also we have: $\dfrac{L_{1,1}}{q_{1,1}} = \dfrac{1}{100\ln 3}$, $\dfrac{L_{1,2}}{q_{1,2}} = \dfrac{1}{200\ln 2}$, $\dfrac{L_{2,1}}{q_{2,1}} = \dfrac{1}{40}$, $\dfrac{L_{2,2}}{q_{2,2}} = \dfrac{1}{80}$, $\dfrac{L_{3,1}}{q_{3,1}} = \dfrac{7}{2250\,|\sin 1|}$ and $\dfrac{L_{3,2}}{q_{3,2}} = \dfrac{7}{2250\,|\sin 2|}$.

2. f_1 satisfies the following conditions:
   - $f_1(t, u_1, u_2, u_3) \le 0.41 \le \dfrac{a}{3p_1M_1} = 0.686708$ for all $u \in [0, 1]$.
   - $f_1(t, u_1, u_2, u_3) \ge 30 \ge \dfrac{b}{2\sum_{d=0}^{2} m_d(\theta)} = 28.1835$ for all $u \in [2, 768]$.
   - $f_1(t, u_1, u_2, u_3) \le 367.931 \le \frac{c}{3p_1M_1} = 580.131$ for all $u \in [0, 844.8]$.

- $\int_0^1 f_1(s, x, y, z)\, ds < +\infty$ for any $x, y, z \in [0, +\infty)$.

3. f_2 satisfies the following conditions:

   - $f_2(t, u_1, u_2, u_3) \leq \frac{\sqrt{3}}{14} = 0.124718 \leq \frac{a}{3p_2M_2} = 0.172602$ for all $u \in [0, 1]$.
   - $f_2(t, u_1, u_2, u_3) \geq 30 \geq \frac{b}{2\sum_{d=0}^{2} m_d(\theta)} = 28.1835$ for all $u \in [2, 768]$.
   - $f_2(t, u_1, u_2, u_3) \leq \ln\left(\frac{3}{2}\right) + 110.001 + \frac{\sqrt{3}}{14} \leq \frac{c}{3p_2M_2} = 213.418$ for all $u \in [0, 844.8]$.
   - $\int_0^1 f_2(s, x, y, z)\, ds < +\infty$ for any $x, y, z \in [0, +\infty)$.

4. $h_{k,i}$ satisfies the following conditions: $h_{1,i}(u_1) \leq \frac{L_{1,i}}{q_{1,i}} v$, $h_{2,i}(u_1) \leq \frac{L_{2,i}}{q_{2,i}} v$ and $h_{3,i}(u_1) \leq \frac{L_{3,i}}{q_{3,i}} v$.

Hence, all assumptions of Theorem 3.3 hold. Then, Theorem 3.3 implies that *problem* (4.1) has at least three positive solutions $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ with $\|\mathbf{u}_1\| < 1$, $\beta(\mathbf{u}_2) > 2$ and $\|\mathbf{u}_3\| > 1$ with $\beta(\mathbf{u}_3) < 2$.

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