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On identifying fuzzy knees in fuzzy multi-criteria optimization problems

Debdas Ghosh[1]

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Abstract
This paper introduces and analyzes the idea of *fuzzy knee* in fuzzy multi-criteria optimization problems. The fuzzy decision feasible region of the problem is constructed under a fuzzy inequality relation that is defined with the help of *same points* in fuzzy geometry. Then, fuzzy criteria feasible region is obtained through the image of the fuzzy decision feasible region by the criteria-vector-valued mapping. For the constructed fuzzy criteria feasible region, we define *fuzzy knee* and then propose a method to capture the fuzzy knee regions, along with the complete fuzzy Pareto set. All the studied ideas and methodologies are supported with suitable examples and pictorial illustrations. An engineering application of the presented method is also given.

Keywords Fuzzy multi-criteria optimization · Same points · Fuzzy inequality · Fuzzy knee

Mathematics Subject Classification 90C70 · 90C29

1 Introduction

In the practical decision making problems, it is mostly observed that a set of conflicting multiple criteria are to be optimized simultaneously. Due to the conflicting nature of the criteria, their optima are evidently attained at different points. Thus, towards the solution concept for multiple-criteria optimization problems (MOPs), the idea of Pareto solution has been introduced [20]. The study on MOPs eventually involves analyzing trade-off between the criteria on a set of Pareto solutions or on a set of satisfiable solutions to the decision maker (DM).

Over the last few decades, many classical methods have been introduced to capture the Pareto solution set of an MOP, such as, weighted sum, ϵ-constraint, normal boundary intersection, normal constraint, direct search domain, ideal cone, etc. All these classical methods attempt to capture the complete Pareto set of MOPs. However, final selection of the problem relies on DM's subjective preference. This final solution is generally singleton. In order to

✉ Debdas Ghosh
debdas.mat@iitbhu.ac.in

1 Department of Mathematical Sciences, Indian Institute of Technology (BHU), Varanasi, Uttar Pradesh 221005, India

guess which solution might be most preferable for a DM, a concept called *knee* of the Pareto set, in the criterion space, has been studied by Das [7], Branke et al. [3], Rachmawati and Srinivasan [21], and Deb and Gupta [9]. However, all these classical methods to solve MOPs are not enough to handle all practical problems because often real-world situations cannot be modeled precisely [17].

In order to deal with imprecise nature of multiple criteria decision making problems, fuzzy multi-criteria optimization problems (FMOPs) are extensively studied after the seminal work by Bellman and Zadeh [2]. Several attempts have been made thereafter to obtain a compromise solution of FMOPs; for instance, see the references [1,4,16,18,19,22,23,25,26], and the references therein. In the literature on solving FMOPs, commonly, the DM ends up with a conventional MOP to get a compromise solution or most preferable solution to the DM. A detailed insightful survey and methodologies on fuzzy multiple-objective decision making can be obtained in [17,27].

In this paper, an attempt is made to obtain fuzzy Pareto set of FMOPs. On solving FMOPs, at first, the fuzzy decision feasible region is constructed under the concept of *same points* [11] in fuzzy geometry [6,11,15]. Next, under the assumption of precise criteria with crisp decision variables, decision feasible region is transferred to criterion space through vector criteria mapping. As the decision feasible region is fuzzy, the criteria feasible region is evidently turns out fuzzy. In the proposed methodology, the entire fuzzy Pareto set along with the newly introduced *fuzzy knees* of FMOPs is obtained using α-cuts of the criteria feasible region. Delineation of the presented work is as follows.

The required preliminaries on fuzzy set theory and on MOPs are given in the immediately next section. A simple technique, the Ideal Cone method [12–14], to obtain the Pareto set of MOPs is briefly sketched in Sect. 3. Section 4 demonstrates the construction procedure of fuzzy decision and fuzzy criteria feasible regions with the help of *same points*. The Sect. 4 also proposes definitions of fuzzy Pareto point, generalized fuzzy Pareto point, fuzzy knee and generalized fuzzy knee for FMOPs. A method to obtain fuzzy Pareto set of FMOPs and its knees are also given in Sect. 4. Two illustrative numerical examples and an application are presented in the Sect. 5. Section 7 includes a brief conclusion and future work of the proposed study.

2 Preliminaries

In this section, the necessary definitions and terminologies which are used throughout this paper, are given. The definitions regarding MOPs are taken from [10] and definitions concerning fuzzy set theory are adopted from [11,23].

2.1 Fuzzy set

Definition 1 (*Fuzzy set* [23]) Let X be a classical set of elements which should be evaluated with regard to a fuzzy statement. Then the set of order pairs

$$\widetilde{A} = \{(x,\ \mu(x|\widetilde{A})) : x \in X\},\ \text{where}\ \mu : X \to [0, 1],$$

is called a fuzzy set in X. The evaluation function $\mu(x|\widetilde{A})$ called the membership function of the fuzzy set $\widetilde{A}$.

Definition 2 (*α-cut of a fuzzy set* [11]) For a fuzzy set $\widetilde{A}$ of $\mathbb{R}^n$, an α-cut of $\widetilde{A}$ is denoted by $\widetilde{A}(\alpha)$ and is defined by

$$\widetilde{A}(\alpha) = \begin{cases} \{x : \mu(x|\widetilde{A}) \geq \alpha\} & \text{if } 0 < \alpha \leq 1 \\ closure\{x : \mu(x|\widetilde{A}) > 0\} & \text{if } \alpha = 0. \end{cases}$$

The sets $\{x : \mu(x|\widetilde{A}) > 0\}$ and $\{x : \mu(x|\widetilde{A}) = 1\}$ are called *support* and *core*, respectively, of the fuzzy set $\widetilde{A}$.

In order to represent the construction of membership function of a fuzzy set $\widetilde{A}$, the notation $\bigvee\{x : x \in \widetilde{A}(0)\}$ is frequently used, which means $\mu(x|\widetilde{A}) = \sup\{\alpha : x \in \widetilde{A}(\alpha)\}$.

Definition 3 (*Fuzzy numbers* [28]) A fuzzy set $\widetilde{A}$ of the real line $\mathbb{R}$ is called a fuzzy number if:

(i) $\widetilde{A}$ is convex, i.e., $\mu\big(\lambda x_1 + (1-\lambda)x_2 \big| \widetilde{A}\big) \geq \min\{\mu(x_1|\widetilde{A}),\ \mu(x_2|\widetilde{A})\}$ for $x_1, x_2 \in \mathbb{R}$ and for all $\lambda \in [0, 1]$,
(ii) there is exactly one $x_0 \in \mathbb{R}$ with $\mu(x_0|\widetilde{A}) = 1$, and
(iii) $\mu(x|\widetilde{A})$ is piece-wise continuous.

Definition 4 (*LR-type fuzzy number* [28]) A function $L : [0, +\infty) \to [0, 1]$ which is non-increasing and satisfies either of the following two

(i) $L(0) = 1$ and $L(1) = 0$
(ii) $L(x) > 0$ for x in $[0, +\infty)$ and $L(+\infty) = 0$

is called a reference function of a fuzzy number.

A fuzzy number $\widetilde{A}$ is called an LR-type fuzzy number if there exist a pair of reference functions L and R, and two positive numbers α and β such that $\mu(x|\widetilde{A})$ can be expressed by

$$\mu(x|\widetilde{A}) = \begin{cases} L(\frac{m-x}{\alpha}) & \text{if } x \leq m \\ R(\frac{x-m}{\beta}) & \text{if } x \geq m. \end{cases}$$

Symbolically, the notation $(m-\alpha/m/m+\beta)_{LR}$ is used to represent an LR-type fuzzy number.

In particular, if $L(x) = R(x) = \max\{0, 1-|x|\}$, then the fuzzy number $\widetilde{A}$ is called a *triangular fuzzy number* and it is denoted by $(m-\alpha/m/m+\beta)$.

Definition 5 (*Same points* [11]) Let x and y be two numbers belonging to the supports of two continuous fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$, respectively. The numbers x and y are said to be same points with respect to $\widetilde{a}$ and $\widetilde{b}$ if

(i) $\mu(x|\widetilde{a}) = \mu(y|\widetilde{b})$ and
(ii) $x \leq a$ and $y \leq b$, or $x \geq a$ and $y \geq b$, where a and b are the cores of $\widetilde{a}$ and $\widetilde{b}$, respectively.

In the next, we give a brief idea on MOPs.

2.2 Conventional MOPs

In mathematical notions, MOPs are defined in the following way

$$\min_{x \in \mathcal{X}} f(x) = \big(f_1(x), f_2(x), \ldots, f_k(x)\big)^T, k \geq 2, \tag{2.1}$$

where $\mathcal{X} = \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0, a \leq x \leq b\}$ is the feasible set; $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^s$ are vector-valued functions; the constant vectors a and b are lower and upper bound, respectively, of the decision vector $x = (x_1, x_2, \ldots, x_n)^T$.

We denote the image of the decision feasible set $\mathcal{X}$ under the criteria-vector-valued mapping f by $\mathcal{Y} := f(\mathcal{X})$. Therefore, $\mathcal{Y}$ is the feasible set in the criterion space. If for each individual i in $\{1, 2, \ldots, k\}$, x_i^* is the point of global minima of the function f_i, the point $y_i^* := f(x_i^*)$, for each $i = 1, 2, \ldots, k$, in the criterion space is said to be an *anchor point*. The point $f^* = (f_1^*, f_2^*, \ldots, f_k^*)^T$, where $f_i(x_i^*) = f_i^*$, is called the *ideal point* or *utopia point*. Without loss of generality, let us redefine $f(x)$ as $f(x) - f^*$. Then,

(i) all criteria will be positive-valued with global minimum value zero,
(ii) the criteria feasible set $\mathcal{Y}$ must be a subset of $\mathbb{R}^k_{\geqq} := \{y \in \mathbb{R}^k : y \geqq 0\}$,
(iii) the origin of $\mathbb{R}^k$ is the ideal point, and
(iv) the anchor points corresponding to i-th criterion must lie on the plane perpendicular to the axis of f_i.

As, in general, the ideal point f^* is not attainable by f, the notion of Pareto optimality being introduced as follows. The definition of weak Pareto optimality is also given subsequently.

Definition of Pareto optimality depends on a dominance structure or componentwise order in the space $\mathbb{R}^k$. In order to represent dominance structure on $\mathbb{R}^k$, the following subsets are usually used. The non-negative orthant of $\mathbb{R}^k$ is represented by $\mathbb{R}^k_{\geqq} := \{y \in \mathbb{R}^k : y \geqq 0\}$. The notation $y \geqq 0$ implies $y_i \geq 0$ for each $i = 1, 2, \ldots, k$. The set $\mathbb{R}^k_{\geq}$ is defined by $\{y \in \mathbb{R}^k : y \geq 0\}$ where $y \geq 0$ means $y \geqq 0$ but $y \neq 0$. The notation $\mathbb{R}^k_{>} := \{y \in \mathbb{R}^k : y > 0\}$ indicates the positive orthant of $\mathbb{R}^k$. Here, $y > 0$ stands for $y_i > 0$ for each $i = 1, 2, \ldots, k$. The relations '$\leqq$', '$\leq$' and '$<$' are defined by: '$y \leqq 0$ if and only if $-y \geqq 0$', '$y \leq 0$ if and only if $-y \leq 0$' and '$y < 0$ if and only if $-y > 0$'. For $\hat{x}, \bar{x} \in \mathcal{X}$, the vector $f(\hat{x})$ is said to dominate another vector $f(\bar{x})$ if $f(\hat{x}) \leq f(\bar{x})$.

Definition 6 (*Pareto optimality* [10]) A feasible solution $\hat{x} \in \mathcal{X}$ is called efficient or Pareto optimal, if there is no other $x \in \mathcal{X}$ such that $f(x) \leq f(\hat{x})$. If $\hat{x}$ is efficient, $f(\hat{x})$ is called non-dominated. The set of all efficient points is denoted by $\mathcal{X}_E$ and the collection of all non-dominated points by $\mathcal{Y}_N$.

Definition 7 (*Weak Pareto optimality* [10]) A feasible solution $\hat{x} \in \mathcal{X}$ is called weakly Pareto optimal if there is no $x \in \mathcal{X}$ such that $f(x) < f(\hat{x})$. The point $\hat{y} = f(\hat{x})$ is then called weakly non-dominated and $\hat{x}$ is called weakly Pareto optimal point.

In the following, a classical method [12–14] to obtain entire Pareto set, and its knees, of the MOP (2.1) is presented.

3 A method to obtain Pareto set and its knees in conventional MOP

In this section, a technique is presented to obtain Pareto points of MOP (2.1). The technique is confined under the following three noteworthy observations on Pareto optimality

- a point $\hat{x} \in \mathcal{X}$ is a Pareto optimal point if and only if

$$f(\mathcal{X}) \cap \left(f(\hat{x}) - \mathbb{R}^k_{\geqq}\right) = \{f(\hat{x})\},$$

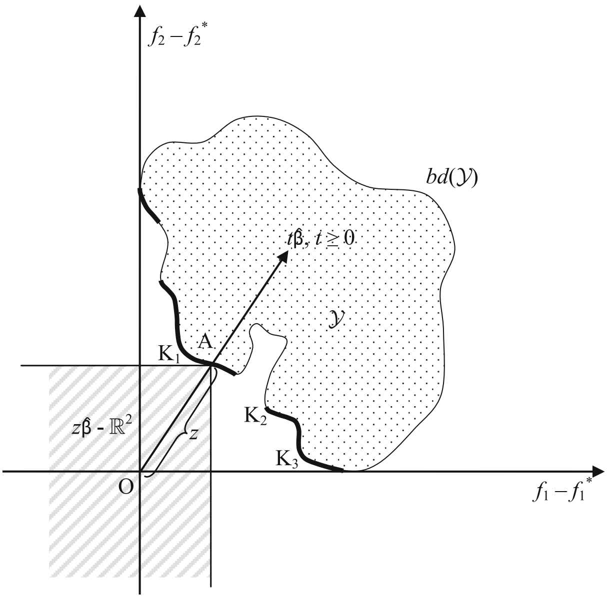

Fig. 1 Illustration of CM($\hat{\beta}$) for a bi-objective problem

- a point $\hat{x} \in \mathcal{X}$ is a weakly Pareto optimal if and only if

$$f(\mathcal{X}) \cap \left(f(\hat{x}) - \mathbb{R}^k_>\right) = \emptyset \text{ and}$$

- sets of non-dominated and weakly non-dominated points must be subsets of the boundary of the criterion feasible region, $bd(\mathcal{Y})$.

The first observation geometrically signifies that—if the criterion feasible region and the translated non-positive orthant $-\mathbb{R}^k_{\geqq}$ whose vertex is being shifted from origin to the point $f(\hat{x})$ have intersection a single point $f(\hat{x})$, then $\hat{x}$ is a Pereto optimal solution. Thus, in order to get a Pareto optimal solution, we may translate the cone of non-positive orthant of the criterion space along a particular direction $\hat{\beta} \in \mathbb{R}^k_{\geqq}$ until this cone touches the criterion feasible region.

If the cone $-\mathbb{R}^k_{\geqq}$ is translated along $\hat{\beta} \in \mathbb{R}^k_{\geqq}$, then it can touch the boundary of the criterion feasible region $\mathcal{Y}$ in two possible ways: either the vertex of the cone touches first or one (or several) boundary plane(s) of the cone touches first. If the first case, the point where the cone touches the criterion feasible region is certainly a global non-dominated point. In the second case, two possibilities may arise: either the touch portion is a single point or a set of points. In the first subcase, the touch point is a Pareto optimal point. In the second subcase, all the points except the extreme points of the touch portion are weakly Pareto optimal solutions [13].

Let us illustrate how the above said touch portion of $z\hat{\beta} - \mathbb{R}^k_{\geqq}$ and $bd(\mathcal{Y})$, for a particular direction $\hat{\beta} \in \mathbb{R}^k_{\geqq}$, can be found. To demonstrate, let us take a graphical perspective of a bi-objective optimization problem. Figure 1 portrays the criterion feasible region, the dotted region, $\mathcal{Y} = f(\mathcal{X})$ for a bi-objective problem and the cone $z\hat{\beta} - \mathbb{R}^k_{\geqq}$ for a specific value of $z = \overline{OA}$. Let us now consider the set $\left\{y : z\hat{\beta} \geqq f(x),\ y = f(x),\ x \in \mathcal{X}\right\}$, $z \in \mathbb{R}$. For each specific value of $z \in \mathbb{R}$, this set represents the intersecting region of $(z\hat{\beta} - \mathbb{R}^k_{\geqq})$ and $f(\mathcal{X})$. For generic $z \in \mathbb{R}$ let us try to reduce the intersecting region between $(z\hat{\beta} - \mathbb{R}^k_{\geqq})$ and $f(\mathcal{X})$ by translating the cone $(z\hat{\beta} - \mathbb{R}^k_{\geqq})$ along $\hat{\beta}$ in such a way that the cone does not leave $f(\mathcal{X})$. In the optimum situation if the intersection $(z\hat{\beta} - \mathbb{R}^k_{\geqq}) \bigcap f(\mathcal{X})$ contains only one point, then that single point is indeed a non-dominated point. We note that minimization of the intersecting region $(z\hat{\beta} - \mathbb{R}^k_{\geqq}) \bigcap f(\mathcal{X})$ eventually involve minimization of the value of z with the constraints $z\hat{\beta} \geqq f(x),\ x \in \mathcal{X}$. It is worthy to note that the above discussion does not depend on the number of criteria. Therefore, to get a non-dominated solution of the MOP (2.1) we solve the following minimization problem:

$$\text{CM}(\hat{\beta}) \quad \begin{cases} \min & z \\ \text{subject to} & z\hat{\beta} \geqq f(x), \\ & x \in \mathcal{X}. \end{cases} \tag{3.1}$$

By solving the problem (3.1) for various $\hat{\beta}$ in $\mathbb{R}^k_{\geqq} \cap \mathbb{S}^{k-1}$, the entire non-dominated set, eventually the weakly non-dominated set, of the considered MOP can be generated; $\mathbb{S}^{k-1}$ represents the k-dimensional unit sphere. It is to observe that any non-dominated point is attainable by the above constructed minimization problem (3.1) (see [13]). For instance, if $y_0 \in \mathcal{Y}_N$ then solution of $\text{CM}(\hat{\beta})$ corresponding to $\hat{\beta} = \frac{y_0}{\|y_0\|}$ is x_0 for which $y_0 = f(x_0)$. In the Fig. 1 we note that the solution of $\text{CM}(\hat{\beta})$ corresponding to $\hat{\beta} = \frac{\overrightarrow{OA}}{\|OA\|} \in \mathbb{R}^2_{\geqq} \cap \mathbb{S}^1$ is the point A which is a Pareto optimal solution of the considered problem. Varying $\hat{\beta}$ for all possible values on $\mathbb{R}^2_{\geqq} \cap \mathbb{S}^1$, all the points in the darken portions of $bd(\mathcal{Y})$ can be obtained. Collection of all these points is the complete Pareto set/non-dominated set of the problem.

Once the set of non-dominated points, i.e, $\mathcal{Y}_N$, is obtained, DM has to perform another decision making job to finally pick a solution from the entire Pareto optimal alternatives $\mathcal{Y}_N$. At this point, an often used process is the method of compromise programming or method of global criteria. In this method, DM has to fix a reference point and a distance metric. The reference point usually signifies the point that DM wishes to ideally obtain. However this ideal solution may not be feasible in the criterion space. Thus, DM may be trying to get a solution as much closer as possible to this ideal solution. Getting closer to reference point/ideal solution eventually imply the distance/deviation minimization of the set of alternatives from the reference point. Obviously this minimum deviation point is essentially member of the *maximum bulge* portion of the boundary of the criteria feasible region towards the ideal point. This maximum bulge is referred as *knee of the Pareto curve* [7]. If we choose reference point as the ideal point 'O', the origin of $\mathbb{R}^k$ space, and the distance metric, $d(x, y)$, as simply the Euclidean distance metric then knee points for MOP (2.1) can be obtained by solving the following minimization problem:

$$\min_{y \in \mathcal{Y}_N} d(O, y). \tag{3.2}$$

Local solutions of this minimization problem are called local knee points and global solution as global knee.

It can be easily perceived that local or global solution of the minimization problem (3.2) are local and global minimum of 'z' values of the subproblems $\text{CM}(\hat{\beta})$ in (3.1). Thus, collectively $\text{CM}(\hat{\beta})$ for all $\hat{\beta} \in \mathbb{R}^k \cap \mathbb{S}^{k-1}$ not only finds the entire $\mathcal{Y}_N$ but also generates all the knee points. In the Fig. 1, we observe that K_1 is the global knee and K_2 and K_3 are local knees of the considered problem.

In the next section, a study on FMOPs and finding its fuzzy Pareto set and fuzzy knee points is given.

4 Solving FMOPs

A general model of a fuzzy multi-criteria optimization problem is described in the following way:

$$\left.\begin{array}{ll} \min & f(x;\widetilde{c}_1,\widetilde{c}_2,\ldots,\widetilde{c}_k) = \Big(f_1(x;\widetilde{c}_1),\, f_2(x;\widetilde{c}_2),\ldots,\, f_k(x;\widetilde{c}_k)\Big)^T,\ k \geq 2 \\ \text{subject to} & \widetilde{C}_i : g_i(x;\widetilde{a}_i) \precsim \widetilde{b}_i,\ i = 1,\, 2, \ldots,\, m \\ & x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_{\geqq}, \end{array}\right\} \tag{4.1}$$

where $\widetilde{c}_j = (\widetilde{c}_{j1}, \widetilde{c}_{j2}, \ldots, \widetilde{c}_{jq_j})$, $j = 1, 2, \ldots, k$ and $\widetilde{a}_i = (\widetilde{a}_{i1}, \widetilde{a}_{i2}, \ldots, \widetilde{a}_{ip_i})$, $i = 1, 2, \ldots, m$. Here, each of $\widetilde{a}_{il}$ and $\widetilde{c}_{jr}$ is a fuzzy set. This paper investigates the FMOPs where all $\widetilde{a}_{il}$ and $\widetilde{b}_i$ fuzzy sets are fuzzy numbers and $\widetilde{c}_i$'s are crisp numbers. We also assume that f_j and g_i functions are continuous when their fuzzy coefficients $\widetilde{c}_j$ and $\widetilde{a}_i$ are assigned to be crisp numbers. Under these assumptions, all the criteria are then continuous crisp functions and $g_i(x;\widetilde{a}_i)$ are fuzzy numbers for each $x \in \mathbb{R}^n_{\geqq}$. Thus, the fuzzy inequality $\precsim$ in each $\widetilde{C}_i$ eventually depends on ordering two fuzzy numbers corresponding to each $x \in \mathbb{R}^n_{\geqq}$.

In this article, we take a new definition of fuzzy inequality $\precsim$ with the help of the concept of *same points* as follows:

$$\widetilde{C}_i : g_i(x;\widetilde{a}_i) \precsim \widetilde{b}_i \iff \bigvee_{\alpha\in[0,1]} \Big\{x : g_i(x; a_{i\alpha}) \ \leq \ b_{i\alpha}\Big\}, \tag{4.2}$$

where $a_{i\alpha} = (a_{i1\alpha}, a_{i2\alpha}, \ldots, a_{ip_i\alpha})$, $b_{i\alpha}$ are same points with respect to the fuzzy numbers $\widetilde{a}_i = (\widetilde{a}_{i1}, \widetilde{a}_{i2}, \ \ldots, \widetilde{a}_{ip_i})$ and $\widetilde{b}_i$. Therefore, the complete fuzzy constraint set, $\widetilde{\mathcal{X}}$ say, can be represented by the collection of crisp points $x \in \mathbb{R}^n_{\geqq}$ with varied membership values as follows:

$$\widetilde{\mathcal{X}} = \bigcap_{i=1}^{m} \bigvee_{\alpha\in[0,1]} \Big\{x \in \mathbb{R}^n_{\geqq} : g_i(x; a_{i\alpha}) \ \leq \ b_{i\alpha}\Big\}.$$

Following numerical example illustrates further detail.

Example 1 Let us consider the following fuzzy bi-criteria optimization problem:

$$\min \begin{pmatrix} \widetilde{2}x_1x_2 - \widetilde{3}x_2^2 \\ \widetilde{1}x_1^2 + \widetilde{4}\sin x_2 \end{pmatrix}$$
$$\text{subject to } \widetilde{C}_1 : \widetilde{2}x_1 + \widetilde{4}x_2 \precsim \widetilde{7},$$
$$x_1 \geq 0, x_2 \geq 0,$$

where $\widetilde{1} = (0.5/1/3)$, $\widetilde{2} = (1/2/3)$, $\widetilde{3} = (2/3/5)$, $\widetilde{4} = (3/4/5)$ and $\widetilde{7} = (5/7/8)$.

The number of criteria in this problem is $k = 2$;
$f_1(x;\widetilde{c}_1) = \widetilde{2}x_1x_2 - \widetilde{3}x_2^2, \widetilde{c}_1 = (\widetilde{2}, -\widetilde{3})$;
$f_2(x;\widetilde{c}_2) = \widetilde{1}x_1^2 + \widetilde{4}\sin x_2, \widetilde{c}_2 = (\widetilde{1}, \widetilde{4})$.

The problem has only one fuzzy constraint, i.e., $m = 1$;

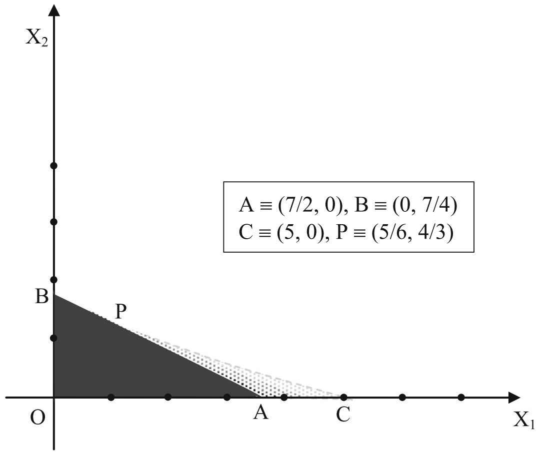

Fig. 2 Fuzzy constraint set of Example 1

$p_1 = 2$, $g_1(x; \widetilde{a}_1) = \widetilde{2}x_1 + \widetilde{4}x_2$, $\widetilde{a}_1 = (\widetilde{2}, \widetilde{4})$ and $\widetilde{b} = \widetilde{7}$.

For each $\alpha \in [0, 1]$, same points with membership value α with respect to the fuzzy numbers $\widetilde{2}$, $\widetilde{4}$ and $\widetilde{7}$ are $1 + \alpha$, $3 + \alpha$ and $5 + 2\alpha$ or $3 - \alpha$, $5 - \alpha$ and $8 - \alpha$, respectively.

Thus, the fuzzy set of the fuzzy inequality in the constraint set is determined by:
$\widetilde{2}x_1 + \widetilde{4}x_2 \precsim \widetilde{7} \Longleftrightarrow$

$$\bigvee_{\alpha \in [0,1]} \Big[\big\{x : (1+\alpha)x_1 + (3+\alpha)x_2 \leq (5+2\alpha)\big\} \cup \big\{x : (3-\alpha)x_1 + (5-\alpha)x_2 \leq (8-\alpha)\big\} \Big].$$

Therefore, the constraint set of the considered problem is:

$$\widetilde{\mathcal{X}} = \bigvee_{\alpha \in [0,1]} \Big[\Big\{x \in \mathbb{R}^2_{\geqq} : (1+\alpha)x_1 + (3+\alpha)x_2 \leq (5+2\alpha)\Big\} \cup \Big\{x \in \mathbb{R}^2_{\geqq} : (3-\alpha)x_1 + (5-\alpha)x_2 \leq (8-\alpha)\Big\} \Big].$$

The fuzzy set $\widetilde{\mathcal{X}}$ is depicted in the Fig. 2. Deeper dark shading portrays higher membership value. The core of $\widetilde{\mathcal{X}}$ is the black triangular region $\triangle OAB$ and support of $\widetilde{\mathcal{X}}$ is the interior and boundary of the region $OACPBO$. The co-ordinates of the specific points are given in the figure.

It is to note that $\widetilde{\mathcal{X}}$ is the intersecting region of $\mathbb{R}^2_{\geqq}$ and union of all the half-planes, that contains origin, of the lines $\frac{x_1}{\frac{5+2\alpha}{1+\alpha}} + \frac{x_2}{\frac{5+2\alpha}{3+\alpha}} = 1$ and $\frac{x_1}{\frac{8-\alpha}{3-\alpha}} + \frac{x_2}{\frac{8-\alpha}{5-\alpha}} = 1$. From the above mathematical expression of $\widetilde{\mathcal{X}}$, we observe that all the points which lie on $\Big\{x \in \mathbb{R}^2_{\geqq} : \frac{x_1}{\frac{5+2\alpha}{1+\alpha}} + \frac{x_2}{\frac{5+2\alpha}{3+\alpha}} \leq 1\Big\}$ or $\Big\{x \in \mathbb{R}^2_{\geqq} : \frac{x_1}{\frac{8-\alpha}{3-\alpha}} + \frac{x_2}{\frac{8-\alpha}{5-\alpha}} \leq 1\Big\}$ must have membership value greater than or equal to α on $\widetilde{\mathcal{X}}$. Therefore, x_1 and x_2-intercepts of the fuzzy linear inequality $\widetilde{C}_1$ may be determined by $\bigvee_{\alpha \in [0,1]} \big\{\frac{5+2\alpha}{1+\alpha}, \frac{8-\alpha}{3-\alpha}\big\}$ and $\bigvee_{\alpha \in [0,1]} \big\{\frac{5+2\alpha}{3+\alpha}, \frac{8-\alpha}{5-\alpha}\big\}$, respectively. These two intercepts determine two fuzzy numbers with support sets as $[\frac{8}{3}, 5]$ and $[\frac{8}{5}, \frac{7}{4}]$, respectively. The core of these fuzzy numbers are $\frac{7}{2}$ and $\frac{7}{4}$, respectively.

From the figure of $\widetilde{\mathcal{X}}$, we observe that on x_2-axis there is no imprecise part of $\widetilde{\mathcal{X}}$. However, on x_1-axis there is an imprecise part, namely AC, of $\widetilde{\mathcal{X}}$. Question may arise about why this has happened. Answer can be obtained from the simple observation that the core point of the fuzzy intercept of the fuzzy inequality on the x_2-axis is $\frac{7}{4}$ and all the points on $\bigcup_{\alpha\in[0,1]}\left\{\frac{5+2\alpha}{3+\alpha}\right\}=[\frac{5}{3},\frac{7}{4}]$ and $\bigcup_{\alpha\in[0,1]}\left\{\frac{8-\alpha}{5-\alpha}\right\}=[\frac{8}{5},\frac{7}{4}]$ are less than or equals to $\frac{7}{4}$. Thus, while taking union of all numbers $\frac{5+2\alpha}{3+\alpha}$ and $\frac{8-\alpha}{5-\alpha}$ with membership value α, no point greater than $\frac{7}{4}$ can be found with positive membership value. Yet question may be kept about the restriction for which some fuzzy part can be obtained towards both the axes. Answer to this question are obtained from the next theorem and its immediate corollary.

Theorem 1 *Let $\widetilde{C}$ be a fuzzy inequality of the form $\widetilde{a}_1 f_1(x)+\widetilde{a}_2 f_2(x)+\cdots+\widetilde{a}_p f_p(x)\,\widetilde{\leq}\,\widetilde{b}$, $x\in\mathbb{R}^n$, where $\widetilde{a}_i=(a_i-\gamma_i/a_i/a_i+\delta_i)_{LR}$, $i=1,2,\ldots,p$ and $\widetilde{b}=(b-\gamma/b/b+\delta)_{LR}$. If for each $i 1,2,\ldots,p$,*

(i) $\frac{b-\gamma}{a_i-\gamma_i}<\frac{b}{a_i}<\frac{b+\delta}{a_i+\delta_i}$ or $\frac{b-\gamma}{a_i-\gamma_i}>\frac{b}{a_i}>\frac{b+\delta}{a_i+\delta_i}$ and
(ii) $0\notin\widetilde{a}_i(0)$ and $\widetilde{b}(0)$,

then

(i) for each i, the set $\bigvee_{\alpha\in[0,1]}\left\{\frac{b_\alpha}{a_{i\alpha}}\right\}$, where $a_{i\alpha}$ and b_α are same points on the fuzzy numbers $\widetilde{a}_i$ and $\widetilde{b}$, respectively, constitutes a fuzzy number, and
(ii) the α-cut of $\widetilde{C}\bigcap\mathbb{R}^n_{\geqq}$, for each $\alpha\in[0,1]$, is the set $\left\{x\in\mathbb{R}^n_{\geqq}:\frac{f_1(x)}{\frac{b_\alpha}{a_{1\alpha}}}+\frac{f_2(x)}{\frac{b_\alpha}{a_{2\alpha}}}+\cdots+\frac{f_p(x)}{\frac{b_\alpha}{a_{p\alpha}}}\leq 1\right\}$.

Proof (i) Let us consider any i in $\{1,2,\ldots,p\}$.
The membership functions of $\widetilde{a}_i$ and $\widetilde{b}$ are, respectively,

$$\mu(t|\widetilde{a}_i)=\begin{cases}L\left(\frac{a_i-t}{\gamma_i}\right) & \text{if } a_i-\gamma_i\leq t\leq a_i,\\ R\left(\frac{t-a_i}{\delta_i}\right) & \text{if } a_i\leq t\leq a_i+\delta_i\end{cases}$$

and

$$\mu(t|\widetilde{b})=\begin{cases}L\left(\frac{b-t}{\gamma}\right) & \text{if } b-\gamma\leq t\leq b,\\ R\left(\frac{t-b}{\delta}\right) & \text{if } b\leq t\leq b+\delta.\end{cases}$$

Same points, with membership value α, with respect to $\widetilde{b}$ and $\widetilde{a}_i$ are $b_\alpha=b-\gamma L^{-1}(\alpha)$, $a_{i\alpha}=a_i-\gamma_i L^{-1}(\alpha)$ and $b_\alpha=b+\delta R^{-1}(\alpha)$, $a_{i\alpha}=a_i+\delta_i R^{-1}(\alpha)$, respectively. Therefore, the fuzzy set $\bigvee_{\alpha\in[0,1]}\{\frac{b_\alpha}{a_{i\alpha}}\}$ becomes $\bigvee_{\alpha\in[0,1]}\left\{\frac{b-\gamma L^{-1}(\alpha)}{a_i-\gamma_i L^{-1}(\alpha)},\frac{b+\delta R^{-1}(\alpha)}{a_i+\delta_i R^{-1}(\alpha)}\right\}$.

We show that under the given conditions of the theorem, this fuzzy set is a fuzzy number.

We note that the quantities $\frac{b-\gamma L^{-1}(\alpha)}{a_i-\gamma_i L^{-1}(\alpha)}$ and $\frac{b+\delta R^{-1}(\alpha)}{a_i+\delta_i R^{-1}(\alpha)}$ are well defined, since $0\notin\widetilde{a}_i(0),\widetilde{b}(0)$.

Let $\alpha,\beta\in[0,1]$ with $\beta\geq\alpha$. We first consider the left spreads of $\widetilde{a}_i$ and $\widetilde{b}$. We note that the inequality

$$\frac{b-\gamma L^{-1}(\alpha)}{a_i-\gamma_i L^{-1}(\alpha)}\leq\frac{b-\gamma L^{-1}(\beta)}{a_i-\gamma_i L^{-1}(\beta)}\leq\frac{b_i}{a}$$

holds true. Since $\frac{b-\gamma L^{-1}(\alpha)}{a_i-\gamma_i L^{-1}(\alpha)} > \frac{b-\gamma L^{-1}(\beta)}{a_i-\gamma_i L^{-1}(\beta)}$ implies that $b\gamma_i - a_i\gamma > 0$, which is contradictory to the assumption that $\frac{b-\gamma}{a_i-\gamma_i} < \frac{b}{a_i}$.
Similarly considering right spreads of $\widetilde{a}_i$ and $\widetilde{b}$, we obtain

$$\frac{b_i}{a} \leq \frac{b+\delta R^{-1}(\beta)}{a_i+\delta_i R^{-1}(\beta)} \leq \frac{b+\delta R^{-1}(\alpha)}{a_i+\delta_i R^{-1}(\alpha)}.$$

Thus,

$$\frac{b-\gamma L^{-1}(\alpha)}{a_i-\gamma_i L^{-1}(\alpha)} \leq \frac{b-\gamma L^{-1}(\beta)}{a_i-\gamma_i L^{-1}(\beta)} \leq \frac{b_i}{a} \leq \frac{b+\delta R^{-1}(\beta)}{a_i+\delta_i R^{-1}(\beta)} \leq \frac{b+\delta R^{-1}(\alpha)}{a_i+\delta_i R^{-1}(\alpha)}.$$

Hence, $\left\{\left[\frac{b-\gamma L^{-1}(\alpha)}{a_i-\gamma_i L^{-1}(\alpha)}, \frac{b+\delta R^{-1}(\alpha)}{a_i+\delta_i R^{-1}(\alpha)}\right]: \alpha \in [0,1]\right\}$ determines a class of nested intervals. Therefore, by Representation Theorem of Fuzzy Numbers [24, p. 129], the fuzzy set $\bigvee_{\alpha\in[0,1]}\{\frac{b_\alpha}{a_{i\alpha}}\}$ is a fuzzy number for each $i = 1, 2, \ldots, p$.

(ii) This part is directly followed from part (i).

□

Corollary 1 *Let* $\widetilde{C}$: $\widetilde{a}_1 x_1 + \widetilde{a}_2 x_2 + \cdots + \widetilde{a}_n x_n \mathrel{\widetilde{\leqq}} \widetilde{b}$ *be a fuzzy linear inequality, where* $\widetilde{a}_i = (a_i - \gamma_i/a_i/a_i + \delta_i)_{LR}$, $i = 1, 2, \ldots, n$ *and* $\widetilde{b} = (b - \gamma/b/b + \delta)_{LR}$. *If for each i,* $\frac{b-\gamma}{a_i-\gamma_i} < \frac{b}{a_i} < \frac{b+\delta}{a_i+\delta_i}$ *or* $\frac{b-\gamma}{a_i-\gamma_i} > \frac{b}{a_i} > \frac{b+\delta}{a_i+\delta_i}$ *and* $0 \notin \widetilde{a}_i(0), \widetilde{b}(0)$, *then*

(i) x_i*-intercept of the fuzzy linear inequality is a fuzzy number,* $i = 1, 2, \ldots, n$, *and*
(ii) for each $\alpha \in [0, 1]$, *the* α*-cut of the fuzzy set representing the part of the fuzzy linear inequality* $\widetilde{C}$ *on* $\mathbb{R}^n_{\geqq}$ *can be obtained by*

$$\left\{x \in \mathbb{R}^n_{\geqq} : \frac{x_1}{\frac{b_\alpha}{a_{1\alpha}}} + \frac{x_2}{\frac{b_\alpha}{a_{2\alpha}}} + \cdots + \frac{x_n}{\frac{b_\alpha}{a_{n\alpha}}} \leq 1\right\},$$

where $a_{i\alpha}$ *and* b_α *are same points on the fuzzy numbers* $\widetilde{a}_i$ *and* $\widetilde{b}$, *respectively.*

After the construction of fuzzy decision feasible region $\widetilde{\mathcal{X}}$, we construct the fuzzy criteria feasible region of the FMOP (4.1). We note that $\widetilde{\mathcal{X}}$ is a collection of crisp points $x \in \mathcal{X}(0)$ with varied membership values and the criteria are considered as crisp functions. Thus, if x is a decision feasible point with membership value α on $\widetilde{\mathcal{X}}$, then $f(x)$ must be a criteria feasible point and, by the sup-min composition of fuzzy set, the membership value of $f(x)$ on the fuzzy criteria feasible region $\widetilde{\mathcal{Y}} = f(\widetilde{\mathcal{X}})$ must be at least α. Here the sup composition is taken since $\widetilde{\mathcal{Y}}$ is the collection, the union, of all the points $y = f(x)$ where $x \in \mathcal{X}$. Thus,

$$\widetilde{\mathcal{Y}} = \bigvee_{\alpha\in[0,1]} \{f(x) : x \in \widetilde{\mathcal{X}}(\alpha)\}.$$

The membership function of the fuzzy criteria feasible region is given by $\mu(y|\widetilde{\mathcal{Y}}) = \sup\{\alpha : y = f(x), \mu(x|\widetilde{\mathcal{X}}) = \alpha\}$.

Under the continuity assumption on the functions f_j and g_i, we note that (see [24, p. 118 and p. 130])

$$f(\widetilde{\mathcal{X}}(\alpha)) = f(\widetilde{\mathcal{X}})(\alpha) = \widetilde{\mathcal{Y}}(\alpha).$$

Thus, corresponding to each $\alpha \in [0, 1]$, defining a crisp MOP, FMOP$_\alpha$ say, as follows

$$\text{FMOP}_\alpha \quad \begin{cases} \min & \left(f_1(x; c_1), f_2(x; c_2), \ldots, f_k(x; c_k)\right)^T, \ k \geq 2 \\ \text{subject to} & x \in \widetilde{\mathcal{X}}(\alpha), \end{cases} \tag{4.3}$$

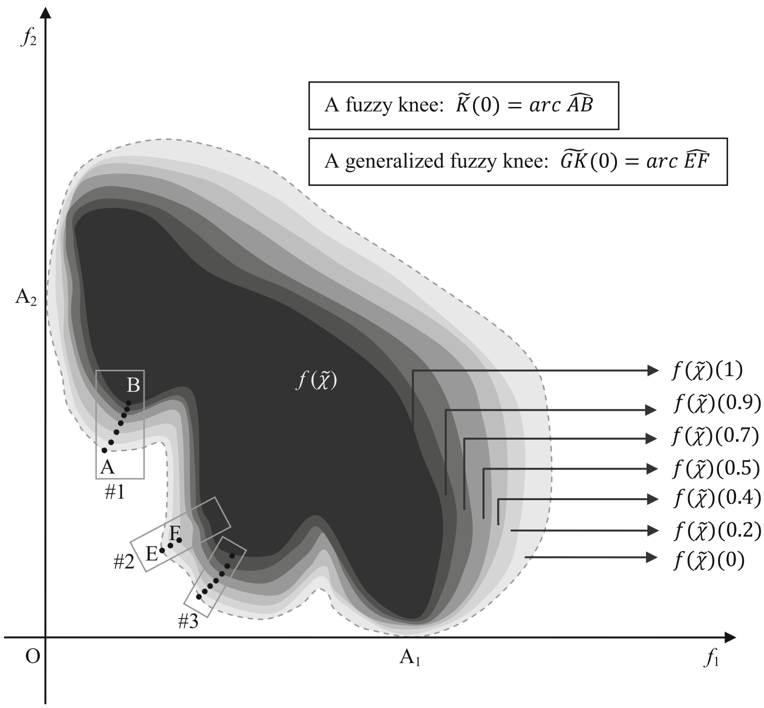

Fig. 3 Explaining fuzzy Pareto point and fuzzy knee point

we obtain the fuzzy decision/criteria feasible region of FMOP (4.1) identical to the union of all decision/criteria feasible region of FMOP$_\alpha$'s.

In the Fig. 3, the fuzzy criteria feasible region of an FMOP is shown. Varied darkness represents varied membership values; more deep dark represents higher membership value. The totally black region is the core of the fuzzy criteria feasible region. Corresponding to seven different values of $\alpha \in [0, 1]$, the set $f(\widetilde{\mathcal{X}})(\alpha)$ depicted in Fig. 3.

A question naturally arises which points/parts of Fig. 3 represent non-dominated points. In the next, we define a concept of fuzzy non-dominated points or fuzzy Pareto optimal points.

Definition 8 (*Fuzzy Pareto optimal point*) A fuzzy subset $\widetilde{P}$ of $\widetilde{\mathcal{Y}}(0)$ is said to be a fuzzy Pareto optimal point of FMOP (4.1) if

(i) $\widetilde{P}$ is a normal fuzzy set, i.e., there exists $p_0 \in \widetilde{P}$ such that $\mu(p_0|\widetilde{P}) = 1$,
(ii) $\mu(p|\widetilde{P})$ is upper semi-continuous[1], and
(iii) for any $p \in \widetilde{P}$, there exists $\alpha \in [0, 1]$ such that p is a Pareto optimal point of FMOP$_\alpha$.

Note 1 Core of a fuzzy Pareto optimal point is a Pareto optimal point of FMOP$_\alpha$ with $\alpha = 1$.

In the Fig. 3, the fuzzy arc $\widehat{AB}$, in the rectangle #1, is a fuzzy Pareto optimal point. Membership value of any point on the fuzzy arc $\widehat{AB}$ is identical to that on the fuzzy set $f(\widetilde{\mathcal{X}})$. Similarly the fuzzy arc in the rectangle #3 is a Pareto optimal point. However, we

[1] A real-valued function μ on a metric space M is called upper semi-continuous if for each real α, the set $\{x \in M : \mu(x) \geq \alpha\}$ is closed in M (see [5, p. 67]).

observe that the fuzzy arc $\overset{\frown}{EF}$, though meets the conditions (ii) and (iii) of the Definition 8, it does not meet the normality condition (i). Hence, fuzzy arc $\overset{\frown}{EF}$ is not a fuzzy Pareto point. We can say this type of fuzzy arc as a generalized fuzzy Pareto optimal point. Mathematically, generalized fuzzy Pareto point is defined as follows.

Definition 9 (*Generalized fuzzy Pareto optimal point*) A fuzzy subset $\widetilde{GP}$ of $\widetilde{\mathcal{Y}}(0)$ is said to be a generalized fuzzy Pareto optimal point of FMOP (4.1) if

(i) $\mu(p|\widetilde{GP})$ is upper semi-continuous, and
(ii) for any $p \in \widetilde{GP}$, there exists $\alpha \in [0, 1]$ such that p is a Pareto optimal point of FMOP$_\alpha$.

Note 2 The term *generalized* is analogous to the concept of fuzzy number and generalized fuzzy number. Generalized fuzzy number differs from fuzzy number in the condition of normality—a normal generalized fuzzy number is a fuzzy number. Likewise, a normal generalized fuzzy Pareto optimal point is a fuzzy Pareto optimal point.

In order to capture the complete fuzzy Pareto set or non-dominated set in an FMOP, it is natural to take union of its all possible fuzzy Pareto points and generalized fuzzy Pareto points. However it can be easily perceived that if $\widetilde{\mathcal{Y}}_N$ is the set of all the Pareto points and $\widetilde{\mathcal{Y}}_{GN}$ is the set of all generalized fuzzy Pareto points then $\widetilde{\mathcal{Y}}_{GN} \subseteq \widetilde{\mathcal{Y}}_N$.[2] Obviously, $\widetilde{\mathcal{X}}_{GE} = f^{-1}(\widetilde{\mathcal{Y}}_{GN}) \subseteq f^{-1}(\widetilde{\mathcal{Y}}_N) = \widetilde{\mathcal{X}}_E$. Thus, to find the complete non-dominated points in an FMOP, we only have to obtain $\widetilde{\mathcal{X}}_E$.

Again if $\mathcal{X}_{E\alpha}$ is the set of Pareto points of FMOP$_\alpha$, then according to the mathematical formulation of FMOP$_\alpha$, following result holds true

$$\widetilde{\mathcal{X}}_E = \bigvee_{\alpha \in [0,1]} \mathcal{X}_{E\alpha}. \tag{4.4}$$

Therefore, to obtain the entire $\widetilde{\mathcal{X}}_E$, we need to evaluate each $\mathcal{X}_{E\alpha}$ for all α in [0, 1]. The $\mathcal{X}_{E\alpha}$'s can be obtained by solving FMOP$_\alpha$ with the help of the classical method in the Sect. 3. Then, the relation (4.4) will be applied.

Once the fuzzy Pareto set $\widetilde{\mathcal{X}}_E$ is being evaluated, the next step will be the final selection of the solution from fuzzy Pareto optimal set of the problem. As explained by different authors [3,7–9,21] knee regions in classical MOP are most interesting or preferable to DM. In FMOP also, the concept of fuzzy knee seems to be promising. Let us now mathematically define fuzzy knee for FMOP (4.1).

Definition 10 (*Fuzzy knee*) A fuzzy set $\widetilde{K}$ of $\widetilde{\mathcal{Y}}(0)$ is said to be a fuzzy knee in FMOP (4.1) if:

(i) $\widetilde{K}$ is a normal fuzzy set, i.e., there exists $y_0 \in \widetilde{\mathcal{Y}}$ such that $\mu(y_0|\widetilde{K}) = 1$,
(ii) $\mu(y|\widetilde{K})$ is upper semi-continuous, and
(iii) for any $y \in \widetilde{K}$, there exists $\alpha \in [0, 1]$ such that y is a knee of FMOP$_\alpha$.

Note 3 The core of a fuzzy knee is a knee of FMOP$_1$.

In the Fig. 3, the fuzzy arc $\overset{\frown}{AB}$, (rectangle #1) is a fuzzy knee. Membership value of any point on the fuzzy arc $\overset{\frown}{AB}$ is same as on the fuzzy set $f(\widetilde{\mathcal{X}})$. Similarly, the fuzzy arc in the rectangle #3 is a fuzzy knee. However, the fuzzy arc $\overset{\frown}{EF}$ in the rectangle #2, though meets

[2] For two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ in X, the relation $\widetilde{A} \subseteq \widetilde{B}$ holds when $\mu(x|\widetilde{A}) \leq \mu(x|\widetilde{B})\ \forall x \in X$.

(ii) and (iii) conditions of the Definition 10, but it does not satisfy the condition of normality. Hence, fuzzy arc $\widehat{EF}$ is not a fuzzy knee. We may call this type of fuzzy arc as generalized fuzzy knee. Mathematically, generalized fuzzy knee is defined as follows.

Definition 11 (*Generalized fuzzy knee*) A fuzzy set $\widetilde{GK}$ of $\widetilde{\mathcal{Y}}(0)$ is said to be a generalized fuzzy knee of the FMOP (4.1) if

(i) $\mu(y|\widetilde{GK})$ is upper semi-continuous and
(ii) for any $y \in \widehat{GP}$, there exists $\alpha \in [0, 1]$ such that y is a knee of FMOP$_\alpha$.

In order to obtain the fuzzy knees in FMOP, it is natural to consider the union of its all possible fuzzy knees and generalized fuzzy knees. It is easy to see here that if $\widetilde{\mathcal{Y}}_K$ and $\widetilde{\mathcal{Y}}_{GK}$ are the set of fuzzy knees and the set of generalized fuzzy knees, respectively, then according to the mathematical formulation of FMOP$_\alpha$, the following result holds true

$$\widetilde{\mathcal{Y}}_K \bigcup \widetilde{\mathcal{Y}}_{GK} = \bigvee_{\alpha\in[0,1]} \left\{ y_{K\alpha} : y_{K\alpha} \text{ is a local solution of } \min_{y\in\mathcal{Y}_{N\alpha}} d(O, y) \right\}, \tag{4.5}$$

where $\mathcal{Y}_{N\alpha}$ is the non-dominated set of FMOP$_\alpha$. Once $\widetilde{\mathcal{Y}}_K \bigcup \widetilde{\mathcal{Y}}_{GK}$ is generated, the final decision making becomes easier since final solution is likely to be appeared in $\widetilde{\mathcal{Y}}_K \bigcup \widetilde{\mathcal{Y}}_{GK}$. Hence, DM may like to choose a point from $\widetilde{\mathcal{Y}}_K \bigcup \widetilde{\mathcal{Y}}_{GK}$ as the final solution, instead of from the set $\widetilde{\mathcal{Y}}_N$.

In order to illustrate the proposed method, in the next section two numerical examples are given.

5 Numerical illustrations

Example 2 We consider the following fuzzy bi-criteria optimization problem

$$\min \begin{pmatrix} (x_1-2)^2 + (x_2-2)^2 \\ \\ \frac{(x_1-4)^2}{2} + \frac{x_2^2}{4} \end{pmatrix}$$

$$\begin{aligned} \text{subject to } \widetilde{C}_1 :\ & (0.5/1/1.5)x_1 + (1/3/4)x_2 \precsim (1/3/6), \\ \widetilde{C}_2 :\ & (2/2.5/3)x_1 + (0.5/1/2)x_2 \precsim (2/2.5/6), \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

At first, we determine the fuzzy decision set $\widetilde{\mathcal{X}} = \widetilde{C}_1 \bigcap \widetilde{C}_2 \bigcap \mathbb{R}^2_{\geqq}$.

According to Eq. (4.2), the fuzzy constraint sets $\widetilde{C}_1$ is determined by

$$\widetilde{C}_1 \equiv \bigvee_{\alpha\in[0,1]} \Big\{ x \in \mathbb{R}^2 : 0.5(1+\alpha)x_1 + (1+2\alpha)x_2 \leq (1+2\alpha) \\ \text{or } \ 0.5(3-\alpha)x_1 + (4-\alpha)x_2 \leq 3(2-\alpha) \Big\}.$$

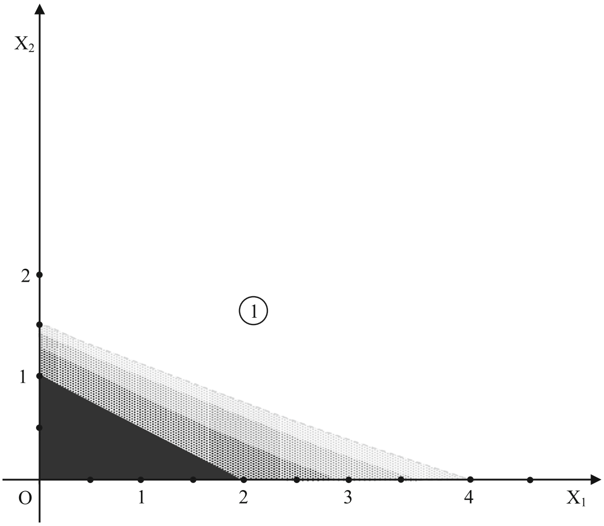

Fig. 4 Fuzzy set $\widetilde{C}_1 \bigcap \mathbb{R}^2_{\geqq}$ of Example 2

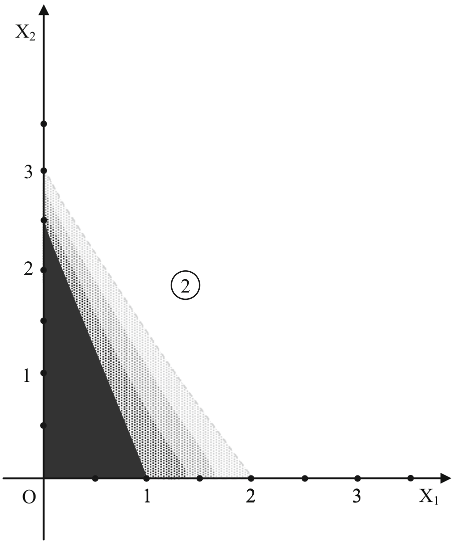

Fig. 5 Fuzzy set $\widetilde{C}_2 \bigcap \mathbb{R}^2_{\geqq}$ of Example 2

The supports of x_1- and x_2-intercepts of $\widetilde{C}_1$ are

$$\bigcup_{\alpha \in [0,1]} \left\{ \frac{1+2\alpha}{0.5(1+\alpha)}, \frac{3(2-\alpha)}{0.5(3-\alpha)} \right\} = [2, 4] \text{ and } \bigcup_{\alpha \in [0,1]} \left\{ 1, \frac{3(2-\alpha)}{4-\alpha} \right\} = [1, 1.5],$$

respectively.

The fuzzy set $\widetilde{C}_1 \bigcap \mathbb{R}^2_{\geqq}$ is depicted in Fig. 4. The core of $\widetilde{C}_1 \bigcap \mathbb{R}^2_{\geqq}$ is depicted by the deep dark region and its imprecise part is shown by grey shading.

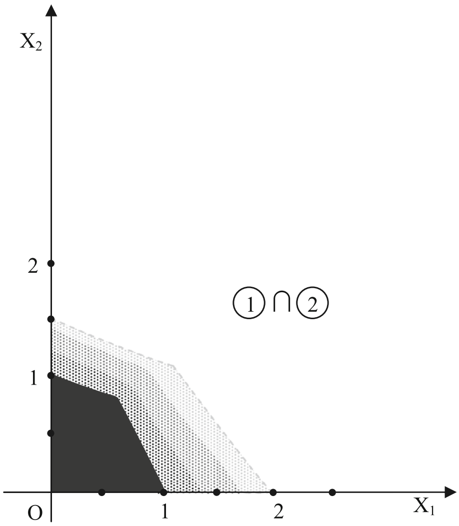

Fig. 6 Fuzzy set $\widetilde{\mathcal{X}}$ of Example 2

Similarly, according to Eq. (4.2), the fuzzy constraint sets $\widetilde{C}_2$ is determined by

$$\widetilde{C}_2 \equiv \bigvee_{\alpha\in[0,1]} \Big\{x \in \mathbb{R}^2 :\ (2+0.5\alpha)x_1 + 0.5(1+\alpha)x_2 \leq (2+0.5\alpha) \\ \text{or}\quad (3-0.5\alpha)x_1 + (2-\alpha)x_2 \leq (6-3.5\alpha)\Big\}.$$

The supports of x_1 and x_2-intercept of $\widetilde{C}_2$ are

$$\bigcup_{\alpha\in[0,1]} \left\{1, \frac{6-3.5\alpha}{3-0.5\alpha}\right\} = [1, 2] \ \text{ and } \bigcup_{\alpha\in[0,1]} \left\{\frac{2+0.5\alpha}{0.5(1+\alpha)}, \frac{6-3.5\alpha}{2-\alpha}\right\} = [2.5, 3],$$

respectively. The fuzzy set $\widetilde{C}_2 \bigcap \mathbb{R}^2_{\geqq}$ is depicted in the Fig. 5. Core of $\widetilde{C}_2 \bigcap \mathbb{R}^2_{\geqq}$ is depicted by the deep dark region and its imprecise part is shown by grey shading.

Entire decision feasible region $\widetilde{\mathcal{X}} = \widetilde{C}_1 \bigcap \widetilde{C}_2 \bigcap \mathbb{R}^2_{\geqq}$ is portrayed in Fig. 6. For each $\alpha \in [0, 1]$, the α-cut of the decision constraint set, i.e., $\widetilde{\mathcal{X}}(\alpha)$ is the set

$$\Big\{x \in \mathbb{R}^2_{\geqq} : (1.5-0.5\alpha)x_1 + (4-\alpha)x_2 \leq 6-3\alpha\Big\} \\ \bigcap \Big\{x \in \mathbb{R}^2_{\geqq} : (2+0.5\alpha)x_1 + 0.5(1+\alpha)x_2 \leq 2+0.5\alpha\Big\}.$$

Therefore, according to the formulation of FMOP$_\alpha$ (4.3), we get

$$\text{FMOP}_\alpha \begin{cases} \min & \begin{pmatrix} (x_1-2)^2 + (x_2-2)^2 \\ \frac{(x_1-4)^2}{2} + \frac{x_2^2}{4} \end{pmatrix} \\ \text{subject to} & (1.5-0.5\alpha)x_1 + (4-\alpha)x_2 \leq 6-3\alpha, \\ & (2+0.5\alpha)x_1 + 0.5(1+\alpha)x_2 \leq 2+0.5\alpha, \\ & x_1 \geq 0, x_2 \geq 0. \end{cases}$$

With the help of the classical method presented in the Sect. 3, Pareto set, $\mathcal{X}_{E\alpha}$ of the problem FMOP$_\alpha$ will be obtained. By taking union, through sup-min composition, of all the Pareto

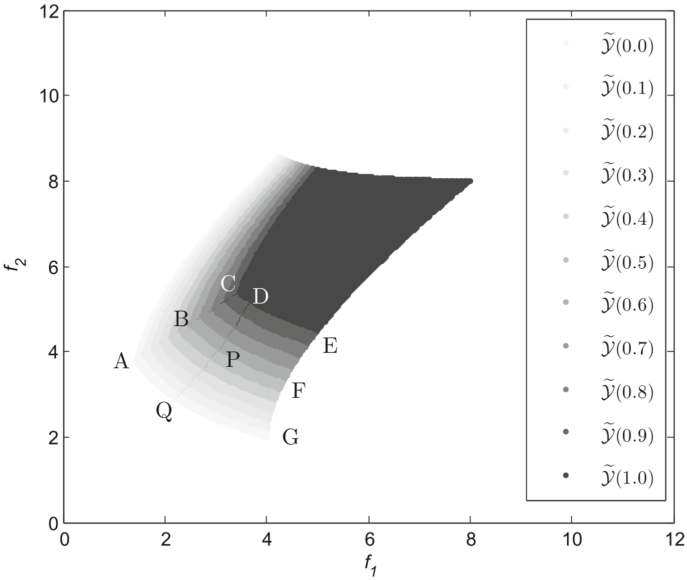

Fig. 7 Fuzzy criteria feasible region $\widetilde{\mathcal{Y}}$ of Example 2

Table 1 Fuzzy knee arc $\widetilde{\mathcal{Y}}_K =$ QPD for Example 2

f_1	f_2	$\mu((f_1, f_2)\vert\widetilde{\mathcal{Y}}_K)$
2.221	2.885	0.0
2.344	3.045	0.1
2.490	3.219	0.2
2.628	3.414	0.3
2.782	3.602	0.4
2.918	3.864	0.5
3.101	4.051	0.6
3.237	4.345	0.7
3.392	4.582	0.8
3.524	4.859	0.9
3.667	5.152	1.0

sets $\mathcal{X}_{E\alpha}$'s, we obtain the fuzzy Pareto set $\widetilde{\mathcal{X}}_E$. Image of the set $\widetilde{\mathcal{X}}_E$ by the vector map f is the fuzzy non-dominated set $\widetilde{\mathcal{Y}}_N$.

The fuzzy criteria feasible region $\widetilde{\mathcal{Y}}$ and the fuzzy non-dominated set $\widetilde{\mathcal{Y}}_N$ are shown in the Fig. 7. The non-dominated set $\widetilde{\mathcal{Y}}_N$ is the interior and boundary of the fuzzy region bounded by $ABCDEFGQA$ on the Fig. 7. Its core is the arc CDE. Coordinates of the points C, D and E are (3.467, 5.377), (3.667, 5.152) and (5, 4.5), respectively.

For the considered problem, the fuzzy arc QPD in the Fig. 7 is obtained as global knee. A discrete approximation of the fuzzy arcs ABC and QPD are displayed in the Table 1 and Table 2, respectively.

In the next, another example is provided. Without any detail, we straightaway explore fuzzy knees for the next problem.

Table 2 Fuzzy arc ABC on criteria feasible region $\widetilde{\mathcal{Y}}$ of Example 2

f_1	f_2	$\mu((f_1, f_2)\mid\widetilde{\mathcal{Y}}_N)$
1.449	3.814	0.0
1.615	3.957	0.1
1.775	4.076	0.2
1.943	4.198	0.3
2.123	4.324	0.4
2.319	4.515	0.5
2.513	4.648	0.6
2.737	4.815	0.7
2.971	4.986	0.8
3.212	5.197	0.9
3.467	5.377	1.0

Example 3 We consider the following fuzzy bi-criteria optimization problem:

$$\min \begin{pmatrix} 46 - 21\sqrt{x_1 + x_2 + 2} \\ \\ 8 - \sin(9x_1 + 8x_2) - (x_1 - x_2)^3 \end{pmatrix}$$

$$\begin{aligned} \text{subject to} \quad & \widetilde{C} : (2/2.5/3)x_1 + (0.5/1/2)x_2 \ \widetilde{\lesssim}\ (2/2.5/6), \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

In this problem, for each $\alpha \in [0, 1]$, the α-cut of the decision constraint set, i.e., $\widetilde{\mathcal{X}}(\alpha)$ is the set (Fig. 5)

$$\left\{x \in \mathbb{R}^2_{\geqq} : (3 - 0.5\alpha)x_1 + (2 - \alpha)x_2 \leq (6 - 3.5\alpha)\right\}.$$

Thus for each $\alpha \in [0, 1]$, FMOP_α is the problem:

$$\text{FMOP}_\alpha \quad \begin{cases} \min & \begin{pmatrix} 46 - 21\sqrt{x_1 + x_2 + 2} \\ 8 - \sin(9x_1 + 8x_2) - (x_1 - x_2)^3 \end{pmatrix} \\ \text{subject to} & x \in \widetilde{\mathcal{X}}(\alpha). \end{cases}$$

The criteria feasible region $\widetilde{\mathcal{Y}}$ is shown in the Fig. 8. Each of the FMOP_α has two knees. Two knees of each FMOP_α for twenty one different values of $\alpha \in [0, 1]$ are given in the Table 3.

Fuzzy knees of this problem are the fuzzy arcs ABC and DEF. The fuzzy arc ABC is a local knee and the fuzzy arc DEF is a global fuzzy knee. The coordinates of different points are: $A \equiv (1.9, 7.001)$, $B \equiv (4.49, 7.7999)$, $C \equiv (5.77, 7.211)$, $D \equiv (4, 0.751)$, $E \equiv (7.564, 5.944)$ and $F \equiv (9.627, 6.588)$.

6 Application

In this section, we apply the proposed technique on an engineering design problem—three-bar truss problem [12]. To demonstrate the problem, we refer to Fig. 9. In the problem, the total volume of the truss and a linear combination of the horizontal and vertical displacements

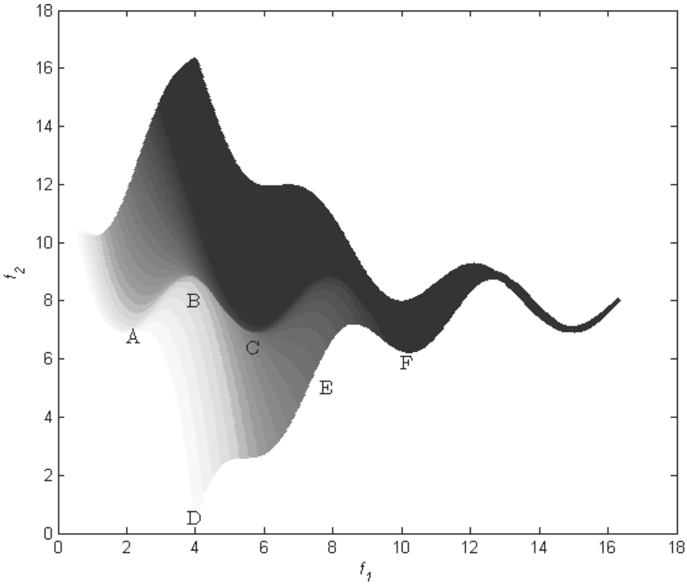

Fig. 8 Fuzzy criteria feasible region $\widetilde{\mathcal{Y}}$ of Example 3

Table 3 Fuzzy knees in Example 3

α	Knee 1 ∈ arc ABC	Knee 2 ∈ arc DEF
0.00	(1.900, 7.001)	(4.000, 0.751)
0.05	(1.950, 7.019)	(4.263, 1.549)
0.10	(2.000, 7.093)	(4.528, 2.125)
0.15	(2.050, 7.231)	(4.795, 2.477)
0.20	(2.201, 7.415)	(5.063, 2.640)
0.25	(2.251, 7.670)	(5.334, 2.683)
0.30	(2.352, 7.968)	(5.606, 2.690)
0.35	(2.504, 8.274)	(5.880, 2.751)
0.40	(2.860, 8.914)	(5.916, 2.832)
0.45	(3.014, 8.931)	(6.155, 2.938)
0.50	(4.490, 7.799)	(6.433, 3.294)
0.55	(5.010, 7.564)	(6.713, 3.821)
0.60	(5.171, 7.359)	(6.994, 4.486)
0.65	(5.660, 7.038)	(7.278, 5.222)
0.70	(5.725, 7.001)	(7.564, 5.944)
0.75	(5.739, 6.997)	(7.852, 6.565)
0.80	(5.748, 7.006)	(8.434, 7.253)
0.85	(5.753, 7.001)	(8.729, 7.278)
0.90	(5.761, 7.047)	(9.026, 7.127)
0.95	(5.767, 7.115)	(9.325, 6.686)
1.00	(5.770, 7.211)	(9.627, 6.588)

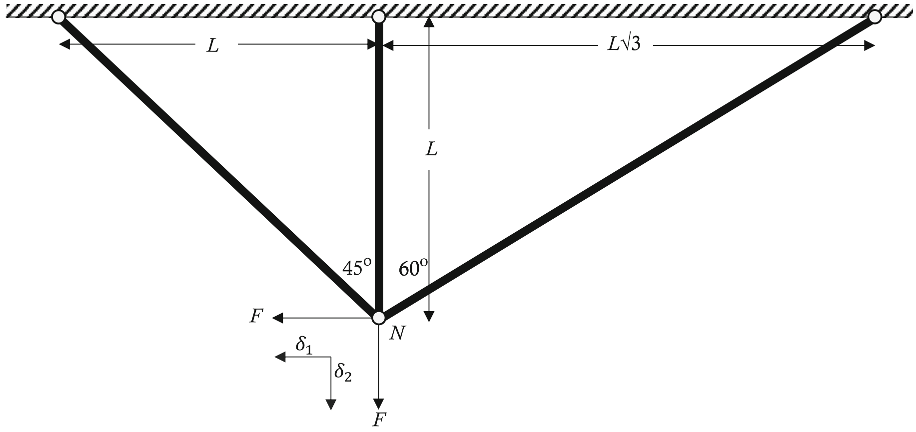

Fig. 9 Three-bar truss under a static loading

(δ_1 and δ_2, respectively) of the node N for a small deformation of the truss are to be minimized simultaneously. We use the subscripts 1, 2 and 3 to refer left, middle and right bar, respectively. The decision variables for this problem are the cross section of the bars: x_1, x_2 and x_3. These three variables are bounded by 0.1 cm^2 and 2 cm^2. Different numerical data of the problem are as follows:

(i) $F = 5\,kN$, $L = 1\,m$,
(ii) Young modulus of the bars $E = 200\,GPa$ and
(iii) the accepted stress in each bar is the triangular fuzzy number $\widetilde{\sigma} = (50/200/400)\,MPa$.

The FMOP of this problem is described by

$$\begin{aligned} \min \quad & \begin{pmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \end{pmatrix} \\ \text{subject to} \quad & \frac{|T_i|}{a_i} \leq \widetilde{\sigma} \\ & 0.1 \times 10^{-4} \leq x_i \leq 2 \times 10^{-4} \\ & i = 1, 2, 3, \end{aligned}$$

where T_i's are tension of the bars which can be calculated as

$$\begin{aligned} T_1 &= \frac{a_1 E}{2L}(\delta_1 - \delta_2), \\ T_2 &= \frac{a_2 E}{L}\delta_2 \text{ and} \\ T_3 &= \frac{a_3 E}{4L}(\delta_1 + \sqrt{3}\delta_2) \end{aligned}$$

and the objective functions are

$$\begin{aligned} f_1(x_1, x_2, x_3) &= L(\sqrt{2}a_1 + a_2 + 2a_3) \text{ and} \\ f_2(x_1, x_2, x_3) &= \tfrac{3}{10}\delta_2 - \tfrac{1}{10}\delta_1. \end{aligned}$$

The displacements δ_1 and δ_2 can be determined from the expression of T_i's and the force balance equations

Table 4 Fuzzy knee in the three-bar truss problem

α	Knee
0.01	$(2.541, 0.527) \times 10^{-4}$
0.20	$(2.241, 1.151) \times 10^{-4}$
0.40	$(2.241, 1.151) \times 10^{-4}$
0.60	$(3.041, 0.599) \times 10^{-4}$
0.80	$(2.541, 1.151) \times 10^{-4}$
1.00	$(3.041, 1.000) \times 10^{-4}$

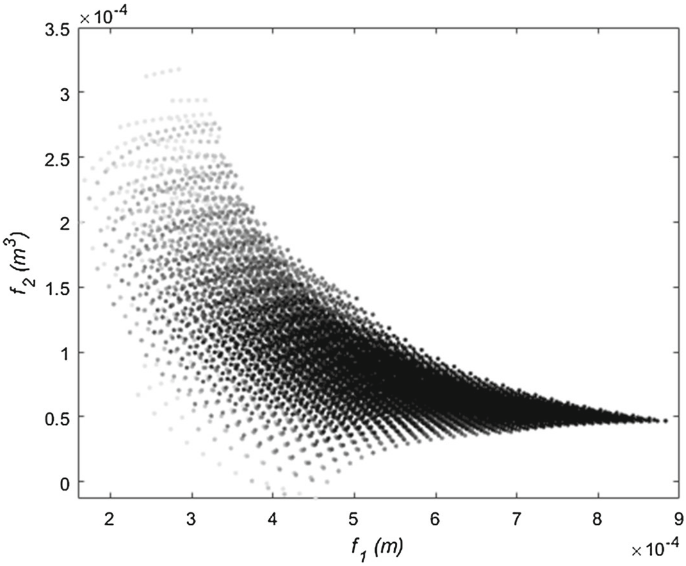

Fig. 10 Fuzzy criteria feasible region of the three-bar truss problem

(i) horizontal: $F = \frac{\sqrt{3}T_3}{2} - \frac{T_1}{\sqrt{2}}$ and
(ii) vertical: $F = T_2 + \frac{T_1}{\sqrt{2}} + \frac{T_3}{2}$.

The criteria feasible region $\widetilde{\mathcal{Y}}$ is shown in the Fig. 10. Each of the FMOP$_\alpha$ has only one knee. This knee for six different values of $\alpha \in [0, 1]$ are given in the Table 4. From the table we note that if the decision maker wants at least 20% satisfaction level (α) of the design, then the cross section of the bars must be chosen so that $(f_1, f_2) = (2.241, 1.151) \times 10^{-4}$. Similarly, for at least 60%, $(f_1, f_2) = (3.041, 0.599) \times 10^{-4}$ must be satisfied and so on.

7 Conclusion

In this paper, two new concepts—*fuzzy Pareto optimality* and *fuzzy knee*— for FMOPs have been introduced. Subsequently, a technique has been proposed to obtain fuzzy knees of the fuzzy Pareto set of an FMOP. The presented technique essentially used a classical method to capture Pareto set of MOPs by considering Pareto solutions of each FMOP$_\alpha$, i.e., the set

$\widetilde{\mathcal{X}}_{E\alpha}$. Then, taking union, by the supremum composition, of all the $\widetilde{\mathcal{X}}_{E\alpha}$ sets, the method has obtained complete fuzzy Pareto set of FMOPs. Similarly, $\widetilde{\mathcal{Y}}_K \bigcup \widetilde{\mathcal{Y}}_{GK}$ is also captured.

As number of points in the fuzzy Pareto set is substantially larger, it is difficult for the DM to select best solution(s). The selection would become more difficult for large number of fuzzy criteria. A proper mathematical construction of DM's preferences while dealing with large number of imprecise criteria and a huge set of imprecise alternatives seems to be really complex. In this situation, the fuzzy knees of the fuzzy Pareto optimal set are likely to be the more relevant to the DM. Thus identification of fuzzy knees may reduce the final selection procedure on a smaller number of potentially more relevant solutions from fuzzy Pareto set.

In this introductory work on our methodology to solve FMOPs, the proposed study has been made on the FMOPs where decision variables and criteria are crisp. Investigation on more generalized FMOPs may be obtained in our future research.

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