SeMA (2018) 75:545–561
https://doi.org/10.1007/s40324-018-0150-8

Newton-like methods with increasing order of convergence and their convergence analysis in Banach space

Janak Raj Sharma[1] · Ioannis K. Argyros[2] · Deepak Kumar[1]

Received: 14 October 2017 / Accepted: 22 February 2018 / Published online: 1 March 2018

Abstract Based on a two-step Newton-like iterative scheme of convergence order $p \geq 3$, we propose a three-step scheme of convergence order $p+3$. Furthermore, on the basis of this scheme a generalized $q+2$-step scheme with increasing convergence order $p+3q$ ($q \in \mathbb{N}$) is presented. Local convergence, including radius of convergence and uniqueness results of the methods, is presented. Theoretical results are verified through numerical experimentation. The performance is demonstrated by the application of the methods on some nonlinear systems of equations. The numerical results, including the elapsed CPU-time, confirm the accurate and efficient character of proposed techniques.

Keywords Newton-like methods · Local convergence · Radius of convergence · Fréchet-derivative · Banach space

Mathematics Subject Classification 65H10 · 65J10 · 41A25 · 49M15

1 Introduction

The construction of fixed point iterative methods for solving nonlinear equations or systems of nonlinear equations is an interesting and challenging task in numerical analysis and many applied scientific branches. The huge importance of this subject has led to the development

✉ Janak Raj Sharma
jrshira@yahoo.co.in

Ioannis K. Argyros
iargyros@cameron.edu

Deepak Kumar
deepak.babbi@gmail.com

[1] Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal, Sangrur 148106, India

[2] Department of Mathematics Sciences, Cameron University, Lawton, OK 73505, USA

of many numerical methods, most frequently of iterative nature (see [4–7,31,32]). With the advancement of computer hardware and software, the problem of solving nonlinear equations by numerical methods has gained an additional importance. In this paper, we consider the problem of approximating a solution x^* of the equation $F(x) = 0$; where $F : \Omega \subseteq B_1 \to B_2$, B_1 and B_2 are Banach spaces and Ω is a nonempty open convex subset of B_1, by iterative methods of a high order of convergence. The solution x^* can be obtained as a fixed point of some function $\Phi : \Omega \subseteq B_1 \to B_2$ by means of fixed point iteration

$$x_{n+1} = \Phi(x_n), \quad n = 0,\ 1,\ 2, \ldots .$$

There are a variety of iterative methods for solving nonlinear equations. A classical method is the quadratically convergent Newton's method [4]

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n), \tag{1}$$

where $F'(x)^{-1}$ is the inverse of first Fréchet derivative $F'(x)$ of the function $F(x)$. This method converges if the initial approximation x_0 is closer to the solution x^* and $F'(x)^{-1}$ exists in an open neighborhood Ω of x^*. In order to attain the higher order of convergence, a number of modified Newton's or Newton-like methods have been proposed in literature, see, for example [1–3,6,8–10,12–30] and references therein.

In this paper, we consider a three-step iterative scheme and its multistep version for solving the nonlinear system $F(x) = 0$. The three-step scheme is given by

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1} F(x_n), \\ z_n &= \varphi_\alpha^{(p)}(x_n, y_n), \\ x_{n+1} &= z_n - \psi(x_n, y_n) F(z_n), \end{aligned} \tag{2}$$

where $\varphi_\alpha^{(p)}(x_n, y_n)$ is any iterative scheme of convergence order $p \geq 3$, $\psi(x_n, y_n) = \big(\beta I + \gamma F'(y_n)^{-1} F'(x_n) + \delta F'(x_n)^{-1} F'(y_n)\big) F'(x_n)^{-1}$ and $\{\alpha,\ \beta,\ \gamma,\ \delta\} \in \mathbb{R}$.

The multistep version of (2), consisting of $q + 2$ steps, is expressed as

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1} F(x_n), \\ z_n &= \varphi_\alpha^{(p)}(x_n, y_n), \\ z_n^{(1)} &= z_n - \psi(x_n, y_n) F(z_n), \\ z_n^{(2)} &= z_n^{(1)} - \psi(x_n, y_n) F(z_n^{(1)}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ z_n^{(q-1)} &= z_n^{(q-2)} - \psi(x_n, y_n) F(z_n^{(q-2)}), \\ z_n^{(q)} &= x_{n+1} = z_n^{(q-1)} - \psi(x_n, y_n) F(z_n^{(q-1)}), \end{aligned} \tag{3}$$

where $q \in \mathbb{N}$ and $z_n^{(0)} = z_n$.

In Sect. 2, we show that for a particular set of values of the parameters $\alpha,\ \beta,\ \gamma$ and δ the methods (2) and (3) possess convergence order $p+3$ and $p+3q$, respectively. In Sect. 3, the local convergence including radius of convergence, computable error bounds and uniqueness results of the proposed methods is presented. In order to verify the theoretical results, some numerical examples are presented in Sect. 4. Finally, in Sect. 5 the methods are applied to solve some systems of nonlinear equations.

2 Convergence-I

We present the convergence of method (2), when $F : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$.

Theorem 1 *Suppose that*

(i) $F : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$ *is a sufficiently many times differentiable mapping.*
(ii) *There exists a solution* $x^* \in \Omega$ *of equation* $F(x) = 0$ *such that* $F'(x^*)$ *is nonsingular.*

Then, sequence $\{x_n\}$ *generated by method* (2) *for* $x_0 \in \Omega$ *converges to* x^* *with order* $p + 3$ *for* $p \geq 3$ *if and only if*

$$\alpha = 1, \quad \beta = -1, \quad \gamma = \frac{3}{2} \quad \text{and} \quad \delta = \frac{1}{2}. \tag{4}$$

Proof Let $e_n = x_n - x^*$. Using Taylor's theorem and the hypothesis $F(x^*) = 0$, we obtain in turn that

$$\begin{aligned} F(x_n) &= F'(x^*)(x_n - x^*) + \frac{1}{2!}F''(x^*)(x_n - x^*)^2 + \frac{1}{3!}F'''(x^*)(x_n - x^*)^3 + O\left(\|x_n - x^*\|^4\right), \\ &= F'(x^*)\left(e_n + T_2(e_n)^2 + T_3(e_n)^3 + O(e_n)^4\right), \end{aligned}$$

where $T_i = \frac{1}{i!}F'(x^*)^{-1}F^{(i)}(x^*)$ and $(e_n)^i = (e_n, e_n, \overset{i}{\ldots}, e_n)$, $e_n \in \mathbb{R}$, $i \in \mathbb{N}$.

Also

$$F'(x_n) = F'(x^*)\left(I + 2T_2 e_n + 3T_3(e_n)^2 + O((e_n)^3)\right), \tag{5}$$

$$F'(x_n)^{-1} = \left(I - 2T_2 e_n + (4T_2^2 - 3T_3)(e_n)^2 + O((e_n)^3)\right) F'(x^*)^{-1}, \tag{6}$$

and

$$F'(x_n)^{-1}F(x_n) = e_n - T_2(e_n)^2 + 2(T_2^2 - T_3)(e_n)^3 + O((e_n)^4). \tag{7}$$

For $\tilde{e}_n = y_n - x^*$, we have that

$$\tilde{e}_n = (1 - \alpha)e_n + \alpha T_2(e_n)^2 - 2\alpha(T_2^2 - T_3)(e_n)^3 + O((e_n)^4).$$

Using again Taylor's theorem on $F'(y_n)$ about $y_n = x^*$, we get in turn that

$$\begin{aligned} F'(y_n) &= F'(x^*)\big(I + 2T_2\tilde{e}_n + 3T_3(\tilde{e}_n)^2 + O((\tilde{e}_n)^3)\big), \\ &= F'(x^*)\big(I + 2(1 - \alpha)T_2 e_n + \big(2\alpha T_2^2 + 3(1 - \alpha)^2 T_3\big)(e_n)^2 + O((e_n)^3)\big), \end{aligned} \tag{8}$$

so

$$\begin{aligned} F'(y_n)^{-1} &= \big(I - 2T_2\tilde{e}_n + (4T_2^2 - 3T_3)(\tilde{e}_n)^2 + O((\tilde{e}_n)^3)\big)F'(x^*)^{-1}, \\ &= \big(I - 2(1 - \alpha)T_2 e_n + \big(2(2 - 5\alpha + 2\alpha^2)T_2^2 - 3(1 - \alpha)^2 T_3\big)(e_n)^2 \\ &\quad + O((e_n)^3)\big)F'(x^*)^{-1}. \end{aligned} \tag{9}$$

It then follows from the Eqs. (5), (6), (8) and (9), respectively that

$$F'(x_n)^{-1}F'(y_n) = I - 2\alpha T_2 e_n + \big(6\alpha T_2^2 - 3\alpha(2 - \alpha)T_3\big)(e_n)^2 + O((e_n)^3)$$

and

$$F'(y_n)^{-1}F'(x_n) = I + 2\alpha T_2 e_n - \big(2\alpha(3 - 2\alpha)T_2^2 + 3\alpha(\alpha - 2)T_3\big)(e_n)^2 + O((e_n)^3).$$

Consequently, summing up we get in turn that

$$\beta I+\gamma F'(y_n)^{-1}F'(x_n)+\delta F'(x_n)^{-1}F'(y_n)= (\beta+\gamma+\delta)I+2\alpha(\gamma-\delta)T_2e_n \\ +\big(2\alpha(3\delta-3\beta+2\alpha\gamma)T_2^2-3\alpha(\alpha-2)(\gamma-\delta)T_3\big)(e_n)^2+O\big((e_n)^3\big).$$

Then

$$\begin{aligned}\psi(x_n,y_n)=&\ \big(\beta I+\gamma F'(y_n)^{-1}F'(x_n)+\delta F'(x_n)^{-1}F'(y_n)\big)F'(x_n)^{-1}\\ =&\ \big((\beta+\gamma+\delta)I-2(\beta+\gamma+\delta-\alpha(\gamma-\delta))T_2e_n\\ &+\big((4\alpha^2\gamma-10\alpha(\gamma-\delta)+4(\beta+\gamma+\delta))T_2^2-3(\beta+\gamma+\delta\\ &+\alpha(\alpha-2)(\gamma-\delta))T_3\big)(e_n)^2+O((e_n)^3)\big)F'(x^*)^{-1}.\end{aligned} \tag{10}$$

By hypothesis $\{z_n\}$ is of order p. Set $\bar{e}_n : z_n-x^*=K((e_n)^p)+O((e_n)^{p+1}),\ K\neq 0$. Then, we have

$$F(z_n)=F'(x^*)(\bar{e}_n+O((\bar{e}_n)^2)). \tag{11}$$

Using (10) and (11) in the third substep of method (2), it follows that

$$\begin{aligned}e_{n+1}=&(1-\beta-\gamma-\delta)\bar{e}_n+2\big(\beta+\gamma+\delta-\alpha(\gamma-\delta)\big)T_2(e_n\bar{e}_n)\\ &-\big(2(2(\alpha^2\gamma+\beta+\gamma+\delta)-5\alpha(\gamma-\delta))T_2^2-3(\beta+\gamma+\delta\\ &+\alpha(\alpha-2)(\gamma-\delta))T_3\big)\big((e_n)^2\bar{e}_n\big)+O\big((e_n)^3\bar{e}_n\big).\end{aligned} \tag{12}$$

Therefore, the order of convergence to x^* is of order $p+3$ $(p\geq 3)$, if and only if, the parameters α, β, γ and δ satisfy

$$\begin{aligned}&\beta+\gamma+\delta=1,\\ &\alpha(\gamma-\delta)=1,\\ &2(\alpha^2\gamma+\beta+\gamma+\delta)-5\alpha(\gamma-\delta)=0,\\ &\beta+\gamma+\delta+\alpha(\alpha-2)(\gamma-\delta)=0,\end{aligned} \tag{13}$$

leading to the unique solutions of the system (13) given in (4).

Note that we have not shown the coefficient of $(e_n)^3\bar{e}_n$ in (12) due to lengthy expression. However, using the values of parameters given in (4), we can write the error equation in simplified form as

$$\begin{aligned}e_{n+1}&=2T_2\big(4T_2^2-T_3\big)\big((e_n)^3\bar{e}_n\big)+O\big((e_n)^4\bar{e}_n\big)\\ &=2KT_2\big(4T_2^2-T_3\big)\big((e_n)^{p+3}\big)+O\big((e_n)^{p+4}\big).\end{aligned}$$

It follows from Theorem 1 that the method to be used from the family (2) is given by

$$\begin{aligned}y_n&=x_n-F'(x_n)^{-1}F(x_n),\\ z_n&=\varphi_1^{(p)}(x_n,y_n),\\ x_{n+1}&=z_n-\Big(-I+\frac{3}{2}F'(y_n)^{-1}F'(x_n)+\frac{1}{2}F'(x_n)^{-1}F'(y_n)\Big)F'(x_n)^{-1}F(z_n).\end{aligned} \tag{14}$$

Next we show that the method (3), on using the values of parameters $\alpha,\ \beta,\ \gamma$ and δ given in (4), possesses convergence order $p+3q$. Thus the following theorem is proved:

Theorem 2 *Under the hypotheses of Theorem 1, the sequence $\{x_n\}$ generated by method (3) for $x_0\in\Omega$ converges to x^* with order $p+3q$ for $p\geq 3$ and $q\in\mathbb{N}$.*

Proof Taylor's expansion of $F(z_n^{(q-1)})$ about x^* yields

$$F(z_n^{(q-1)}) = F'(x^*)\Big((z_n^{(q-1)} - x^*) + T_2(z_n^{(q-1)} - x^*)^2 + \cdots\Big). \tag{15}$$

Then, we have that

$$\begin{aligned}\psi(x_n, y_n)F(z_n^{(q-1)}) = &\ \big(I - 2(4T_2^3 - T_2T_3)(e_n)^3 + O((e_n)^4))\big)F'(x^*)^{-1}\\ &\times F'(x^*)\big((z_n^{(q-1)} - x^*) + T_2(z_n^{(q-1)} - x^*)^2 + \cdots\big)\\ &= (z_n^{(q-1)} - x^*) - 2(4T_2^3 - T_2T_3)(e_n)^3(z_n^{(q-1)} - x^*)\\ &+ T_2(z_n^{(q-1)} - x^*)^2 + \cdots .\end{aligned} \tag{16}$$

Using (16) in (3), we obtain

$$z_n^{(q)} - x^* = 2(4T_2^3 - T_2T_3)(e_n)^3(z_n^{(q-1)} - x^*) + T_2(z_n^{(q-1)} - x^*)^2 + \cdots . \tag{17}$$

As we know that $z_n^{(1)} - x^* = 2KT_2(4T_2^2 - T_3)(e_n)^{p+3} + O((e_n)^{p+4})$, therefore, from (17) for $q = 2, 3$, we have

$$\begin{aligned}z_n^{(2)} - x^* &= 2(4T_2^3 - T_2T_3)(e_n)^3(z_n^{(1)} - x^*) + \cdots\\ &= 2^2KT_2^2(4T_2^2 - T_3)^2(e_n)^{p+6} + O\big((e_n)^{p+7}\big)\end{aligned}$$

and

$$\begin{aligned}z_n^{(3)} - x^* &= 2(4T_2^3 - T_2T_3)(e_n)^3(z_n^{(2)} - x^*) + \cdots\\ &= 2^3KT_2^3(4T_2^2 - T_3)^3(e_n)^{p+9} + O\big((e_n)^{p+10}\big).\end{aligned}$$

Proceeding by induction, we have

$$e_{n+1} = z_n^{(q)} - x^* = 2^qKT_2^q(4T_2^2 - T_3)^q(e_n)^{p+3q} + O\big((e_n)^{p+3q+1}\big).$$

This completes the proof of Theorem 2. □

3 Convergence-II

In this section we study the convergence of new methods in Banach space settings. Let $w_0 : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ be a continuous and nondecreasing function with $w_0(0) = 0$. Let also ϱ be the smallest positive solution of equation

$$w_0(t) = 1. \tag{18}$$

Consider, function $w : [0, \varrho) \to \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing with $w(0) = 0$. Define functions g_1 and h_1 on the interval $[0, \varrho)$ by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)} \tag{19}$$

and

$$h_1(t) = g_1(t) - 1.$$

We have $h_1(0) = -1 < 0$ and $h_1(t) \to +\infty$ as $t \to \varrho^-$. The intermediate value theorem guarantees that equation $h_1(t) = 0$ has solutions in $(0, \varrho)$. Denote by ϱ_1 the smallest such solution. Let $\lambda \geq 1$ and $g_2 : [0, \varrho_1) \to \mathbb{R}_+ \cup \{0\}$ be a continuous and nondecreasing function. Define function h_2 on $[0, \varrho_1)$ by

$$h_2(t) = g_2(t)t^{\lambda-1} - 1. \tag{20}$$

Suppose that $g_2(t)t^{\lambda-1} - 1 \to +\infty$ or a positive number as $t \to \varrho_1^-$.

Then, we get that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ or a positive number as $t \to \varrho_1^-$. Denote by ϱ_2 the smallest solution in $(0, \varrho_1)$ of equation $h_2(t) = 0$. If $\lambda = 1$, suppose instead of (20) that

$$g_2(0) < 1 \tag{21}$$

and $g_2(t) - 1 \to +\infty$ or a positive number as $t \to \varrho_1^-$. Denote again by ϱ_2 the smallest solution of equation $h_2(t) = 0$.

Let $v : (0, \varrho_1) \to \mathbb{R}_+ \cup \{0\}$ be a continuous and nondecreasing function. Define functions g_3 and h_3 on the interval $(0, \varrho_1)$ by

$$\begin{aligned} g_3(t) = \Bigg(& \frac{\int_0^1 w((1-\theta)g_2(t)t^\lambda)d\theta}{1 - w_0(g_2(t)t^\lambda)} \\ & + \frac{(w_0(t) + w_0(g_1(t)t))\int_0^1 v(\theta g_2(t)t^\lambda)d\theta}{(1-w_0(t))(1-w_0(g_1(t)t))} \\ & + \frac{1}{2}\frac{(w_0(g_2(t)t^\lambda) + w_0(g_1(t)t))\int_0^1 v(\theta g_2(t)t^\lambda)d\theta}{(1-w_0(g_2(t)t^\lambda))(1-w_0(g_1(t)t))} \\ & + \frac{1}{2}\frac{\left(\frac{v(t)}{1-w_0(g_1(t)t)} + \frac{v(g_2(t)t)}{1-w_0(t)}\right)v(g_1(t)t)\int_0^1 v(\theta g_2(t)t^\lambda)d\theta}{(1-w_0(g_2(t)t^\lambda))(1-w_0(t))}\Bigg)g_2(t)t^{\lambda-1} \end{aligned} \tag{22}$$

and

$$h_3(t) = g_3(t) - 1.$$

We obtain that $h_3(0) = -1 < 0$ and $h_3(t) \to \infty$ as $t \to \varrho_2^-$. Denote by ϱ_3 the smallest solution of equation $h_3(t)$ in $(0, \varrho_2)$. Then, we have that for each $t \in [0, \varrho)$

$$0 \leq g_i(t) < 1, \quad i = 1, 2, 3.$$

Denote by $U(\mu, \varepsilon) = \{x \in B_1 : \|x - \mu\| < \varepsilon\}$ the ball with center $\mu \in B_1$ and of radius $\varepsilon > 0$. Moreover, $\bar{U}(\mu, \varepsilon)$ denotes the closure of $U(\mu, \varepsilon)$. We shall show the local convergence analysis of method (14) in a Banach space setting under hypotheses (A):

(a1) $F : \Omega \subseteq B_1 \to B_2$ is a continuously Fréchet-differentiable operator.

(a2) There exists $x^* \in \Omega$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in \mathfrak{L}(B_2, B_1)$.

(a3) There exists function $w_0 : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing with $w_0(0) = 0$ such that for each $x \in \Omega$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq w_0(\|x - x^*\|).$$

(a4) Let $\Omega_0 = \Omega \cap U(x^*, \varrho)$, where ϱ was defined previously. There exist functions $w : [0, \varrho) \to \mathbb{R}_+ \cup \{0\}$, $v : [0, \varrho) \to \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing with $w(0) = 0$ such that for each $x, y \in \Omega_0$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq w(\|x - y\|)$$

and

$$\|F'(x^*)^{-1}F'(x)\| \leq v(\|x - x^*\|).$$

(a5) There exists function $g_2 : [0, \varrho_1) \to \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing and $\lambda \geq 1$ satisfying (20) if $\lambda > 1$ and (21), if $\lambda = 1$ such that

$$\left\| \varphi_{\alpha}^{(p)}\big(x, x - F'(x)^{-1}F'(x)\big) - x^* \right\| \leq g_2(\|x - x^*\|)\|x - x^*\|^{\lambda}.$$

(a6) $\bar{U}(x^*, \varrho_3) \subseteq \Omega$.

(a7) Let $\varrho^* \geq \varrho_3$ and set $\Omega_1 = \Omega \cap \bar{U}(x^*, \varrho^*)$, $\int_0^1 w_0(\theta\varrho^*)d\theta < 1$.

Theorem 3 *Suppose that the hypotheses (A) are satisfied. Then, the sequence* $\{x_n\}$ *generated for* $x_0 \in U(x^*, \varrho_3) - \{x^*\}$ *by method (14) is well defined in* $U(x^*, \varrho_3)$*, remains in* $U(x^*, \varrho_3)$ *for all* $n = 0, 1, 2, \ldots$ *and converges to* x^**, so that*

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < \varrho_3, \tag{23}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\|^{\lambda} \leq \|x_n - x^*\| \tag{24}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{25}$$

where the functions $g_i,\ i = 1, 2, 3$ *are defined previously. Moreover, the vector* x^* *is the only solution of equation* $F(x) = 0$ *in* Ω_1.

Proof We shall show estimates (23)–(25) using mathematical induction. By hypothesis (a2) and for $x \in U(x^*, \varrho_3)$, we have that

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq w_0(\|x - x^*\|) \leq w_0(\varrho_3) < 1. \tag{26}$$

By the Banach perturbation Lemma [4,6] and (26) we get that $F'(x)^{-1} \in \mathfrak{L}(B_2, B_1)$ and

$$\|F'(x)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x - x^*\|)}. \tag{27}$$

In particular, (27) holds for $x = x_0$, since $x_0 \in U(x^*, \varrho) - \{x^*\}$ and y_0, z_0 are well defined by the first and second substep of method (14) for $n = 0$. We can write by the first substep of method (14) and (a2) that

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= \int_0^1 F'(x_0)^{-1}\big(F'(x^* + \theta(x_0 - x^*)) - F'(x_0)\big)(x_0 - x^*)d\theta. \end{aligned} \tag{28}$$

Then, using (22) (for $i = 1$), the first condition in (a4), (27) (for $x = x_0$) and (28) we get in turn that

$$\begin{aligned} \|y_0 - x^*\| &= \|F'(x_0)^{-1}F'(x^*)\| \left\| \int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*) - F'(x_0)](x_0 - x^*)d\theta \right\| \\ &\leq \frac{\int_0^1 w((1-\theta)\|x_0 - x^*\|)d\theta\|x_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < \varrho_3, \end{aligned} \tag{29}$$

which implies (23) for $n = 0$ and $y_0 \in U(x^*, \varrho_3)$.

Using (a5) and (22) for $i = 2$, we get that

$$\|z_0 - x^*\| = \|\varphi_1(x_0, y_0) - x^*\| \leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\|^{\lambda} \leq \|x_0 - x^*\| < \varrho_3, \tag{30}$$

so (24) holds for $n = 0$ and $z_0 \in U(x^*, \varrho_3)$. Notice that since $y_0, z_0 \in U(x^*, \varrho_3)$, we have that

$$\|F'(y_0)^{-1}F'(x^*)\| \le \frac{1}{1 - w_0(\|y_0 - x^*\|)} \le \frac{1}{1 - w_0(g_1(\|x_0 - x^*\|)\|x_0 - x^*\|)} \tag{31}$$

and

$$\|F'(z_0)^{-1}F'(x^*)\| \le \frac{1}{1 - w_0(\|z_0 - x^*\|)} \le \frac{1}{1 - w_0(g_2(\|x_0 - x^*\|)\|x_0 - x^*\|^\lambda)}. \tag{32}$$

Moreover, x_1 is well defined by the third substep of method (14) for $n = 0$. We can write by the third substep of method (14) for $n = 0$

$$\begin{aligned} x_1 - x^* &= z_0 - x^* - F'(z_0)^{-1}F(z_0) + F'(z_0)^{-1}F(z_0) + F'(x_0)^{-1}F(z_0) \\ &\quad - \frac{3}{2}F'(y_0)^{-1}F(z_0) - \frac{1}{2}F'(x_0)^{-1}F'(y_0)F'(x_0)^{-1}F(z_0) \\ &= (z_0 - x^* - F'(z_0)^{-1}F(z_0)) + (F'(x_0)^{-1} - F'(y_0)^{-1})F(z_0) \\ &\quad + \frac{1}{2}(F'(z_0)^{-1} - F'(y_0)^{-1})F(z_0) \\ &\quad + \frac{1}{2}(F'(z_0)^{-1} - F'(x_0)^{-1}F'(y_0)F'(x_0)^{-1})F(z_0) \\ &= (z_0 - x^* - F'(z_0)^{-1}F(z_0)) + F'(x_0)^{-1}[(F'(y_0) - F'(x^*)) \\ &\quad + (F'(x^*) - F'(x_0))]F'(y_0)^{-1}F(z_0) \\ &\quad + \frac{1}{2}F'(z_0)^{-1}\left[(F'(y_0) - F'(x^*)) + (F'(x^*) - F'(z_0))\right]F'(y_0)^{-1}F(z_0) \\ &\quad + \frac{1}{2}F'(z_0)^{-1}\left[F'(x_0)F'(y_0)^{-1} - F'(z_0)F'(x_0)^{-1}\right]F'(y_0)F'(x_0)^{-1}F(z_0). \end{aligned} \tag{33}$$

Using (22) (for $i = 3$), (27), (a3), (29)–(33), and the triangle inequality, we obtain in turn that

$$\begin{aligned} \|x_1 - x^*\| &\le \Bigg(\frac{\int_0^1 w(1-\theta)\|z_0 - x^*\|d\theta}{1 - w_0(\|z_0 - x^*\|)} \\ &\quad + \frac{[w_0(\|x_0 - x^*\|) + w_0(\|y_0 - x^*\|)]\int_0^1 v(\theta\|z_0 - x^*\|)d\theta}{(1 - w_0(\|x_0 - x^*\|))(1 - w_0(\|y_0 - x^*\|))} \\ &\quad + \frac{1}{2}\frac{[w_0(\|z_0 - x^*\|) + w_0(\|y_0 - x^*\|)]\int_0^1 v(\theta\|z_0 - x^*\|)d\theta}{(1 - w_0(\|z_0 - x^*\|))(1 - w_0(\|y_0 - x^*\|))} \\ &\quad + \frac{1}{2}\frac{[\frac{v\|x_0 - x^*\|)}{1 - w_0(\|y_0 - x^*\|)} + \frac{v\|z_0 - x^*\|)}{1 - w_0(\|x_0 - x^*\|)}]v(\|y_0 - x^*\|)\int_0^1 v(\theta\|z_0 - x^*\|)d\theta}{(1 - w_0(\|z_0 - x^*\|))(1 - w_0(\|x_0 - x^*\|))} \Bigg) \\ &\quad \times \|z_0 - x^*\| \le g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| < \varrho_3, \end{aligned}$$

which shows (25) for $(k = 0)$ and $x_1 \in U(x^*, \varrho_3)$. The induction for estimates (23)–(25) is completed by simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates. Then, from estimate

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < \varrho_3, \quad \text{where } c = g_3(\|x_0 - x^*\|) \in [0, 1),$$

we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, \varrho_3)$.

The uniqueness part is shown using (a3) and (a7) as follows:

Define operator Q by $Q = \int_0^1 F'(x^{**} + \theta(x^* - x^{**}))d\theta$ for some $x^{**} \in \Omega_1$ with $F(x^{**}) = 0$. Then, we have that

$$\begin{aligned}\|F'(x^*)^{-1}\big(Q - F'(x^*)\big)\| &\leq \int_0^1 w_0(\theta\|x^* - x^{**}\|)d\theta \\ &\leq \int_0^1 w_0(\theta\varrho^*)d\theta < 1,\end{aligned}$$

so $Q^{-1} \in \mathfrak{L}(B_2, B_1)$. Then, from the identity

$$0 = F(x^*) - F(x^{**}) = Q(x^* - x^{**}),$$

we conclude that $x^* = x^{**}$. □

Next, we present the local convergence analysis of method (3) along the same lines of method (14). Define functions $\bar{g}_2$, λ, μ and h_μ on the interval $[0, \varrho_2)$ by

$$\begin{aligned}\bar{g}_2(t) &= \frac{1}{1 - w_0(t)} + \frac{3}{2}\frac{1}{1 - w_0(g_1(t)t)} + \frac{1}{2}\frac{\int_0^1 v(\theta g_1(t)t)d\theta}{(1 - w_0(t))^2}, \\ \lambda(t) &= 1 + \bar{g}_2(t)\int_0^1 v\big(\theta g_2(t)t^\lambda\big)d\theta, \\ \mu(t) &= \lambda^q(t)g_2(t)t^{\lambda-1}\end{aligned} \tag{34}$$

and

$$h_\mu(t) = \mu(t) - 1.$$

We have that $h_\mu(0) < 0$. Suppose that

$$\mu(t) \to +\infty \quad \text{or a positive number as } t \to \varrho_2^-. \tag{35}$$

Denote by $\varrho^{(q)}$ the smallest zero of function h_μ on the interval $(0, \varrho_2)$. Define the radius of convergence ϱ^* by

$$\varrho^* = \min\{\varrho_1, \varrho^{(q)}\}. \tag{36}$$

Denote by (A') the conditions (A) but with ϱ^* replacing ϱ together with condition (35).

Proposition 1 *Suppose that the conditions (A') hold. Then, sequence $\{x_n\}$ generated for $x_0 \in U(x^*, \varrho^*) - \{x^*\}$ by method (3) is well defined in $U(x^*, \varrho^*)$, remains in $U(x^*, \varrho^*)$ and converges to x^*. Moreover, the following estimates hold*

$$\begin{aligned}\|y_k - x^*\| &\leq g_1(\|x_k - x^*\|)\|x_k - x^*\| \leq \|x_k - x^*\| < \varrho^*, \\ \|z_k - x^*\| &\leq g_2(\|x_k - x^*\|)\|x_k - x^*\|^\lambda \leq \|x_k - x^*\|, \\ \|z_k^{(i)} - x^*\| &\leq \lambda^i(\|x_k - x^*\|)\|z_k^{(i-1)} - x^*\| \\ &\leq \lambda^i(\|x_k - x^*\|)g_2(\|x_k - x^*\|)\|x_k - x^*\|^\lambda \\ &\leq \|x_k - x^*\|, \quad i = 1, 2, \ldots, q-1\end{aligned} \tag{37}$$

and

$$\begin{aligned}\|x_{k+1}-x^*\| = \|z_k^{(q)}-x^*\| &\le \lambda^q(\|x_k-x^*\|)\|z_k^{(q-1)}-x^*\|\\ &\le \mu(\|x_k-x^*\|)\|x_k-x^*\|, \end{aligned} \tag{38}$$

where the functions λ and μ are defined previously. Furthermore, the vector x^* is the only solution of equation $F(x)=0$ in Ω_1.

Proof We shall only show new estimates (37) and (38). Using the proof of Theorem 3, we show the first two estimates. Then, we can obtain that

$$\begin{aligned}\|\psi(x_k,y_k)F'(x^*)\| &\le \|F'(x_k)^{-1}F'(x^*)\| + \frac{3}{2}\|F'(y_k)^{-1}F'(x^*)\|\\ &\quad + \frac{1}{2}\|F'(x_k)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F'(y_k)\|\|F'(x_k)^{-1}F'(x^*)\|\\ &\le \frac{1}{1-w_0(\|x_k-x^*\|)} + \frac{3}{2}\frac{1}{1-w_0(\|y_k-x^*\|)}\\ &\quad + \frac{1}{2}\frac{\int_0^1 v(\theta\|y_k-x^*\|)d\theta}{(1-w_0(\|x_k-x^*\|))^2} \le \bar{g}_2(\|x_k-x^*\|).\end{aligned}$$

Moreover, we have in turn the estimates

$$\begin{aligned}\|z_k^{(1)}-x^*\| &= \|z_k-x^*-\psi(x_k,y_k)F(z_k)\|\\ &\le \|z_k-x^*\| + \|\psi(x_k,y_k)F'(x^*)\|\|F'(x^*)^{-1}F(z_k)\|\\ &\le \|z_k-x^*\| + \bar{g}_2(\|x_k-x^*\|)\int_0^1 v(\theta\|z_k-x^*\|)d\theta\|z_k-x^*\|\\ &\le \lambda(\|x_k-x^*\|)\|z_k-x^*\|\\ &\le \mu(\|x_k-x^*\|)\|x_k-x^*\|.\end{aligned}$$

Similarly, we get that

$$\begin{aligned}\|z_k^{(2)}-x^*\| &\le \lambda(\|x_k-x^*\|)\|z_k^{(1)}-x^*\|\\ &\le \lambda^2(\|x_k-x^*\|)\|z_k-x^*\|\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ \|z_k^{(i)}-x^*\| &\le \lambda^i(\|x_k-x^*\|)\|z_k^{(i-1)}-x^*\|\\ \|x_{k+1}-x^*\| = \|z_k^{(q)}-x^*\| &\le \lambda^q(\|x_k-x^*\|)\|z_k^{(q-1)}-x^*\|\\ &\le \mu(\|x_k-x^*\|)\|x_k-x^*\|.\end{aligned}$$

That is we have $x_k, z_k, z_k^{(i)} \in U(x^*, \varrho^*)$, $i=1,2,\ldots,q$ and

$$\|x_{k+1}-x^*\| \le \bar{c}\|x_k-x^*\|,$$

where $\bar{c}=\mu(\|x_0-x^*\|)\in[0,1)$, so $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1}\in U(x^*,\varrho^*)$. □

Remark (a) The result obtained here can be used for operators F satisfying autonomous differential equation [5] of the form

$$F'(x) = T(F(x)),$$

where T is a known continuous operator. Since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose: $T(x) = x + 1$.

(b) It is worth noticing that methods (14) and (3) do not change when we use the conditions of Theorem 3 instead of stronger conditions used in Theorems 1 and 2. Moreover, we can compute the computational order of convergence (COC) [32] defined by

$$COC = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) \Big/ \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right), \quad n = 1, 2, \ldots \tag{39}$$

or the approximate computational order of convergence (ACOC) [14], given by

$$ACOC = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) \Big/ \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right). \quad n = 1, 2, \ldots \tag{40}$$

This way we obtain in practice the order of convergence.

(c) Numerous choices for function $\varphi_\alpha^{(p)}$ are possible. Let us choose, e.g. $p = 4$, $\alpha = 1$ and

$$\varphi_1^{(4)}(x_n, y_n) = y_n - F'(y_n)^{-1} F(y_n), \tag{41}$$

which is a fourth order iteration function. Then, we can have as in (29) that

$$\begin{aligned}\|z_n - x^*\| &\le \frac{\int_0^1 w(1-\theta)\|y_n - x^*\|)d\theta\|y_n - x^*\|}{1 - w_0(\|y_n - p\|)} \\ &\le \frac{\int_0^1 w(1-\theta)g_1(\|x_n - x^*\|)\|x_n - x^*\|)d\theta g_1(\|x_n - x^*\|)\|x_n - x^*\|}{1 - w_0(g_1(\|x_n - x^*\|)\|x_n - x^*\|)}.\end{aligned}$$

So, we can choose

$$g_2(t) = \frac{\int_0^1 w((1-\theta)g_1(t)t)g_1(t)d\theta}{1 - w_0(g_1(t)t)} \text{ and } \lambda = 1.$$

Then function g_3 is given by Eq. (22).

It is worth noticing that the definition of function g_2 (and consequently of function g_3) is not unique. Indeed, let $w_0(t) = l_0\, t$, $w(t) = l\, t$. Then we get $g_1(t) = \frac{lt}{2(1-l_0 t)}$, $g_2(t) = \left(\frac{l}{2}\right)^3 \frac{1}{(1-l_0 t)^2}$ and $\lambda = 4$ (see also Example 2 for $l_0 = 15$ and $l = 30$).

4 Numerical examples

Here, we shall demonstrate the theoretical results which we have proved in section 3. For this, the methods of the family (3) chosen, with the choices g_1, g_2, g_3 and $\varphi_1^{(4)}(x_n, y_n)$ given in Remark (c), are of order seven and ten that now we denote by M_7 and M_{10}, respectively. We consider two numerical examples, which are defined as follows:

Example 1 Let $B_1 = B_2 = C[0, 1]$. Consider the equation

$$x(s) = \int_0^1 T(s,t)\left(\frac{1}{2}x(t)^{\frac{3}{2}} + \frac{x(t)^2}{8}\right)dt, \tag{42}$$

where the kernel T is the Green's function defined on the interval $[0, 1] \times [0, 1]$ by

$$T(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases} \tag{43}$$

Table 1 Numerical results of Example 1

M_7	M_{10}
$\varrho_1 = 10.5212$	$\varrho_1 = 10.5212$
$\varrho^{(1)} = 6.00895$	$\varrho^{(2)} = 4.05263$
$\varrho^* = 6.00895$	$\varrho^* = 4.05263$

Table 2 Numerical results of Example 2

M_7	M_{10}
$\varrho_1 = 0.033333$	$\varrho_1 = 0.033333$
$\varrho^{(1)} = 0.021401$	$\varrho^{(2)} = 0.012603$
$\varrho^* = 0.021401$	$\varrho^* = 0.012603$

Define operator $F : C[0, 1] \rightarrow C[0, 1]$ by

$$F(x)(s) = x(s) - \int_0^1 T(s,t)\left(\frac{1}{2}x(t)^{\frac{3}{2}} + \frac{x(t)^2}{8}\right)dt,$$

then,

$$F'(x)\mu(s) = \mu(s) - \int_0^1 T(s,t)\left(\frac{3}{4}x(t)^{\frac{1}{2}} + \frac{x(t)}{4}\right)\mu(t)dt. \tag{44}$$

Notice that $x^*(s) = 0$ is a solution of (42). Using (43), we obtain

$$\left\| \int_0^1 T(s,t)dt \right\| \leq \frac{1}{8}.$$

Then, by (43) and (a4), we have that

$$\|F'(x) - F'(y)\| \leq \frac{1}{32}\left(3\|x-y\|^{\frac{1}{2}} + \|x-y\|\right). \tag{45}$$

Hence, we can set $w_0(t) = w(t) = \frac{1}{32}\left(3t^{\frac{1}{2}} + t\right)$ and $v(t) = 1 + w_0(t)$. Numerical results are displayed in Table 1. From the numerical values we observe that $\varrho* > 1$. But, since we cannot go outside the unit ball, we choose $\varrho*$ to be the maximum available value which is 1. to be the maximum available value which is 1. Thus, for both methods M_7 and M_{10}, we have $\varrho* = 1$.

Thus the convergence of the methods M_7 and M_{10} to $x^*(s) = 0$ is guaranteed, provided that $x_0 \in U(x^*, \varrho^*)$. Notice that in view of (45) earlier results using hypotheses on the second derivative or higher cannot be used to solve this problem [3,4,26].

Example 2 Let $B_1 = B_2 = C[0, 1]$, be the space of continuous functions defined on the interval $[0, 1]$ and be equipped with max norm. Let $\Omega = \bar{U}(0, 1)$. Define function F on Ω by

$$F(\varphi)(x) = \phi(x) - 10\int_0^1 x\theta\varphi(\theta)^3 d\theta.$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 30\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \quad for\ \ each\ \ \xi \in \Omega.$$

Then for $x^* = 0$ we have $w_0(t) = 15t$, $w(t) = 30t$, $v(t) = 2$. The parameters are shown in Table 2.

Thus, the methods M_7 and M_{10} converge to $x^* = 0$, provided that $x_0 \in U(x^*, \varrho^*)$.

5 Applications

We apply the methods M_7 and M_{10} of the proposed family (3) to solve systems of nonlinear equations in $\mathbb{R}^m$. A comparison between the performance of present methods with existing higher order methods is also shown. For example, we choose the fifth order method proposed by Madhu et al. [23]; sixth order methods by Parhi and Gupta [26], Esmaeili and Ahmadi [15], Behl et al. [9] and Grau et al. [17]; eighth order method by Sharma and Arora [28]. These methods are given as follows:

Madhu–Babajee–Jayaraman method (MBJ_5):

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n),\\ x_{n+1} &= y_n - HF'(x_n)^{-1}F(y_n), \end{aligned}$$

where $H = 2I - t(x_n) + \frac{5}{4}(t(x_n) - 1)^2$ and $t(x_n) = F'(x_n)^{-1}F'(y_n)$.

Behl–Cordero–Motsa–Torregrosa method ($BCMT_6$)

$$\begin{aligned} y_n &= x_n - aF'(x_n)^{-1}F(x_n),\\ z_n &= y_n - \big(bF'(x_n)^{-1} + (cF'(x_n) + dF'(y_n))^{-1}\big)F(x_n),\\ x_{n+1} &= z_n - \big(gF'(x_n)^{-1} + (eF'(x_n) + hF'(y_n))^{-1}\big)F(z_n), \end{aligned}$$

where $a = \frac{2}{3}$, $b = -\frac{1}{6}$, $c = -1$, $d = 3$, $g = \frac{1}{2}$, $e = -\frac{2g+1}{2(g-1)^2}$ and $h = \frac{3}{2(g-1)^2}$.

Parhi–Gupta method (PG_6):

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n),\\ z_n &= y_n - 2\big(F'(x_n) + F'(y_n)\big)^{-1}F(x_n),\\ x_{n+1} &= z_n - F'(x_n)^{-1}F(z_n)(3F'(y_n) - F'(x_n))^{-1}(F'(x_n) + F'(y_n)). \end{aligned}$$

Esmaeili–Ahmadi method (EA_6):

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n),\\ z_n &= y_n + \frac{1}{3}\big(F'(x_n)^{-1} + 2(F'(x_n) - 3F'(y_n))^{-1}\big)F(x_n),\\ x_{n+1} &= z_n + \frac{1}{3}\big(-F'(x_n)^{-1} + 4(F'(x_n) - 3F'(y_n))^{-1}\big)F(z_n). \end{aligned}$$

Grau–Grau–Noguera method (GGN_6):

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n),\\ z_n &= y_n - \big(2[y_n, x_n; F] - F'(x_n)\big)^{-1}F(y_n),\\ x_{n+1} &= z_n - \big(2[y_n, x_n; F] - F'(x_n)\big)^{-1}F(z_n). \end{aligned}$$

Sharma–Arora method (SA_8):

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n),\\ z_n &= y_n - \left(\frac{13}{4}I - G_n\left(\frac{7}{2}I - \frac{5}{4}G_n\right)\right)F'(x_n)^{-1}F(y_n),\\ x_{n+1} &= z_n - \left(\frac{7}{2}I - G_n\left(4I - \frac{3}{2}G_n\right)\right)F'(x_n)^{-1}F(z_n), \end{aligned}$$

where $G_n = F'(x_n)^{-1}F'(y_n)$.

The programs are performed in the processor, AMD A8-7410 APU with AMU Radeon R5 Graphics @ 2.20 GHz(64 bit Operating System) Microsoft Window 10 Ultimate 2016 and are complied by *Mathematica* using multi-precision arithmetics. For every method, we record the number of iterations (n) needed to converge to the solution such that the stopping criterion

$$||x_{n+1} - x_n|| + ||F(x_n)|| < 10^{-400}$$

is satisfied. In order to verify the theoretical order of convergence, we calculate the approximate computational order of convergence (ACOC) using the formula (40). In the comparison of performance of methods, we also include CPU time utilized in the execution of program which is computed by the *Mathematica* command "TimeUsed[ ]".

We consider the nonlinear integral equation $F(x) = 0$ where

$$F(x)(s) = x(s) - 1 + \frac{1}{2}\int_0^1 s\cos(x(t))dt, \tag{46}$$

wherein $s \in [0, 1]$ and $x \in \Omega = U(0, 2) \subset X$. Here, $X = C[0, 1]$ is the space of continuous functions on [0, 1] with the max-norm,

$$\|x\| = \max_{s\in[0,1]}|x(s)|.$$

Integral equation of this kind is called Chandrasekhar equation (see [11]) which arises in the study of radiative transfer theory, neutron transport problems and kinetic theory of the gases.

Using the trapezoidal rule of integration with step $h = 1/m$ to discretize (46), we obtain the following system of nonlinear equations

$$0 = x_i - 1 + \frac{s_i}{2m}\left(\frac{1}{2}\cos(x_0) + \sum_{j=1}^{m-1}\cos(x_j) + \frac{1}{2}\cos(x_m)\right), \quad i = 0, 1, \ldots, m, \tag{47}$$

where $s_i = t_i = i/m$ and $x_i = x(t_i)$ with $x_0 = 1/2$. We apply the methods to solve (47) for the size $m = 8, 25, 50, 100$ by selecting the initial value $\{x_1, x_2, \ldots, x_m\}^T = \{\frac{1}{10}, \frac{1}{10}, \overset{m-times}{\cdots\cdots}, \frac{1}{10}\}^T$ towards the required solutions of the systems. The corresponding solutions are given by

$$\{0.9565\ldots, 0.9130\ldots, 0.8696\ldots, 0.8261\ldots, 0.7827\ldots, 0.7392\ldots, 0.6958\ldots, 0.6523\ldots\}^T,$$

$$\begin{aligned}\{&0.9864\ldots, 0.9728\ldots, 0.9593\ldots, 0.9457\ldots, 0.9321\ldots, 0.9186\ldots, 0.9050\ldots, 0.8915\ldots, 0.8779\ldots, 0.8643\ldots,\\ &0.8508\ldots, 0.8372\ldots, 0.8237\ldots, 0.8101\ldots, 0.7965\ldots, 0.7830\ldots, 0.7694\ldots, 0.7559\ldots, 0.7423\ldots, 0.7287\ldots,\\ &0.7152\ldots, 0.7016\ldots, 0.6881\ldots, 0.6745\ldots, 0.6609\ldots\}^T,\end{aligned}$$

$$\begin{aligned}\{&0.9932\ldots, 0.9865\ldots, 0.9797\ldots, 0.9730\ldots, 0.9663\ldots, 0.9595\ldots, 0.9528\ldots, 0.9460\ldots, 0.9393\ldots, 0.9326\ldots,\\ &0.9258\ldots, 0.9191\ldots, 0.9123\ldots, 0.9056\ldots, 0.8989\ldots, 0.8921\ldots, 0.8854\ldots, 0.8786\ldots, 0.8719\ldots, 0.8652\ldots,\\ &0.8584\ldots, 0.8517\ldots, 0.8449\ldots, 0.8382\ldots, 0.8315\ldots, 0.8247\ldots, 0.8180\ldots, 0.8112\ldots, 0.8045\ldots, 0.7978\ldots,\\ &0.7910\ldots, 0.7843\ldots, 0.7775\ldots, 0.7708\ldots, 0.7641\ldots, 0.7573\ldots, 0.7506\ldots, 0.7439\ldots, 0.7371\ldots, 0.7304\ldots,\\ &0.7236\ldots, 0.7169\ldots, 0.7102\ldots, 0.7034\ldots, 0.6967\ldots, 0.6899\ldots, 0.6832\ldots, 0.6765\ldots, 0.6697\ldots, 0.6630\ldots\}^T.\end{aligned}$$

Table 3 Comparison of performance of methods for problem (47)

Methods	MBJ_5	$BCMT_6$	PG_6	EA_6	GGN_6	SA_8	M_7	M_{10}
$m = 8$								
n	5	4	4	4	4	4	4	3
$\|\|x_{n+1} - x_n\|\|$	1.15(−222)	5.51(−218)	1.82(−179)	7.66(−214)	4.08(−216)	3.97(−370)	7.44(−300)	3.03(−81)
ACOC	5.000	6.000	6.000	6.000	6.000	8.000	7.000	10.000
CPU-time	13.58	14.95	12.05	15.28	16.64	12.33	11.92	11.20
$m = 25$								
n	5	4	4	4	4	4	4	3
$\|\|x_{n+1} - x_n\|\|$	8.33(−216)	1.33(−219)	7.52(−175)	3.26(−212)	1.37(−207)	2.10(−363)	2.24(−294)	2.10(−79)
ACOC	5.000	6.000	6.000	6.000	6.000	8.000	7.000	10.000
CPU-time	137.84	149.64	125.36	149.02	176.87	138.70	124.72	123.84
$m = 50$								
n	5	4	4	4	4	4	4	3
$\|\|x_{n+1} - x_n\|\|$	3.98(−214)	3.97(−220)	1.14(−173)	7.42(−212)	1.72(−231)	1.09(−361)	5.92(−293)	7.22(−79)
ACOC	5.000	6.000	6.000	6.000	6.000	8.000	7.000	10.000
CPU-time	583.82	660.15	598.59	725.77	654.31	585.81	573.38	495.71
$m = 100$								
n	5	4	4	4	4	4	4	3
$\|\|x_{n+1} - x_n\|\|$	3.25(−213)	2.34(−220)	5.27(−173)	1.27(−211)	1.62(−230)	9.49(−361)	3.67(−292)	1.60(−78)
ACOC	5.000	6.000	6.000	6.000	6.000	8.000	7.000	10.000
CPU-time	2470.94	2754.74	2461.00	2443.42	2921.77	2709.58	2436.52	2245.48

and

$$\{0.9966\ldots, 0.9932\ldots, 0.9899\ldots, 0.9865\ldots, 0.9832\ldots, 0.9798\ldots, 0.9764\ldots, 0.9731\ldots, 0.9697\ldots, 0.9664\ldots,$$
$$0.9630\ldots, 0.9596\ldots, 0.9563\ldots, 0.9529\ldots, 0.9496\ldots, 0.9462\ldots, 0.9428\ldots, 0.9395\ldots, 0.9361\ldots, 0.9328\ldots,$$
$$0.9294\ldots, 0.9260\ldots, 0.9227\ldots, 0.9193\ldots, 0.9160\ldots, 0.9126\ldots, 0.9092\ldots, 0.9059\ldots, 0.9025\ldots, 0.8992\ldots,$$
$$0.8958\ldots, 0.8924\ldots, 0.8891\ldots, 0.8857\ldots, 0.8824\ldots, 0.8790\ldots, 0.8756\ldots, 0.8723\ldots, 0.8689\ldots, 0.8656\ldots,$$
$$0.8622\ldots, 0.8589\ldots, 0.8555\ldots, 0.8521\ldots, 0.8488\ldots, 0.8454\ldots, 0.8421\ldots, 0.8387\ldots, 0.8353\ldots, 0.8320\ldots,$$
$$0.8286\ldots, 0.8253\ldots, 0.8219\ldots, 0.8185\ldots, 0.8152\ldots, 0.8118\ldots, 0.8085\ldots, 0.8051\ldots, 0.8017\ldots, 0.7984\ldots,$$
$$0.7950\ldots, 0.7917\ldots, 0.7883\ldots, 0.7849\ldots, 0.7816\ldots, 0.7782\ldots, 0.7749\ldots, 0.7379\ldots, 0.7345\ldots, 0.7312\ldots,$$
$$0.7715\ldots, 0.7681\ldots, 0.7648\ldots, 0.7614\ldots, 0.7581\ldots, 0.7547\ldots, 0.7513\ldots, 0.7480\ldots, 0.7446\ldots, 0.7413\ldots,$$
$$0.7278\ldots, 0.7245\ldots, 0.7211\ldots, 0.7178\ldots, 0.7144\ldots, 0.7110\ldots, 0.7077\ldots, 0.7043\ldots, 0.7010\ldots, 0.6976\ldots,$$
$$0.6942\ldots, 0.6909\ldots, 0.6875\ldots, 0.6842\ldots, 0.6808\ldots, 0.6774\ldots, 0.6741\ldots, 0.6707\ldots, 0.6674\ldots, 0.6640\ldots\}^T.$$

Numerical results are displayed in Table 3, which include:

- The dimension (m) of the system of equations.
- The required number of iterations (n).
- The error $||x_{n+1} - x_n||$ of approximation to the corresponding solution of considered problems, where $A(-h)$ denotes $A \times 10^{-h}$ in each table.
- The approximate computational order of convergence (ACOC) calculated by the formula (40).
- The elapsed CPU time (CPU-time) in seconds.

It is clear from the numerical results displayed in Table 3 that the new methods like the existing methods show stable convergence behavior. Also, observe that at same iteration the absolute value of error of approximating solution obtained by the higher order methods is smaller than the error by the lower order methods which justifies the superiority of higher order methods. From the calculation of computational order of convergence, it is also verified that the theoretical order of convergence is preserved. The CPU time used in the execution of program shows the efficient nature of proposed methods as compared to other methods. Similar numerical experimentations, carried out for a number of problems of different type, confirmed the above conclusions to a large extent.

References

1. Amat, S., Busquier, S., Plaza, S.: Dynamics of the King and Jarratt iterations. Aequationes Math. **69**, 212–223 (2005)
2. Amat, S., Busquier, S., Plaza, S.: Chaotic dynamics of a third-order Newton-type method. J. Math. Anal. Appl. **366**, 24–32 (2010)
3. Amat, S., Hernández, M.A., Romero, N.: A modified Chebyshev's iterative method with at least sixth order of convergence. Appl. Math. Comput. **206**, 164–174 (2008)
4. Argyros, I.K.: Convergence and Applications of Newton-type Iterations. Springer, New York (2008)
5. Argyros, I.K.: Computational Theory of Iterative Methods. In: Chui, C.K., Wuytack, L. (eds.) Studies in Computational Mathematics, vol. 15. Elsevier Publ. Co., New York (2007)
6. Argyros, I.K., Magreñán, Á.A.: Ball convergence theorems and the convergence planes of an iterative method for nonlinear equations. SeMa **71**, 39–55 (2015)
7. Argyros, I.K., Magreñán, Á.A.: Iterative Methods and Their Dynamics with Applications. CRC Press, New York (2017)
8. Argyros, I.K., Ren, H.: Improved local analysis for certain class of iterative methods with cubic convergence. Numer. Algorithms **59**, 505–521 (2012)
9. Behl, R., Cordero, A., Motsa, S.S., Torregrosa, J.R.: Stable high-order iterative methods for solving nonlinear models. Appl. Math. Comput. **303**, 70–88 (2017)

10. Candela, V., Marquina, A.: Recurrence relations for rational cubic methods I: the Halley method. Computing **44**, 169–184 (1990)
11. Chandrasekhar, S.: Radiative Transfer. Dover, New York (1960)
12. Chun, C., Stănică, P., Neta, B.: Third-order family of methods in Banach spaces. Comput. Math. Appl. **61**, 1665–1675 (2011)
13. Cordero, A., Ezquerro, J.A., Hernández-Veron, M.A., Torregrosa, J.R.: On the local convergence of a fifth-order iterative method in Banach spaces. Appl. Math. Comput. **251**, 396–403 (2015)
14. Cordero, A., Torregrosa, J.R.: Variants of Newton's method using fifth-order quadrature formulas. Appl. Math. Comput. **190**, 686–698 (2007)
15. Esmaeili, H., Ahmadi, M.: An efficient three-step method to solve system of non linear equations. Appl. Math. Comput. **266**, 1093–1101 (2015)
16. Ezquerro, J.A., Hernández, M.A.: New iterations of R-order four with reduced computational cost. BIT Numer. Math. **49**, 325–342 (2009)
17. Grau-Sánchez, M., Grau, Á., Noguera, M.: Ostrowski type methods for solving systems of nonlinear equations. Appl. Math. Comput. **218**, 2377–2385 (2011)
18. Hernández, M.A.: Chebyshev's approximation algorithms and applications. Comput. Math. Appl. **41**, 433–455 (2001)
19. Hernández, M.A., Martínez, E.: On the semilocal convergence of a three steps Newton-type process under mild convergence conditions. Numer. Algorithms **70**, 377–392 (2015)
20. Homeier, H.H.H.: A modified Newton method with cubic convergence: the multivariate case. J. Comput. Appl. Math. **169**, 161–169 (2004)
21. Kou, J.: A third-order modification of Newton method for system of nonlinear equation. Appl. Math. Comput. **191**, 117–121 (2007)
22. Lukić, T., Ralević, N.M.: Geometric mean Newton's method for simple and multiple roots. Appl. Math. Lett. **21**, 30–36 (2008)
23. Madhu, K., Babajee, D.K.R., Jayaraman, J.: An improvement to double-step Newton method and its multi-step version for solving system of nonlinear equations and its applications. Numer. Algorithms **74**, 593–607 (2017)
24. Neta, B.: A sixth order family of methods for nonlinear equations. Int. J. Comput. Math. **7**, 157–161 (1979)
25. Parhi, S.K., Gupta, D.K.: Recurrence relations for a Newton-like method in Banach spaces. J. Comput. Appl. Math. **206**, 873–887 (2007)
26. Parhi, S.K., Gupta, D.K.: A sixth order method for nonlinear equations. Appl. Math. Comput. **203**, 50–55 (2008)
27. Ren, H., Wu, Q., Bi, W.: New variants of Jarratt's method with sixth-order convergence. Numer. Algorithms **52**, 585–603 (2009)
28. Sharma, J.R., Arora, H.: Improved Newton-like methods for solving systems of nonlinear equations. SeMa **74**, 147–163 (2017)
29. Sharma, J.R., Guha, R.K., Sharma, R.: An efficient fourth-order weighted-Newton method for system of nonlinear equations. Numer. Algorithms **62**, 307–323 (2013)
30. Sharma, J.R., Sharma, R., Bahl, A.: An improved Newton–Traub composition for solving systems of nonlinear equations. Appl. Math. Comput. **290**, 98–110 (2016)
31. Traub, J.F.: Iterative Methods for the Solution of Equations. Chelsea Publishing Company, New York (1982)
32. Weerakoon, S., Fernando, T.G.I.: A variant of Newton's method with accelerated third order convergence. Appl. Math. Lett. **13**, 87–93 (2000)