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Fixed point theorems of integral contraction type mappings in fuzzy metric space

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Abstract In this paper, we show an existence and uniqueness of fixed point for contractive mappings of integral type using altering distance functions in fuzzy metric spaces. Moreover, we give examples to support our results. Our results generalize corresponding results given in the literature.

Keywords Altering distance · Contractive mappings of integral type · Fuzzy metric space

AMS subject classification 47H10 · 47H07

1 Introduction

Fixed point results have been studied in many contexts, one of which is the fuzzy setting. It is well thought that the fuzzy set concept plays an significant role in plenty of scientific and engineering applications. In 1965, the notion of fuzzy sets was presented by Zadeh [13]. After that, in 1975, Kramosil and Michalek [10] offered the notion of fuzzy metric spaces. It is emerge that a fuzzy metric space is a great extension of the metric space. Thereafter, many authors continued the study of Kramosil and achieve many fixed point results for contractive mappings in fuzzy metric spaces. See,e.g [6,7,10].

Branciari [4] was the first to study the existence of fixed points for the contractive mappings of integral type. He appointed good integral prescription of the Banach contraction principle [3]. The authors [8,9] and others continued the study of Branciari and showed the existence of fixed point theorems using a general contractive condition of integral type in complete metric spaces.

Lately, Shen et al. [12] presented the concept of altering distance in fuzzy metric spaces and showed a fixed point theorem in complete and compact fuzzy metric spaces.

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In this paper, we introduce two families of functions and obtain an existence and uniqueness theorems for contractive mappings of integral type using altering distance functions in fuzzy metric spaces. Throughout this paper, we assume that $\mathbf{R}^+ = [0, \infty)$ and Φ is the family of mappings on $\mathbf{R}^+$ such that are Lebesgue integrable, summable on each compact subset and for each $\epsilon > 0$, $\int_0^\epsilon \phi(t)\mathbf{dt} > 0$.

The following lemmas and definitions will be needed in the sequel.

Definition 1.1 [1] A binary operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous triangular norm (t-norm) if the following conditions hold:

1. $*$ is associative and commutative;
2. $*$ is continuous;
3. $c * 1 = c$ for all $c \in [0, 1]$;
4. $c * f \leq e * d$, whenever $c \leq e$ and $f \leq d$, for all $c, f, e, d \in [0, 1]$.

Basic t-norms [7] are: $e *_1 f = \min\{e, f\}$, $e *_2 f = e.f$, $e *_3 f = \max\{e + f - 1, 0\}$ and $e *_4 f = \frac{ef}{\max\{e,f,\lambda\}}$ for $\lambda \in (0, 1)$.

Definition 1.2 [1] A fuzzy metric space is a triple $(X, M, *)$, where X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty)$, satisfying, for all $e, f \in X$, the following properties:

1. $M(e, f, t) > 0$ for all $e, f \in X$ and $t > 0$;
2. $M(e, f, t) = 1$ for all $t > 0$ if and only if $e = f$;
3. $M(e, f, t) = M(f, e, t)$ for all $e, f \in X$ and $t > 0$;
4. $M(e, k, t + s) \geq M(e, f, t) * M(f, k, s)$ for all $k \in X$ and $t, s > 0$;
5. $M(e, f, .) : (0, \infty) \rightarrow [0, 1]$ is continuous for all $e, f \in X$.

Lemma 1.3 [1] $M(e, f, .)$ is non-decreasing for all $e, f \in X$.

Definition 1.4 [1] Let $(X, M, *)$ be a fuzzy metric space. Then

1. A sequence $\{y_n\}_{n\in\mathbf{N}}$ convergence to $y \in X$, that is $\lim_{n\rightarrow\infty} y_n = y$, if $\lim_{n\rightarrow\infty} M(y_n, y, t) = 1$ for all $t > 0$.
2. A sequence $\{y_n\}_{n\in\mathbf{N}}$ is called M-cauchy, if for each $\epsilon \in (0, 1)$ and $t > 0$ there exists $n_0 \in \mathbf{N}$ such that $M(y_n, y_m, t) > 1 - \epsilon$ for all $m, n \geq n_0$.
3. A sequence $\{y_n\}_{n\in\mathbf{N}}$ is called G-cauchy, if $\lim_{n\rightarrow\infty} M(y_n, y_{n+m}, t) = 1$ for all $t > 0$ and $m \in \mathbf{N}$.

A fuzzy metric space $(X, M, *)$ is called M-complete(G-complete) if every M-cauchy(G-cauchy) sequence is convergent.

Lemma 1.5 [8] Let $\gamma \in \Phi$ and $\{c_n\}_{n\in\mathbf{N}}$ be a nonnegative sequence with $\lim\limits_{n\rightarrow\infty} c_n = c$. Then

$$\lim_{n\rightarrow\infty} \int_0^{c_n} \gamma(t)\mathbf{dt} = \int_0^{c} \gamma(t)\mathbf{dt}.$$

Lemma 1.6 [9] Let $\gamma \in \Phi$ and $\{c_n\}_{n\in\mathbf{N}}$ be a nonnegative sequence. Then

$$\lim_{n\rightarrow\infty} \int_0^{c_n} \gamma(t)\mathbf{dt} = 0,$$

if and only if $\lim_{n\rightarrow\infty} c_n = 0$.

Definition 1.7 [2] A mapping $\varphi : [0,1] \to [0,1]$ is an altering distance if

1. φ is strictly decreasing and continuous;
2. $\varphi(\gamma) = 0$ if and only if $\gamma = 1$.

Lemma 1.8 [5] Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0, \infty)$.

2 Main results

Now we give the following definition.

Definition 2.1 Suppose that ζ_1 is the set of mapping $g : [0,\infty)^3 \to (0,\infty)$ satisfying the following conditions:

1. g is continuous;
2. If $\int_0^v \gamma(s)\mathbf{ds} \le k(t)\int_0^{g(v,u,u)} \gamma(s)\mathbf{ds}$, then $\int_0^v \gamma(s)\mathbf{ds} \le k(t)\int_0^u \gamma(s)\mathbf{ds}$;
3. If $\int_0^u \gamma(s)\mathbf{ds} \le k(t)\int_0^{g(0,u,0)} \gamma(s)\mathbf{ds}$ or $\int_0^u \gamma(s)\mathbf{ds} \le k(t)\int_0^{g(0,0,u)} \gamma(s)\mathbf{ds}$, then $u = 0$;

where $k : (0,\infty) \to (0,1)$ is a mapping and $\gamma \in \Phi$.

For example, the following functions belong to ζ_1.

1. If $g(a,b,c) = \max\{a,b,c\}$, then $g \in \zeta_1$.

   Obviously, g is continuous. If
   $$\int_0^v \gamma(s)\mathbf{ds} \le k(t)\int_0^{\max\{v,u,u\}} \gamma(s)\mathbf{ds} = k(t)\max\left\{\int_0^u \gamma(s)\mathbf{ds}, \int_0^v \gamma(s)\mathbf{ds}\right\},$$
   then we have $\int_0^v \gamma(s)\mathbf{ds} \le k(t)\int_0^u \gamma(s)\mathbf{ds}$.
   Also if
   $$\int_0^u \gamma(s)\mathbf{ds} \le k(t)\int_0^{\max\{0,0,u\}} \gamma(s)\mathbf{ds} = k(t)\max\left\{\int_0^0 \gamma(s)\mathbf{ds}, \int_0^u \gamma(s)\mathbf{ds}\right\} = k(t)\int_0^u \gamma(s)\mathbf{ds},$$
   so $(1-k(t))\int_0^u \gamma(s)\mathbf{ds} \le 0$, then $u = 0$. If
   $$\int_0^u \gamma(s)\mathbf{ds} \le k(t)\int_0^{\max\{0,u,0\}} \gamma(s)\mathbf{ds} = k(t)\max\left\{\int_0^0 \gamma(s)\mathbf{ds}, \int_0^u \gamma(s)\mathbf{ds}\right\} = k(t)\int_0^u \gamma(s)\mathbf{ds},$$
   so $(1-k(t))\int_0^u \gamma(s)\mathbf{ds} \le 0$ which implies that $u = 0$.
2. $g(a,b,c) = \frac{c+b}{2}$;
3. $g(a,b,c) = a$;
4. $g(a,b,c) = c$.

Now we prove our first result.

Theorem 2.2 *Suppose that $(X, M, *)$ is a G-complete fuzzy metric space and ϕ is an altering distance mapping and $f : X \to X$ is a mapping satisfying*
$$\int_0^{\phi(M(fx,fy,t))} \gamma(s)\mathbf{ds} \le k(t)\int_0^{g(\phi(M(x,fx,t)),\phi(M(y,fy,t)),\phi(M(x,y,t)))} \gamma(s)\mathbf{ds}, \quad x, y \in X,$$
where $g \in \zeta_1$ and $\gamma \in \Phi$ and let $k : (0,\infty) \to (0,1)$ be a function. Then f has a unique fixed point.

Proof Consider $x_{n+1} = fx_n$, for all $n \in \mathbf{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbf{N}$, hence x_n is a fixed point of f. Let $x_n \neq x_{n+1}$, for all $n \in \mathbf{N}$. First we show that the sequence $M_n = \{M(x_n, x_{n+1}, t)\}$ is an increasing. Contrariwise, suppose that there exists $n_0 \in \mathbf{N}$ such that $M_{n_0} \leq M_{n_0-1}$. Since ϕ is strictly decreasing, so $\phi(M_{n_0}) > \phi(M_{n_0-1})$. We have

$$\begin{aligned}\int_0^{\phi(M_{n_0-1})} \gamma(s)\mathbf{ds} \leq \int_0^{\phi(M_{n_0})} \gamma(s)\mathbf{ds} &= \int_0^{\phi(M(x_{n_0},x_{n_0+1},t))} \gamma(s)\mathbf{ds}\\ &= \int_0^{\phi(M(fx_{n_0-1},fx_{n_0},t))} \gamma(s)\mathbf{ds}\\ &\leq k(t)\int_0^{g(\phi M(x_{n_0-1},x_{n_0},t),\phi M(x_{n_0},x_{n_0+1},t),\phi M(x_{n_0-1},x_{n_0},t))} \gamma(s)\mathbf{ds}\end{aligned}$$

so definition 2.1, follows that

$$\int_0^{\phi(M_{n_0-1})} \gamma(s)\mathbf{ds} \leq \int_0^{\phi(M_{n_0})} \gamma(s)\mathbf{ds} \leq k(t)\int_0^{\Phi(M_{n_0-1})} \gamma(s)\mathbf{ds},$$

that is a contradiction. Thus the sequence $\{M_n\}$ is an increasing sequence of positive real numbers in [0, 1]. Now we show that $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$. We have

$$\begin{aligned}\int_0^{\phi(M(x_{n+1},x_n,t))} \gamma(s)\mathbf{ds} &= \int_0^{\phi(M(fx_n,fx_{n-1},t))} \gamma(s)\mathbf{ds}\\ &\leq k(t)\int_0^{g(\phi M(x_n,fx_n,t),\phi M(x_{n-1},fx_{n-1},t),\phi M(x_n,x_{n-1},t))} \gamma(s)\mathbf{ds}\\ &= k(t)\int_0^{g(\phi M(x_n,x_{n+1},t),\phi M(x_{n-1},x_n,t),\phi M(x_{n-1},x_n,t))} \gamma(s)\mathbf{ds},\end{aligned} \tag{2.1}$$

so by definition 2.1

$$\int_0^{\phi(M(x_{n+1},x_n,t))} \gamma(s)\mathbf{ds} \leq k(t)\int_0^{\phi(M(x_{n-1},x_n,t))} \gamma(s)\mathbf{ds},$$

again

$$\begin{aligned}\int_0^{\phi(M(x_n,x_{n-1},t))} \gamma(s)\mathbf{ds} &= \int_0^{\phi(M(fx_{n-1},fx_{n-2},t))} \gamma(s)\mathbf{ds}\\ &\leq k(t)\int_0^{g(\phi M(x_{n-1},x_n,t),\phi M(x_{n-2},x_{n-1},t),\phi M(x_{n-1},x_{n-2},t))} \gamma(s)\mathbf{ds},\end{aligned} \tag{2.2}$$

then definition 2.1 follows that

$$\int_0^{\phi(M(x_n,x_{n-1},t))} \gamma(s)\mathbf{ds} \leq k(t)\int_0^{\phi(M(x_{n-2},x_{n-1},t))} \gamma(s)\mathbf{ds},$$

by 2.2 and 2.1, we get

$$\int_0^{\phi(M(x_{n+1},x_n,t))} \gamma(s)\mathbf{ds} \leq k^2(t)\int_0^{\phi(M(x_{n-2},x_{n-1},t))} \gamma(s)\mathbf{ds},$$

similarly we have

$$\int_0^{\phi(M(x_{n+1},x_n,t))} \gamma(s)\mathbf{ds} \leq k^n(t) \int_0^{\phi(M(x_0,x_1,t))} \gamma(s)\mathbf{ds},$$

since $0 < k(t) < 1$, by letting $n \to \infty$ we have

$$\lim_{n\to\infty} \int_0^{\phi(M(x_{n+1},x_n,t))} \gamma(s)\mathbf{ds} \leq 0,$$

by lemma 1.6, $\lim_{n\to\infty} \phi(M(x_{n+1}, x_n, t)) = 0$, by definition 1.7, we have $\lim_{n\to\infty}$ $M(x_{n+1}, x_n, t)$
$= 1$. For fixed $s \in \mathbf{N}$, we have

$$M(x_n, x_{n+s}, t) \geq M\left(x_n, x_{n+1}, \frac{t}{s}\right) * M\left(x_{n+1}, x_{n+2}, \frac{t}{s}\right) * \cdots * M\left(x_{n+s-1}, x_{n+s}, \frac{t}{s}\right),$$

thus

$$M(x_n, x_{n+s}, t) \geq 1 * 1 * \cdots * 1 = 1,$$

as $n \to \infty$ and thus $\{x_n\}$ is a G-cauchy sequence. Therefore $\{x_n\}$ converges to x for some $x \in X$. We have

$$\int_0^{\phi(M(x_n,fx,t))} \gamma(s)\mathbf{ds} = \int_0^{\phi(M(fx_{n-1},fx,t))} \gamma(s)\mathbf{ds} \leq k(t) \int_0^{g(\phi M(x_{n-1},x_n,t),\phi M(x,fx,t),\phi M(x_{n-1},x,t))} \gamma(s)\mathbf{ds}, \quad (2.3)$$

since g, ϕ are continuous, by letting $n \to \infty$ and lemma 1.5, we get

$$\lim_{n\to\infty} \int_0^{\phi(M(x_n,fx,t))} \gamma(s)\mathbf{ds} \leq k(t) \int_0^{g(\lim_{n\to\infty} \phi M(x_{n-1},x_n,t), \lim_{n\to\infty} \phi M(x,fx,t), \lim_{n\to\infty} \phi M(x_{n-1},x,t))} \gamma(s)\mathbf{ds},$$

so we obtain

$$\int_0^{\lim_{n\to\infty} \phi(M(x_n,fx,t))} \gamma(s)\mathbf{ds} \leq k(t) \int_0^{g(0,\phi(M(x,fx,t)),0)} \gamma(s)\mathbf{ds},$$

so

$$\int_0^{\phi(M(x,fx,t))} \gamma(s)\mathbf{ds} \leq k(t) \int_0^{g(0,\phi(M(x,fx,t)),0)} \gamma(s)\mathbf{ds}.$$

Therefore from definition 2.1, $\phi(M(x, fx, t)) = 0$. Then $M(x, fx, t) = 1$, i.e $x = fx$. We will show that x is a unique fixed point. Suppose that y is a another fixed point of f, i.e $y = fy$ with $y \neq x$. We have

$$\int_0^{\phi(M(x,y,t))} \gamma(s)\mathbf{ds} = \int_0^{\phi(M(fx,fy,t))} \gamma(s)\mathbf{ds} \leq k(t) \int_0^{g(\phi M(x,fx,t),\phi M(y,fy,t),\phi M(x,y,t))} \gamma(s)\mathbf{ds}, \quad (2.4)$$

hence

$$\int_0^{\phi(M(x,y,t))} \gamma(s)\mathbf{ds} \leq k(t) \int_0^{g(0,0,\phi(M(x,y,t)))} \gamma(s)\mathbf{ds},$$

then applying definition 2.1 we get $\phi(M(x, y, t)) = 0$, then $M(x, y, t) = 1$, i.e $x = y$. □

Now we give an example to support our theorem.

Example 2.3 Suppose that $X = \{A, B, C, D, E\}$ is the subset of $\mathbf{R}^2$, where $A = (0, 0)$, $B = (1, 0)$, $C = (1, 2)$, $D = (0, 1)$, $E = (1, 4)$ and $f : X \to X$ is a mapping such that $f(A) = f(B) = f(C) = f(D) = A$ and $f(E) = B$. Also let $k : (0, \infty) \to (0, 1)$

$$k(t) = \begin{cases} \frac{t}{\frac{1}{5}+t}, & t \in (0, 4] \\ 1 - \mathbf{e}^{\frac{-5}{t}}, & t \in (4, \infty) \end{cases}$$

and $\gamma(s) = 2s$ and define $\phi : [0, 1] \to [0, 1]$ by $\phi(r) = 1 - \sqrt{r}$. It is easy to check that $M(x, y, t) = \frac{t}{t+d(x,y)}$ is a fuzzy metric space , $t > 0$, where by $d(x, y)$ is metric max in $\mathbf{R}^2$, let $g : [0, \infty)^3 \to (0, \infty)$ be $g(a, b, c) = c$. Note that, by $(X, M, *)$ is given a complete fuzzy metric space with respect to the t-norm $* = x.y$. We have

$$g(\phi M(x, fx, t), \phi M(y, fy, t), \phi M(x, y, t)) = \phi M(x, y, t) = 1 - \sqrt{\frac{t}{t + \max\{0, 4\}}} = 1 - \sqrt{\frac{t}{t+4}},$$

and

$$M(fx, fy, t) = \frac{t}{t+1}.$$

Therefore

$$\int_0^{\phi(M(fx,fy,t))} \gamma(s)\mathbf{ds} = \int_0^{1-\sqrt{\frac{t}{t+1}}} 2s\mathbf{ds} = \left(1 - \sqrt{\frac{t}{t+1}}\right)^2,$$

and

$$\int_0^{\phi(M(x,y,t))} \gamma(s)\mathbf{ds} = \int_0^{1-\sqrt{\frac{t}{t+4}}} 2s\mathbf{ds} = \left(1 - \sqrt{\frac{t}{t+4}}\right)^2.$$

We have two cases: If $t \in (0, 4]$, then

$$\int_0^{\phi(M(fx,fy,t))} \gamma(s)\mathbf{ds} \leq \left(\frac{t}{\frac{1}{5}+t}\right) \int_0^{\phi(M(x,y,t))} \gamma(s)\mathbf{ds}.$$

If $t \in (4, \infty)$, then

$$\int_0^{\phi(M(fx,fy,t))} \gamma(s)\mathbf{ds} \leq \left(1 - \mathbf{e}^{\frac{-5}{t}}\right) \int_0^{\phi(M(x,y,t))} \gamma(s)\mathbf{ds}.$$

Therefore all conditions of theorem 2.2 are satisfied. Then f has a unique fixed point in X. We see that $A \in X$ is the unique fixed point of f.

In theorem 2.2, set $g(a, b, c) = c$ and $\gamma(s) = 1$, then we can obtain the following theorem.

Theorem 2.4 *Suppose that* (X, M, T) *is a G-complete fuzzy metric space and* $f : X \to X$ *is a mapping and let* $\varphi : [0, 1] \to [0, 1]$ *be altering distance function. Furthermore, let* k

be a function from $(0, \infty)$ *into* $(0, 1)$. *If for any* $t > 0$, *mapping* f *satisfies the following condition*

$$\varphi(M(fx, fy, t)) \leq k(t).\varphi(M(x, y, t)),$$

where $x, y \in X$ *and* $x \neq y$, *then* f *has a unique fixed point.*

Also in theorem 2.2, set $g(a, b, c) = \max\{a, b, c\}$ and $\gamma(s) = 1$, then we can obtain the following theorem.

Theorem 2.5 *Suppose that* (X, M, T) *is a G-complete fuzzy metric space and* $f : X \to X$ *is a mapping and assume that* $\varphi : [0, 1] \to [0, 1]$ *is a altering distance function. Furthermore, let k be a function from* $(0, \infty)$ *into* $(0, 1)$. *If for any* $t > 0$, *mapping* f *satisfies the following condition*

$$\varphi(M(fx, fy, t)) \leq k(t). \max\{\varphi(M(x, fx, t)), \varphi(M(y, fy, t)), \varphi(M(x, y, t))\},$$

where $x, y \in X$ *and* $x \neq y$, *then* f *has a unique fixed point.*

In this section, we prove our next theorem. Assume that [7] Φ_1 is the family of all right continuous mappings, and $\epsilon : [0, \infty) \to [0, \infty)$, with $\epsilon(r) < r$, for all $r > 0$. Note that for every function $\epsilon \in \Phi_1$, we have $\lim_{n\to\infty} \epsilon^n(r) = 0$. Now, we mention the following definition.

Definition 2.6 Suppose that ζ_2 is the family of functions $g : [0, \infty)^3 \to (0, \infty)$ satisfying the following conditions:

1. g is continuous;
2. If $\int_0^v \gamma(s)\mathbf{ds} \leq \epsilon\left(\int_0^{g(v,u,u)} \gamma(s)\mathbf{ds}\right)$, then $\int_0^v \gamma(s)\mathbf{ds} \leq \epsilon\left(\int_0^u \gamma(s)\mathbf{ds}\right)$;
3. If $\int_0^u \gamma(s)\mathbf{ds} \leq \epsilon\left(\int_0^{g(0,u,0)} \gamma(s)\mathbf{ds}\right)$ or $\int_0^u \gamma(s)\mathbf{ds} \leq \epsilon\left(\int_0^{g(0,0,u)} \gamma(s)\mathbf{ds}\right)$ for all $\epsilon \in \Phi_1$ then $u = 0$;

Similar to the previous, the following functions belong to ζ_2.

1. $g(a, b, c) = \max\{a, b, c\}$;
2. $g(a, b, c) = \frac{b+c}{2}$;
3. $g(a, b, c) = c$.

Definition 2.7 [7] Suppose that $(X, M, *)$ is a fuzzy metric space. We say that $f : X \to X$ is β-admissible if there exists a function $\beta : X \times X \times (0, \infty) \to (0, \infty)$ such that, for all $t > 0$,

$$x, y \in X, \quad \beta(x, y, t) \geq 1 \quad \text{implies} \quad \beta(fx, fy, t) \geq 1.$$

Now, we prove our next result.

Theorem 2.8 *Assume that* $(X, M, *)$ *is a G-complete fuzzy metric space and* ϕ *is an altering distance function and* $f : X \to X$ *is a* β*-admissible mapping satisfying*

$$\beta(x, y, t) \int_0^{\phi(M(fx,fy,t))} \gamma(s)\mathbf{ds} \leq \epsilon\left(\int_0^{g(\phi(M(x,fx,t)),\phi(M(y,fy,t)),\phi(M(x,y,t)))} \gamma(s)\mathbf{ds}\right), \quad x, y \in X,$$

where $g \in \zeta$, $\gamma \in \Phi$ *and* $\epsilon \in \Phi_1$. *Also there exists* $x_0 \in X$ *such that* $\beta(x_0, fx_0, t) \geq 1$, *for all* $t > 0$. *Then* f *has a fixed point.*

Proof Consider $x_0 \in X$ such that $\beta(x_0, fx_0, t) \geq 1$, for all $t > 0$. Define $x_{n+1} = fx_n$, for all $n \in \mathbf{N}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbf{N}$, then x_{n_0} is a fixed point of f. Assume that $x_n \neq x_{n+1}$, for all $n \in \mathbf{N}$. Since f is β-admissible, we have

$$\beta(x_0, x_1, t) = \beta(x_0, fx_0, t) \geq 1,$$

again

$$\beta(fx_0, fx_1, t) = \beta(x_1, x_2, t) \geq 1.$$

By induction, we deduce that

$$\beta(x_n, x_{n+1}, t) \geq 1, \quad \text{for all} \quad n \in \mathbf{N} \quad \text{and} \quad t > 0.$$

First we show that the sequence $M_n = \{M(x_n, x_{n+1}, t)\}$ is an increasing. On the opposite, suppose that there exists $n_0 \in \mathbf{N}$ such that $M_{n_0} \leq M_{n_0-1}$. Since ϕ is strictly decreasing, so $\phi(M_{n_0}) > \phi(M_{n_0-1})$. We have

$$\begin{aligned}
\int_0^{\phi(M_{n_0-1})} \gamma(s)\mathbf{ds} \leq \int_0^{\phi(M_{n_0})} \gamma(s)\mathbf{ds} &= \int_0^{\phi(M(x_{n_0}, x_{n_0+1}, t))} \gamma(s)\mathbf{ds} \\
&\leq \beta(x_{n_0-1}, x_{n_0}, t) \int_0^{\phi(M(fx_{n_0-1}, fx_{n_0}, t))} \gamma(s)\mathbf{ds} \\
&\leq \epsilon\left(\int_0^{g(\phi M(x_{n_0-1}, x_{n_0}, t), \phi M(x_{n_0}, x_{n_0+1}, t), \phi M(x_{n_0-1}, x_{n_0}, t))} \gamma(s)\mathbf{ds}\right)
\end{aligned}$$

therefore by definition 2.6

$$\int_0^{\phi(M_{n_0-1})} \gamma(s)\mathbf{ds} \leq \int_0^{\phi(M_{n_0})} \gamma(s)\mathbf{ds} \leq \epsilon\left(\int_0^{\phi(M_{n_0-1})} \gamma(s)\mathbf{ds}\right),$$

since $\epsilon(r) < r$, we get

$$\int_0^{\phi(M_{n_0-1})} \gamma(s)\mathbf{ds} < \int_0^{\phi(M_{n_0-1})} \gamma(s)\mathbf{ds},$$

that is a inconsistency. Thus the sequence $\{M_n\}$ is an increasing sequence of positive real numbers in [0, 1]. In this section, we show that $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$. We have

$$\begin{aligned}
\int_0^{\phi(M(x_{n+1}, x_n, t))} \gamma(s)\mathbf{ds} &= \int_0^{\phi(M(fx_n, fx_{n-1}, t))} \gamma(s)\mathbf{ds} \\
&\leq \beta(x_n, x_{n-1}, t) \int_0^{\phi(M(fx_n, fx_{n-1}, t))} \gamma(s)\mathbf{ds} \\
&\leq \epsilon\left(\int_0^{g(\phi M(x_{n+1}, x_n, t), \phi M(x_{n-1}, x_n, t), \phi M(x_{n-1}, x_n, t))} \gamma(s)\mathbf{ds}\right),
\end{aligned} \tag{2.5}$$

so by definition 2.6, we have

$$\int_0^{\phi(M(x_{n+1}, x_n, t))} \gamma(s)\mathbf{ds} \leq \epsilon\left(\int_0^{\phi(M(x_{n-1}, x_n, t))} \gamma(s)\mathbf{ds}\right), \tag{2.6}$$

again

$$\begin{aligned}\int_0^{\phi(M(x_n,x_{n-1},t))}\gamma(s)\mathbf{ds} &= \int_0^{\phi(M(fx_{n-1},fx_{n-2},t))}\gamma(s)\mathbf{ds}\\ &\leq \beta(x_{n-1},x_{n-2},t)\int_0^{\phi(M(fx_{n-1},fx_{n-2},t))}\gamma(s)\mathbf{ds}\\ &\leq \epsilon\left(\int_0^{g(\phi M(x_{n-1},x_n,t),\phi M(x_{n-2},x_{n-1},t),\phi M(x_{n-1},x_{n-2},t))}\gamma(s)\mathbf{ds}\right),\end{aligned} \tag{2.7}$$

then definition 2.6 follows that

$$\int_0^{\phi(M(x_n,x_{n-1},t))}\gamma(s)\mathbf{ds} \leq \epsilon\left(\int_0^{\phi(M(x_{n-2},x_{n-1},t))}\gamma(s)\mathbf{ds}\right), \tag{2.8}$$

by 2.6 and 2.8, we get

$$\int_0^{\phi(M(x_{n+1},x_n,t))}\gamma(s)\mathbf{ds} \leq \epsilon^2\left(\int_0^{\phi(M(x_{n-2},x_{n-1},t))}\gamma(s)\mathbf{ds}\right),$$

similarly

$$\int_0^{\phi(M(x_{n+1},x_n,t))}\gamma(s)\mathbf{ds} \leq \epsilon^n\left(\int_0^{\phi(M(x_0,x_1,t))}\gamma(s)\mathbf{ds}\right),$$

since $\lim_{n\to\infty}\epsilon^n(r) = 0$, by letting $n \to \infty$ we get

$$\lim_{n\to\infty}\int_0^{\phi(M(x_{n+1},x_n,t))}\gamma(s)\mathbf{ds} \leq 0,$$

by lemma 1.6, $\lim_{n\to\infty}\phi(M(x_{n+1},x_n,t)) = 0$, by definition 1.7, we have $\lim_{n\to\infty} M(x_{n+1},x_n,t)$
$= 1$. Now, we prove that

$$\lim_{n\to\infty} M(x_n,x_{n+s},t) = 1.$$

For fixed $s \in \mathbf{N}$, we have

$$\begin{aligned}M(x_n,x_{n+s},t) &\geq M\left(x_n,x_{n+1},\frac{t}{s}\right) * M\left(x_{n+1},x_{n+2},\frac{t}{s}\right)\\ &\quad * \cdots * M\left(x_{n+s-1},x_{n+s},\frac{t}{s}\right),\end{aligned}$$

thus

$$M(x_n,x_{n+s},t) \geq 1 * 1 * \cdots * 1,$$

as $n \to \infty$ and thus $\{x_n\}$ is a G-cauchy sequence. Therefore $\{x_n\}$ converges to x for some $x \in X$. We have

$$\begin{aligned}\int_0^{\phi(M(x_n,fx,t))}\gamma(s)\mathbf{ds} &= \int_0^{\phi(M(fx_{n-1},fx,t))}\gamma(s)\mathbf{ds}\\ &\le \beta(x_{n-1},x,t)\int_0^{\phi(M(fx_{n-1},fx,t))}\gamma(s)\mathbf{ds}\\ &\le \epsilon\left(\int_0^{g(\phi M(x_{n-1},x_n,t),\phi M(x,fx,t),\phi M(x_{n-1},x,t))}\gamma(s)\mathbf{ds}\right), \qquad (2.9)\end{aligned}$$

since g,ϕ are continuous, by letting $n\to\infty$ and by lemma 1.5 we get

$$\lim_{n\to\infty}\int_0^{\phi(M(x_n,fx,t))}\gamma(s)\mathbf{ds}\le\epsilon\left(\int_0^{g(0,\phi(M(x,fx,t)),0)}\gamma(s)\mathbf{ds}\right),$$

hence

$$\int_0^{\lim_{n\to\infty}\phi(M(x_n,fx,t))}\gamma(s)\mathbf{ds}\le\epsilon\left(\int_0^{g(0,\phi(M(x,fx,t)),0)}\gamma(s)\mathbf{ds}\right),$$

so

$$\int_0^{\phi(M(x,fx,t))}\gamma(s)\mathbf{ds}\le\epsilon\left(\int_0^{g(0,\phi(M(x,fx,t)),0)}\gamma(s)\mathbf{ds}\right).$$

Therefore by definition 2.1, $\phi(M(x,fx,t))=0$. Then $M(x,fx,t)=1$, i.e $x=fx$. □

In the next theorem, we give a condition for the uniqueness of the fixed point.

Theorem 2.9 *Assume that all assumptions of theorem 2.8 are satisfied. Furthermore let for all $x,y\in X$ and for all $t>0$, there exist $z\in X$ such that $\beta(x,z,t)\ge 1$ and $\beta(y,z,t)\ge 1$. Then f has a unique fixed point.*

Proof Suppose that y and x are fixed points of f, with $y\neq x$. i.e $y=fy$ and $x=fx$. Assume that $\beta(x,z,t)\ge 1$, since f is β-admissible, $\beta(x,f^{n-1}z,t)\ge 1$. we have

$$\begin{aligned}\int_0^{\phi(M(x,f^nz,t))}\gamma(s)\mathbf{ds} &= \int_0^{\phi(M(fx,f(f^{n-1})z,t))}\gamma(s)\mathbf{ds}\\ &\le \beta(x,f^{n-1}z,t)\int_0^{\phi(M(fx,f(f^{n-1})z,t))}\gamma(s)\mathbf{ds}\\ &\le \epsilon\left(\int_0^{g(\phi M(x,fx,t),\phi M(f^{n-1}z,f^nz,t),\phi M(x,f^{n-1}z,t))}\gamma(s)\mathbf{ds}\right), \qquad (2.10)\end{aligned}$$

since g is continuous, by letting $n\to\infty$ and using lemma 1.5, we get

$$\begin{aligned}\lim_{n\to\infty}\int_0^{\phi(M(x,f^nz,t))}\gamma(s)\mathbf{ds} &= \int_0^{\lim_{n\to\infty}\phi(M(fx,f(f^{n-1})z,t))}\gamma(s)\mathbf{ds}\\ &\le \lim_{n\to\infty}\beta(x,f^{n-1}z,t)\int_0^{\lim_{n\to\infty}\phi(M(fx,f(f^{n-1})z,t))}\gamma(s)\mathbf{ds}\\ &\le \epsilon\left(\int_0^{g(0,0,\lim_{n\to\infty}\phi M(x,f^{n-1}z,t))}\gamma(s)\mathbf{ds}\right), \qquad (2.11)\end{aligned}$$

put $u = \lim_{n\to\infty} \phi M(x, f^{n-1}z, t) = \lim_{n\to\infty} \phi M(x, f^n z, t)$, then

$$\int_0^u \gamma(s)\mathbf{ds} \leq \epsilon \left(\int_0^{g(0,0,u)} \gamma(s)\mathbf{ds} \right),$$

by definition 2.6, we get $u = \lim_{n\to\infty} \phi M(x, f^n z, t) = 0$. Then $\lim_{n\to\infty} M(x, f^n z, t) = 1$, i.e $f^n z \to x$. Similarly, we can deduce that $f^n z \to y$. Therefore by uniqueness of the limit, we obtain $x = y$. □

Example 2.10 Suppose that $X = \{A, B, C, D, E\}$ is the subset of $\mathbf{R}^2$, where $A = (0, 0)$, $B = (1, 0)$, $C = (1, 2)$, $D = (0, 1)$, $E = (1, 4)$ and $f : X \to X$ be a mapping such that $f(A) = f(B) = f(C) = f(D) = A$ and $f(E) = B$. Also suppose that $\epsilon : [0, \infty) \to [0, \infty)$ is $\epsilon(r) = \sqrt{r}$ and for every $t > 0$

$$\beta(x, y, t) = \begin{cases} 1, & x, y \in X \\ 0, & o.w. \end{cases}$$

and $\gamma(s) = 2s$ and $\phi : [0, 1] \to [0, 1]$ is $\phi(r) = 1 - \sqrt{r}$. Also $M(x, y, t) = \frac{t}{t+d(x,y)}$, $t > 0$, where $d(x, y)$ is denoted Euclidean distance in $\mathbf{R}^2$ and define $g : [0, \infty)^3 \to (0, \infty)$ by $g(a, b, c) = c$. Note that, (X, M, T) is the complete fuzzy metric space with respect to the t-norm $* = x.y$. We have

$$g(\phi M(x, fx, t), \phi M(y, fy, t), \phi M(x, y, t)) = \phi M(x, y, t) = 1 - \sqrt{\frac{t}{t + \max\{0, 4\}}} = 1 - \sqrt{\frac{t}{t+4}},$$

and

$$M(fx, fy, t) = \frac{t}{t+1}.$$

Therefore

$$\int_0^{\phi(M(fx,fy,t))} \gamma(s)\mathbf{ds} = \int_0^{1-\sqrt{\frac{t}{t+1}}} 2s\mathbf{ds} = \left(1 - \sqrt{\frac{t}{t+1}}\right)^2,$$

and

$$\int_0^{\phi(M(x,y,t))} \gamma(s)\mathbf{ds} = \int_0^{1-\sqrt{\frac{t}{t+4}}} 2s\mathbf{ds} = \left(1 - \sqrt{\frac{t}{t+4}}\right)^2.$$

We have

$$\left(1 - \sqrt{\frac{t}{t+1}}\right)^2 \leq \left(1 - \sqrt{\frac{t}{t+4}}\right) = \epsilon \left(1 - \sqrt{\frac{t}{t+4}}\right)^2,$$

then

$$\int_0^{\phi(M(fx,fy,t))} \gamma(s)\mathbf{ds} \leq \epsilon \left(\int_0^{\phi(M(x,y,t))} \gamma(s)\mathbf{ds} \right).$$

So all conditions of theorem 2.2 are satisfied. Therefore f has a fixed point in X. We see that $A \in X$ is the fixed point of f.

In theorem 2.2, set $g(a, b, c) = c$, $\gamma(s) = 1$ and $\phi(r) = \frac{1}{r} - 1$, then we can obtain the following theorem.

Theorem 2.11 *[7] Suppose that (X, M, T) is a G-complete fuzzy metric space and $f : X \to X$ is a mapping. If for any $t > 0$ and $x, y \in X$ mapping f satisfies the following condition*

$$\beta(x, y, t)\left(\frac{1}{M(fx, fy, t)} - 1\right) \leq \epsilon\left(\frac{1}{M(x, y, t)} - 1\right),$$

where $\epsilon \in \beta$ and f is β-admissible, continuous and there exists $x_0 \in X$ such that $\beta(x_0, fx_0, t) \geq 1$, then f has a fixed point.

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