

A quasi-Newton method with rank-two update to solve fuzzy optimization problems

Debdas Ghosh¹  · Debdulal Ghosh² ·
Sushil Kumar Bhuiya²

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Abstract In this article, we develop a *quasi-Newton method* to obtain *nondominated solutions* of fuzzy optimization problems. The objective function of the optimization problem under consideration is a fuzzy-number-valued function. The notion of generalized Hukuhara difference of fuzzy numbers, and hence *generalized Hukuhara differentiability* for multi-variable fuzzy-number-valued functions are used to develop the *quasi-Newton method*. The proposed technique produces a sequence of positive definite inverse Hessian approximations to generate the iterative points. A sequential algorithm and the convergence result of the proposed method are also given. It is obtained that the sequence in the proposed method has *superlinear convergence rate*. The method is also found to have quadratic termination property. Two numerical examples are provided to illustrate the developed technique.

Keywords Quasi-Newton method · Generalized-Hukuhara differentiability · Fuzzy optimization · Nondominated solution

Mathematics Subject Classification 90C70 · 90C29

List of symbols

\mathbb{R} The set of all real numbers

✉ Debdas Ghosh
debdas.email@gmail.com

Debdulal Ghosh
debdulal.email@gmail.com

Sushil Kumar Bhuiya
skbhuiya@maths.iitkgp.ernet.in

¹ Department of Mathematical Sciences, Indian Institute of Technology (BHU), Varanasi, Uttar Pradesh 221005, India

² Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur, West Bengal 721302, India

$\tilde{A}, \tilde{B}, \dots$ and $\tilde{a}, \tilde{b}, \dots$	Fuzzy sets on \mathbb{R}
$\mathcal{F}(\mathbb{R})$	The set of all fuzzy numbers on \mathbb{R}
$\mu(\cdot \tilde{A})$	Membership function of the fuzzy set \tilde{A}
$\tilde{A}(\alpha)$	α -cut of the fuzzy set \tilde{A}
$(l/a/r)$	A triangular fuzzy number at a
$[\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$	α -cut of the triangular fuzzy number \tilde{a}
\oplus	Extended addition
\ominus	Extended subtraction
\odot	Extended multiplication
\ominus_{gH}	Generalized Hukuhara difference
$[\tilde{f}_\alpha^L(x), \tilde{f}_\alpha^U(x)]$	α -cut of the fuzzy function \tilde{f} at x
\tilde{f}_α^L	Lower function of the α -cut of the fuzzy function \tilde{f}
\tilde{f}_α^U	Upper function of the α -cut of the fuzzy function \tilde{f}
$\frac{\partial \tilde{f}}{\partial x_i}(x)$	Partial gH -derivative of \tilde{f} with respect to x_i at x
$\nabla \tilde{f}(x)$	gH -gradient of \tilde{f} at x
$\nabla^2 \tilde{f}(x)$	gH -Hessian of \tilde{f} at x
$\tilde{a} \leq \tilde{b}$	$\tilde{a}_\alpha^L \leq \tilde{b}_\alpha^L$ and $\tilde{a}_\alpha^U \leq \tilde{b}_\alpha^U$ for all $\alpha \in [0, 1]$
$\tilde{a} < \tilde{b}$	$\tilde{a} \leq \tilde{b}$ and $\exists \beta \in [0, 1]$ such that $\tilde{a}_\beta^L < \tilde{b}_\beta^L$ or $\tilde{a}_\beta^U < \tilde{b}_\beta^U$.

1 Introduction

After the seminal work by Bellman and Zadeh [7], in 1970, fuzzy optimization problems (FOPs) are extensively studied to tackle the decision-making problems that are emanated from the imprecise environment. Recently, Luhandjula [28] projected a condensed and selective look at the known landscape of the theory and applications of fuzzy optimization. The edited monograph by Słowiński [36] and the article by Cadenas and Verdegay [8] and the references therein are the main stream of this topic. An insightful overview on the development of FOPs has been reported in the survey books by Lai and Hwang [24,25] and by Lodwick and Kacprzyk [27].

Wu [41–43], Zhong [52] and Cheng [11] have studied duality theory for FOPs. Gong and Li [23] and Wu [48] have found the saddle point optimality conditions for FOPs. Recently, Ghosh and Chakraborty reported fuzzy geometrical view [10,16,17] on FOPs [18–21]. In order to solve an unconstrained FOP, Pirzada and Pathak [32] and Chalco-Cano et al. [9] developed a Newton method. Much similar to fuzzy optimization, Ghosh [14] derived a Newton method and quasi-Newton method [15] for interval optimization problems.

The practical situations of mathematical optimization models often encounter the problem of optimizing a fuzzy function over real data set. Precisely, this problem can be stated as ‘ $\min_{x \in \mathbb{R}^n} \tilde{f}(x)$ ’, where \tilde{f} is a fuzzy-number-valued function. In classical optimization theory, there are various numerical algorithms [1,30,51] for unconstrained optimization problems. However, those methods are evidently not applicable to solve FOPs. *In this article, we attempt to develop a quasi-Newton method to solve an unconstrained FOP.*

In the context of conventional optimization problems, towards searching the optimal points [4,30] of a smooth optimization problem, it is natural to employ the derivative of the objective function. However, to find the optimal points of an FOP, not only the differentiability of fuzzy

functions but also an appropriate ordering of fuzzy numbers has to be considered to find the optimal points, since unlike to the real numbers, fuzzy numbers are not linearly ordered [35].

There have been a wide literature on ordering of fuzzy numbers including the research articles referred in [12, 26, 29, 34, 39, 49]. Cheng [12] proposed a ranking of fuzzy numbers with the help of a distance method. Sun and Wu [38] presented a ranking based on fuzzy simulation analysis method. Peng et al. [31] studied ranking fuzzy variables in terms of credibility measure. Yao and Wu [50] defined ranking fuzzy numbers based on decomposition principle and signed distance. Wang [40] proposed a ranking of L - R fuzzy numbers based on a deviation degree. References of [40] reports the main stream on ordering of fuzzy numbers.

In this paper, the fuzzy-max ordering of fuzzy numbers of Ramík and Rimanek [34] will be used. There are two reasons behind this selection. First, it is a partial ordering in the space of fuzzy numbers [32]. Second, it has insightful association [44, 45] with the optimality notion on FOPs.

In order to find the notion of differentiability of fuzzy functions, Dubois and Prade [13] used the extension principle. Dubois and Prade [13] have also reported that a fuzzy extension of the usual definition of differentiation is of no use. As a result of this fact, Goetschel and Voxman [22] used a so-called 'opposite' of a fuzzy number to define fuzzy differentiation in analogy to the usual definition of differentiation. Recently, Hukuhara differentiability (H -differentiability) [3] received substantial attention in fuzzy optimization theory. Banks and Jacobs [3] developed H -differentiability for multifunctions. Puri and Ralescu [33] further extended the H -differentiability for fuzzy functions. The concept of H -fuzzy differentiation is rigorously discussed in [2] for the univariate and multivariate fuzzy functions. The H -differentiation is also studied by several authors in different fields. For instance, Wu et al. [47] applied the H -differentiability in fuzzy differential equations. In fuzzy optimization, Wu employed the H -derivative in order to find optimality conditions for multi-objective [46] and single objective [44, 45] fuzzy-valued functions. Bede and Gal [5, 37] proposed generalizations of H -differentiability (gH -differentiability) and its application in fuzzy differential equations. Bede and Stefanini [6] gave a generalized H -differentiability of fuzzy-valued functions. Pirzada and Pathak [32] derived a Newton method for FOP using H -derivative. Chalco-Cano et al. [9] used gH -differentiability to modify the Newton method in [32].

Since gH -derivative is the most general concept [9] for differentiability of fuzzy functions, in this paper, we employ the gH -derivative and its calculus [9, 44, 45] to derive a quasi-Newton method for solving an FOP.

It is to note that in the derivation of Newton method for FOPs [9, 32], the generating equation for iterative points involves computation of inverse of the concerned Hessian matrix. Computation of inverse-Hessian is truly cost effective. Thus, in this study, *we attempt to propose a quasi-Newton method to sidestep the inherent computational expense in computing inverse-Hessian in the existing Newton method for FOPs*. This sidestepping is done through a rank-two modification to an appropriate approximation of the pertaining inverse-Hessian.

The work in this study is organized as follows. Immediately next of this paragraph, list of symbols those are used throughout the article is given. In Sect. 2, the notations and terminologies are given which are used throughout the paper. In Sect. 3, we derive a quasi-Newton method for FOPs. Convergence analysis of the proposed method is presented in Sect. 4. Section 5 includes two illustrative numerical examples. Finally, we give a brief conclusions and scopes for future research in Sect. 6.

2 Preliminaries

This section provides basic definitions and notations which are employed in this article. Upper and lower case letters with a tilde ($\tilde{A}, \tilde{B}, \tilde{C}, \dots$ and $\tilde{a}, \tilde{b}, \tilde{c}, \dots$) denote fuzzy subsets of \mathbb{R} . The membership function of a fuzzy set \tilde{A} of \mathbb{R} is represented by $\mu(x|\tilde{A})$, for x in \mathbb{R} , with $\mu(\mathbb{R}) \subseteq [0, 1]$.

2.1 Fuzzy numbers

Definition 1 (*α -cut of a fuzzy set [16]*) The α -cut of a fuzzy set \tilde{A} of \mathbb{R} , denoted $\tilde{A}(\alpha)$, is defined by

$$\tilde{A}(\alpha) = \begin{cases} \{x : \mu(x|\tilde{A}) \geq \alpha\} & \text{if } 0 < \alpha \leq 1 \\ \text{closure}\{x : \mu(x|\tilde{A}) > 0\} & \text{if } \alpha = 0. \end{cases}$$

The sets $\{x : \mu(x|\tilde{A}) > 0\}$ and $\{x : \mu(x|\tilde{A}) = 1\}$ are respectively called the *support* and the *core* of the fuzzy set \tilde{A} .

Definition 2 (*Fuzzy number [16]*) A fuzzy set \tilde{N} of \mathbb{R} is called a fuzzy number if its membership function μ has the following properties:

- (i) $\mu(x|\tilde{N})$ is upper semi-continuous,
- (ii) $\mu(x|\tilde{N}) = 0$ outside some interval $[a, d]$, and
- (iii) there exist real numbers b and c such that $a \leq b \leq c \leq d$ and $\mu(x|\tilde{N})$ is increasing on $[a, b]$ and decreasing on $[c, d]$, and $\mu(x|\tilde{N}) = 1$ for each x in $[b, c]$.

In particular, when $b = c$, and the parts of the membership functions $\mu(x|\tilde{N})$ in $[a, b]$ and $[c, d]$ are linear, then the fuzzy number is called a triangular fuzzy number and is denoted by $(a/c/d)$. We denote the set of all fuzzy numbers on \mathbb{R} by $\mathcal{F}(\mathbb{R})$.

Since $\mu(x|\tilde{a})$ is upper semi-continuous for a fuzzy number \tilde{a} , the set $\{x : \mu(x|\tilde{a}) \geq \alpha\}$ is closed for each α in \mathbb{R} . Thus, the α -cut of a fuzzy number \tilde{a} (the set $\tilde{a}(\alpha)$) is a closed and bounded interval of \mathbb{R} for all α in $[0, 1]$. We write $\tilde{a}(\alpha) = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$.

Let \oplus and \odot denote the extended addition and multiplication. According to the extension principle [53], the membership function of $\tilde{a} \otimes \tilde{b}$ ($\otimes = \oplus$ or \odot) is defined by

$$\mu(z|\tilde{a} \otimes \tilde{b}) = \sup_{x \times y = z} \min \{ \mu(x|\tilde{a}), \mu(y|\tilde{b}) \}.$$

For any \tilde{a} and \tilde{b} in $\mathcal{F}(\mathbb{R})$, the α -cut (for any α in $[0, 1]$) of their addition and scalar multiplication can be obtained as [53]:

$$\begin{aligned} (\tilde{a} \oplus \tilde{b})(\alpha) &= [\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U] \text{ and} \\ (\lambda \odot \tilde{a})(\alpha) &= \begin{cases} [\lambda \tilde{a}_\alpha^L, \lambda \tilde{a}_\alpha^U] & \text{if } \lambda \geq 0, \\ [\lambda \tilde{a}_\alpha^U, \lambda \tilde{a}_\alpha^L] & \text{if } \lambda < 0. \end{cases} \end{aligned}$$

Definition 3 (*Generalized Hukuhara difference [37]*) Let \tilde{a} and \tilde{b} be two fuzzy numbers. If there exists a fuzzy number \tilde{c} such that $\tilde{c} \oplus \tilde{b} = \tilde{a}$ or $\tilde{b} = \tilde{a} \ominus \tilde{c}$, then \tilde{c} is said to be generalized Hukuhara difference (*gH-difference* for short) between \tilde{a} and \tilde{b} . If such \tilde{c} exists, then it is unique. Hukuhara difference between \tilde{a} and \tilde{b} is denoted by $\tilde{a} \ominus_{gH} \tilde{b}$.

In terms of α -cut, for all $\alpha \in [0, 1]$, we have

$$(\tilde{a} \ominus_{gH} \tilde{b})(\alpha) = \left[\min \{ \tilde{a}_\alpha^L - \tilde{b}_\alpha^L, \tilde{a}_\alpha^U - \tilde{b}_\alpha^U \}, \max \{ \tilde{a}_\alpha^L - \tilde{b}_\alpha^L, \tilde{a}_\alpha^U - \tilde{b}_\alpha^U \} \right].$$

2.2 Fuzzy functions

Let $\tilde{f} : \mathbb{R}^n \rightarrow \mathcal{F}(\mathbb{R})$ be a fuzzy function. We present the α -cuts of this fuzzy function by

$$\tilde{f}(x)(\alpha) = \left[\tilde{f}_\alpha^L(x), \tilde{f}_\alpha^U(x) \right] \quad \text{for all } \alpha \in [0, 1].$$

Here the functions \tilde{f}_α^L and \tilde{f}_α^U are two real valued functions and are called lower and upper functions, respectively.

With the help of gH -difference between two fuzzy numbers, gH -differentiability of a fuzzy function is defined as follows.

Definition 4 (*gH -differentiability of fuzzy functions [5]*) Let $\tilde{f} : \mathbb{R}^n \rightarrow \mathcal{F}(\mathbb{R})$ be a fuzzy function and $x_0 = (x_1^0, x_2^0, \dots, x_n^0)$ be an element of \mathbb{R}^n . We define a fuzzy function $\tilde{h}_i : \mathbb{R} \rightarrow \mathcal{F}(\mathbb{R})$ for each $i = 1, 2, \dots, n$ as follows. Consider

$$\tilde{h}_i(x_i) = \tilde{f}(x_1^0, \dots, x_{i-1}^0, x_i, x_{i+1}^0, \dots, x_n^0).$$

We say \tilde{h}_i is gH -differentiable if the following limit exists

$$\lim_{t_i \rightarrow 0} \frac{\tilde{h}_i(x_i^0 + t_i) \ominus_{gH} \tilde{h}_i(x_i^0)}{t_i}.$$

If \tilde{h}_i is gH -differentiable, then we say that \tilde{f} has the i th partial gH -derivative at x_0 and is denoted by $\frac{\partial \tilde{f}}{\partial x_i}(x_0)$.

The function \tilde{f} is said to be gH -differentiable at $x_0 \in \mathbb{R}^n$ if all the partial gH -derivatives $\frac{\partial \tilde{f}}{\partial x_1}(x_0), \frac{\partial \tilde{f}}{\partial x_2}(x_0), \dots, \frac{\partial \tilde{f}}{\partial x_n}(x_0)$ exists on some neighborhood of x_0 and are continuous at x_0 .

Proposition 1 (See [9]) *If a fuzzy function $\tilde{f} : \mathbb{R}^n \rightarrow \mathcal{F}(\mathbb{R})$ is gH -differentiable at $x_0 \in \mathbb{R}^n$, then for each $\alpha \in [0, 1]$, the real valued function $\tilde{f}_\alpha^L + \tilde{f}_\alpha^U$ is differentiable at x_0 . Moreover,*

$$\frac{\partial \tilde{f}_\alpha^L}{\partial x_i}(x_0) + \frac{\partial \tilde{f}_\alpha^U}{\partial x_i}(x_0) = \frac{\partial (\tilde{f}_\alpha^L + \tilde{f}_\alpha^U)}{\partial x_i}(x_0).$$

Definition 5 (*gH -gradient [9]*) The gH -gradient of a fuzzy function $\tilde{f} : \mathbb{R}^n \rightarrow \mathcal{F}(\mathbb{R})$ at a point $x_0 \in \mathbb{R}^n$ is defined by

$$\left(\frac{\partial \tilde{f}(x_0)}{\partial x_1}, \frac{\partial \tilde{f}(x_0)}{\partial x_2}, \dots, \frac{\partial \tilde{f}(x_0)}{\partial x_n} \right)^t.$$

We denote this gH -gradient by $\nabla \tilde{f}(x_0)$.

We define an m -times *continuously gH -differentiable* fuzzy function \tilde{f} as a function whose all of the partial gH -derivatives of order m exist and are continuous. Then, we have the following immediate result.

Proposition 2 (See [9]) *Let \tilde{f} be a fuzzy function. Let at $x_0 \in \mathbb{R}^n$, the function \tilde{f} be m -times gH -differentiable. Then the real-valued function $\tilde{f}_\alpha^L + \tilde{f}_\alpha^U$ is m -times differentiable at x_0 .*

Definition 6 (*gH -Hessian [9]*) Let the fuzzy function \tilde{f} be twice gH -differentiable at x_0 . Then, for each i , the function $\frac{\partial \tilde{f}}{\partial x_i}$ is gH -differentiable at x_0 . The second order partial gH -derivative can be calculated through computing all the $\frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j}$'s. The gH -Hessian of \tilde{f} at x_0 can be captured by the square matrix

$$\nabla^2 \tilde{f}(x_0) = \left[\frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j} (x_0) \right]_{n \times n}.$$

2.3 Optimality concept

As it is mentioned in Sect. 1 that in order to search the optimal solution of an FOP, we need a partial order relation between fuzzy numbers. Recently, fuzzy-max ordering of fuzzy numbers are extensively used [32,44,45] in fuzzy optimization, since it is a partial order relation in $\mathcal{F}(\mathbb{R})$ [32]. Fuzzy-max ordering is defined as follows.

Definition 7 (*Dominance relation between fuzzy numbers* [32]) Let \tilde{a} and \tilde{b} be two fuzzy numbers. For any $\alpha \in [0, 1]$, let $\tilde{a}(\alpha) = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$ and $\tilde{b}(\alpha) = [\tilde{b}_\alpha^L, \tilde{b}_\alpha^U]$. We say \tilde{a} dominates \tilde{b} if $\tilde{a}_\alpha^L \leq \tilde{b}_\alpha^L$ and $\tilde{a}_\alpha^U \leq \tilde{b}_\alpha^U$ for all $\alpha \in [0, 1]$. If \tilde{a} dominates \tilde{b} , then we write $\tilde{a} \preceq \tilde{b}$. The fuzzy number \tilde{b} is said to be strictly dominated by \tilde{a} , if $\tilde{a} \preceq \tilde{b}$ and there exists $\beta \in [0, 1]$ such that $\tilde{a}_\beta^L < \tilde{b}_\beta^L$ or $\tilde{a}_\beta^U < \tilde{b}_\beta^U$. If \tilde{a} strictly dominates \tilde{b} , then we write $\tilde{a} < \tilde{b}$.

With the help of Definition 7, the concept of non-dominated solution for FOPs is defined as follows.

Definition 8 (*Non-dominated solution for fuzzy optimization* [32]) Let $\tilde{f} : \mathbb{R}^n \rightarrow \mathcal{F}(\mathbb{R})$ be a fuzzy function and we intend to find a solution of ‘ $\min_{x \in \mathbb{R}^n} \tilde{f}(x)$ ’. A point $\tilde{x} \in \mathbb{R}^n$ is said to be a locally non-dominated solution if there exists no $x \in N_\epsilon(\tilde{x})$ such that $\tilde{f}(x) \preceq \tilde{f}(\tilde{x})$, where $N_\epsilon(\tilde{x})$ denotes ϵ -neighborhood of \tilde{x} . A local non-dominated solution is called a local solution of ‘ $\min_{x \in \mathbb{R}^n} \tilde{f}(x)$ ’.

For local non-dominated solution, the following result is proved in [9].

Proposition 3 (See [9]) *Let $\tilde{f} : \mathbb{R}^n \rightarrow \mathcal{F}(\mathbb{R})$ be a fuzzy function. If x^* is a local minimizer of the real valued $\tilde{f}_\alpha^L + \tilde{f}_\alpha^U$ for all $\alpha \in [0, 1]$, then x^* is a locally non-dominated solution of ‘ $\min_{x \in \mathbb{R}^n} \tilde{f}(x)$ ’.*

3 Quasi-Newton method

In this section, we consider to solve the following unconstrained Fuzzy Optimization Problem:

$$(FOP) \quad \min_{x \in \mathbb{R}^n} \tilde{f}(x),$$

where $\tilde{f} : \mathbb{R}^n \rightarrow \mathcal{F}(\mathbb{R})$ is a fuzzy-number-valued function. On finding a nondominated solution of the problem, we note that the existing Newton method (see [9,32]) requires computation of the inverse of the concerned Hessian. However, computation of the inverse of Hessian is cost effective. Thus in this article, we intend to develop a quasi-Newton method to sidestep the computational difficulty of the Newton method. Towards this end, for the FOP we assume that at each of the following generated sequential points x_k , the function \tilde{f} and its gH -gradient and gH -Hessian are well-defined. Therefore, according to Propositions 1 and 2, we can calculate $\tilde{f}_\alpha^L(x_k)$, $\tilde{f}_\alpha^U(x_k)$, $\nabla \tilde{f}_\alpha^L(x_k)$, $\nabla \tilde{f}_\alpha^U(x_k)$, $\nabla^2 \tilde{f}_\alpha^L(x_k)$ and $\nabla^2 \tilde{f}_\alpha^U(x_k)$ for all $\alpha \in [0, 1]$, for all $k = 0, 1, 2, \dots$. Hence, we can have a quadratic approximations of the the lower and the upper functions \tilde{f}_α^L and \tilde{f}_α^U .

Let the quadratic approximation models of \tilde{f}_α^L and \tilde{f}_α^U at x_{k+1} be

$$h_\alpha^L(x) = f_\alpha^L(x_{k+1}) + \nabla f_\alpha^L(x_{k+1})^t(x - x_{k+1}) + \frac{1}{2}(x - x_{k+1})^t \nabla^2 f_\alpha^L(x_{k+1})(x - x_{k+1})$$

and

$$h_\alpha^U(x) = f_\alpha^U(x_{k+1}) + \nabla f_\alpha^U(x_{k+1})^t(x - x_{k+1}) + \frac{1}{2}(x - x_{k+1})^t \nabla^2 f_\alpha^U(x_{k+1})(x - x_{k+1}),$$

which satisfy the interpolating conditions

$$h_\alpha^L(x_{k+1}) = f_\alpha^L(x_{k+1}), \quad h_\alpha^U(x_{k+1}) = f_\alpha^U(x_{k+1}), \quad \nabla h_\alpha^L(x_{k+1}) = \nabla f_\alpha^L(x_{k+1}),$$

and $\nabla h_\alpha^U(x_{k+1}) = \nabla f_\alpha^U(x_{k+1})$.

The derivatives of h_α^L and h_α^U yield

$$\begin{cases} h_\alpha^L(x) = \nabla f_\alpha^L(x_{k+1}) + \nabla^2 f_\alpha^L(x_{k+1})(x - x_{k+1}) \text{ and} \\ h_\alpha^U(x) = \nabla f_\alpha^U(x_{k+1}) + \nabla^2 f_\alpha^U(x_{k+1})(x - x_{k+1}). \end{cases} \tag{3.1}$$

In the next, the Newton method (see [9,32]) attempts to find x in terms of x_{k+1} as follows

$$x = x_{k+1} - [\nabla^2 \phi(x_{k+1})]^{-1} \nabla \phi(x_{k+1}),$$

where $\phi(x) = \int_0^1 (f_\alpha^L(x) + f_\alpha^U(x)) d\alpha$.

However, due to inherent computational difficulty to find $[\nabla^2 \phi(x_{k+1})]^{-1}$, it is often suggested that consider an appropriate approximation.

Let A_{k+1} be an approximation of $[\nabla^2 \phi(x_{k+1})]^{-1}$. Then, from (3.1), setting $x = x_k$, $\delta_k = x_{k+1} - x_k$, $\beta_{\alpha k}^L = \nabla f_\alpha^L(x_{k+1}) - \nabla f_\alpha^L(x_k)$ and $\beta_{\alpha k}^U = \nabla f_\alpha^U(x_{k+1}) - \nabla f_\alpha^U(x_k)$, we obtain

$$\begin{aligned} \beta_{\alpha k}^L &= \nabla^2 f_\alpha^L(x_{k+1})\delta_k \quad \text{and} \quad \beta_{\alpha k}^U = \nabla^2 f_\alpha^U(x_{k+1})\delta_k \\ \implies \beta_{\alpha k}^L + \beta_{\alpha k}^U &= \left(\nabla^2 f_\alpha^L(x_{k+1}) + \nabla^2 f_\alpha^U(x_{k+1}) \right) \delta_k \quad \text{for all } \alpha \in [0, 1] \\ \implies \int_0^1 (\beta_{\alpha k}^L + \beta_{\alpha k}^U) d\alpha &= \left(\int_0^1 \nabla^2 (f_\alpha^L + f_\alpha^U)(x_{k+1}) d\alpha \right) \delta_k \\ \implies \int_0^1 (\nabla \phi_\alpha(x_{k+1}) - \nabla \phi_\alpha(x_k)) d\alpha &= \left(\int_0^1 \nabla^2 \phi_\alpha(x_{k+1}) d\alpha \right) \delta_k, \\ &\text{where } \phi_\alpha(x) = f_\alpha^L(x) + f_\alpha^U(x) \\ \implies \nabla \phi(x_{k+1}) - \nabla \phi(x_k) &= \nabla^2 \phi(x_{k+1})\delta_k, \quad \text{where } \phi(x) = \int_0^1 \phi_\alpha(x) d\alpha \\ \implies A_{k+1}\Phi_k &= \delta_k, \quad \text{where } \Phi_k = \nabla \phi(x_{k+1}) - \nabla \phi(x_k). \end{aligned}$$

According to Proposition 3, to obtain nondominated solutions of (FOP), we need to have solutions of the equation $\nabla(f_\alpha^L + f_\alpha^U)(x) = 0$.

In order to capture solutions of this equation, much similar to the Newton method [32], the above procedure clearly suggests to consider the following generating sequence

$$\begin{aligned} \delta_k &= x_{k+1} - x_k = A_{k+1}\Phi_k \\ \implies x_{k+1} &= x_k + A_{k+1}\Phi_k. \end{aligned}$$

If A_k is an appropriate approximation of the inverse Hessian $[\nabla^2\phi(x_{k+1})]^{-1}$ and $\nabla(f_\alpha^L + f_\alpha^U)(x_{k+1}) \approx 0$, then the equation $x_{k+1} = x_k + A_{k+1}\Phi_k$ reduces to

$$x_{k+1} = x_k - [\nabla^2\phi(x_{k+1})]^{-1} \nabla\phi(x_k),$$

which is the generating equation of the Newton method [32] and hence obviously will converge to the minimizer of $f_\alpha^L + f_\alpha^U$.

As we observe, the key point of the above method is to appropriately generate A_k 's. Due to the inherent computational difficulty to find the inverse of the Hessian $\nabla^2\phi(x_{k+1})$ we consider an approximation that should satisfy

$$A_{k+1}\Phi_k = \delta_k. \tag{3.2}$$

In this article we attempt to introduce a simple *rank-two update* of the sequence $\{A_k\}$ that satisfy the above *quasi-Newton equation* (3.2).

Let A_k be the approximation of the k th iteration. We attempt to update A_k into A_{k+1} by adding two symmetric matrices, each of rank one as follows:

$$A_{k+1} = A_k + p_k v_k v_k^t + q_k w_k w_k^t$$

where u_k and v_k are two vectors in \mathbb{R}^n , and p_k and q_k are two scalars which are to be determined by the quasi-Newton equation (3.2). Therefore, we now have

$$A_k \Phi_k + p_k v_k v_k^t \Phi_k + q_k w_k w_k^t \Phi_k = \delta_k. \tag{3.3}$$

Evidently, v_k and w_k are not uniquely determined, but their obvious choices are $v_k = \delta_k$ and $w_k = A_k \Phi_k$. Putting this values in (3.3), we obtain

$$p_k = \frac{1}{v_k^t \Phi_k} = \frac{1}{\delta_k^t \Phi_k} \quad \text{and} \quad q_k = -\frac{1}{w_k^t \Phi_k} = -\frac{1}{\Phi_k^t A_k \Phi_k}.$$

Therefore,

$$A_{k+1} = A_k + \frac{\delta_k \delta_k^t}{\delta_k^t \Phi_k} - \frac{A_k \Phi_k \Phi_k^t A_k}{\Phi_k^t A_k \Phi_k}. \tag{3.4}$$

We consider this equation to generate the sequence of above mentioned inverse Hessian approximation.

Therefore, accumulating all, we follow the following sequential way (Algorithm 1) to obtain an efficient solution of the considered (FOP) $\min_{x \in \mathbb{R}^n} \tilde{f}(x)$.

4 Convergence analysis

In this section, the convergence analysis of the proposed quasi-Newton method is performed. At first, we show the *superlinear convergence rate* of the proposed method. Then, we explore that the used search directions are descent and the Hessian approximations are positive definite. Finally, the quadratic termination property of the method is proved.

Theorem 1 (Superlinear convergence rate)

Let $\tilde{f} : \mathbb{R}^n \rightarrow \mathcal{F}(\mathbb{R})$ be thrice continuously gH -differentiable on \mathbb{R}^n . Suppose that \bar{x} in \mathbb{R}^n satisfies the following three conditions:

- (i) \bar{x} is a local minimizer of \tilde{f}_α^L and \tilde{f}_α^U ,
- (ii) $\int_0^1 \nabla^2(f_\alpha^L(x) + f_\alpha^U(x))d\alpha$ is symmetric positive definite, and

Algorithm 1 Quasi-Newton Method with Rank-two Modification to Solve FOP

Require: Given \tilde{f} , the objective function

x_0 , the initial point

ϵ , a termination scalar

A_0 , a symmetric positive definite matrix

1: Compute f_α^L, f_α^U and $\phi(x) = \int_0^1 (f_\alpha^L(x) + f_\alpha^U(x))d\alpha$

2: Set $k = 0$

3: If $\|\nabla\phi(x_k)\| < \epsilon$, then Stop

4: Compute the search direction $d_k = -A_k\nabla\phi(x_k)$

5: Compute the step length $\alpha_k := \operatorname{argmin}_{\alpha \geq 0} \phi(x_k + \alpha d_k)$

6: Evaluate

$$\begin{aligned} \delta_k &= \alpha_k d_k \\ x_{k+1} &= x_k + \delta_k \\ \Phi_k &= \nabla\phi(x_{k+1}) - \nabla\phi(x_k), \text{ and} \\ A_{k+1} &= A_k + \frac{\delta_k \delta_k^t}{\delta_k^t \Phi_k} - \frac{A_k \Phi_k \Phi_k^t A_k}{\Phi_k^t A_k \Phi_k}. \end{aligned}$$

7: Set $k = k + 1$, go to Step 1.

(iii) $\int_0^1 \nabla^2 (f_\alpha^L(x) + f_\alpha^U(x))d\alpha$ is Lipschitzian with constant $\gamma^L + \gamma^U$.

Then the iteration sequence $\{x_k\}$ in Algorithm 1 converges to \bar{x} superlinearly if and only if

$$\lim_{k \rightarrow \infty} \frac{\| [A_{k+1}^{-1} - \nabla^2\phi(\bar{x})] \delta_k \|}{\|\delta_k\|} = 0,$$

where $\phi(x) = \int_0^1 (f_\alpha^L(x) + f_\alpha^U(x)) d\alpha$.

Proof According to the hypothesis (i), at \bar{x} we have

$$\nabla\phi(\bar{x}) = \int_0^1 \nabla (f_\alpha^L(\bar{x}) + f_\alpha^U(\bar{x})) d\alpha = 0.$$

From hypothesis (ii), the Hessian matrix

$$\nabla^2\phi(\bar{x}) = \int_0^1 \nabla^2 (f_\alpha^L(\bar{x}) + f_\alpha^U(\bar{x})) d\alpha$$

is positive definite and a symmetric matrix.

With the help of hypothesis (iii), the function

$$\nabla^2\phi(x) = \int_0^1 \nabla^2 (f_\alpha^L(x) + f_\alpha^U(x)) d\alpha$$

is found to be Lipschitzian at \bar{x} with constant $\gamma^L + \gamma^U$. Mathematically, there exists a neighborhood $N_\epsilon(\bar{x})$ where

$$\| \nabla^2\phi(x) - \nabla^2\phi(\bar{x}) \| \leq (\gamma^L + \gamma^U) \|x - \bar{x}\| \quad \forall x \in N_\epsilon(\bar{x}).$$

Towards proving the result, the following equivalence will be proved.

$$\lim_{k \rightarrow \infty} \frac{\| [A_{k+1}^{-1} - \nabla^2\phi(\bar{x})] \delta_k \|}{\|\delta_k\|} = 0$$

$$\begin{aligned} &\iff \lim_{k \to \infty} \frac{\|\Phi_{k+1}\|}{\|\delta_k\|} = 0 \\ &\iff \lim_{k \to \infty} \frac{\|x_{k+1} - \bar{x}\|}{\|x_k - \bar{x}\|} = 0. \end{aligned}$$

With the help of quasi-Newton equation (3.1), we have

$$\begin{aligned} &\left[A_{k+1}^{-1} - \nabla^2\phi(\bar{x}) \right] [x_{k+1} - x_k] \\ &= -\Phi_k - \nabla^2\phi(\bar{x})(x_{k+1} - x_k) \\ &= (\Phi_{k+1} - \Phi_k - \nabla^2\phi(\bar{x})(x_{k+1} - x_k)) - \Phi_{k+1}. \end{aligned}$$

Therefore,

$$\frac{\|\Phi_{k+1}\|}{\|\delta_k\|} \leq \frac{1}{\|\delta_k\|} \left[\left\| \left(A_{k+1}^{-1} - \nabla^2\phi(\bar{x}) \right) \delta_k \right\| + \left\| \Phi_{k+1} - \Phi_k - \nabla^2\phi(\bar{x})\delta_k \right\| \right].$$

It is evident to note that

$$\begin{aligned} \|\Phi_{k+1} - \Phi_k - \nabla^2\phi(\bar{x})\delta_k\| &= \left\| \left(\int_0^1 \nabla^2 \left(f_\alpha^L + f_\alpha^U \right) (x_{k+1} + t(x_{k+1} - x_k)) dt \right) \delta_k \right. \\ &\quad \left. - \nabla^2\phi(\bar{x})\delta_k \right\| \\ &= \left\| \left(\int_0^1 \left(\nabla^2\phi(x_{k+1} + t\delta_k) - \nabla^2\phi(\bar{x}) \right) dt \right) \delta_k \right\| \\ &\leq (\gamma^L + \gamma^U) (\|x_{k+1} - \bar{x}\| + \|x_k - \bar{x}\|). \end{aligned}$$

Since $\nabla^2\phi(\bar{x})$ is positive definite, we have $\xi > 0$ and $m \in \mathbb{N}$ such that for all $k \geq m$

$$\|\Phi_{k+1}\| = \|\Phi_{k+1} - \nabla\phi(\bar{x})\| \geq \xi \|x_{k+1} - \bar{x}\|.$$

Hence we now have

$$\frac{\|\Phi_{k+1}\|}{\|\delta_k\|} \geq \frac{\xi \|x_{k+1} - \bar{x}\|}{\|x_{k+1} - \bar{x}\| + \|x_k - \bar{x}\|} = \xi \frac{c_k}{1 + c_k},$$

where $c_k = \frac{\|x_{k+1} - \bar{x}\|}{\|x_k - \bar{x}\|}$. This inequality gives

$$\begin{aligned} &\lim_{k \to \infty} \frac{c_k}{1 + c_k} = 0 \\ &\implies \lim_{k \to \infty} c_k = 0 \\ &\implies \lim_{k \to \infty} \frac{\|x_{k+1} - \bar{x}\|}{\|x_k - \bar{x}\|} = 0. \end{aligned}$$

This completes the proof of superlinear convergence of the sequence $\{x_k\}$ in Algorithm 1. Conversely, since $\{x_k\}$ converges superlinearly to \bar{x} and $\nabla\phi(\bar{x}) = 0$, we must have $\beta > 0$ and $p \in \mathbb{N}$ such that

$$\|\Phi_{k+1}\| \leq \beta \|x_{k+1} - \bar{x}\| \quad \text{for all } k \geq p.$$

Again due to superlinear convergence of $\{x_k\}$, we have

$$0 = \lim_{k \to \infty} \frac{\|x_{k+1} - \bar{x}\|}{\|x_k - \bar{x}\|} \geq \lim_{k \to \infty} \frac{\|\Phi_{k+1}\|}{\beta \|x_k - \bar{x}\|} = \lim_{k \to \infty} \frac{1}{\beta} \frac{\|\Phi_{k+1}\|}{\|x_{k+1} - x_k\|} \frac{\|x_{k+1} - x_k\|}{\|x_k - \bar{x}\|}.$$

Since $\lim_{k \rightarrow \infty} \frac{\|x_{k+1} - x_k\|}{\|x_k - \bar{x}\|} = 1$, this inequality implies $\lim_{k \rightarrow \infty} \frac{\|\Phi_k\|}{\|x_{k+1} - x_k\|} = 0$. Hence the result follows. \square

Theorem 2 (Positive definiteness of the inverse Hessian approximation)

Let $x_0 \in \mathbb{R}^n$ and the objective function \tilde{f} in Algorithm 1 is twice continuously gH -differentiable. If

$$\int_0^1 \nabla \left(\tilde{f}_\alpha^L(x_k) + \tilde{f}_\alpha^U(x_k) \right) d\alpha \neq 0,$$

then the inverse Hessian approximations A_k 's are positive definite and the search directions $d_k = -A_k \nabla \phi(x_k)$ are descent for all $k = 1, 2, \dots$.

Proof If all A_k 's are positive definite, then according to the hypothesis, the search directions $d_k = -A_k \nabla \phi(x_k)$ are trivially descent, since

$$\nabla \phi(x_{k+1})^t d_{k+1} = -\nabla \phi(x_{k+1})^t A_{k+1} \nabla \phi(x_{k+1}) < 0.$$

Thus, we prove that all A_k 's are positive definite.

According to Algorithm 1, the matrix A_0 is assumed to be positive definite.

Let A_1, A_2, \dots, A_m are positive definite. We will show that A_{m+1} is positive definite. Then, by the principle of mathematical induction, the result will follow.

Let $x \neq 0$ be a vector in \mathbb{R}^n . With the help of the rank-two update of A_k 's in Algorithm 1, we obtain

$$x^t A_{m+1} x = x^t A_m x + \frac{(x^t \delta_m)^2}{\delta_m^t \Phi_m} - \frac{(x^t A_m \Phi_m)^2}{(\Phi_m^t A_m \Phi_m)}. \tag{4.1}$$

Since A_m is a positive definite matrix, by Cholesky decomposition there exists a positive definite symmetric matrix D_m such that $A_m = D_m D_m$.

Let $a = D_m x$ and $b = D_m \Phi_m$. Then

$$x^t A_{m+1} x = a^t a, \quad \Phi_m^t A_m \Phi_m = b^t b, \quad \text{and} \quad x^t A_m \Phi_m = a^t b.$$

We note that

- (i) $\delta_m^t \Phi_m = \alpha_m d_m^t (\nabla \phi(x_{m+1}) - \nabla \phi(x_m)) = \alpha_m \nabla \phi(x_m)^t A_m \nabla \phi(x_m) > 0$, since $d_m = -A_m \nabla \phi(x_m)$ is a descent direction, and hence $\alpha_m > 0$, and $d_m^t \nabla \phi(x_{m+1}) = 0$, and
- (ii) $b^t b > 0$, since $\Phi_m \neq 0$.

Eq. (4.1) gives

$$\begin{aligned} x^t A_{m+1} x &= a^t a - \frac{(a^t b)^2}{b^t b} + \frac{(x^t \delta_m)^2}{\delta_m^t \Phi_m} \\ &= \frac{(a^t a)(b^t b) - (a^t b)^2}{b^t b} + \frac{(x^t \delta_m)^2}{\delta_m^t \Phi_m} \\ &\geq 0, \text{ by Cauchy-Schwarz inequality.} \end{aligned}$$

Below we prove that out of the two terms $\frac{(a^t a)(b^t b) - (a^t b)^2}{b^t b}$ and $\frac{(x^t \delta_m)^2}{\delta_m^t \Phi_m}$ in $x^t A_{m+1} x$ at least one is strictly positive.

Since $x \neq 0$, the equality in Cauchy-Schwarz inequality holds only when $a \parallel b$, equivalently, $x \parallel \Phi_m$.

Then there exists $\lambda \neq 0$ such that $x = \lambda \Phi_m$ and hence $\frac{(x^t \delta_m)^2}{\delta_m^t \Phi_m} = \lambda^2 \delta_m^t \Phi_m > 0$. This yields the result. \square

Theorem 3 (Quadratic termination property)

Let $\tilde{f}(x) = \frac{1}{2} x^t \tilde{A}_{n \times n} x \oplus \tilde{B}_{n \times 1}^t x \oplus \tilde{C}_{1 \times 1}$ be a quadratic fuzzy function with the end point matrices of $\tilde{A}(\alpha)$ are such that the Hessian matrices $\int_0^1 \nabla^2 f_\alpha^L(x) d\alpha$ and $\int_0^1 \nabla^2 f_\alpha^U(x) d\alpha$ are positive definite. Let H be the Hessian of

$$\phi(x) = \int_0^1 \left(f_\alpha^L(x) + f_\alpha^U(x) \right) d\alpha.$$

Then with the help of exact line search, the generated sequence $\{\delta_k\}$ in Algorithm 1 satisfies the following properties for $i = 0, 1, \dots, m$ where $m \leq n - 1$:

- (i) $A_{i+1} \Phi_j = \delta_j, j = 0, 1, 2, \dots, i,$
- (ii) $\delta_i^t H \delta_j = 0, j = 0, 1, 2, \dots, i - 1,$ and
- (iii) The method terminates at $m + 1 \leq n$ steps. If $m = n - 1,$ then $A_n = H^{-1}.$

Proof We note that according to the hypothesis,

$$H = \nabla^2 \int_0^1 \left(f_\alpha^L(x) + f_\alpha^U(x) \right) d\alpha$$

is positive definite.

With the help of induction principle, we show the parts (i) and (ii).

The proof is trivial for $i = 0.$ Let the parts (i) and (ii) be true for some $i.$ We will show that the results hold for $i + 1.$

As $\nabla \phi(x_{i+1}) \neq 0,$ by exact line search, and $\Phi_k = \nabla \phi(x_{k+1}) - \nabla \phi(x_k) = H(x_{k+1} - x_k) = H \delta_k, k = 1, 2, \dots, i,$ the induction hypothesis implies that for $j \leq i,$

$$\begin{aligned} \nabla \phi(x_{i+1}) \delta_j &= \nabla \phi(x_{i+1})^t \delta_j + \sum_{k=j+1}^i (\nabla \phi(x_{k+1}) - \nabla \phi(x_k))^t \delta_j \\ &= \nabla \phi(x_{i+1})^t \delta_j + \sum_{k=j+1}^i \Phi_k^t \delta_j \\ &= \sum_{k=j+1}^i \delta_k^t H \delta_j \\ &= 0. \end{aligned}$$

Thus, with the help of $\delta_{i+1} = -\alpha_{i+1} A_{i+1} \nabla \phi(x_{i+1}),$ it follows that

$$\delta_{i+1}^t H \delta_j = -\alpha_{i+1} \nabla \phi(x_{i+1})^t A_{i+1} \Phi_j = -\alpha_{i+1} \nabla \phi(x_{i+1})^t \delta_j = 0,$$

which completes the proof of the part (ii) holding for $i + 1.$

Below we show that part (i) holds for $i + 1,$ i.e.,

$$A_{i+2} \delta_j = \delta_j, \quad j = 0, 1, 2, \dots, i + 1.$$

From Algorithm 1, the rank-two update of A_k 's clearly shows

$$A_{i+2} \Phi_{i+1} = \delta_{i+1}.$$

Thus, part (i) is true when $j = i + 1.$

From part (ii), it follows from the induction hypothesis that for $j \leq i,$

$$\delta_{i+1}^t \Phi_j = \delta_{i+1}^t H \delta_j = 0 \quad \text{and}$$

$$\Phi_{i+1}^t A_{i+1} \Phi_j = \Phi_{i+1}^t \delta_j = \delta_{i+1}^t H \delta_j = 0.$$

Therefore,

$$\begin{aligned} A_{i+2} \Phi_j &= A_{i+1} \Phi_j + \frac{\delta_{i+1} \delta_{i+1}^t \Phi_j}{\delta_{i+1}^t \Phi_{i+1}} - \frac{A_{i+1} \Phi_{i+1} \Phi_{i+1}^t A_{i+1} \Phi_j}{\Phi_{i+1}^t A_{i+1} \Phi_{i+1}} \\ &= A_{i+1} \Phi_j \\ &= \delta_j. \end{aligned}$$

Hence, the part (i) follows.

Lastly, from part (ii), we note that the directions $\delta_i, i = 0, 1, 2, \dots, m$ are conjugate. Hence, Algorithm 1 must terminate after $m \leq n$ steps. If $m = n - 1$, linear independence of δ_i 's, $i = 0, 1, \dots, n - 1$ clearly shows

$$A_n H \delta_j = A_n \Phi_j = \delta_j, \quad j = 0, 1, 2, \dots, n - 1,$$

which implies $A_n = H^{-1}$. □

5 Illustrative examples

In this section, two illustrative examples are presented to explore the computational procedure of Algorithm 1. In the first example, we consider a fuzzy quadratic form as the objective function. In the second, a general nonlinear fuzzy function is taken as the objective function.

Example 1 Consider the following quadratic fuzzy optimization problem:

$$\min_{(x_1, x_2) \in \mathbb{R}^2} \left(\frac{1}{2}/\frac{3}{2} \right) x_1^2 \oplus \left(0/\frac{1}{2}/1 \right) x_2^2 \oplus \left(\frac{1}{2}/1/\frac{3}{2} \right) x_1 x_2 \oplus \left(0/\frac{1}{2}/1 \right) x_1 \ominus \left(0/\frac{1}{2}/1 \right) x_2.$$

Let us consider the initial approximation to the minimizer as $x_0 = (x_1^0, x_2^0) = (0, 0)$. With the help of fuzzy arithmetic, the lower and the upper function can be obtained as

$$f_\alpha^L(x_1, x_2) = \begin{cases} \frac{1+\alpha}{2} x_1^2 + \frac{\alpha}{2} x_2^2 + \frac{1+\alpha}{2} x_1 x_2 + \frac{\alpha}{2} x_1 - (1 - \frac{\alpha}{2}) x_2 & \text{if } x_1 \geq 0, x_2 \geq 0 \\ \frac{1+\alpha}{2} x_1^2 + \frac{\alpha}{2} x_2^2 + \frac{3-\alpha}{2} x_1 x_2 + \frac{\alpha}{2} x_1 - \frac{\alpha}{2} x_2 & \text{if } x_1 \geq 0, x_2 \leq 0 \\ \frac{1+\alpha}{2} x_1^2 + \frac{\alpha}{2} x_2^2 + \frac{3-\alpha}{2} x_1 x_2 + (1 - \frac{\alpha}{2}) x_1 - (1 - \frac{\alpha}{2}) x_2 & \text{if } x_1 \leq 0, x_2 \geq 0 \\ \frac{1+\alpha}{2} x_1^2 + \frac{\alpha}{2} x_2^2 + \frac{1+\alpha}{2} x_1 x_2 + (1 - \frac{\alpha}{2}) x_1 - \frac{\alpha}{2} x_2 & \text{if } x_1 \leq 0, x_2 \leq 0 \end{cases}$$

and

$$\begin{aligned} f_\alpha^U(x_1, x_2) &= \begin{cases} \frac{3-\alpha}{2} x_1^2 + (1 - \frac{\alpha}{2}) x_2^2 + \frac{3-\alpha}{2} x_1 x_2 + (1 - \frac{\alpha}{2}) x_1 - \frac{\alpha}{2} x_2 & \text{if } x_1 \geq 0, x_2 \geq 0 \\ \frac{3-\alpha}{2} x_1^2 + (1 - \frac{\alpha}{2}) x_2^2 + \frac{1+\alpha}{2} x_1 x_2 + (1 - \frac{\alpha}{2}) x_1 - (1 - \frac{\alpha}{2}) x_2 & \text{if } x_1 \geq 0, x_2 \leq 0 \\ \frac{3-\alpha}{2} x_1^2 + (1 - \frac{\alpha}{2}) x_2^2 + \frac{1+\alpha}{2} x_1 x_2 + \frac{\alpha}{2} x_1 - \frac{\alpha}{2} x_2 & \text{if } x_1 \leq 0, x_2 \geq 0 \\ \frac{3-\alpha}{2} x_1^2 + (1 - \frac{\alpha}{2}) x_2^2 + \frac{3-\alpha}{2} x_1 x_2 + \frac{\alpha}{2} x_1 - (1 - \frac{\alpha}{2}) x_2 & \text{if } x_1 \leq 0, x_2 \leq 0. \end{cases} \end{aligned}$$

Therefore,

$$\phi(x_1, x_2) = \int_0^1 \left(f_\alpha^L(x_1, x_2) + f_\alpha^U(x_1, x_2) \right) d\alpha = 2x_1^2 + x_2^2 + 2x_1 x_2 + x_1 - x_2.$$

Here

$$\nabla\phi(x_1, x_2) = \begin{bmatrix} 4x_1 + 2x_2 + 1 \\ 2x_1 + 2x_2 - 1 \end{bmatrix}.$$

Considering the initial matrix $A_0 = I_2$, we calculate the sequence $\{x_k\}$, $x_k = (x_1^k, x_2^k)$, through the following equations:

$$\left\{ \begin{array}{l} d_k = -A_k \begin{bmatrix} 4x_1^k + 2x_2^k + 1 \\ 2x_1^k + 2x_2^k - 1 \end{bmatrix} \\ \alpha_k = \operatorname{argmin}_{\alpha \geq 0} \phi(x_k + \alpha d_k) \\ \delta_k = \alpha_k d_k \\ x_{k+1} = x_k + \delta_k \\ \Phi_k = \nabla\phi(x_{k+1}) - \nabla\phi(x_k) = \begin{bmatrix} 4(x_1^{k+1} - x_1^k) + 2(x_2^{k+1} - x_1^k) \\ 2(x_1^{k+1} - x_1^k) + 2(x_2^{k+1} - x_1^k) \end{bmatrix}, \text{ and} \\ A_{k+1} = A_k + \frac{\delta_k \delta_k^t}{\delta_k^t \Phi_k} - \frac{A_k \Phi_k \Phi_k^t A_k}{\Phi_k^t A_k \Phi_k}. \end{array} \right.$$

The initial iteration ($k = 0$)

$$\left\{ \begin{array}{l} x_0 = (0, 0) \\ A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \|\nabla\phi(x_0)\| = \|(1, -1)^t\| = \sqrt{2} \neq 0 \\ d_0 = -\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \alpha_0 = 1 \\ x_1 = (-1, 1) \\ \delta_1 = (-1, 1) \\ \Phi_1 = (-2, 0) \\ A_1 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}. \end{array} \right.$$

The second iteration ($k = 1$)

$$\left\{ \begin{array}{l} x_1 = (-1, 1) \\ d_1 = (0, 1) \\ \alpha_1 = \frac{1}{2} \\ x_2 = (-1, \frac{3}{2}) \\ \nabla\phi(x_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{array} \right.$$

Hence $\bar{x} = (-1, \frac{3}{2})$ is a nondominated solution of the considered problem. It is important to note that the method converged at the second iteration since the objective function is a quadratic fuzzy function. This convergence at second iteration supports the result in Theorem 3.

Example 2 Consider the following nonlinear fuzzy optimization problem:

$$\min_{(x_1, x_2) \in \mathbb{R}^2} (0/\frac{1}{2}/1)x_1^4 \ominus (3/4/5)x_1^3 \oplus (\frac{23}{2}/\frac{25}{2}/\frac{27}{2})x_1^2 \\ \ominus (1/2/3)x_1^2x_2^2 \oplus (1/2/3)x_2^2 \ominus (15/16/17)x_1 \oplus (6/8/10).$$

With the help of fuzzy arithmetic, the lower and the upper function can be obtained as

$$f_\alpha^L(x_1, x_2) = \begin{cases} \frac{\alpha}{2}x_1^4 - (5 - \alpha)x_1^3 + \frac{23+\alpha}{2}x_1^2 - (3 - \alpha)x_1x_2 + (1 + \alpha)x_2^2 \\ \quad - (17 - \alpha)x_1 + (6 + 2\alpha) & \text{if } x_1 \geq 0, x_2 \geq 0 \\ \frac{\alpha}{2}x_1^4 - (5 - \alpha)x_1^3 + \frac{23+\alpha}{2}x_1^2 - (1 + \alpha)x_1x_2 + (1 + \alpha)x_2^2 \\ \quad - (17 - \alpha)x_1 + (6 + 2\alpha) & \text{if } x_1 \geq 0, x_2 \leq 0 \\ \frac{\alpha}{2}x_1^4 - (3 + \alpha)x_1^3 + \frac{23+\alpha}{2}x_1^2 - (1 + \alpha)x_1x_2 + (1 + \alpha)x_2^2 \\ \quad - (15 + \alpha)x_1 + (6 + 2\alpha) & \text{if } x_1 \leq 0, x_2 \geq 0 \\ \frac{\alpha}{2}x_1^4 - (3 + \alpha)x_1^3 + \frac{23+\alpha}{2}x_1^2 - (3 - \alpha)x_1x_2 + (1 + \alpha)x_2^2 \\ \quad - (15 + \alpha)x_1 + (6 + 2\alpha) & \text{if } x_1 \leq 0, x_2 \leq 0 \end{cases}$$

and

$$f_\alpha^U(x_1, x_2) = \begin{cases} (1 - \frac{\alpha}{2})x_1^4 - (3 + \alpha)x_1^3 + \frac{27-\alpha}{2}x_1^2 - (1 + \alpha)x_1x_2 + (3 - \alpha)x_2^2 \\ \quad - (15 + \alpha)x_1 + (10 - 2\alpha) & \text{if } x_1 \geq 0, x_2 \geq 0 \\ (1 - \frac{\alpha}{2})x_1^4 - (3 + \alpha)x_1^3 + \frac{27-\alpha}{2}x_1^2 - (3 - \alpha)x_1x_2 + (3 - \alpha)x_2^2 \\ \quad - (15 + \alpha)x_1 + (10 - 2\alpha) & \text{if } x_1 \geq 0, x_2 \leq 0 \\ (1 - \frac{\alpha}{2})x_1^4 - (5 - \alpha)x_1^3 + \frac{27-\alpha}{2}x_1^2 - (3 - \alpha)x_1x_2 + (3 - \alpha)x_2^2 \\ \quad - (17 - \alpha)x_1 + (10 - 2\alpha) & \text{if } x_1 \leq 0, x_2 \geq 0 \\ (1 - \frac{\alpha}{2})x_1^4 - (5 - \alpha)x_1^3 + \frac{27-\alpha}{2}x_1^2 - (1 + \alpha)x_1x_2 + (3 - \alpha)x_2^2 \\ \quad - (17 - \alpha)x_1 + (10 - 2\alpha) & \text{if } x_1 \leq 0, x_2 \leq 0 \end{cases}$$

Therefore,

$$\phi(x_1, x_2) = \int_0^1 (f_\alpha^L(x_1, x_2) + f_\alpha^U(x_1, x_2)) d\alpha \\ = x_1^4 - 8x_1^3 + 25x_1^2 - 4x_1x_2 + 4x_2^2 - 32x_1 + 16.$$

Here

$$\nabla\phi(x_1, x_2) = \begin{bmatrix} 4x_1^3 - 24x_1^2 + 50x_1 - 4x_2 - 32 \\ 8x_2 - 4x_1 \end{bmatrix}.$$

The following Table 1 exhibits the performance of Algorithm 1 on this example with the initial point as $x_0 = (0, 3)$, initial approximation of the Hessian as $A_0 = I_2$ and the termination scalar as $\epsilon = 0.01$. It is found that $(2.12, 1.06)$ is a solution of the considered problem.

6 Conclusion

In this paper, a quasi-Newton method with rank-two modification has been derived to find a non-dominated solution of an unconstrained fuzzy optimization problem. In the optimality concept, the fuzzy-max ordering of a pair of fuzzy numbers has been used. The

Table 1 Performance of quasi-Newton Algorithm 1 on Example 2

k	x_k	$\nabla\phi(x_k)$	$\ \nabla\phi(x_k)\ $	A_k	d_k	α_k	x_{k+1}	δ_k	Φ_k	A_{k+1}
0	(0, 3)	$\begin{bmatrix} -44 \\ 24 \end{bmatrix}$	50.12	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -44 \\ 24 \end{bmatrix}$	0.06	(2.70, 1.51)	(2.70, -1.49)	$\begin{bmatrix} 46.76 \\ -21.52 \end{bmatrix}$	$\begin{bmatrix} 0.25 & 0.38 \\ 0.38 & 0.81 \end{bmatrix}$
1	(2.70, 1.51)	$\begin{bmatrix} 0.73 \\ 1.28 \end{bmatrix}$	1.47	$\begin{bmatrix} 0.25 & 0.38 \\ 0.38 & 0.81 \end{bmatrix}$	$\begin{bmatrix} -0.67 \\ 1.31 \end{bmatrix}$	0.22	(2.55, 1.22)	(-0.15, -0.29)	$\begin{bmatrix} 1.09 \\ -5.88 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
2	(2.55, 1.22)	$\begin{bmatrix} 0.89 \\ -0.44 \end{bmatrix}$	0.98	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.89 \\ 0.44 \end{bmatrix}$	0.11	(2.45, 1.27)	(-0.10, 0.05)	$\begin{bmatrix} -0.50 \\ -4.12 \end{bmatrix}$	$\begin{bmatrix} 0.65 & 0.45 \\ 0.45 & 0.46 \end{bmatrix}$
3	(2.45, 1.27)	$\begin{bmatrix} 0.18 \\ 0.36 \end{bmatrix}$	0.40	$\begin{bmatrix} 0.65 & 0.45 \\ 0.45 & 0.46 \end{bmatrix}$	$\begin{bmatrix} -0.28 \\ -0.25 \end{bmatrix}$	0.64	(2.27, 1.11)	(-0.18, -0.16)	$\begin{bmatrix} 0.35 \\ -4.56 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
4	(2.27, 1.11)	$\begin{bmatrix} 0.18 \\ -0.20 \end{bmatrix}$	0.27	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.18 \\ 0.20 \end{bmatrix}$	0.10	(2.25, 1.13)	(-0.02, 0.02)	$\begin{bmatrix} -0.74 \\ -4.32 \end{bmatrix}$	$\begin{bmatrix} 0.80 & 0.38 \\ 0.38 & 0.31 \end{bmatrix}$
5	(2.25, 1.13)	$\begin{bmatrix} 0.04 \\ 0.04 \end{bmatrix}$	0.06	$\begin{bmatrix} 0.80 & 0.38 \\ 0.38 & 0.31 \end{bmatrix}$	$\begin{bmatrix} -0.05 \\ -0.03 \end{bmatrix}$	2.64	(2.12, 1.05)	(-0.13, 0.08)	$\begin{bmatrix} -0.66 \\ -4.08 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
6	(2.12, 1.05)	$\begin{bmatrix} 0.05 \\ -0.08 \end{bmatrix}$	0.09	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.05 \\ -0.08 \end{bmatrix}$	0.10	(2.12, 1.06)	(0, 0.01)	$\begin{bmatrix} -1.34 \\ -4.20 \end{bmatrix}$	$\begin{bmatrix} 0.11 & 0.32 \\ 0.32 & 0.82 \end{bmatrix}$
7	(2.12, 1.06)	$\begin{bmatrix} 0.004 \\ 0.004 \end{bmatrix}$	0.006							

gH -differentiability of fuzzy functions have been employed to find the non-dominated solution point. An algorithmic implementation and the convergence analysis of the proposed technique has also been presented. The technique is found to have superlinear convergence rate and quadratic termination property. Two numerical examples have been explored to illustrate the proposed technique.

It is to note that unlike Newton method for fuzzy optimization problems [9,32], the proposed method is derived without using the inverse of the concerned Hessian matrix. Instead, a sequence of positive definite inverse Hessian approximation $\{A_k\}$ is used which satisfies the *quasi-Newton equation* $A_{k+1}\Phi_k = \delta_k$. Thus the derived method sidestepped the inherent computational difficulty to compute inverse of the concerned Hessian matrix. In this way, the proposed method is made as more efficient than the existing Newton method [9,32].

The method can be easily observed as much similar to the classical DFP method [30] for the conventional optimization problem. In the similar way of the presented technique, it can be further extended to a BFGS-like method [32] for fuzzy optimization. Instead of the rank-two modification in the approximation of the inverse of the Hessian matrix, a rank-one modification can also be done. It is also to be observed that in Algorithm , we make use the exact line search technique along the descent direction $d_k = -A_k \nabla \phi(x_k)$. However, an inexact line search technique [30] could have also been used. A future research on this rank-one modification and inexact line search technique for fuzzy optimization problem can be performed and implemented on Newton and quasi-Newton method.

It is important to observe that the proposed technique generates only one non-dominated solution for a fuzzy optimization problem. Development of a method which can capture complete nondominated solution set is an important issue for fuzzy optimization problem. Our next step of research would be focused on capturing complete nondominated solution of a fuzzy optimization problem with the help of the proposed quasi-Newton method.

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