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Sinc-Galerkin solution to the clamped plate eigenvalue problem

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Abstract We propose an accurate and computationally efficient numerical technique for solving the biharmonic eigenvalue problem. The technique is based on the sinc-Galerkin approximation method to solve the clamped plate problem. Numerical experiments for plates with various aspect ratios are reported, and comparisons are made with other methods in literature. The calculated results accord well with those published earlier, which proves the accuracy and validity of the proposed method.

Keywords Sinc functions · Sinc-Galerkin · Biharmonic problem · Eigenvalues

Mathematics Subject Classification 65N25 · 31A30

1 Introduction

Eigenvalue problems such as plate vibration problems have attracted much research using a wide range of methods. However, exact solutions are available only for certain boundary conditions and domain configurations, hence approximate solutions are of great importance when analytical methods fail or become too cumbersome.

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A large number of numerical methods have been developed to obtain solutions for many rectangular plate problems with different boundary conditions [23,26,29,37–39]. A boundary homotopy method was used in [46] to obtain strict bounds for the N lowest eigenvalues of the clamped plate equation in the unit square. Some of these methods can only provide upper bounds for the eigenvalues. For example, spectral Legendre-Galerkin method [6] was used to provide highly accurate solution to the biharmonic eigenvalue problem for the clamped unit-square plate and buckling plate problems. Another method that always gives upper bounds for the eigenvalues is the Rayleigh-Ritz method [48]. Recently, Gavalas and El-Raheb [21] extended the method for eigenvalue problems with discontinuous boundary conditions applied to vibration of rectangular plates. Also, the method was applied in [8] for the vibration analysis of exponential functionally graded rectangular plates in thermal environment. On the other hand, the superposition method developed by Gorman [24] gives lower bound results to the same problem. This method has been successfully applied for the analysis of undamped out-of-plane vibrations of single isotropic plates [22]. Other successful numerical methods include the spline finite strip method by Fan and Cheung [19], the Galerkin approach by Chia [11] and Leipholz [27], the least squares technique [45], meshless methods [4,13], and finite element methods [1,10]. Differential quadrature (DQ) methods [2,9] have been successfully applied in the vibration analysis. A generalized differential quadrature (GDQ) method was introduced by Shu and Richards [41] to simplify the calculation of the weighting coefficients of the derivatives approximation.

The biharmonic boundary value problem

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = f(x, y), \quad (x, y) \in \Omega \equiv (a, b) \times (c, d),$$

subject to the nonhomogeneous boundary conditions

$$u|_{\partial\Omega} = g(x, y), \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = h(x, y)$$

(where $\frac{\partial u}{\partial n}$ is the outward normal derivative) was solved using the sinc-Galerkin method in [18]. In this paper, we apply the sinc-Galerkin method to solve the biharmonic eigenvalue problem

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = \lambda u, \quad (x, y) \in \Omega \equiv (a, b) \times (c, d),$$

subject to the following boundary conditions for a clamped plate

$$u|_{\partial\Omega} = \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0.$$

In recent years, a lot of attention has been devoted to the study of the sinc method to investigate various scientific models. It is possible to solve two point boundary value problems [5,34], initial-value problems [3], fourth-order differential equations [40], sixth-order boundary-value problems [17], nonlinear higher-order boundary-value problems [16], partial differential equations [32], eigenvalue problems, singular problem-like Poisson [47], linear Fredholm integro-differential equations [33], linear and nonlinear Volterra integro-differential equations [35], linear and nonlinear system of second-order boundary value problems [14], as well as Troesch's problem [15] by using sinc methods. The comparison of finite difference, spectral and sinc-convolution treatments was considered in [12].

The outline of the paper is as follows. Section 2, contains notations, definitions and some results of sinc function theory. In Sect. 3, the sinc-Galerkin approach to the clamped plate eigenvalue problem is presented. In Sect. 4, we verify the reliability of the proposed algorithm by numerical results obtained and comparisons with published results in literature. Conclusions are given in Sect. 5.

2 Preliminaries and fundamentals

The books [31,43] provide excellent overviews of methods based on sinc functions for solving ordinary and partial differential equations and integral equations. The goal of this section is to recall notations and definitions of the sinc function, state some known results, and derive useful formulas that are important for this paper.

The sinc function is defined on the whole real line by

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \quad -\infty < x < \infty.$$

For $h > 0$, the translated sinc functions with evenly spaced nodes are given as

$$S(k,h)(x) = \operatorname{sinc}\left(\frac{x-kh}{h}\right), \quad \mathrm{k} = 0, \pm 1, \pm 2, \ldots$$

If f is defined on the real line, then for $h > 0$ the series

$$C(f,h) = \sum_{k=-\infty}^{\infty} f(hk)\operatorname{sinc}\left(\frac{x-hk}{h}\right),$$

is called the Whittaker cardinal expansion of f whenever this series converges. The properties of Whittaker cardinal expansions have been studied and are thoroughly surveyed in [43]. These properties are derived in the infinite strip D_d of the complex plane where for $d > 0$

$$D_d = \left\{\zeta = \xi + i\eta : |\eta| < d \leq \frac{\pi}{2}\right\}. \tag{2.1}$$

To construct approximations on the interval (a, b) which are used in this paper, we consider the conformal map [43]

$$\phi(z) = \ln\left(\frac{z-a}{b-z}\right), \tag{2.2}$$

The map ϕ carries the eye-shaped region

$$D_E = \left\{z = x + iy : \left|\arg\left(\frac{z-a}{b-z}\right)\right| < d \leq \frac{\pi}{2}\right\}, \tag{2.3}$$

onto the infinite strip D_d.

The "mesh sizes" h represent the mesh sizes in D_d for the uniform grids $\{kh\}$, $k = 0, \pm 1, \pm 2, \ldots$. The sinc grid points $z_k \in (a, b)$ in D_E will be denoted by x_k because they are real, and are given by

$$x_k = \phi^{-1}(kh) = \frac{a + b\, e^{kh}}{1 + e^{kh}}, \tag{2.4}$$

The class of functions suitable for sinc interpolation and quadrature is denoted by $B(D)$ and defined below.

Definition 2.1 [43] Let $B(D)$ be the class of functions F that are analytic in D, satisfy

$$\int_{\psi(L+t)} |F(z)dz| \to 0, \quad \text{as } t = \pm\infty,$$

where

$$L = \left\{ iy : |y| < d \leq \frac{\pi}{2} \right\},$$

and on the boundary of D (denoted ∂D) satisfy

$$N(F) = \int_{\partial D_E} |F(z)dz| < \infty.$$

The following theorem provides the error bounds of sinc interpolation and quadrature formulae for functions in $B(D)$.

Theorem 2.1 *[43]Let Γ be (a, b). Let $F \in B(D)$ and $\tau_j = \psi(jh) = \phi^{-1}(jh)$, $j = 0, \pm 1, \pm 2, \ldots,$. Let there exist positive constants α, β and C such that*

$$\left| \frac{F(\tau)}{\phi'(\tau)} \right| \leq C \begin{cases} \exp(-\alpha|\phi(\tau)|), & \tau \in \psi((-\infty, 0)), \\ \exp(-\beta|\phi(\tau)|), & \tau \in \psi((0, \infty)). \end{cases} \tag{2.5}$$

then the error bound is

$$\left| \int_\Gamma F(\tau)d\tau - h \sum_{j=-M}^{N} \frac{F(\tau_j)}{\phi'(\tau_j)} \right| \leq C \left(\frac{e^{-\alpha M h}}{\alpha} + \frac{e^{-\beta N h}}{\beta} \right) + |I_F|. \tag{2.6}$$

Making the selections

$$h = \sqrt{\frac{\pi d}{\alpha M}}, \quad \text{and} \quad N \equiv \left[\left| \frac{\alpha}{\beta} M + 1 \right| \right],$$

where $[x]$ is the integer part of x, then

$$\int_\Gamma F(\tau)d\tau = h \sum_{j=-M}^{N} \frac{F(\tau_j)}{\phi'(\tau_j)} + O\left(e^{-(\pi \alpha d M)^{1/2}}\right).$$

The sinc-Galerkin method requires that the derivatives of composite sinc functions be evaluated at the nodes. We need the following lemma.

Lemma 2.1 *[31,43] Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_d, given by (2.2). Then*

$$\delta_{jk}^{(0)} = [S(j,h) \circ \phi(x)]\,|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{2.7}$$

$$\delta_{jk}^{(1)} = h \frac{d}{d\phi} [S(j,h) \circ \phi(x)]\,|_{x=x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases} \tag{2.8}$$

$$\delta_{jk}^{(2)} = h^2 \frac{d^2}{d\phi^2} [S(j,h) \circ \phi(x)]\,|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k. \end{cases} \tag{2.9}$$

$$\begin{aligned} \delta_{jk}^{(3)} &= h^3 \left[\frac{d^3}{d\phi^3} [S(j,h) \circ \phi(x)] \right]_{x=x_k} \\ &= \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{(k-j)^3} \left[6 - \pi^2(k-j)^2 \right], & j \neq k, \end{cases} \end{aligned} \tag{2.10}$$

$$\delta_{jk}^{(4)} = h^4 \left[\frac{d^4}{d\phi^4} \left[S(j,h) \circ \phi(x) \right] \right]_{x=x_k} = \begin{cases} \frac{\pi^4}{5}, & j = k, \\ \frac{-4(-1)^{k-j}}{(k-j)^4} \left[6 - \pi^2 (k-j)^2 \right], & j \neq k, \end{cases} \tag{2.11}$$

3 The Sinc-Galerkin approach to the biharmonic eigenvalue problem

The equation of motion for the undamped free vibration of a plate may be written as [44]

$$\frac{\partial^4 u}{\partial \bar{x}^4} + 2\frac{\partial^4 u}{\partial \bar{x}^2 \partial \bar{y}^2} + \frac{\partial^4 u}{\partial \bar{y}^4} + \frac{\rho}{D}\frac{\partial^2 u}{\partial t^2} = 0 \tag{3.1}$$

where u is the transverse displacement at a point defined by the coordinates $(\bar{x}, \bar{y}) \in (0, a) \times (0, b)$ where a and b are the plate dimensions, at any given time t, D is the flexural rigidity of the plate and ρ is the mass of the plate per unit area of its surface.

For a plate of constant thickness σ and material properties E (Young's modulus of elasticity) and ν (Poisson's ratio), the flexural rigidity D is given by

$$D = \frac{E\sigma^3}{12(1-\nu^2)}$$

Assuming harmonic vibration, we may write

$$u(\bar{x}, \bar{y}, t) = U(\bar{x}, \bar{y}) sin(\omega t) \tag{3.2}$$

where $U(\bar{x}, \bar{y})$ is a shape function satisfying the fully clamped plate boundary conditions and describing the shape of the deflected middle surface of the vibrating plate, and ω is a natural circular frequency of the plate. Substituting for u in Eq. (3.1), we obtain

$$\frac{\partial^4 U}{\partial \bar{x}^4} + 2\frac{\partial^4 U}{\partial \bar{x}^2 \partial \bar{y}^2} + \frac{\partial^4 U}{\partial \bar{y}^4} - \frac{\rho\omega^2}{D} U = 0 \tag{3.3}$$

For convenience, the governing Eq. (3.3) is expressed in dimensionless form. Define the dimensionless coordinates x and y, where $x = \bar{x}/a$ and $y = \bar{y}/b$. Equation (3.3) may be then written as

$$LU \equiv \frac{\partial^4 U}{\partial y^4} + 2\Phi^2 \frac{\partial^4 U}{\partial x^2 \partial y^2} + \Phi^4 \frac{\partial^4 U}{\partial x^4} - \Phi^4 \lambda^2 U = 0, \tag{3.4}$$

where $(\Phi = b/a)$ is the plate aspect ratio, and the non-dimensional frequency parameter, λ of the plate may be expressed as

$$\lambda = \omega a^2 \sqrt{\frac{\rho}{D}} \tag{3.5}$$

The assumed sinc approximate solution to the eigenvalue problem (3.4) takes the form:

$$U_n(x, y) = \sum_{j=-M}^{N} \sum_{i=-M}^{N} U_{ij} \, S_{ij}(x, y), \quad n = M + N + 1 \tag{3.6}$$

where the basis functions $\left\{S_{ij}(x, y)\right\}$ for $-M \leq i, j \leq N$ are given as simple product basis functions of one dimensional sinc basis

$$\begin{aligned} S_{ij}(x,y) &= S_i(x)S_j(y) \\ &= [S(i,h_x)\circ\phi_1(x)][S(j,h_y)\circ\phi_2(y)] \end{aligned} \tag{3.7}$$

where ϕ be as before.

The assumed approximate solution satisfies the clamped plate boundary conditions

$$U|_\Gamma = \left.\frac{\partial U}{\partial n}\right|_\Gamma = 0$$

where Γ is the boundary of the new dimensionless domain, $\Omega \equiv (0,1)\times(0,1)$, and n is the outward normal to the boundary.

We use the Galerkin scheme to determine the unknown coefficients $\{U_{ij}\}$ in (3.6). First, we define the inner product of two functions f and g by

$$\langle f, g\rangle = \int_0^1\int_0^1 f(x,y)\,g(x,y)\,w(x)\,v(y)\,dx\,dy,$$

where $w(x) = \frac{1}{[\phi_1'(x)]^2}$ and $v(y) = \frac{1}{[\phi_2'(x)]^2}$ are the weight functions in the direction of the x-axis and y-axis, respectively.

The discrete Galerkin system is then given by

$$\langle LU_n, S_{kl}\rangle = \Phi^4\lambda^2\langle U_n, S_{kl}\rangle, \quad -M \le k,l \le N \tag{3.8}$$

Instead of substituting the approximate solution given by (3.6) into (3.8), we first analyze the equation

$$\left\langle\Phi^4 U_{xxxx}, S_k\,S_l\right\rangle + \left\langle 2\,\Phi^2 U_{xxyy}, S_k S_l\right\rangle + \left\langle U_{yyyy}, S_k\,S_l\right\rangle = \left\langle\Phi^4\lambda^2 U, S_k\,S_l\right\rangle \tag{3.9}$$

The method of approximating the integrals in (3.9) begins by integrating by parts to transfer all derivatives from U to S_{kl}. We are lead to the following theorem

Theorem 3.1 *The following relations hold*

$$\left\langle\Phi^4 U_{xxxx}, S_k\,S_l\right\rangle \approx h_x h_y \Phi^4 \frac{v(y_l)}{\phi_2'(y_l)} \sum_{i=-M}^{N}\sum_{j=0}^{4} \frac{U(x_i,y_l)}{\phi_1'(x_i)}\left[\frac{1}{h_x^j}\delta_{ki}^{(j)}\mu_j\right], \tag{3.10}$$

$$\left\langle U_{yyyy}, S_k S_l\right\rangle \approx h_x h_y \frac{w(x_k)}{\phi_1'(x_k)} \sum_{i=-M}^{N}\sum_{j=0}^{4} \frac{U(x_k,y_i)}{\phi_2'(y_i)}\left[\frac{1}{h_y^j}\delta_{li}^{(j)}\eta_j\right], \tag{3.11}$$

$$\left\langle 2\,\Phi^2 u_{xxyy}, S_k S_l\right\rangle \approx 2\Phi^2 h_x\,h_y \sum_{j=-M}^{N}\sum_{i=-M}^{N}\sum_{r=0}^{2}\sum_{p=0}^{2} \frac{\tau_r\,\xi_p\,U(x_i,y_j)}{h_x^r\,h_y^p\,\phi_1'(x_i)\,\phi_2'(y_i)}\,\delta_{ki}^{(r)}\,\delta_{lj}^{(p)}, \tag{3.12}$$

and

$$\left\langle\Phi^4\lambda^2 U, S_k S_l\right\rangle \approx \Phi^4\lambda^2\,h_x\,h_y \frac{w(x_k)\,U(x_k,y_l)\,v(y_l)}{\phi_1'(x_k)\,\phi_2'(y_j)} \tag{3.13}$$

for some functions μ_j, η_j, ξ_p and τ_r to be determined.

Proof The proof is given in Appendix 1. □

Replacing each term of (3.9) with the corresponding approximations defined in (3.10), (3.11), (3.12) and (3.13) and replacing $U(x_k, y_l)$ by U_{kl} and dividing by $h_x\,h_y$, we obtain the following theorem

Theorem 3.2 *If the assumed approximate solution of the boundary-value problem (3.1) is (3.6), then the discrete sinc-Galerkin system for the determination of the unknown coefficients* $\{U_{kl},\ k = -M, \ldots, N, \quad l = -M, \ldots, N\}$ *is given by*

$$\Phi^4 \frac{v(y_l)}{\phi_2'(y_l)} \sum_{i=-M}^{N} \sum_{j=0}^{4} \frac{U_{il}}{\phi_1'(x_i)} \left[\frac{1}{h_x^j} \delta_{ki}^{(j)} \mu_j \right]$$
$$+ 2\,\Phi^2 \sum_{j=-M}^{N} \sum_{i=-M}^{N} \sum_{r=0}^{2} \sum_{p=0}^{2} \frac{\tau_r\, \xi_p\, \delta_{ki}^{(r)}\, \delta_{lj}^{(p)}}{h_x^r\, h_y^p\, \phi_1'(x_i)\, \phi_2'(y_j)} U_{ij}$$
$$+ \frac{w(x_k)}{\phi_1'(x_k)} \sum_{i=-M}^{N} \sum_{j=0}^{4} \frac{U_{ki}}{\phi_2'(y_i)} \left[\frac{1}{h_y^j} \delta_{li}^{(j)} \eta_j \right] = \Phi^4 \lambda^2 \frac{w(x_k)\, U(x_k, y_l)\, v(y_l)}{\phi_1'(x_k)\, \phi_2'(y_l)} \tag{3.14}$$

Recall the notation of Toepleitz matrices [25]. Let $I_n^{(P)}$, $P = 0, 1, 2, 3, 4$ be the $n \times n$ matrices $I^{(P)}$, with jk-th entry $\delta_{jk}^{(P)}$ as given by equations (2.7)–(2.11). Further, $D(g_x)$ is an $n \times n$ diagonal matrix whose diagonal entries are $[g_{-M}, g_{-M+1}, \ldots, g_N]^T$. Lastly, the $n \times n$ matrix $\mathbf{U}$ has kl-th entries given by U_{kl}. Introducing this notation in Eq. (3.14) leads to the matrix form

$$\mathbf{A}\,\mathbf{X} + \mathbf{C}\,\mathbf{X}\,\mathbf{E} + \mathbf{X}\,\mathbf{B} = \lambda^2\, \mathbf{X} \tag{3.15}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{E}$ and $\mathbf{X}$ are matrices of size $n \times n$, and given by

$$\mathbf{A} = \mathbf{D}(\phi_1') \sum_{i=0}^{4} \left[\frac{1}{h_x^i} \mathbf{I}_n^{(i)} \mathbf{D}\left(\frac{\mu_i}{\phi_1'^{\,2} w} \right) \right] \mathbf{D}(\phi_1'),$$
$$\mathbf{B} = \left(\frac{1}{\Phi^4} \right) \mathbf{D}(\phi_2') \sum_{i=0}^{4} \left[\frac{1}{h_y^i} \mathbf{I}_n^{(i)} \mathbf{D}\left(\frac{\eta_i}{\phi_2'^{\,2} v} \right) \right]^T \mathbf{D}(\phi_2'),$$
$$\mathbf{C} = \left(\frac{2}{\Phi^2} \right) \mathbf{D}(\phi_1') \sum_{i=0}^{2} \left[\frac{1}{h_x^i} \mathbf{I}_n^{(i)} \mathbf{D}\left(\frac{\tau_i}{\phi_1'^{\,2} w} \right) \right] \mathbf{D}(\phi_1')$$
$$\mathbf{E} = \mathbf{D}(\phi_2') \sum_{i=0}^{2} \left[\frac{1}{h_y^i} \mathbf{I}_n^{(i)} \mathbf{D}\left(\frac{\xi_i}{\phi_2'^{\,2} v} \right) \right]^T \mathbf{D}(\phi_2')$$

and

$$\mathbf{X} = \mathbf{D}(w)\, \mathbf{U}\, \mathbf{D}(v),$$

The last step is to convert the matrix equation (3.15) to a matrix eigenvalue problem. This is done via vectorization of (3.15) using Kronecker matrix products [36]. This yields the algebraic eigenvalue problem

$$\mathbf{M}\, z = \lambda^2\, z \tag{3.16}$$

where

$$\mathbf{M} \ = \mathbf{I}_n \otimes \mathbf{A} + \mathbf{E}^T \otimes \mathbf{C} + \mathbf{B}^T \otimes \mathbf{I}_n$$

From the above equation, the values of λ defined in Eq. (3.5) can be obtained from the eigenvalues of matrix $\mathbf{M}$.

4 Numerical results and discussions

For purposes of comparison, contrast and performance, we consider the computation of λ for the square plate and rectangular plates of various aspect ratios, Φ. For all cases, we have made the selections, $d = \pi/2$, $\alpha = 0.5$ and $h = \pi/\sqrt{M}$.

- *Case 1* In this case, a clamped square plate is considered. In Table 1, we report the calculated values of λ_1, λ_2, λ_3 and λ_4 using different number of sinc basis functions with $M = 5,\ 10,\ 15, \ldots, 50$.

It is worth noting that clamped plate eigenvalue problem has no exact solution. Hence, we further use the results obtained in Table 1 to find the limit values $\{\lambda_1^*,\ \lambda_2^*,\ \lambda_3^*,\ \lambda_4^*\}$ using the minimal polynomial extrapolation (MPE) approach [7,42]. The obtained limit values are reported in Table 2 along with the strict lower bounds, $\underline{\lambda}_j$ and upper bounds, $\overline{\lambda}_j$ obtained in [46].

Since we are using the Galerkin scheme, we note from the results in Table 1 that our method converges to the accurate upper bound. We define an approximate relative error (ARE) by

$$e_i = \frac{\left|\lambda_{\text{sinc}} - \lambda_i^*\right|}{\lambda_i^*}, \qquad i = 1, \ldots, 4. \tag{4.1}$$

The exponential convergence rate shown in Fig. 1 for the ARE of the first four frequency paramters verifies the validity and accuracy of the proposed scheme. The shapes of the corresponding eigenmodes are shown in Fig. 2.

The results in Table 3 also show good agreement with those obtained by other methods; the Rayleigh–Ritz method with displacement components expressed in simple algebraic

Table 1 Convergence of the computed values of λ for the square plate using different M

M	λ_1	λ_2	λ_3	λ_4
5	35.31842943502423	75.663083586124	75.66308358612369	108.99701174386280
10	36.01186902491406	75.189800637716	75.18980063773876	111.77565118066765
15	35.97852996553088	73.298707870932	73.29870787093215	107.83109310267633
20	35.98630157083767	73.462784551881	73.46278455205049	108.37576563864191
25	35.98491854966878	73.379810693652	73.37981069999269	108.17595847912698
30	35.98526594894749	73.399756043821	73.39975604382076	108.23138796521422
35	35.98516958251560	73.391942503801	73.39194272596092	108.21143908446292
40	35.98519840348746	73.394621971451	73.39462578813033	108.21847745722619
45	35.98518863479417	73.393542486825	73.39354248682497	108.21572730381321
50	35.98519148325810	73.393983806872	73.39399978580005	108.21684351123402

Table 2 Minimal polynomial extrapolation limits of λ for the square plate compared to the strict lower and upper bounds of [46]

j	$\underline{\lambda}_j$ [46]	λ_j^*	$\overline{\lambda}_j$ [46]
1	35.98519056501	35.985190478245	35.98519123195
2	73.39384442854	73.393857389441	73.39384579105
3	73.39384442854	73.393878120887	73.39384579105
4	108.2164973560	108.21651715621	108.2165019764

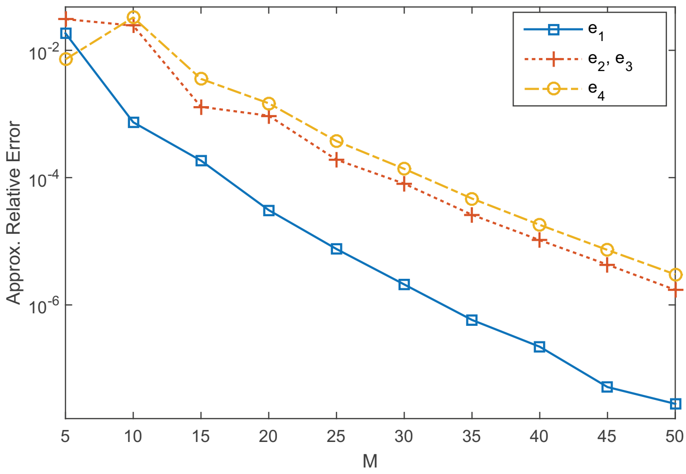

Fig. 1 Approximate relative error of the calculated λ_i, $i = 1, \ldots, 4$

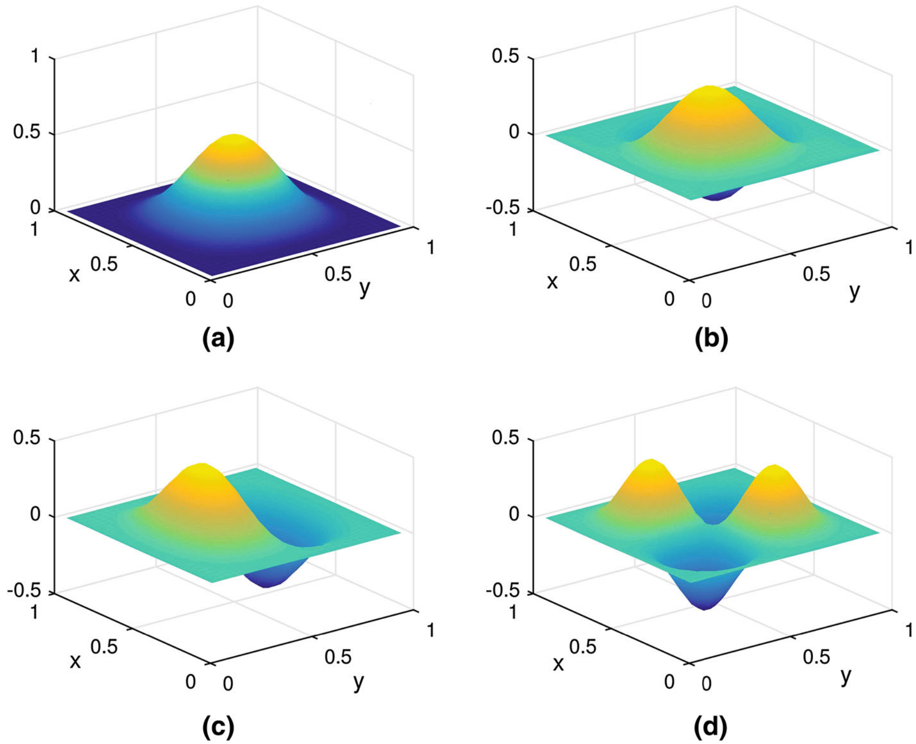

Fig. 2 Eigenmodes of the clamped square plate: **a** The first mode, $\lambda_1 \simeq 35.985$. **b** The second mode, $\lambda_2 \simeq 73.394$. **c** The third mode, $\lambda_3 \simeq 73.394$. **d** The fourth mode, $\lambda_4 \simeq 108.217$

polynomial forms [8], the Rayleigh–Ritz method together with natural co-ordinate regions and normalized beam characteristic orthogonal polynomials [20], the Ritz method with 36 terms containing the products of beam functions [28], and Rayleigh–Ritz procedure for minimization of the energy function derived using Mindlin's plate theory [30].

Table 3 Comparison of the first four frequency parameters for clamped square plate

	λ_1	λ_2	λ_3	λ_4
Present	35.985191	73.393857	73.393878	108.216517
Ref. [8]	35.9888	73.3989	73.3989	108.2653
Ref. [20]	35.9852	73.3939	73.3939	108.2166
Ref. [28]	35.990	73.390	473.390	108.220
Ref. [30]	35.9875	73.3943	73.3943	108.2172

Table 4 Convergence of the computed values of λ for clamped rectangular plate with an aspect ratio $\Phi = 2/3$

M	λ_1	λ_2	λ_3	λ_4
30	60.76122010739	93.83998365940	148.7920510151	149.6466009464
35	60.76106350275	93.83128876907	148.7758755388	149.6708398564
40	60.76111043487	93.83433147164	148.7813199810	149.6720005497
45	60.76109450428	93.83312644706	148.7791053127	149.6742219060
50	60.76109920513	93.83361184351	148.7799710311	149.6739596225
Limit values (MPE)	λ_1^*	λ_2^*	λ_3^*	λ_4^*
	60.76113900726	93.83378370844	148.7799435960	149.6834918113

Table 5 Convergence of the computed values of λ for clamped rectangular plate with an aspect ratio $\Phi = 1.5$

M	λ_1	λ_2	λ_3	λ_4
30	27.00498671452	41.70665938377	66.12980042868	66.50960041874
35	27.00491711008	41.70279501987	66.12261089919	66.52037319215
40	27.00493799774	41.70414730246	66.12503096920	66.52088922261
45	27.00493098404	41.70361190873	66.12404356175	66.52187622539
50	27.00493294635	41.70382454475	66.12442801675	66.52175983594
Limit values (MPE)	λ_1^*	λ_2^*	λ_3^*	λ_4^*
	27.00495453929	41.70394873204	66.12446219173	66.52670293706

- *Case 2* In this case, we consider clamped rectangular plates with different aspect ratios $\Phi = 2/3$, 1.5 and 2.5. The calculated values of λ_i, $i = 1, \ldots, 4$ for $\Phi = 2/3$ and 1.5 are reported in Tables 4 and 5, respectively.

The mode shapes for the rectangular plate with an aspect ration, $\Phi = 1.5$ are shown in Fig. 3.

In Table 6, the values of λ for the case of a clamped rectangular plate with $\Phi = 2.5$ are reported. The calculated values of the first four frequency parameters for the cases of $\Phi = 2/3$, 1.5 and 2.5 are listed in Table 7, compared with those obtained by other approaches in [8,20,28,30].

Based on the limit values obtained using the MPE method, the approximate relative errors are defined for each case of $\Phi = 2/3$, 1.5 and 2.5 by (4.1). The values of the AREs are listed in Table 8.

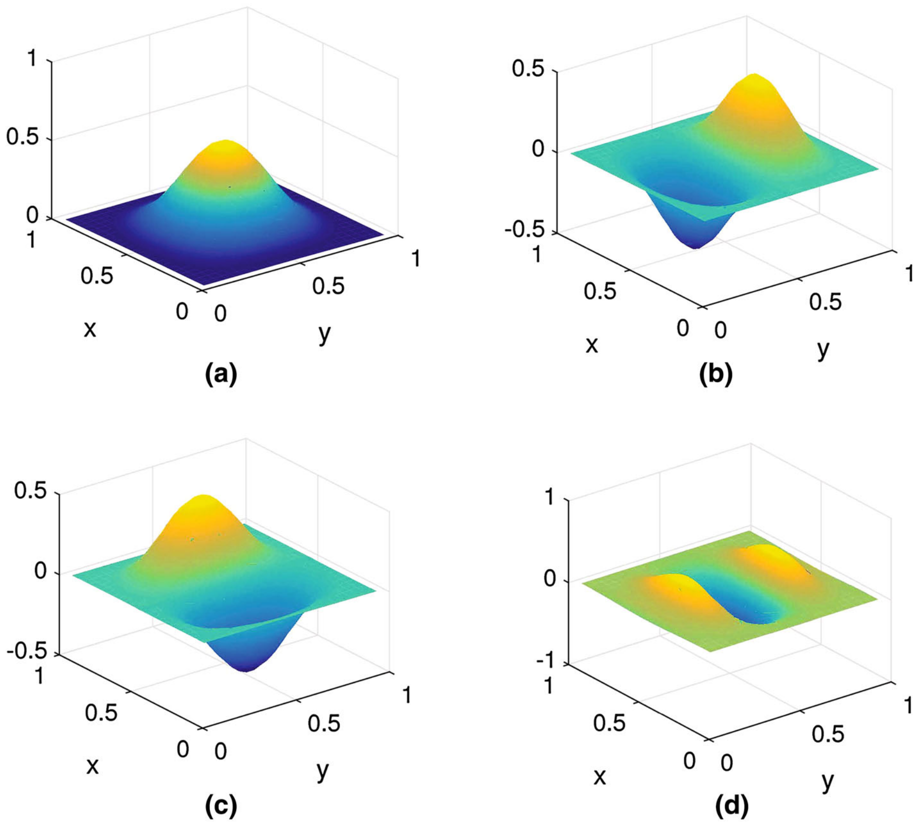

Fig. 3 Mode shapes of clamped rectangular plate with $\Phi = 1.5$: **a** The first mode, $\lambda_1 \simeq 27.005$. **b** The second mode, $\lambda_2 \simeq 41.704$. **c** The third mode, $\lambda_3 \simeq 66.125$. **d** The fourth mode, $\lambda_4 \simeq 66.527$

Table 6 Convergence of the computed values of λ for clamped rectangular plate with an aspect ratio $\Phi = 2.5$

M	λ_1	λ_2	λ_3	λ_4
30	23.6438225808	27.80804081952	35.41407342015	46.70972717425
35	23.6437666850	27.80647344327	35.41693379346	46.70942071288
40	23.6437835022	27.80702770197	35.41686532809	46.67057617978
45	23.6437780212	27.80681452817	35.41716005961	46.67725263736
50	23.6437803938	27.80690327282	35.41710051486	46.67011375938
Limit values (MPE)	λ_1^*	λ_2^*	λ_3^*	λ_4^*
	23.64383887648	27.80745144512	35.41872102840	46.63730533502

5 Conclusion

In this paper, the sinc-Galerkin method was applied to solve the biharmonic eigenvalue problem. Clamped thin square and rectangular plates with various aspect ratios were considered. The calculated results for these cases accord well with those published earlier. In

Table 7 Comparison of the first four frequency parameters for clamped rectangular plates with different aspect ratios

Φ		λ_1	λ_2	λ_3	λ_4
2/3	Present	60.761139	93.833784	148.779944	149.683492
	Ref. [8]	60.7626	93.8415	148.7881	149.6842
	Ref. [28]	60.772	93.860	148.820	149.740
	Ref. [30]	60.7662	93.8390	148.7798	149.6770
1.5	Present	27.004955	41.703949	66.124462	66.526703
	Ref. [8]	27.0075	41.7073	66.1280	66.6235
	Ref. [20]	27.0050	41.7038	66.1245	66.5225
	Ref. [28]	27.0100	41.7160	66.1430	66.5520
2.5	Present	23.643839	27.807451	35.418721	46.637305
	Ref. [8]	23.6442	27.8095	35.4201	46.8183
	Ref. [20]	23.6438	27.8070	35.4179	46.6762
	Ref. [28]	23.648	27.817	35.446	46.702
	Ref. [30]	23.6428	27.8056	35.4158	46.6687

Table 8 Approximate relative errors for clamped plates with different aspect ratios

Φ	M	e_1	e_2	e_3	e_4
2/3	30	1.335E−06	6.607E−05	8.138E−05	2.465E−04
	35	1.243E−06	2.659E−05	2.734E−05	8.452E−05
	40	4.702E−07	5.838E−06	9.251E−06	7.677E−05
	45	7.323E−07	7.005E−06	5.634E−06	6.193E−05
	50	6.551E−07	1.832E−06	1.844E−07	6.368E−05
1.5	30	1.191E−06	6.499E−05	8.073E−05	2.571E−04
	35	1.386E−06	2.766E−05	2.798E−05	9.515E−05
	40	6.125E−07	4.761E−06	8.602E−06	8.739E−05
	45	8.723E−07	8.077E−06	6.331E−06	7.255E−05
	50	7.996E−07	2.978E−06	5.168E−07	7.429E−05
2.5	30	6.892E−07	2.119E−05	1.312E−04	1.553E−03
	35	3.053E−06	3.517E−05	5.046E−05	1.546E−03
	40	2.342E−06	1.524E−05	5.239E−05	7.134E−04
	45	2.574E−06	2.289E−05	4.407E−05	8.566E−04
	50	2.473E−06	1.971E−05	4.575E−05	7.035E−04

addition, compared to the strict lower and upper bounds available for the square plate, the sinc-Galerkin has a high convergence rate. This proves the accuracy and validity of the sinc-Galerkin method.

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Appendix 1: Proof of Theorem 3.1

For U_{xxxx}, the inner product with sinc basis element is given by

$$\left\langle \Phi^4 U_{xxxx}, S_k S_l \right\rangle = \Phi^4 \int_0^1 \int_0^1 U_{xxxx}(x, y) S_k(x) S_l(y) w(x) v(y) dx dy.$$

Integrating by parts to remove the fourth order derivatives from the dependent variable U leads to the equality.

$$\left\langle \Phi^4 U_{xxxx}, S_k S_l \right\rangle = B_x + \Phi^4 \int_0^1 \int_0^1 U(x, y) \left[S_k(x) S_l(y) w(x) v(y)\right]_{xxxx} dx dy. \tag{5.1}$$

where the boundary term is

$$B_x = \Phi^4 \int_0^1 \left[U_{xxx}(S_k S_l w v) - U_{xx}(S_k S_l w v)_x + U_x(S_k S_l w v)_{xx} - U(S_k S_l w v)_{xxx}\right]_0^1 dy.$$

The boundary terms in Eq. (5.1) vanished. Continuing only with the remaining integral in (5.1) and expanding the derivative results in

$$\left\langle \Phi^4 U_{xxxx}, S_k S_l \right\rangle = \Phi^4 \int_0^1 \int_0^1 \sum_{i=0}^{4} U(x, y) \left[S_k^{(i)} \mu_i\right] S_l v(y) dx dy. \tag{5.2}$$

where $S_k^{(i)}$ denotes the ith derivative of S_k with respect to the ϕ_1 and

$$\begin{aligned}
\mu_4 &= (\phi_1')^4 w, \\
\mu_3 &= 6(\phi_1')^2 \phi_1'' w + 4(\phi_1')^3 w', \\
\mu_2 &= 3(\phi_1'')^2 w + 4\phi_1' \phi_1''' w + 12\phi_1' \phi_1'' w' + 6(\phi_1')^2 w'', \\
\mu_1 &= \phi_1'''' w + 4\phi_1''' w' + 6\phi_1'' w'' + 4\phi_1' w'''.
\end{aligned}$$

and

$$\mu_0 = w''''$$

Applying the sinc quadrature in the x-domain and y-domain to Eq. (5.2) yields Eq. (3.10).

The inner product for U_{yyyy} may be handled in a similar manner. This gives the expression (3.11) where

$$\begin{aligned}
\eta_4 &= (\phi_2')^4 v, \\
\eta_3 &= 6(\phi_2')^2 \phi_2'' v + 4(\phi_2')^3 v', \\
\eta_2 &= 3(\phi_2'')^2 v + 4\phi_2' \phi_2''' v + 12\phi_2' \phi_2'' v' + 6(\phi_2')^2 v'', \\
\eta_1 &= \phi_2'''' v + 4\phi_2''' v' + 6\phi_2'' v'' + 4\phi_2' v'''.
\end{aligned}$$

and

$$\eta_0 = v''''$$

For U_{xxyy}, the inner product with sinc basis element is given by

$$\left\langle 2\, \Phi^2 u_{xxyy}, S_k\, S_l \right\rangle = 2\, \Phi^2 \int_0^1 \int_0^1 U_{xxyy}(x, y)\, S_k(x)\, S_l(y)\, w(x)\, v(y)\, dx\, dy.$$

Integrating by parts to remove the fourth derivatives from the dependent variable U leads to the equality

$$\left\langle 2\,\Phi^2 U_{xxyy},\, S_k\, S_l \right\rangle = B_{xy} + \Phi^2 \int_0^1 \int_0^1 U(x,y)\ [2\, S_k(x)\, S_l(y)\, w(x)\, v(y)]_{xxyy}\ dx\, dy. \tag{5.3}$$

where the boundary term is

$$\begin{aligned} B_{xy} &= \Phi^2 \int_0^1 \left[U_{xyy}(S_k\, S_l\, w\, v) - U_{yy}(S_k\, S_l\, w\, v)_x \right]_0^1 dy \\ &\quad + \int_0^1 \left[U_y(S_k\, S_l\, w\, v)_{xx} - U(S_k\, S_l\, w\, v)_{xxy} \right]_0^1 dx = 0 \end{aligned}$$

Continuing with the remaining integral in (5.3) and expanding the derivative result in

$$\left\langle 2\,\Phi^2 U_{xxyy},\, S_k S_l \right\rangle = 2\,\Phi^2 \int_0^1 \int_0^1 \sum_{r=0}^{2} \sum_{p=0}^{2} U(x,y) S_k^{(r)}\, S_l^{(p)}\, \tau_r\, \xi_p\ dx\, dy. \tag{5.4}$$

where

$$\tau_2 = (\phi_1')^2\, w, \quad \tau_1 = 2\phi_1' w' + \phi_1'' w, \quad \tau_0 = w'',$$

and

$$\xi_2 = (\phi_2')^2 v, \quad \xi_1 = 2\phi_2' v' + \phi_2'' v, \quad \xi_0 = v''$$

Applying the sinc quadrature in the x-domain and y-domain to the Eq. (5.4) yields Eq. (3.12).

For $\Phi^4 \lambda^2 U(x,y)$, the inner product is

$$\left\langle \Phi^4 \lambda^2 U,\, S_k S_l \right\rangle = \Phi^4 \int_0^1 \int_0^1 \lambda^2 U(x,y) S_k S_l w(x) v(y) dx dy \tag{5.5}$$

Applying the sinc quadrature to (5.5) yields

$$\left\langle \Phi^4 \lambda^2 U,\, S_k S_l \right\rangle \approx \Phi^4 \lambda^2 h_x h_y \frac{w(x_k) U(x_k, y_l) v(y_l)}{\phi_1'(x_k) \phi_2'(y_l)} \tag{5.6}$$

as given in Eq. (3.13).

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