SeMA (2017) 74:37–56
DOI 10.1007/s40324-016-0079-8

An efficient technique for finding the eigenvalues and the eigenelements of fourth-order Sturm-Liouville problems

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Received: 26 October 2015 / Accepted: 15 March 2016 / Published online: 4 April 2016

Abstract In this paper an efficient method based on Legendre-Galerkin method for computing the eigenvalues of fourth-order Sturm-Liouville problem subject to a kind of fixed boundary conditions is developed. Properties of Legendre polynomials are first presented, these properties are then utilized to reduce the eigenvalues of fourth-order Sturm-Liouville problem to some linear algebraic equations. The method is computationally attractive, and applications are demonstrated through an illustrative example and a comparisons with other methods are made.

Keywords Legendre · Galerkin · Eigenvalues · Eigenfunctions · Sturm-Liouville

Mathematics Subject Classification Primary 65L15; Secondary 34L10

1 Introduction

The present work describes Legendre-Galerkin method for finding the eigenvalues and the eigenelements of fourth-order Sturm-Liouville problems of the form

$$\left[p_1(x)\,\frac{d^2u}{dx^2}\right]'' - \left[s(x)\,\frac{du}{dx}\right]' = [\lambda w(x) - q(x)]\,u(x), \quad a < x < b, \tag{1.1}$$

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subject to one of the following pairs of homogenous boundary conditions:

$$\begin{aligned}
&(1)\ u(a) = u'(a) = u(b) = u'(b) = 0\\
&(2)\ u(a) = u''(a) = u(b) = u''(b) = 0\\
&(3)\ u''(a) = u'''(a) = u''(b) = u'''(b) = 0\\
&(4)\ u(a) = u'(a) = u(b) = u''(b) = 0
\end{aligned} \tag{1.2}$$

where λ is a parameter is independent on x and $p_1(x), q(x), s(x)$ and $w(x)$ are piecewise continuous functions and $p_1(x)$ and $w(x)$ are positive.

The Sturm-Liouville problems arise throughout applied mathematics, classical and quantum mechanics. Most of physical phenomena, can describe by PDEs in several dimensions. This leads to a Sturm-Liouville problem when the equations are separable. The Sturm-Liouville boundary value problems for ODEs play an important role in both the theory and applications of physical, biological and chemical phenomena [10].

In elasticity, this equation is associated to the steady-state Euler-Bernoulli beam equation for the deflection u of a vibrating beam, with the other quantities involved having physical meaning, e.g. $p > 0$ is the flexural rigidity of the beam, $p_1 u''$ is the bending moment and $(\lambda w - q)$ is the frequency of vibration.

It is well known that (1.1) has an infinite sequence of eigenvalues $(\lambda_k)_{k \geq 1}$ which are bounded from below by a constant λ_0, i.e.

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \tag{1.3}$$

with $\lim_{k\to\infty} \lambda_k = \infty$ and each eigenvalues has multiplicity at most 2. For more details see [11,12]. In applications, there are more types of boundary conditions commonly used with (1.1) discussed in [5,13].

In most cases, it is not possible to obtain the eigenvalues of the problem analytically. However, there are various approximate methods for solving special types of Sturm-Liouville problems as, for example, Adomian decomposition method [4], the homotopy analysis method [2], variational iteration method [3], spectral parameter power series method [14], Extended sampling method [9], finite difference and Numerov's methods [6], differential quadrature method [1], Chebyshev spectral collocation method [10], Haar wavelet method [13] and polynomial expansion method [5].

In recent years, a lot of attention has been devoted to the study of Legendre methods to investigate various scientific models. Using these methods made it possible to solve differential equations of Lane-Emden type [16], second and fourth order equations [18], Cahn-Hilliard equations with Neumann boundary conditions [19], Fredholm integral [21], Helmholtz equation [22], second kind Volterra integral equations [20], high-order linear Fredholm integro-differential [23] and Abels integral equation [17].

Legendre methods for ordinary differential equations has many salient features due to the properties of the basis functions and the manner in which the problem is discretized. The approximating discrete system depends only on parameters of the differential equation. The approximation rate of Legendre polynomials is n^{-k} where n is the number of Legendre polynomial elements used, and k is a positive constant. The efficiency of the method has been formally proved by many researchers [15,24,25].

The paper is organized as follows: in Sect. 2, we present the preliminaries of Legendre polynomials. Section 3 presents the convergence of Legendre polynomials and error estimation for Legendre-Galerkin method. Section 4 is devoted to derivation of the discrete system. Section 5 shows the accuracy of the proposed method using numerical examples and making comparisons with other methods. Section 6 gives a brief conclusion. Note that we have computed the numerical results by Mathematica programming.

2 Preliminaries and fundamentals

Orthogonal polynomials have very useful properties in the solution of mathematical and physical problems. They provide a natural way to solve, expand, and interpret solutions to many types of important differential and integro-differential equations. Legendre polynomials are one of the most common orthogonal polynomial set.

Legendre polynomials $P_n(x)$ satisfy Legendre differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right]+n(n+1)P_n(x)=0, -1\leq x\leq 1, n\geq 0, \tag{2.1}$$

with recurrence relations

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x), \tag{2.2}$$

and the orthogonality on $[-1, 1]$

$$\int_{-1}^{1} P_m(x)P_n(x)dx=\begin{cases}1, & \text{if } m=n,\\ 0, & \text{if } m\neq n.\end{cases} \tag{2.3}$$

Theorem 2.1 [15] *Let n and m be any two integer numbers such that $n, m \leq N$, then*

(i)

$$\int_{-1}^{1} P_n'(x)P_m(x)dx=\begin{cases}2, & if\ n=m+i,\\ 0, & if\ n\neq m+i\ or\ m\geq n.\end{cases}$$

(ii)

$$\int_{-1}^{1} xP_n'(x)P_m(x)dx=\begin{cases}\frac{2n}{2n+1}, & if\ m=n,\\ 0, & if\ n=m+i\ or\ m>n,\\ 2, & if\ n\neq m+i.\end{cases}$$

(iii)

$$\int_{-1}^{1} xP_n(x)P_m(x)dx=\begin{cases}\frac{2(m-1)}{(2m-1)(2m-3)}, & if\ n=m-1,\\ \frac{2(n-1)}{(2n-1)(2n-3)}, & if\ n=m+1.\end{cases}$$

(iv)

$$\int_{-1}^{1} P_n''(x)P_m(x)dx=\begin{cases}n(n+1)-m(m+1), & if\ n\neq m+i,\\ 0, & if\ n=m+i\ or\ m\geq n.\end{cases}$$

(v)

$$\int_{-1}^{1} xP_n''(x)P_m(x)dx=\begin{cases}n(n+1)-m(m+1)-2, & if\ n=m+i,\\ 0, & if\ n\neq m+i\ or\ m\geq n.\end{cases}$$

(vi)

$$\int_{-1}^{1} x^2 P_n''(x) P_m(x) dx = \begin{cases} \frac{2n(n-1)}{2n+1}, & if\ m = n, \\ n(n+1) - m(m+1) - 4, & if\ n = m + i + 1, \\ 0, & if\ n \neq m + i + 1\ or\ m > n. \end{cases}$$

(vii)

$$\int_{-1}^{1} x^2 P_n(x) P_m(x) dx = \begin{cases} \dfrac{4n^2 + 4n - 2}{(2n+3)(2n+1)(2n-1)}, & if\ n = m, \\ \dfrac{2(n+2)(n+1)}{(2n+5)(2n+3)(2n+1)}, & if\ n = m - 2, \\ \dfrac{2n(n-1)}{(2n+1)(2n-1)(2n-3)}, & if\ n = m + 2. \end{cases}$$

where $i = 1, 3, 5, \ldots, 2k + 1 \leq N - m$.

Theorem 2.2 *Let* n *and* m *be any two integer numbers such that* $n, m \leq N$*, then*

(i)

$$\int_{-1}^{1} x^3 P_n(x) P_m(x) dx$$
$$= \begin{cases} \frac{2n(m-2)(m-1)}{(2m-1)(2m-3)(2m-5)(2n-1)}, & if\ n = m - 3\ or\ m = n - 3, \\ \frac{2\left(m(6n^3-19n^2+9n+7)-2n^3+5n^2-n-2\right)}{(2m+1)(2m-1)(2m-3)(2m-5)(2n-1)}, & if\ n = m - 1\ or\ m = n - 1. \end{cases}$$

(ii)

$$\int_{-1}^{1} x^4 P_n(x) P_m(x) dx$$
$$= \begin{cases} \frac{2m(m+1)(n-2)(n-1)}{(2n-3)(2n-1)(2n+3)(2n+1)(2n-1)}, & if\ n = m + 4\ or\ m = n+4, \\ \frac{\Lambda(n,m)}{(2m-3)(2m-1)(2m+1)(2n-3)(2n-1)(2n+1)}, & if\ n = m+2\ or\ m = n+2, \\ \frac{\Upsilon(n,m)}{(2n-1)^2(2n-3)(2n+1)(2n+3)(2m-5)(2m-3)(2m-1)(2m+1)}, & if\ n = m. \end{cases}$$

where

$$\Upsilon(n,m) = 2\left(m^3(4n^2 - 4n - 2) + m^2(4n^3 - 12n^2 + 4n + 5) + m(-4n^3 + 4n^2 + 4n - 1) + (-2n^3 + 5n^2 - n - 2)\right),$$

$$\Lambda(n,m) = 2\Big(8n^5(2 + m - 5n^2 + 2n^3) + 2mn(27 + m - 29m^2 + 10m^3) + n^2(189 - 102m - 670m^2 + 690m^3 - 164m^4) + 3(-9 + 25m^2 - 20m^3 + 4m^4) - 2n^3(22 + 41m - 57m^2 - 2m^3 + 8m^4) + 4n^4(-21 + 18m + 74m^2 - 82m^3 + 20m^4))\Big),$$

and $i = 1, 3, 5, \ldots, 2k + 1 \leq N - m - 1$.

Proof By recalling Eq. (2.2) twice for (i) and three times for (ii) beside the orthogonality relation (2.3), Theorem 2.2 can be proved. □

To solve the fourth-order equation, we need the following theorem.

Theorem 2.3 [26] *Let n and m be any two integer numbers such that $n, m \leq N$, then*

(i)

$$\int_{-1}^{1} P_n'''(x) P_m(x) dx = \begin{cases} \frac{1}{4}\prod_{i=0}^{3}(n-i+2) - \sum_{k=1}^{[\frac{m}{2}]}(4k-1)\,[n(n+1) \\ -2k(2k-1)]\,, \quad m = \text{even}, n = \text{odd}, 2k-1 < n, \\ \frac{1}{4}\prod_{i=0}^{3}(n-i+2) - \sum_{k=0}^{[\frac{m}{2}]}(4k+1)\,[n(n+1) \\ -2k(2k+1)]\,, \quad m = \text{odd}, n = \text{even}, 2k < n, \\ 0, \quad \text{otherwise.} \end{cases}$$

(ii)

$$\int_{-1}^{1} P_n''''(x) P_m(x) dx = \begin{cases} \frac{1}{24}\prod_{i=0}^{5}(n-i+3) - \sum_{k=1}^{[\frac{m}{2}]}(4k-1)\Big[\frac{1}{4}\prod_{i=0}^{3}(n-i+2) \\ - \sum_{r=0}^{[\frac{2k-1}{2}]}(4r+1)\,(n(n+1)-2r(2r+1))\Big], \\ \qquad m \text{ and } n \text{ are even}, 2r < n, \\ \frac{1}{24}\prod_{i=0}^{5}(n-i+3) - \sum_{k=0}^{[\frac{m}{2}]}(4k+1)\Big[\frac{1}{4}\prod_{i=0}^{3}(n-i+2) \\ - \sum_{r=1}^{k}(4r-1)\,(n(n+1)-2r(2r-1))\Big], \\ \qquad m \text{ and } n \text{ are odd}, 2r-1 < n, \\ 0, \quad \text{otherwise.} \end{cases}$$

3 Convergence and error estimation

3.1 Convergence of Legendre polynomial

Lemma 3.1 [24,25] *Let $x(t) \in H^k(-1,1)$ (a Sobolev space) and let $x_n(t) = \sum_{i=0}^{n} a_i P_i(t)$ be the best approximation polynomial of $x(t)$ in the L^2-norm, then*

$$||x(t) - x_n(t)||_{L^2[-1,1]} \leq c_0 n^{-k} ||x(t)||_{H^k(-1,1)},$$

where

$$||x(t)||_{L_2[-1,1]} = \left(\int_{-1}^{1} x^2(t)dt\right)^{1/2},$$

$$||x(t)||_{H^k(-1,1)} = \left(\sum_{i=0}^{k}\int_{-1}^{1} |x^{(i)}(t)|^2 dt\right)^{1/2}.$$

and c_0 is a positive constant, which depends on the selected norm and is independent of $x(t)$ and n.

By regard to the Lemma 3.1 we conclude that approximation rate of Legendre polynomials is n^{-k}

3.2 Error estimation of Legendre-Galerkin method

In this subsection an error estimator for Legendre-Galerkin approximate solution of a Sturm-Liouville equation is obtained. Let us call $e_n(x) = u(x) - u_n(x)$ as the error function of Legendre approximation $u_n(x)$ to $u(x)$, where $u(x)$ is the exact solution of (1.1) with one boundary from (1.2). Hence, $u_n(x)$ satisfies the following problem

$$\left(p_1(x)u_n''(x)\right)'' - \left(s(x)u_n'(x)\right)' = (\lambda w(x) - q(x))\, u_n(x) + H_n(x), \;\; x \in (a, b), \tag{3.1}$$

with one boundary condition from (1.2)

$$u_n(a) = u_n'(a) = u_n(b) = u_n'(b) = 0, \tag{3.2}$$

where H_n is a perturbation term associated with $u_n(x)$ and can be obtained by substituting $u_n(x)$ into the equation

$$H_n(x) = \left(p_1(x)u_n''(x)\right)'' - \left(s(x)u_n'(x)\right)' - (\lambda w(x) - q(x))\, u_n(x). \tag{3.3}$$

We proceed to find an approximation $e_{n,N}(x)$ to the $e_n(x)$ in the same way as in Sect. 4 for the solution (1.1) and (1.2).

Subtracting (3.1) and (3.2) from (1.1) and (1.2), respectively, the error function $e_n(x)$ satisfies the equation

$$\left(p_1(x)e_n''(x)\right)'' - \left(s(x)e_n'(x)\right)' = (\lambda w(x) - q(x))\, e_n(x) + H_n(x), \tag{3.4}$$

with boundary conditions

$$\begin{array}{l} (1)\;\; e_n(a) = e_n'(a) = e_n(b) = e_n'(b) = 0. \\ (2)\;\; e_n(a) = e_n''(a) = e_n(b) = e_n''(b) = 0. \\ (3)\;\; e_n''(a) = e_n'''(a) = e_n''(b) = e_n'''(b) = 0. \\ (4)\;\; e_n(a) = e_n'(a) = e_n(b) = e_n''(b) = 0. \end{array} \tag{3.5}$$

By solving this problem in the same way as Sect. 4, we get the approximation $e_{n,N}(x)$. It should be noted that in order to construct Legendre approximation $e_{n,N}(x)$ to $e_n(x)$.

4 Legendre-Galerkin method

4.1 Eigenvalues computation

In this subsection, we explain how Legendre bases are used beside Galerkin method to find the eignvalues of (1.1) with one boundary conditions from (1.2).

First, let us rewrite (1.1) in the following form

$$u^{(4)}(x) + \mu_3(x)u'''(x) + \mu_2(x)u''(x) + \mu_1(x)u'(x) + \mu_0(x)u(x) = \lambda W(x)u(x), \tag{4.1}$$

where

$$\mu_3(x) = 2\frac{p_1'(x)}{p_1(x)}, \;\; \mu_2(x) = \frac{p_1''(x) - s(x)}{p_1(x)}, \;\; \mu_1(x) = -\frac{s'(x)}{p_1(x)}, \;\; \mu_0(x) = \frac{q(x)}{p_1(x)}$$

$$\text{and} \;\; W(x) = \frac{w(x)}{p_1(x)}$$

are defined on the interval $a \leq x \leq b$ and λ is a parameter is independent on x.

Second, we must transform the solution domain from $[a, b]$ to $[-1, 1]$ by using the linear transformation $x = \frac{b-a}{2}X + \frac{b+a}{2}$. So, the Eq. (1.1) can be written as

$$\left(\frac{2}{b-a}\right)^4 u^{(4)}(X) + 2\left(\frac{2}{b-a}\right)^3 \mu_3(X)u'''(X) + \left(\frac{2}{b-a}\right)^2 \mu_2(X)u''(X) + \left(\frac{2}{b-a}\right) \mu_1(X)u'(X) + \mu_0(X)u(X) = \lambda W(X)u(X), \quad -1 \leq X \leq 1, \tag{4.2}$$

subject to one of the following homogenous boundary conditions

$$\begin{array}{l} 1.\ u(-1) = u'(-1) = u(1) = u'(1) = 0. \\ 2.\ u(-1) = u''(-1) = u(1) = u''(1) = 0. \\ 3.\ u''(-1) = u'''(-1) = u''(1) = u'''(1) = 0. \\ 4.\ u(-1) = u'(-1) = u(1) = u''(1) = 0. \end{array} \tag{4.3}$$

where $X := (\frac{2}{b-a})x - 1$.

Assume the solution of the Eq. (4.2) is approximate by the finite expansion of Legendre basis function with $n + 1$ undetermined coefficients $\{c_0, c_1, \ldots, c_n\}$

$$u_n(X) = \sum_{j=0}^{n} c_j P_j(X). \tag{4.4}$$

We can reduce the Eq. (4.2) by orthogonalizing the residual with respect to the basis functions

$$\left(\frac{2}{b-a}\right)^4 \langle u^{(4)}(X), P_r(X)\rangle + 2\left(\frac{2}{b-a}\right)^3 \langle \mu_3(X)u'''(X), P_r(X)\rangle + \left(\frac{2}{b-a}\right)^2 \langle \mu_2(X)u''(X), P_r(X)\rangle + \left(\frac{2}{b-a}\right) \langle \mu_1(X)u'(X), P_r(X)\rangle + \langle \mu_0(X)u(X), P_r(X)\rangle = \lambda\langle W(X)u(X), P_r(X)\rangle, \tag{4.5}$$

where

$$\langle \zeta, \eta\rangle = \int_{-1}^{1} \zeta \cdot \eta \, dX.$$

The method of approximating the integrals in (4.5) begins by integrating by parts to transfer all derivatives from u to P_r. The approximation of the last four inner products on the left-hand side of (4.5) has been thoroughly treated in [15]. We will list them for convenience

$$\big\langle \mu_2(X)\, u''(X), P_r(X)\big\rangle = B_{T,2} + \int_{-1}^{1} u(X)[\mu_2(X)P_r(X)]'' dX, \tag{4.6}$$

where

$$B_{T,2} = \begin{cases} 0, & \text{in case (1)}, \\ \left[\mu_2(X)u'(X)P_r(X)\right]_{-1}^{1}, & \text{in case (2)}, \\ \left[\mu_2(X)u'(X)P_r(X)\right]_{-1}^{1} - \left[u'(X)\,(\mu_2(X)P_r(X))'\right]_{-1}^{1}, & \text{in case (3)}, \\ \mu_2(1)u(1)P_r(1), & \text{in case (4)}. \end{cases}$$

$$\big\langle \mu_1(X)\, u'(X), P_r(X)\big\rangle = B_{T,1} - \int_{-1}^{1} u(X)[\mu_1(X)P_r(X)]' dX, \tag{4.7}$$

where

$$B_{T,1} = \begin{cases} [\mu_1(X)u(X)P_r(X)]_{-1}^{1}, & \text{in case (3)},\\ 0, & \text{in case (1), (2) and (4)}. \end{cases}$$

and

$$\langle S(X)\,u(X),\, P_r(X)\rangle = \int_{-1}^{1} \mu(X)\,u(X)\,P_r(X)\,dX, \tag{4.8}$$

To solve the Eqs. (1.1)–(1.2), we need the following lemma

Lemma 4.1 *The following relations hold*

(i)

$$\left\langle u^{(4)}(X),\, P_r(X)\right\rangle = \sum_{k=2}^{3}(-1)^{k+1}\left[u^{(k)}(X)P_r^{(3-k)}(X)\right]_{-1}^{1} + \int_{-1}^{1} u(X)P_r^{(4)}(X)dX, \tag{4.9}$$

(ii)

$$\left\langle \mu_3(X)\,u'''(X),\, P_r(X)\right\rangle = B_{T,3} - \int_{-1}^{1} u(X)[\mu_3(X)P_r(X)]'''dX, \tag{4.10}$$

where

$$B_{T,3} = \begin{cases} \left[\mu_3(X)u''(X)P_r(X)\right]_{-1}^{1}, & \text{in case (1)},\\ -\left[(\mu_3(X)P_r(X))'u'(X)\right]_{-1}^{1}, & \text{in case (2)},\\ -\left[(\mu_3(X)P_r(X))'u'(X)\right]_{-1}^{1} + \left[(\mu_3(X)P_r(X))''u(X)\right]_{-1}^{1}, & \text{in case (3)},\\ -\mu_3(-1)u(-1)P_r(-1) - (\mu_3(1)P_r(1))'u'(1), & \text{in case (4)}. \end{cases}$$

Proof For $u^{(4)}$, the inner product with Legendre basis elements is given by

$$\left\langle u^{(4)},\, P_r(X)\right\rangle = \int_{-1}^{1} u^{(4)}P_r(X)dX.$$

Integrating by parts to remove the fourth derivatives from the dependent variable u leads to the equality

$$\left\langle u^{(4)}(X),\, P_r(X)\right\rangle = B_{T,4} + b_{T,4} + \int_{-1}^{1} u(X)P_r^{(4)}(X)dX, \tag{4.11}$$

where the boundary terms are

Case (1)

$$B_{T,4} = \left[\sum_{k=0}^{1}(-1)^{k+1}\,u^{(k)}(X)P_r^{(3-k)}(X)\right]_{-1}^{1} = 0,$$

$$b_{T,4} = \sum_{k=2}^{3}(-1)^{k+1}\left[u^{(k)}(X)P_r^{(3-k)}(X)\right]_{-1}^{1}.$$

Case (2)

$$B_{T,4} = \left[\sum_{k=0}^{2} - u^{(k)}(X)P_r^{(3-k)}(X)\right]_{-1}^{1} = 0,\ k = even,$$

$$b_{T,4} = \sum_{k=0}^{2} \left[u^{(k+1)}(X) P_r^{(2-k)}(X) \right]_{-1}^{1}, \; k = even.$$

Case (3)

$$B_{T,4} = \left[\sum_{k=2}^{3} (-1)^{k+1} \, u^{(k)}(X) P_r^{(3-k)}(X) \right]_{-1}^{1} = 0,$$

$$b_{T,4} = \sum_{k=0}^{1} (-1)^{k+1} \left[u^{(k)}(X) P_r^{(3-k)}(X) \right]_{-1}^{1}.$$

Case (4)

$$B_{T,4} = \left[u(X) P_r'''(X) \right]_{-1}^{1} - \left[u'(-1) P_r''(-1) \right] - \left[u''(1) P_r'(1) \right] = 0,$$

$$b_{T,4} = \left[u'''(X) P_r(X) \right]_{-1}^{1} + \left[u''(-1) P_r'(-1) \right] + \left[u'(1) P_r''(1) \right].$$

Then (4.11) may be written as (4.9). In the same way, Lemma 4.1(ii) can be proved. □

Replacing each term of (4.5) with the approximation defined in (4.6)–(4.10), we obtain the following theorem

Theorem 4.2 *If the assumed approximate solution of* (1.1) *with the boundary-value problem case (1) is* (4.4)*, then the discrete Legendre-Galerkin system for the determination of the unknown coefficients* $\{c_j\}_{j=0}^{n}$ *is given by*

$$\begin{aligned}
&\sum_{j=0}^{n} \Bigg[\left(\frac{2}{b-a} \right)^4 \left[\sum_{k=2}^{3} (-1)^{k+1} \left[P_j^{(k)}(X) P_r^{(3-k)}(X) \right]_{-1}^{1} + \int_{-1}^{1} P_j(X) P_r''''(X) \, dX \right] \\
&+ \left(\frac{2}{b-a} \right)^3 \left[\left[\mu_3(X) P_j''(X) P_r(X) \right]_{-1}^{1} - \int_{-1}^{1} P_j(X) [\mu_3(X) P_r(X)]''' dX \right] \\
&+ \left(\frac{2}{b-a} \right)^2 \int_{-1}^{1} P_j(X) \, [\mu_2(X) P_r(X)]'' \, dX \\
&+ \frac{2}{b-a} \int_{-1}^{1} P_j(X) \, [\mu_1(X) P_r(X)]' \, dX \\
&+ \int_{-1}^{1} P_j(X) \mu_0(X) P_r(X) dX \Bigg] c_j = \lambda \sum_{j=0}^{n} \left[\int_{-1}^{1} W(X) P_j(X) P_r(X) dX \right] c_j. \quad (4.12)
\end{aligned}$$

The system in (4.12) takes the matrix form

$$\mathbf{A}\mathbf{c} = \lambda \, \mathbf{B}\mathbf{c}, \tag{4.13}$$

where

$$\mathbf{A} = \begin{pmatrix} e_{0,0} + v_{0,0} + w_{0,0} & e_{1,0} + v_{1,0} + w_{1,0} & \dots & e_{n,0} + v_{n,0} + w_{n,0} \\ e_{0,1} + v_{0,1} + w_{0,1} & e_{1,1} + v_{1,1} + w_{1,1} & \dots & e_{n,1} + v_{n,1} + w_{n,1} \\ e_{0,2} + v_{0,2} + w_{0,2} & e_{1,2} + v_{1,2} + w_{1,2} & \dots & e_{n,2} + v_{n,2} + w_{n,2} \\ e_{0,3} + v_{0,3} + w_{0,3} & e_{1,3} + v_{1,3} + w_{1,3} & \dots & e_{n,3} + v_{n,3} + w_{n,3} \\ \vdots & \vdots & \ddots & \vdots \\ e_{0,n} + v_{0,n} + w_{0,n} & e_{1,n} + v_{1,n} + w_{1,n} & \dots & e_{n,n} + v_{n,n} + w_{n,n} \end{pmatrix}, \tag{4.14}$$

and

$$\mathbf{B} = \sum_{j=0}^{n} \int_{-1}^{1} W(X) P_j(X) P_r(X) dX,$$

$$e_{j,r} = \left(\frac{2}{b-a}\right)^4 \int_{-1}^{1} P_j(X) P_r''''(X) dX - 2\left(\frac{2}{b-a}\right)^3 \int_{-1}^{1} P_j(X)[\mu_3(X) P_r(X)]''' dX$$
$$+ \left(\frac{2}{b-a}\right)^2 \int_{-1}^{1} P_j(X)\,[\mu_2(X) P_r(X)]''\, dX - \frac{2}{b-a} \int_{-1}^{1} P_j(X)\,[\mu_1(X) P_r(X)]'\, dX$$
$$+ \int_{-1}^{1} P_j(X) \mu_0(X) P_r(X) dX,$$

$$v_{j,r} = \left(\frac{2}{b-a}\right)^4 \left[\sum_{k=2}^{3} (-1)^{k+1} P_j^{(k)}(X) P_r^{(3-k)}(X)\right]_{-1}^{1},$$

$$w_{j,r} = 2\left(\frac{2}{b-a}\right)^3 \left[\mu_3(X) P_j''(X) P_r(X)\right]_{-1}^{1}.$$

$e_{j,r}$ can be evaluated from theorems and lemmas in Sect. 2 and the boundary term and $v_{j,r}$ can be calculated as

$$\sum_{k=2}^{3} (-1)^k \left[P_j^{(k)}(X) P_r^{(3-k)}(X)\right]_{-1}^{1} = \frac{1}{16}\left[(r(r+1))(1-(-1)^{n+r+j-1})\right] \prod_{i=0}^{3} (j-i+2)$$
$$- \frac{1}{48}\left[\left(1+(-1)^{n+r+j}\right)\right] \prod_{i=0}^{5} (j-i+3).$$

By multiplying (4.13) by B^{-1} yields the equivalent system

$$\Phi \mathbf{c} = \lambda\, \mathbf{c}, \tag{4.15}$$

where

$$\Phi = \mathbf{B^{-1} A}.$$

From Eq. (4.15), the values of λ can be obtained from the eigenvalues of matrix Φ. This can be done by using various methods.

In the same manner, to solve (1.1) with other boundary of (1.2), we can rewrite the Theorem 4.2 after applying the remainder three homogeneous boundary value problem.

4.2 Eigenfunction computation

This subsection illustrates how eigenfunctions can be computed. The main point of calculation associate an initial condition to problem (1.1). We compute the eigenfunctions as solutions to the initial value problems with the initial conditions

$$\left(u(-1), u'(-1), u''(-1), u'''(-1)\right) = \begin{cases} (0, 0, 1, \beta), \textit{ for case (1) and (4)}, \\ (0, 1, 0, \beta), \textit{ for case (2)}, \\ (1, \beta, 0, 0), \textit{ for case (3)}. \end{cases}$$

We normalize the eigenfunction using $u'(x) = 1$ for case (2), $u(x) = 1$ for case (3) and $u''(x) = 1$ for case (1) and (4). If we denote $u_{1n}(x, \lambda)$ and $u_{2n}(x, \lambda)$ the solutions corresponding to initial conditions (0, 0, 1, 0) and (0, 0, 1, 1) respectively for case (1) and (4), (0, 1, 0, 0) and (0, 1, 0, 1) respectively for case (2) and (1, 0, 0, 0) and (1, 1, 0, 0) respectively for case (3). Then the eigenfunctions $u(x)$ given by

$$u_n(x) = u_{1n}(x) + \beta u_{2n}(x).$$

By applying the boundary conditions, β can be evaluated as

$$\beta = \begin{cases} -\dfrac{u_{1n}(1,\lambda)}{u_{2n}(1,\lambda)}, \textit{ for case (1), (2) and (4),} \\ -\dfrac{u''_{1n}(1,\lambda)}{u''_{2n}(1,\lambda)}, \textit{ for case (3).} \end{cases}$$

So, we can write the eigenfunctions $u_n(x)$ as

$$u_n(x) = \begin{cases} u_{1n}(x,\lambda) - \dfrac{u_{1n}(1,\lambda)}{u_{2n}(1,\lambda)} u_{2n}(x,\lambda), \textit{ for case (1), (2) and (4),} \\ u_{1n}(x,\lambda) - \dfrac{u''_{1n}(1,\lambda)}{u''_{2n}(1,\lambda)} u_{2n}(x,\lambda), \textit{ for case (3).} \end{cases}$$

5 Numerical examples

The six examples included in this section were selected in order to illustrate the performance of Legendre-Galerkin method to find the eigenvalues of the regular Sturm-Liouville problem. We compare our method once with the exact solution and other with the methods introduced in [1–10]. It is shown that Legendre-Galerkin method yields better results. Here, we used of the Mathematica 10 package to calculate the eigenvalues and eigenfunction problem (1.1) with one boundary condition of (1.2). The computational codes were conducted on an Intel(R) Core(TM) i3 CPU with power 2.40 GHz, equipped with 4 GB of RAM.

Example 1 [1–5] Consider the eigenvalue problem

$$u^{(4)}(x) = \lambda u(x), \qquad 0 \le x \le 1,$$

subject to the boundary conditions

$$u(0) = u'(0) = 0,$$
$$u(1) = u''(1) = 0.$$

The exact eigenvalues in the latter case can be obtained by solving

$$\tanh \sqrt[4]{\lambda} - \tan \sqrt[4]{\lambda} = 0.$$

CPU time of calculating the eigenfunction and the comparison between Legendre-Galerkin method with other methods is tabulated in Table 1. The first four eigenfunctions have been computed using Legendre-Galerkin method and are displayed in Fig. 1. When the estimate error $e_n(x)$ is calculated for the first four eigenfunctions, we find $e_n(x) = 0$.

Example 2 [3,5] Consider the eigenvalue problem

$$u^{(4)}(x) = \lambda u(x), \qquad 0 \le x \le 1,$$

Table 1 Comparison of eigenvalues and CPU time (s) for Example 1

k	λ_k^{Exact}	$\lambda_k^{\text{Legendre-Galerkin}}$	Homotopy analysis method [2]
1	237.7210675311166	237.7210675311166	237.72106753
2	2496.487437856831	2496.487437856831	2496.48743785
3	10867.58221697888	10867.58221697888	10867.58221697
4	31780.09645408107	31780.09645408107	31780.09645427
CPU	–	0.015625	–
k	Variational iteration methods [3]		Adomian decomposition method [4]
1	237.7210675352447		237.72106753111657
2	2496.487438430018		2496.4874378489776
3	10867.582216996850		10867.593671455183
4	31780.096507847312		31475.483550381574
CPU	–		–

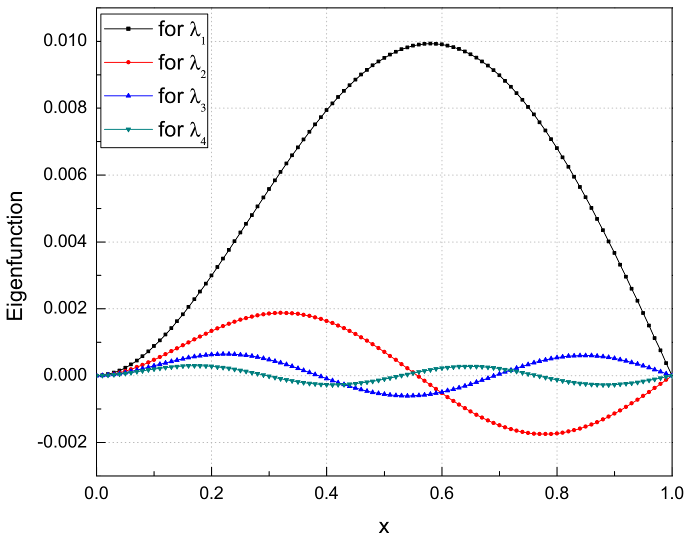

Fig. 1 The first four eigenfunctions for Example 1

subject to the boundary conditions

$$u''(0) = u'''(0) = 0,$$
$$u''(1) = u'''(1) = 0.$$

The exact eigenvalues in the latter case can be obtained by solving

$$\cos \sqrt[4]{\lambda} \cosh \sqrt[4]{\lambda} - 1 = 0.$$

The comparison between the exact value of the eigenvalues and Legendre-Galerkin method with other methods is tabulated in Table 2. CPU time of calculating the eigenfunction listed in Table 2. We are normalizing the eigenfunctions using $u'(0) = 1$. Figure 2 produces

Table 2 Comparison of eigenvalues and CPU time (s) for Example 2

k	λ_k^{Exact}	$\lambda_k^{\text{Legendre-Galerkin}}$	Variational iteration method [3]	Polynomial expansion method [5]
1	500.5639017404325	500.5639017404325	500.5639017568876	500.563902
2	3803.537080497866	3803.537080497866	3803.5370804978857	3803.53708
3	14617.63013112234	14617.63013112234	14617.630131122345	14617.6301
4	39943.79900570930	39943.79900570930	39943.799005710076	39943.7994
5	89135.40765718032	89135.40765718032	89135.40765718037	89135.4223
CPU	–	0.015625	–	–

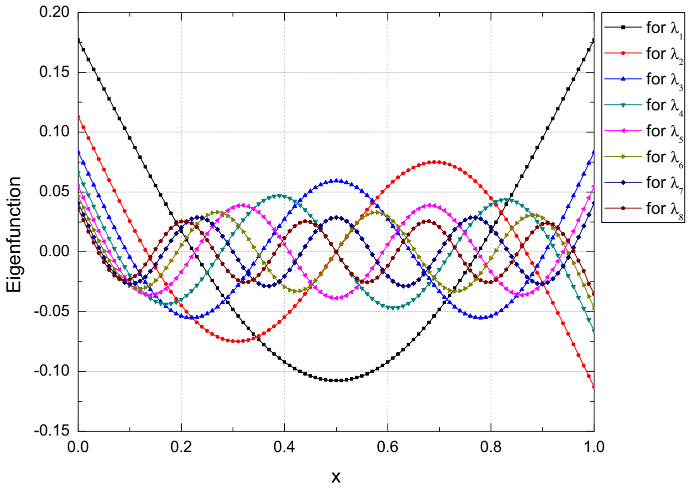

Fig. 2 The first eight eigenfunctions for Example 2

the first eight eigenfunction. When the estimate error $e_n(x)$ is calculated for the first four eigenfunctions, we find $e_n(x) = 0$.

Example 3 [6] Let us consider the fourth order Sturm-Liouville problems of order four

$$u^{(4)}(x) + u(x) = \lambda u(x), \qquad 0 < x < 1,$$

subject to the boundary conditions

$$\begin{aligned} u(0) &= u(1) = 0, \\ u''(0) &= u''(1) = 0. \end{aligned}$$

The exact eigenvalues of this problem are given in

$$\lambda_k = (k\pi)^4, \quad k = 1, 2, 3, \ldots$$

Table 3 lists the CPU time in second for the proposed method beside the comparison between Legendre-Galerkin method (LGM) and each of the methods in [6] with the exact value of the eigenvalues. These methods are finite difference method (FDM), modified

Table 3 Comparison of eigenvalues for Example 3

Method	λ_1	$\lambda_2(\times 10^3)$	$\lambda_3(\times 10^3)$	$\lambda_4(\times 10^4)$
FDM	98.40909079949648	1.559545456837260	7.891136372791951	2.493772730528151
MNM	98.40909053149949	1.559545454217382	7.891136377248733	2.493772730341868
BVM	98.40908696311035	1.559545454175871	7.891136371808981	2.493772730319019
LGM	98.40909103400243	1.559545456544039	7.891136373754197	2.493772730470462
Exact	98.40909103400243	1.559545456544039	7.891136373754197	2.493772730470462
Method	$\lambda_{10}(\times 10^5)$	$\lambda_{50}(\times 10^8)$	$\lambda_{100}(\times 10^9)$	$\lambda_{200}(\times 10^{11})$
FDM	9.740919103371439	6.088068199625164	9.740909104400242	1.558545456554039
MNM	9.740919103339907	6.088068199625074	9.740909104400244	1.558545456554039
BVM	9.740919103368853	6.088068199625131	9.740909104400244	1.558545456554040
LGM	974091.9103400243	6.088068199625152	9.740909104400244	1.558545456554039
Exact	9.740919103400243	6.088068199625152	9.740909104400244	1.558545456554039

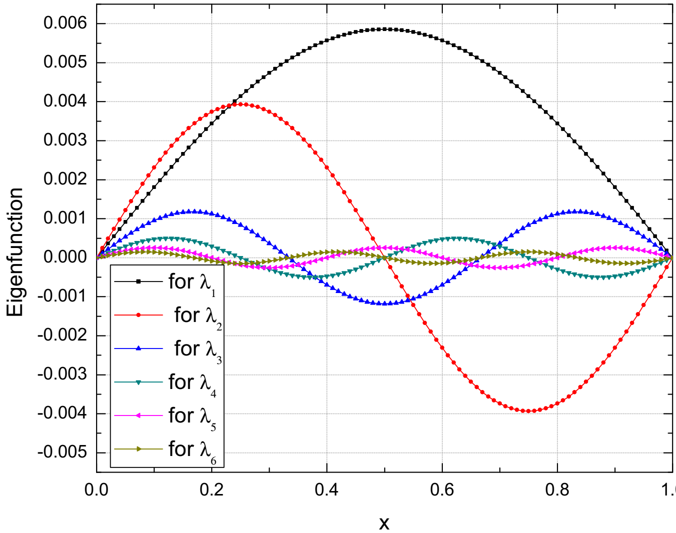

Fig. 3 The first six eigenfunctions for Example 3

Numerovs method (MNM) and boundary value method (BVM) of order 10. Figure 3 presents the first six eigenfunctions. The estimate error $e_n(x) = 0$ for the first six eigenfunctions.

Example 4 [5,7] Let us consider the fourth order Sturm-Liouville problems of order four

$$\left((1+ax)^3 u''(x)\right)'' - \lambda(1+ax)u(x) = 0, \quad 0 \le x \le 1,$$

subject to the boundary conditions

$$\begin{aligned} u(0) &= u(1) = 0, \\ u'(0) &= u'(1) = 0, \end{aligned}$$

Table 4 Comparison of eigenvalues and CPU time (s) for Example 4

Method	$a=-0.1$			CPU
	1	2	3	
The modified Rayleigh-Ritz method [7]	–	–	–	–
Polynomial expansion method [5]	21.2409778	58.5500546	114.7802778	–
Legendre-Galerkin method	21.2409777	58.5500545	114.7802416	0.062500
Method	$a=0.1$			CPU
	1	2	3	
The modified Rayleigh-Ritz method [7]	23.479607	64.721086	126.87804	–
Polynomial expansion method [5]	23.479607	64.721067	126.87805	–
Legendre-Galerkin method	23.479607	64.721067	126.87801	0.062500
Method	$a=0.2$			CPU
	1	2	3	
The modified Rayleigh-Ritz method [7]	24.563418	67.704755	132.72398	–
Polynomial expansion method [5]	24.563417	67.704755	132.72406	–
Legendre-Galerkin method	24.563417	67.704755	132.72397	0.062500

where a is a parameter. Table 4 shows the CPU time of Legendre-Galerkin for calculating eigenvalues and the comparison of the approximations of first three eigenvalues square root $\sqrt{\lambda}$ between Legendre-Galerkin method (LGM) and each of the methods in [5,7].

Example 5 [8] Let us consider the fourth order Sturm-Liouville problems which is simplified Cahn-Hilliard equation of order four

$$\left((1.1-x^2)u''(x)\right)''+20u''(x)=\lambda u(x), \qquad -1\le x\le 1,$$

subject the Dirichlet boundary conditions

$$\begin{aligned} u(-1) &= u(1)=0,\\ u'(-1) &= u'(1)=0. \end{aligned}$$

Table 5 presents the comparison of the approximations of eigenvalues between Legendre-Galerkin method (LGM) andexact dynamic stiffness method analogy in [8]. CPU time for Legendre-Galerkin method is listed in Table 5. Figure 4 shows the first six eigenfunctions. Figure 4 displays the estimate error $e_n(x)$ for first three eigenfunctions.

Table 5 Comparison of eigenvalues and CPU time (s) for Example 5

k	$\lambda_k^{\text{Legendre-Galerkin}}$	Exact dynamic stiffness method analogy [8]
1	−77.89968895	−77.89968895
2	−43.13822158	−43.13822158
3	81.02449654	81.02449680
4	703.9992915	703.9992919
5	2182.636239	2182.636239
6	4991.260832	4991.260833
7	9702.727093	9702.727093
8	16985.85788	16985.85788
9	27605.35265	27605.35265
10	42421.71721	42421.71719
CPU	0.796875	–

Example 6 [1,4,5,9,10] Next we consider the following fourth-order eigenvalue problems with variable coefficients

$$u^{(4)}(x) - 0.02x^2u''(x) - 0.04xu'(x) + (0.0001x^4 - 0.02)u(x) = \lambda u(x), \qquad 0 \le x \le 5,$$

subject the Dirichlet boundary conditions

$$\begin{aligned} u(0) &= u(5) = 0, \\ u'(0) &= u'(5) = 0. \end{aligned}$$

The numerical results of first five eigenvalues λ obtained by the Adomian decomposition method (ADM) [4], polynomial-based differential quadrature method (PDQM) [1], the polynomial expansion method (PEM) [5], the extended sampling method (ESM) [9] and Chebyshev differentiation matrices method (CDMM) [10] are listed in Table 6 beside Legendre-Galerkin Method (LGM). As compared to the existing results, we can find that the accuracy of proposed method is very satisfactory. The CPU time for the current work is compared with the program 40, page 151 [27] and Chebyshev spectral collocation method [10]. As it is shown in Table 6, the CPU time for the proposed method in this work has less cost than the one introduced in Refs. [10,27]. Figure 5a shows the first six eigenfunctions. The estimate error $e_n(x)$ for first three eigenfunctions displays in Fig. 5b.

6 Conclusion

In this paper, Legendre-Galerkin method has been successfully used for finding the solution of linear fourth-order Sturm-Liouville problems. The results in the previous section indicate that our procedure can be used to obtain accurate numerical solutions of linear Sturm-Liouville problems with very little computational effort. The effectiveness of the method has been confirmed by comparing our eigenvalues with the exact values and/or other numerical methods. It was shown that the method is easy to implement and produces accurate results for eigenfunctions. The estimate error $e_n(x)$ for eigenfunctions is calculated to prove the accuracy of Legender-Galerkin method for finding eigenfunctions.

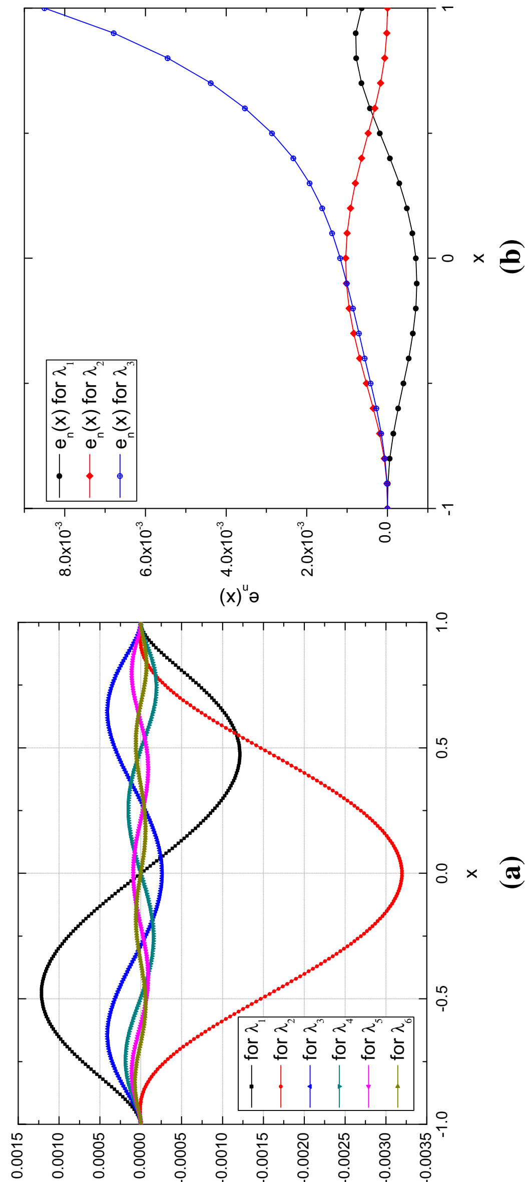

Fig. 4 **a** The first six eigenfunctions for Example 5. **b** The estimate error for first three eigenfunctions for Example 5

Table 6 Comparison of eigenvalues and CPU time (s) for Example 6

k	1	2	3	4	CPU
LGM	0.866902503962611	6.357686448012739	23.99274684746457	64.97866759427632	0.078125
PEM [5]	0.866902502399560	6.357686447869980	23.99274694653769	64.97869559403952	–
ADM [4]	0.866902502399710	6.357686448145815	23.99274685028137	64.97866759571622	–
ESM [9]	0.866902502399465	6.357686448174460	23.99274697506674	64.97863591597007	–
PDQM [1]	0.866902502602292	6.357686448439836	23.99274686509660	64.97866761311830	–
CDMM [10]	0.866902502391964	6.357686448143859	23.99274685032633	64.97866759484156	0.138014
Trefethen [27]	0.866902502426100	6.357686448126560	23.99274685028240	64.97866759501310	0.226369

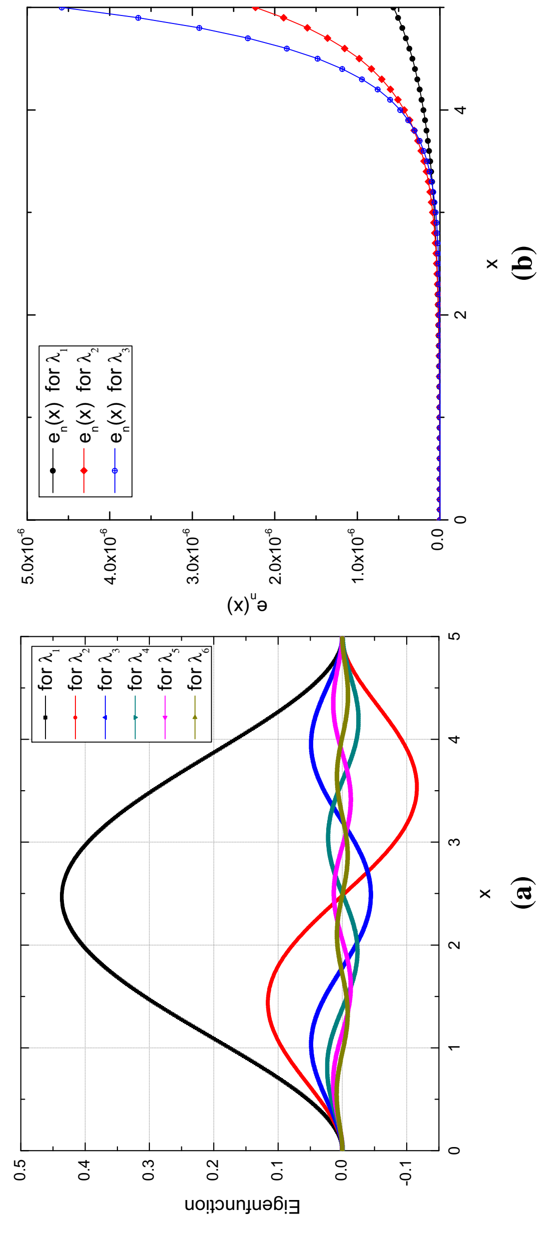

Fig. 5 **a** The first six eigenfunctions for Example 6. **b** The estimate error for first three eigenfunctions for Example 6

Acknowledgements The authors are grateful for the referees for their valuable comments and suggestions on the original manuscript.

References

1. Yücel, U., Boubaker, K.: Differential quadrature method (DQM) and Boubaker Polynomials Expansion Scheme (BPES) for efficient computation of the eigenvalues of fourth-order Sturm-Liouville problems. Appl. Math. Model. **36**, 158–167 (2012)
2. Abbasbandy, S., Shirzadi, A.: A new application of the homotopy analysis method: solving the Sturm-Liouville problems. Commun. Nonlinear Sci. Numer. Simul. **16**, 112–126 (2011)
3. Syam, M., Siyyam, H.: An efficient technique for finding the eigenvalues of fourth-order Sturm-Liouville problems. Chaos Solitons Fractals **39**, 659–665 (2009)
4. Attili, B., Lesnic, D.: An efficient method for computing eigenelements of Sturm-Liouville fourth-order boundary value problems. Appl. Math. Comput. **182**, 1247–1254 (2006)
5. Huang, Y., Chen, J., Luo, Q.: A simple approach for determining the eigenvalues of the fourth-order Sturm-Liouville problem with variable coefficients. Appl. Math. Lett. **26**, 729–734 (2013)
6. Rattana, A., Böckmanna, C.: Matrix methods for computing eigenvalues of Sturm-Liouville problems of order four. J. Comput. Appl. Math. **249**, 144–156 (2013)
7. Abrate, S.: Vibration of non-uniform rods and beams. J. Sound Vibr. **185**, 703–716 (1995)
8. Yuan, S., Ye, K., Xiao, C., Kennedy, D., Williams, F.: Solution of regular second-and fourth-order Sturm-Liouville problems by exact dynamic stiffness method analogy. J. Eng. Math. **86**, 157–173 (2014)
9. Chanane, B.: Accurate solutions of fourth order Sturm-Liouville problems. J. Comput. Appl. Math. **234**, 3064–3071 (2010)
10. Taher, A., Malek, A., Momeni-Masuleh, S.: Chebyshev differentiation matrices for efficient computation of the eigenvalues of fourth-order Sturm-Liouville problems. Appl. Math. Model. **37**, 4634–4642 (2013)
11. Amara, J.: Sturm theory for the equation of vibrating beam. J. Math. Anal. Appl. **349**, 1–9 (2009)
12. Amara, J.: Oscillation properties for the equation of vibrating beam with irregular boundary conditions. J. Math. Anal. Appl. **360**, 7–13 (2009)
13. Shi, Z., Cao, Y.: Application of Haar wavelet method to eigenvalue problems of high order differential equations. Appl. Math. Model. **36**, 4020–4026 (2012)
14. Khmelnytskaya, K., Kravchenko, V., Baldenebro-Obeso, J.: Spectral parameter power series for fourth-order Sturm-Liouville problems. Appl. Math. Comput. **219**, 3610–3624 (2012)
15. Fathy, M., El-Gamel, M., El-Azab, M.: Legendre-Galerkin method for the linear Fredholm integro-differential equations. Appl. Math. Comput. **243**, 789–800 (2014)
16. Yousefi, S.: Legendre wavelets method for solving differential equations of Lane-Emden type. Appl. Math. Comput. **181**, 1417–1422 (2006)
17. Yousefi, S.: Numerical solution of Abels integral equation by using legendre wavelets. Appl. Math. Comput. **175**, 574–580 (2006)
18. Shen, J.: Efficient spectral-Galerkin method i. direct solvers for second- and fourth-order equations by using Legendre polynomials. SIAM J. Sci. Comput. **15**, 1489–1505 (1994)
19. Ye, X.: The legendre collocation method for the Cahn-Hilliard equation. Appl. Math. Model. **150**, 87–108 (2003)
20. Zhengsu, W., Yanping, C., Yunqing, H.: Legendre spectral Galerkin method for second-kind Volterra integral equations. Front. Math. China **4**, 181–193 (2009)
21. Khater, A., Shamardan, A., Callebaut, D., Sakran, M.: Numerical solutions of integral and integro-differential equations using legendre polynomials. Numer. Algorithms **46**, 195–218 (2007)
22. Bialecki, B., Karageorghis, A.: Legendre Gauss spectral collocation for the Helmholtz equation on a rectangle. Numer. Algorithms **36**, 203–227 (2004)
23. Yalçinbaş, S., Sezer, M., Sorkun, H.: Legendre polynomial solutions of high-order linear fredholm integro-differential equations. Appl. Math. Comput. **210**, 334–349 (2009)
24. Maleknejad, K., Nouri, K., Yousefi, M.: Discussion on convergence of Legendre polynomial for numerical solution of integral equations. Appl. Math. Comput. **193**, 335–339 (2007)
25. Canuto, C., Hussaini, M., Al-Shara, S.: Spectral Methods. Springer, New York (2006)
26. El-Gamel, M., El-Azab, M., Fathy, M.: The numerical solution of sixth-order boundary-value problems value problems by the Legendre-Galerkin method. Acta Univ. Apulensis **40**, 145–165 (2014)
27. Trefethen, L.: Spectral Methods in Matlab. SIAM Press, Philadelphia (2000)