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A survey for the Muskat problem and a new estimate

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Abstract This paper shows a summary of mathematical results about the Muskat problem. The main concern is well-posed scenarios which include the possible formation of singularities in finite time or existence of solutions for all time. These questions are important in mathematical physics but also have a strong mathematical interest. Stressing some recent results of the author, we also give a new estimate for the problem in the last section. Initial data with L^2 decay and slope less than one provide weak solutions which satisfy a parabolic inequality as in the linear regime.

Keywords Porous medium · Darcy's law · Muskat problem · Maximum principle

Mathematics Subject Classification 35 · 76

1 Introduction

The mathematical analysis of fluid mechanics models in PDEs is a classical topic of research since Euler's 1757 paper, where the evolution equation of an ideal flow was first derived. For the well established models, such as Navier-Stokes and Euler, the incompressible case presents basic and important open questions such as global regularity and finite time singularity formation of the solutions. It is a current area of mathematical research of fundamental interest in particular due to its relevance in Physics and wide applicability.

In the analysis of PDEs from fluid mechanics, an outstanding class of problems are those in which the evolution of fluids of different nature are modeled. The interaction between the fluids provides the dynamics of their common free boundary that evolves with the flow. It gives rise to long standing problems such as vortex-patch [14], vortex-sheet [4,50], water waves [44], viscous waves evolution [41], interface flows in porous media and Hele-Shaw

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cells [9,37], as well as atmospheric front dynamics [23], among others. These free boundary dynamics problems are modeled by fluid mechanics PDEs such as Euler, Navier-Stokes, Darcy momentum equation and quasi-geostrophic systems. In all of them fundamental questions are local-in-time existence, global-in-time regularity of solutions or finite time singularity formation in well-posed scenarios.

In this manuscript we focus on the classical Muskat problem [45]. It considers contour dynamics problems for incompressible fluids of different nature permeating a porous medium. Recently, computer evidence has shown how singularities may developed in Muskat [12]. With recent new techniques, it is now possible to prove different types of finite time singularity formation for those scenarios [10,11,13]. These are the first analytic proofs of blow-up for incompressible fluid in well-posed situations.

We introduce now the equations of the problem, considering an active scalar $\rho(x,t)$, depending on time $t \geq 0$ and position $x \in \mathbb{R}^2$. Here we will pick the two dimensional case for simplicity of exposition. The fluid velocity is incompressible

$$\nabla \cdot u(x,t) = 0, \tag{1}$$

and the scalar $\rho(x,t)$ satisfies a general transport evolution equation for incompressible flows

$$\rho_t(x,t) + u(x,t) \cdot \nabla \rho(x,t) = 0. \tag{2}$$

That we are dealing with two different fluids is reflected in the configuration of $\rho(x,t)$, which is a discontinuous function with constant values in two complementary connected sets $D^1(t)$ and $D^2(t) = \mathbb{R}^2 \backslash D^1(t)$:

$$\rho(x,t) = \begin{cases} \rho^1, & x \in D^1(t), \\ \rho^2, & x \in D^2(t). \end{cases} \tag{3}$$

The constants ρ^1 and ρ^2 represent the density of each fluid that occupy the sets $D^1(t)$ and $D^2(t)$, respectively. Therefore Eq. (2) becomes the conservation of mass and it is understood in a weak sense. The main concern is about the dynamics of the free boundary of the fluids $\partial D^j(t)$, $j = 1, 2$, which is parameterized by the curve $z(\alpha,t)$ as follows:

$$\partial D^j(t) = \{z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)) \,:\, \alpha \in \mathbb{R}\}.$$

Above the curve $z(\alpha,t)$ is asymptotically flat: $z(\alpha,t) - (\alpha,0) \to 0$ as $\alpha \to \infty$, and we will consider also the case of a 2π-periodic contour in the x_1 direction: $z(\alpha+2\pi,t) = z(\alpha,t) + (2\pi,0)$. The fluid with density ρ^2 essentially lies below the fluid of density ρ^1 in such a way that there is a constant $M > 1$ big enough so that $\mathbb{R} \times (-\infty, -M] \subset D^2(t)$.

For the Muskat problem, the most common example for applications is the dynamics of water and oil [6]. This is a classical topic of investigation dating back to Muskat's 1934 paper [45]. In consequence the fluids can also have different constant viscosities, given by

$$\mu(x,t) = \begin{cases} \mu^1, & x \in D^1(t), \\ \mu^2, & x \in D^2(t). \end{cases} \tag{4}$$

Finally, the system is closed using Darcy's law

$$\frac{\mu(x,t)}{\kappa} u(x,t) = -\nabla p(x,t) - g(0, \rho(x,t)), \tag{5}$$

which relates the incompressible velocity with the pressure [31], considering that the fluids saturate the porous media. The permeability and the gravity constants are given by κ and g respectively.

In [47], a completely different physical scenario is studied, with comparable mathematical properties. This describes the flow evolution in Hele-Shaw cells, where the fluids are confined between two parallel plates that are close together. The evolution is essentially in 2D, and it is governed by the equation

$$\frac{12}{b^2}\mu(x,t)u(x,t) = -\nabla p(x,t) - g(0,\rho(x,t)),$$

where b is the distance between the plates. Since these two pioneering works, these different physical phenomena have been extensively studied from a mathematical point of view.

2 Contour evolution equation

The Muskat problem can be considered taking into account many more peculiarities as boundary effects [27] and three dimensional flows [2,20]. The framework picked in this presentation allows us to reduce the problem from its original Eulerian variables formulation (Eqs. 1–5) to the self-evolution of an interface, hence the name contour evolution equation. It provides a simple way to linearize the system of equations to illustrate in a non-technical manner what is going on at the nonlinear level.

Darcy's law (5) shows that the velocity has to be irrotational

$$\partial_{x_1} u_2(x,t) - \partial_{x_2} u_1(x,t) = 0,$$

in the interior of each domain $D^j(t)$, $j = 1, 2$. For that reason the vorticity is given by a measure on the free interface as follows

$$\partial_{x_1} u_2(x,t) - \partial_{x_2} u_1(x,t) = \omega(\alpha,t)\delta(x = z(\alpha,t)),$$

defined in a distributional sense as follows:

$$< u, (\partial_{x_2}\varphi, -\partial_{x_1}\varphi) >= \int \omega(\alpha,t)\varphi(z(\alpha,t))d\alpha, \tag{6}$$

with $\varphi(x)$ a regular test function. Using the Biot-Savart law

$$u(x,t) = (-\partial_{x_2}, \partial_{x_1})\Delta^{-1}(\partial_{x_1} u_2 - \partial_{x_2} u_1)(x,t),$$

it is possible to recover the velocity from the vorticity. It is given by the partial derivatives of the Newton potential as follows:

$$u(x,t) = \frac{1}{2\pi} PV \int \frac{(x - z(\alpha,t))^{\perp}}{|x - z(\alpha,t)|^2}\omega(\alpha,t)d\alpha, \tag{7}$$

for $x \neq z(\alpha,t)$ where PV denotes principal value (as it is necessary at infinity) and $(x_1,x_2)^{\perp} = (-x_2,x_1)$. Taking limits by approaching the free boundary in the normal direction, we can obtain the velocity at the interface with a discontinuity. It reads

$$\begin{aligned} u^2(z(\alpha,t),t) &= BR(z,\omega)(\alpha,t) + \frac{1}{2}\frac{\omega(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t), \\ u^1(z(\alpha,t),t) &= BR(z,\omega)(\alpha,t) - \frac{1}{2}\frac{\omega(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t), \end{aligned} \tag{8}$$

where $u^j(z(\alpha,t),t)$ denotes the limit obtained from inside $D^j(t)$. Above BR stands for the Birkhoff-Rott integral, which is given by

$$BR(z,\omega)(\alpha,t)=\frac{1}{2\pi}PV\int\frac{(z(\alpha,t)-z(\beta,t))^{\perp}}{|z(\alpha,t)-z(\beta,t)|^2}\omega(\beta,t)d\beta. \tag{9}$$

In the above contour operator it is easy to see the importance of the arc-chord condition in order for the Birkhoff-Rott integral to make sense. A one-to-one curve satisfies the arc-chord condition if

$$|z(\alpha,t)-z(\beta,t)|\geq C_{ac}(t)|\alpha-\beta|,\quad \forall\alpha,\ \beta\in\mathbb{R},\quad C_{ac}(t)>0. \tag{10}$$

The discontinuity of the velocity at the free boundary, which is produced by the vorticity configuration (6), is given in the tangential direction and consequently it does not give any insight on the evolution of the shape for $z(\alpha,t)$. In fact, the normal velocity describes the dynamics and it is continuous on $z(\alpha,t)$ (see (8)). Darcy's law implies

$$\Delta p(x,t)=-\mathrm{div}\left(\frac{\mu(x,t)}{\kappa}u(x,t)+g(0,\rho(x,t))\right),$$

where

$$\Delta p(x,t)=RT(\alpha,t)\delta(x-z(\alpha,t)),$$

and the function $RT(\alpha,t)$ is given by

$$RT(\alpha,t)=\frac{\mu^2-\mu^1}{\kappa}u(z(\alpha,t),t)\cdot\partial_\alpha z^{\perp}(\alpha,t)+g(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t). \tag{11}$$

Above

$$\begin{aligned}u(z(\alpha,t),t)\cdot\partial_\alpha z^{\perp}(\alpha,t)&=u^2(z(\alpha,t),t)\cdot\partial_\alpha z^{\perp}(\alpha,t)\\&=u^1(z(\alpha,t),t)\cdot\partial_\alpha z^{\perp}(\alpha,t),\end{aligned} \tag{12}$$

due to (8). Recovering the pressure through the Newton potential,

$$p(x,t)=\frac{1}{2\pi}\int\ln|x-z(\alpha,t)|RT(\alpha,t)d\alpha,$$

for $x\neq z(\alpha,t)$, it is possible to obtain the continuity of the pressure at the free boundary

$$p^2(z(\alpha,t),t)=p^1(z(\alpha,t),t), \tag{13}$$

which is just a mathematical consequence of Darcy's law. Let us introduce the following notation:

$$[\mu u](\alpha,t)=\left(\mu^2u^2(z(\alpha,t),t)-\mu^1u^1(z(\alpha,t),t)\right)\cdot\partial_\alpha z(\alpha,t).$$

Taking limits in Darcy's law provides

$$\begin{aligned}\frac{[\mu u](\alpha,t)}{\kappa}&=-\left(\nabla p^2(z(\alpha,t),t)-\nabla p^1(z^1(\alpha,t),t)\right)\cdot\partial_\alpha z(\alpha,t)-g(\rho^2-\rho^1)\,\partial_\alpha z_2(\alpha,t)\\&=-\partial_\alpha\left(p^2(z(\alpha,t),t)-p^1(z(\alpha,t),t)\right)-g(\rho^2-\rho^1)\,\partial_\alpha z_2(\alpha,t)\\&=-g(\rho^2-\rho^1)\,\partial_\alpha z_2(\alpha,t),\end{aligned}$$

which allows us to relate the vorticity amplitude ω with the unknown curve through the following implicit identity

$$\omega(\alpha,t)+2\frac{\mu^2-\mu^1}{\mu^2+\mu^1}BR(z,\omega)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)=-2g\kappa\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_\alpha z_2(\alpha,t). \tag{14}$$

The dynamics is given by the velocity with the following evolution equation

$$z_t(\alpha, t) = BR(z, \omega)(\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t), \tag{15}$$

where the subscript t denote partial derivative in time and $c(\alpha, t)$ is the function which provides parametrization freedom. It is worth mentioning that considering different $c(\alpha, t)$ the geometry of the curve is the same, as the evolution is described by the normal direction of the velocity [42]. It is usual to pick c as the function zero, but different choices provide different advantages in the analysis of the evolution equation. In particular, it could be chosen in such a way that the interface is parameterized as a graph (see formula (23) below). In conclusion, the contour dynamics equation is now closed and given by (9, 14, 15).

It is possible to write

$$BR(z, \omega)(\alpha, t) \cdot \partial_\alpha z(\alpha, t) = \partial_\alpha \int \arctan\left(\frac{z_2(\alpha, t) - z_2(\beta, t)}{z_1(\alpha, t) - z_1(\beta, t)}\right) \omega(\beta, t) d\beta,$$

so that, in the asymptotically flat and 2π-periodic cases, the amplitude of the vorticity is found to have mean zero,

$$\int \omega(\alpha, t) d\alpha = 0,$$

by integrating identity (14). Hence formula (7) provides

$$u(x, t) = \frac{x^\perp}{|x|^2} \int \omega(\alpha, t) d\alpha + O(|x|^{-2}) = O(|x|^{-2}),$$

for $|x| \to +\infty$. In consequence the velocity is in L^2

$$\int |u(x, t)|^2 dx < \infty,$$

the finite energy and physically relevant scenario.

On the other hand, the velocity can be given through a potential in the interior of each domain $D^j(t)$ as follows

$$u(x, t) = \nabla\phi^j(x, t), \quad x \in \text{int}\,(D^j(t)),$$

due to the irrotationality condition. It gives the possibility of integrating Darcy's law (5) in the interior of each domain $D^j(t)$,

$$\frac{\mu^j}{\kappa}\phi^j(x, t) = -p(x, t) - g\rho^j x_2, \quad x \in \text{int}\,(D^j(t)). \tag{16}$$

The incompressibility gives

$$\begin{aligned} 0 = \frac{\mu^2}{\kappa} \int_{D^2(t)} \Delta\phi^2(x, t)\phi^2(x, t) dx &= -\frac{\mu^2}{\kappa} \int_{D^2(t)} |\nabla\phi^2(x, t)|^2 dx \\ &\quad + \int_{\partial D^2(t)} \nabla\phi^2(x, t) \cdot n(t) \frac{\mu^2}{\kappa} \phi^2(x, t) d\sigma(x, t), \end{aligned}$$

where $n(t) = \partial_\alpha z^\perp(\alpha, t)/|\partial_\alpha z(\alpha, t)|$. A similar identity can be obtained in $D^1(t)$ so that adding them and using formula (16) together with (12) and (13) it is possible to get

$$0 = -\sum_{j=1,2} \frac{\mu^j}{\kappa} \int_{D^j(t)} |u(x, t)|^2 dx - (\rho^2 - \rho^1) g \int_{\partial D^2(t)} u(x, t) \cdot n(t) x_2 d\sigma(x, t).$$

In the last integral above one finds the time derivative of the potential energy

$$\int_{\partial D^2(t)} u(x,t)\cdot n(t) x_2 d\sigma(x,t) = \int \partial_t z(\alpha,t)\cdot\partial_\alpha z^{\perp}(\alpha,t) z_2(\alpha,t)d\alpha$$
$$= \frac{d}{dt}\frac{1}{2}\int |z_2(\alpha,t)|^2\partial_\alpha z_1(\alpha,t)d\alpha = \frac{d}{dt}E_p(t),$$

so that an energy balance follows:

$$(\rho^2-\rho^1) g E_p(t) + \sum_{j=1,2}\frac{\mu^j}{\kappa}\int_0^t\int_{D^j(s)}|u(x,s)|^2 dx ds = (\rho^2-\rho^1) g E_p(0). \qquad (17)$$

3 Mathematical results

The Muskat problem has a rich variety of features which have been studied with a wide diversity of techniques. Interesting scenarios consider 3D fluids, multi-phase flows, boundary effects or permeability discontinuities (see for example [5,43]), etc. Different methods interact, raging from analytic to computer-assisted proofs [38]. In what follows, the permeability κ is considered to be equal to one.

A very significant peculiarity of the problem is that Muskat develops instabilities [46]. If the system of Eqs. (1–5) is satisfied in a week sense, some scenarios yield non-uniqueness of solutions [49]. In the contour evolution setting (9, 14, 15), those unstable cases give rise to ill-possed equations. These phenomena can be understood through the Rayleigh-Taylor condition. Considering the jump across the normal direction of the gradient pressures, it is possible to find

$$-\big(\nabla p^2(z(\alpha,t),t) - \nabla p^1(z(\alpha,t),t)\big) = RT(\alpha,t),$$

with RT the Rayleigh-Taylor function given in (11). The Rayleigh-Taylor condition is said to be satisfied if $RT(\alpha,t) > c > 0$. By linearizing the Eqs. (9, 14, 15) near the steady state $z(\alpha,t) = (\alpha, 0)$, it is possible to obtain

$$f_t^L(\alpha,t) = -(\mu^2+\mu^1)^{-1} RT^L \Lambda f^L(\alpha,t), \qquad (18)$$

where $(\alpha, f^L(\alpha,t))$ represents the linearized free boundary, and the constant RT^L is the linear version of the Rayleigh-Taylor function. The operator Λ is the minus square root of the Laplacian, $\Lambda = (-\Delta)^{1/2}$, also given by a kernel representation and using the Fourier transform as follows:

$$\Lambda f^L(\alpha) = \frac{1}{\pi} PV\int \frac{f^L(\alpha)-f^L(\beta)}{(\alpha-\beta)^2}d\beta, \quad \widehat{\Lambda f^L}(\xi) = |\xi|\widehat{f^L}(\xi). \qquad (19)$$

Now the importance of the Rayleigh-Taylor is disclosed. The case $RT^L > 0$ turns the Muskat problem into a parabolic system at the linear level. For $RT^L < 0$ the character of the equation changes dramatically, giving an ill-posed system. This fact is easy to understand by using the Fourier transform in space to solve (18) obtaining

$$\widehat{f^L}(\xi,t) = \widehat{f^L}(\xi,0)\exp\Big(-(\mu^2+\mu^1)^{-1} RT^L|\xi|t\Big).$$

At the nonlinear level, the Rayleigh-Taylor function (see Eq. (11)) implicates the normal velocity of the fluids with viscosity jump and the geometry of the contour for different densities. Basically, the unstable case arises in the viscosity jump situation when a less viscous fluid pushes a more viscous one. This case was studied in [48], where the contour dynamic equation is proved to be ill-possed. In the density jump case ($\mu^2 = \mu^1$), the unstable regime holds when the more dense fluid lies above the interface and the less dense fluid lies below it. The contour dynamics equation is shown to be ill-posed in this scenario [21]. On the other hand, the lost of derivative in the contour equation is of order one, so that it is possible to find solutions of the system with analytic initial data even in the unstable case [12,32]. At the linear level this fact can be checked using Eq. (18) and the theory of the Fourier transform for analytic functions.

Besides gravity, the evolution Muskat problem can be driven by capillary force. In that case surface tension effects are considered, and the discontinuity of the pressure on the interface is proportional to its curvature as follows:

$$p^2(z(\alpha,t),t) - p^1(z(\alpha,t),t) = -\tau \frac{\partial_\alpha^2 z(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^3}, \tag{20}$$

where $\tau > 0$ is the surface tension coefficient. In this case Eq. (14) is replaced by

$$\begin{aligned}\omega(\alpha,t) + 2\frac{\mu^2-\mu^1}{\mu^2+\mu^1} BR(z,\omega)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)\\ = -2g\kappa\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_\alpha z_2(\alpha,t) - \tau\partial_\alpha\left(\frac{\partial_\alpha^2 z\cdot\partial_\alpha^\perp z}{|\partial_\alpha z|^3}\right)(\alpha,t).\end{aligned} \tag{21}$$

The linearization of the system in this case is given by

$$f_t^{L\tau}(\alpha,t) = -(\mu^2+\mu^1)^{-1} RT^L\Lambda f^{L\tau}(\alpha,t) + \tau\Lambda\partial_\alpha^2 f^{L\tau}(\alpha,t), \tag{22}$$

showing a high parabolic regularizing effect for the graph $(\alpha, f^{L\tau}(\alpha,t))$. The local-in-time existence for the nonlinear problem without gravity and surface tension ($g = 0$ and $RT^L = 0$ in above linear interpretation) and in the one fluid case ($\mu^2 = 0 = \rho^2$) was given in [33]. See also [29] for the boundary value problem. Same type of results for the two fluids case were given in [35]. Besides this surface tension regularizing mechanism, Rayleigh-Taylor instabilities still play a crucial role considering the force of gravity. In [40] initial small perturbation are shown to be unstable under small time evolution for low order norms. In [34] finger shaped stationary-states are found using bifurcation theory which are unstable. On the other hand, when the Rayleigh-Taylor condition is satisfied initially, surface tension solutions approach to solutions without surface tension effects as the coefficient τ vanishes [3].

Without surface tension ($\tau = 0$), the positivity of the nonlinear Rayleigh-Taylor function have been shown to be crucial to disclose a local-in-time existence result [1,51]. The system is proved to be well-posed in the case of equal viscosity $\mu^1 = \mu^2 = \mu$ for the stable case [21] by using energy estimates on the contour dynamics equation through the chain of Sobolev norms. In this situation, the Rayleigh-Taylor condition is satisfied only if the free boundary is represented by the graph of a function $(\alpha, f(\alpha,t))$ and $\rho^2 > \rho^1$. The contour evolution equation is given in this situation by

$$f_t(\alpha,t) = \frac{g(\rho^2-\rho^1)}{2\mu\pi}\int \frac{\beta(\partial_\alpha f(\alpha,t) - \partial_\alpha f(\alpha-\beta,t))}{\beta^2 + (f(\alpha,t)-f(\alpha-\beta,t))^2}d\beta, \tag{23}$$

and $RT(\alpha,t) = g(\rho^2-\rho^1)$ due to $\partial_\alpha z_1(\alpha,t) = 1$. The case with different viscosities and densities was shown to be well-posed in [19]. In that proof it is crucial to get control of the norm of the implicit operator given in (14) involved in the definition of the amplitude of the vorticity ω. The arguments rely upon quantitative bounds of Hilbert transforms in variable domains in the plane. It requires a harmonic analysis approach involving the Hopf maximum principle, conformal mappings and Harnack inequalities. A local-in-time control of the positivity of the Rayleigh-Taylor sign condition is indispensable to reach legitimate energy estimates, as for a general parametrization $RT(\alpha, t)$ does not need to be positive. Finally we would like to quote some recent articles where local-in-time existence is shown of classical solution for large and low regular initial data. For the one fluid case ($\mu^1 = \rho^1 = 0$) see [30] and [17] for the density jump case.

If $\mu^2 = \mu^1 = \mu$ and $\tau = 0$, it is possible to obtain decay of the L^∞ norm of the interface for arbitrary initial data (see [22]). The graph interface evolves by (23) giving

$$\left\| f - \frac{1}{2\pi}\int_{-\pi}^{\pi} f_0 d\alpha \right\|_{L^\infty}(t) \leq \left\| f_0 - \frac{1}{2\pi}\int_{-\pi}^{\pi} f_0 d\alpha \right\|_{L^\infty} e^{-Ct},$$

for 2π-periodic f and

$$\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty}(1+Ct)^{-1},$$

in the asymptotically flat case, where $f(\alpha, 0) = f_0(\alpha)$ and $C = C(f_0) > 0$. These maximum principles are sharp as they provide the same rate of decay as Eq. (18) for f^L. On the other hand, the L^2 norm evolution allows to control half a derivative of f^L in (18) due to the identity

$$\left\| f^L \right\|_{L^2}^2(t) + \frac{g(\rho^2-\rho^1)}{\mu}\int_0^t \left\| \Lambda^{1/2} f^L \right\|_{L^2}^2(s) ds = \left\| f_0^L \right\|_{L^2}^2,$$

or equivalently

$$\left\| f^L \right\|_{L^2}^2(t) + \frac{g(\rho^2-\rho^1)}{2\mu\pi}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{f^L(\alpha,s)-f^L(\beta,s)}{\alpha-\beta}\right)^2 d\beta d\alpha ds = \|f_0\|_{L^2}^2, \quad (24)$$

using the integral formula (19) and that $RT^L = g(\rho^2 - \rho^1)$. For the nonlinear problem, the identity

$$\|f\|_{L^2}^2(t) + \frac{g(\rho^2-\rho^1)}{2\mu\pi}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} \ln\left(1+\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^2\right) d\alpha d\beta ds = \|f_0\|_{L^2}^2 \quad (25)$$

holds [16], which does not give a chance of gaining any regularity at the level of f (compare with (17) in the case of viscosity jump). This can be easily shown by the bound

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \ln\left(1+\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^2\right) d\alpha d\beta \leq C\|f\|_{L^1}(s), \quad (26)$$

which allows to control the nonlinear term with zero derivatives.

In the case with small initial data, it is possible to use the parabolic character of the equation in the stable state (see (18) and (22) for the lineal interpretation) to prove global in time regularity in different situations. For purely surface tension driven fluids ($g = 0$) see results in [18,29]. Without surface tension ($\tau = 0$), global existence for the viscosity jump case was proven in [48] and extended to the density jump case in [21], showing in both papers instant analyticity of the solutions. For gravity and surface tension interaction with boundary values see [34]. Those global existence results have been extended in some situations assuming initial smallness for critical norms with respect to the scaling [16,39], and showing instant analyticity in [7]. In works [15,16] some results of global in time regularity of classical solutions are shown with $\mu^1 = \mu^2$, $\tau = 0$ and medium-size initial slope in the Wiener algebra, i.e

$$\int |\xi||\hat{f}(\xi)|d\xi \leq c_0$$

with c_0 an explicit constant. In particular, the terminology medium-size is used to emphasize that the constant c_0 is of size $O(1)$ and independent of the physical constants g, κ, ρ^j, and μ^j $(j = 1, 2)$. Those papers show global existence of Lipschitz weak solutions with initial slope less than 1 and gradient less than 1/3 in 3D. Using Eq. (23), multiplying by a test function and integrating by parts, it is possible to find a weak formulation of the system. In fact, we say that the graph $(\alpha, f(\alpha, t))$ is a weak solution of the Muskat problem if the following identity is satisfied

$$\int_0^T \int_{\mathbb{R}} \eta_t(\alpha, t) f(\alpha, t) d\alpha dt + \int_{\mathbb{R}} \eta(\alpha, 0) f_0(\alpha) d\alpha$$
$$= \int_0^T \int_{\mathbb{R}} \partial_\alpha \eta(\alpha, t) \frac{g(\rho^2 - \rho^1)}{2\mu\pi} PV \int_{\mathbb{R}} \arctan\left(\frac{f(\alpha, t) - f(\beta, t)}{\alpha - \beta}\right) d\beta d\alpha dt, \quad (27)$$

for any $\eta \in C_c^\infty([0, T) \times \mathbb{R})$.

A fascinating behavior of Muskat solution, which can be proved analytically, is finite time singularity formation starting from regular stable initial data. In [13], it is proved that in the case $\mu^1 = \mu^2$ and $\tau = 0$ there are solutions of the Muskat equation with initial interfaces being certain smooth stable graphs, which enter the unstable regime, where the interface is no longer a graph, in finite time. In particular there exists a time t_p in which

$$\lim_{t \to t_p^+} \|\partial_\alpha f\|_{L^\infty}(t) = +\infty,$$

for solutions of Eq. (23). In other words, the interface evolves into a non-graph in finite time (see Fig. 1 for an example). For some contour dynamics problems these "wave-turning" effects are not dramatic, it is just a breakdown in the parametrization as a graph. But for

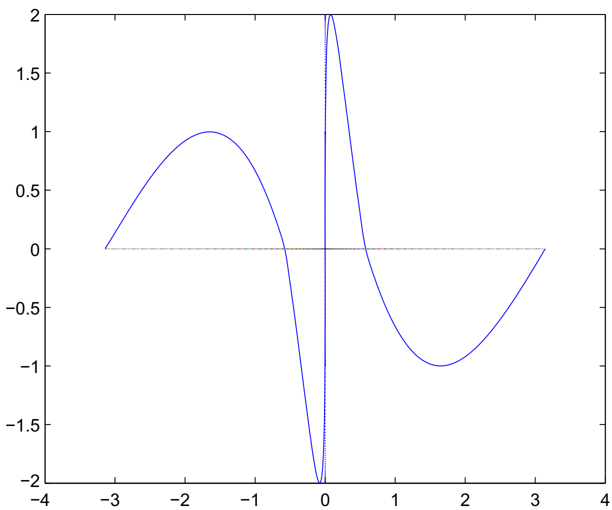

Fig. 1 An example of wave-turning for Muskat. At $(x, y) = (0, 0)$ the slope of the function is $+\infty$

the Muskat problem this is a strong change in the character of the equation. In particular the significance of a wave-turning is that the Rayleigh-Taylor condition breaks down. At some branch in theinterface it is possible to localize the heavy fluid on top of the lighter one. An important reason why this phenomenon arises is that, even for large initial data, Muskat solutions become instantly analytic [13]. So that, despite the interface is about to reach an unstable regime, the analyticity remains by the time the wave-turning occurs.

In fact, the Muskat curve solution exists and remains analytic for some time after the turnover, even in the unstable regime. Furthermore, global existence can be false for certain scenarios with large initial data. In [10], it is shown that some of these smooth initial interfaces in the stable regime turn to the unstable regime and later blow-up; i.e. for some time $t_s > t_p$ there is a lost of regularity in the interface. Therefore Muskat develops finite time singularities starting from well-posed scenarios. This is the first case of singularity formation in contour dynamics of incompressible fluids in an initially well-posed problem. The pattern of these initial data is far from trivial: numerical simulations performed in [25] show that there exists initial data with steep slopes for which a regularizing effect appears. Even more, some analytic unstable solutions can reach a stable regime and some later time become unstable [26]. If the contour evolution remains regular in the stable regime is not known, but a finite time singularity formation characterization is given in [17] in terms of the interface slope.

A different kind of singularities which could breakdown the dynamics of incompressible fluid interfaces is finite time self-intersection. In this scenario two different particles of the fluid interface collide. For Muskat with density jump ($\mu^1 = \mu^2$) the self-intersection can not occur along a curve of points if the interface remains regular. This type of collision is called "squirt" or "splat" singularity (see Fig. 2 below).

The result is given in [24], using extra cancelation of the operators that relates the velocity and the density in Darcy's law (5) for constant viscosity. They are given by singular integrals with even kernels, yielding a velocity in L^∞ for regular interfaces (see [8] where the extra cancelation was found). The lack of squirt singularities was extended to the case of viscosity jump in [28] showing that the interface becomes instantly analytic.

A less severe type of singularity is given by the collapse at just a single point while the interface preserves regularity. The term used for this scenario is "splash" singularity. This

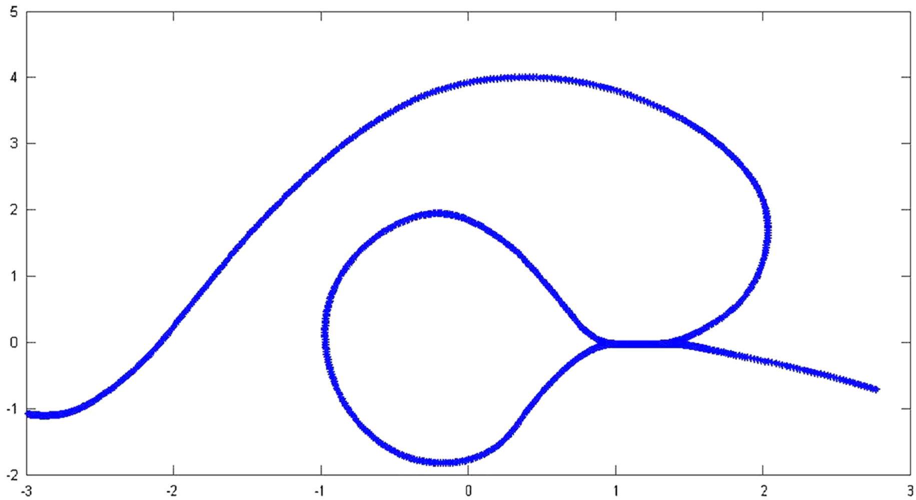

Fig. 2 A splat singularity: regular curve with self-intersection along an arc

type of blow-up can be removed for the case of equal viscosities [36]. The proof is based on a cancelation in the kernel of the integral in (23), which is of degree zero for bounded curvature contours. On the other hand, for the dynamics of one fluid ($\mu^1 = \rho^1 = 0$) it is possible to prove finite time splash singularity formation for some particular geometries. In this case the Rayleigh-Taylor condition holds all the way up to the blow-up time.

This result is achieved by two different ideas. First of all, it is possible to convert the geometry problem with the "splash" singularity into a new contour dynamics equation with a conformal map P. It transforms the equations with P given by a square root in complex variables whose discontinuity branch passes through the collision point x_s. We denote this new problem by P(Muskat). The transformation gives P(Muskat) with no self-intersecting points of the interface. An important point here to have in mind is that before to the "splash" time Muskat and P(Muskat) are equivalent, but at the "splash" singularity time P(Muskat) makes sense. For this reason it is possible to find local existence P(Muskat) and go further in time. We pick an initial contour $z^l(\alpha, 0)$ for Muskat with one pointwise collapse as a splash (see Fig. 3). We transform this initial contour with P and use it as initial datum for P(Muskat). Next key idea is to obtain a stability result for P(Muskat) which does not depend on the arc-chord condition for $z^l(\alpha, 0)$ but it may depend on the arc-chord condition of the contour of P(Muskat). We denote this solution by $P(z^l(\alpha, t))$. Due to the transformation, the arc-chord constant for $P(z^l(\alpha, t))$ is going to be big. Then, the stability reads

$$\begin{aligned}\|P(z(\alpha,t)) - P(z^l(\alpha,t))\| &\leq C(t)\|P(z(\alpha,0)) - P(z^l(\alpha,0))\| \\ &\leq C(t)\|z(\alpha,0) - z^l(\alpha,0)\|,\end{aligned}$$

where $z(\alpha, 0)$ is a $z^l(\alpha, 0)$ perturbation, $P(z(\alpha, t))$ is the P(Muskat) solution with $P(z(\alpha, 0))$ as initial datum, $C(t)$ is a controlled constant and $\|\cdot\|$ is an appropriate norm. It is possible to show that the velocity for $z^l(\alpha, 0)$ gives that the two branches on the interface with the common intersection point are going to cross as time goes forward. Then we take an initial datum $z(\alpha, 0)$ which is a small perturbation of $z^l(\alpha, 0)$ but without pointwise intersection. Because the time of existence for P(Muskat) is independent of the smallness of $\|z(\alpha, 0) -$

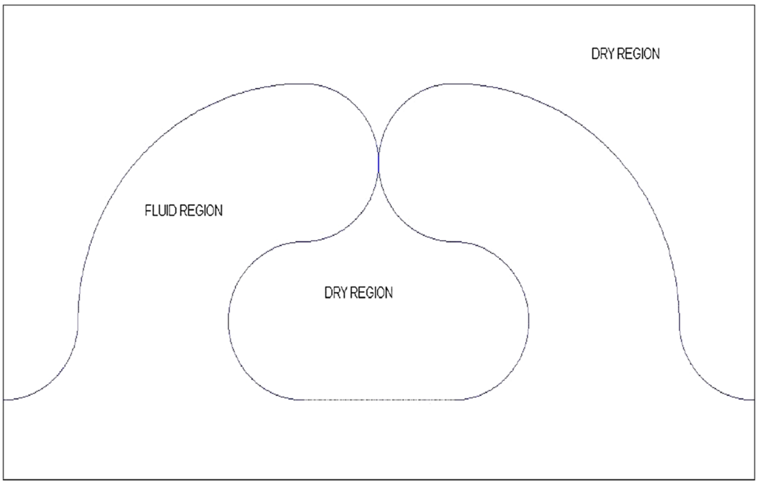

Fig. 3 An appropriate geometry for a splash singularity in Muskat with a dry region

$z^l(\alpha, 0)\|$ we can conclude that, due to the fact that z^l self-intersects at a point, there exists a finite time such that z has to break down with a splash singularity.

4 The new estimate

This section is devoted to show a new inequality for weak Muskat solutions. This result was announced at the Special Session "Analysis of free boundary problems" in the 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications, in Madrid (Spain), July 2014. Below we provide details.

We define the spaces $L^\infty([0,T]; W^{1,\infty})$, $L^\infty([0,T]; L^2)$ and $L^2([0,T]; H^{1/2})$ with norms

$$\|f\|_{L^\infty(0,T;W^{1,\infty})} = \text{ess sup}_{(0,T)}(\|f\|_{L^\infty}(t) + \|\partial_\alpha f\|_{L^\infty}(t)),$$

$$\|f\|_{L^\infty(0,T;L^2)} = \text{ess sup}_{(0,T)}\|f\|_{L^2}(t),$$

$$\|f\|^2_{L^2(0,T;H^{1/2})} = \int_0^T \left(\|f\|^2_{L^2}(t) + \|\Lambda^{1/2} f\|^2_{L^2}(t)\right) dt,$$

and $C([0,T]\times\mathbb{R})$ is the space of continuous function with $(t,\alpha) \in [0,T]\times\mathbb{R}$.

Lemma 4.1 *For $f_0 \in L^2$ and $\|\partial_\alpha f_0\|_{L^\infty} < 1$, there exist weak solutions of* (23) *with*

$$f(\alpha,t) \in C([0,T]\times\mathbb{R}) \cap L^\infty(0,T; W^{1,\infty}), \quad \forall T > 0,$$

satisfying the following estimate

$$\|f\|^2_{L^2}(t) + \frac{g(\rho^2-\rho^1)}{4\mu\pi}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^2 d\beta d\alpha ds \le \|f_0\|^2_{L^2}, \qquad (28)$$

for any $t \in [0,T]$.

Remark 4.2 With this estimate we reproduce an analogous feature to the linearized system (see (24)), giving a solution with $f \in L^\infty(0,T; L^2) \cap L^2(0,T; H^{1/2})$.

Proof Using the method in [16], it is easy to find the weak solution of Muskat in the space $C([0,T]\times\mathbb{R}) \cap L^\infty(0,T; W^{1,\infty})$. It is given passing to the limit into the regularized system

$$\begin{aligned} f^\epsilon_t(\alpha,t) = & \frac{g(\rho^2-\rho^1)}{2\mu\pi}\partial_\alpha PV\int_{\mathbb{R}} \arctan\left(\frac{f(\alpha,t)-f(\alpha-\beta,t)}{\beta|\beta|^{-\varepsilon}}\right) d\beta \\ & -\varepsilon C\Lambda^{1-\varepsilon} f^\epsilon(\alpha,t) + \varepsilon f^\epsilon_{\alpha\alpha}(\alpha,t), \end{aligned} \qquad (29)$$

where $C > 0$ is a universal constant, the operator $\Lambda^{1-\varepsilon} f$ is given by

$$\Lambda^{1-\varepsilon} f(\alpha) = c_1(\varepsilon)\int_{\mathbb{R}} \frac{f(\alpha)-f(\alpha-\beta)}{|\beta|^{2-\varepsilon}} d\beta,$$

with $0 < c_m \le c_1(\varepsilon) \le c_M$, and $\varepsilon > 0$ small enough. The initial data $f_0 \in L^2$ with $\|\partial_x f_0\|_{L^\infty(\mathbb{R})} < 1$ is regularized with a mollifier to find global-in-time regular solutions of (29). The convergence as $\varepsilon \to 0^+$ is, up to a subsequence, strong in L^∞ on compact sets of $[0, T] \times \mathbb{R}$ and weak-start in $L^\infty(0, T; W^{1,\infty})$. Furthermore, it is possible to find (see discussion in [16] above Remark 4.4 for more details)

$$\frac{d}{dt}\|f^\varepsilon\|^2_{L^2}(t) = -\frac{g(\rho^2-\rho^1)}{2\mu\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1-\varepsilon}{|\alpha-\beta|^\varepsilon}\ln\left(1+\left(\frac{f^\varepsilon(\alpha,t)-f^\varepsilon(\beta,t)}{(\alpha-\beta)|\alpha-\beta|^{-\varepsilon}}\right)^2\right)d\beta d\alpha$$
$$-2C\varepsilon\|\Lambda^{(1-\varepsilon)/2}f^\varepsilon\|^2_{L^2}(t) - 2\varepsilon\|f^\varepsilon_\alpha\|^2_{L^2}(t).$$

Therefore, integration in time provides

$$\|f^\varepsilon\|^2_{L^2}(t) + \frac{g(\rho^2-\rho^1)}{2\mu\pi}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1-\varepsilon}{|\alpha-\beta|^\varepsilon}\ln\left(1+\left(\frac{f^\varepsilon(\alpha,s)-f^\varepsilon(\beta,s)}{(\alpha-\beta)|\alpha-\beta|^{-\varepsilon}}\right)^2\right)d\beta d\alpha ds$$
$$\leq \|f_0\|^2_{L^2}.$$

The strong convergence of a subsequence of ε and Fatou's lemma allows us to find that identity (25) is satisfied for the weak Muskat solution in an inequality form

$$\|f\|^2_{L^2}(t)+\frac{g(\rho^2-\rho^1)}{2\mu\pi}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\ln\left(1+\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^2\right)d\alpha d\beta ds \leq \|f_0\|^2_{L^2}.$$

In general, this inequality does not yield any gain of regularity as it is disclosed in (26). But in this case, due to

$$\|\partial_\alpha f\|_{L^\infty}(t) \leq \|\partial_\alpha f_0\|_{L^\infty} < 1$$

it is possible to expand the $\ln(1+x^2)$ function, to find

$$S = \ln\left(1+\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^2\right) = \sum_{j=1}^\infty(-1)^{j+1}\frac{1}{j}\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^{2j}.$$

In the infinite sum we gather terms so that

$$S = \sum_{j=1}^\infty\left[\frac{1}{2j-1}\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^{4j-2} - \frac{1}{2j}\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^{4j}\right]$$

and therefore

$$S = \sum_{j=1}^\infty\frac{1}{2j-1}\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^{4j-2}\left[1-\frac{2j-1}{2j}\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^2\right]. \quad (30)$$

At this point it is easy to get

$$\frac{1}{2j} \leq 1 - \frac{2j-1}{2j}\|\partial_\alpha f\|^2_{L^\infty}(s) \leq 1 - \frac{2j-1}{2j}\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^2$$

and therefore all the addends in S given by (30) are positive. Then

$$\frac{1}{2}\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^2 \leq S.$$

This yields finally

$$\|f\|^2_{L^2}(t) + \frac{g(\rho^2-\rho^1)}{4\mu\pi}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^2 d\alpha d\beta ds \leq \|f_0\|^2_{L^2}.$$

□

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