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Existence and regularity results for p-Laplacian boundary value problems

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Abstract In this paper we introduce some of the main tools to study non-linear boundary value problems whose simplest model is

$$\begin{cases} -\Delta_p(u) = -\text{div}(|\nabla u|^{p-2}\nabla u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

and $f(x)$ belongs to $\mathrm{L}^m(\Omega)$, $m \geq 1$.

Keywords Nonlinear boundary value problems · Weak solutions · Distributional solutions

Mathematics Subject Classification 35J60

1 Introduction

The main purpose of these lectures is to introduce some of the main tools to study nonlinear boundary value problems. In particular, we are concerned with the Dirichlet problem for the p-Laplace operator which is the simplest example of these ones. To be more precise, given Ω a bounded open set in $I\!R^N$, $N \geq 2$, we consider the problem

$$\begin{cases} -\Delta_p(u) = -\text{div}(|\nabla u|^{p-2}\nabla u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

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or the more general

$$\begin{cases} A(u) = -\text{div}(a(x)|\nabla u|^{p-2}\nabla u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.1}$$

where

$$f \in L^m(\Omega), \quad m \geq 1, \tag{1.2}$$

$$1 < p < N, \tag{1.3}$$

$$0 < \alpha \leq a(x) \leq \beta, \tag{1.4}$$

for some constants $0 < \alpha \leq \beta$.

The classical theory of nonlinear elliptic equations states that $W_0^{1,p}(\Omega)$ is the natural functional space framework to find weak solutions of (1.1), if the function f belongs to the dual space of $W_0^{1,p}(\Omega)$. However, for the model problem (1.1), the existence of $W_0^{1,p}(\Omega)$ solutions fails if the right hand side is a function which does not belong to the dual space of $W_0^{1,p}(\Omega)$. It is possible to find distributional solutions in function spaces larger than $W_0^{1,p}(\Omega)$ but contained in $W_0^{1,1}(\Omega)$. Keeping this in mind, these lecture notes are divided into four sections. After this introductory section, the second one deals with existence and regularity results when the right hand side belongs to the dual space of $W_0^{1,p}(\Omega)$. In this case, the model problem (1.1) is a variational boundary value problem. In Sects. 3 and 4 we consider the problem (1.1) when the right hand side is a function which does not belong to the dual space $W_0^{1,p}(\Omega)$. In the former, we study the existence of distributional solutions belonging to a function space strictly contained in $W_0^{1,1}(\Omega)$. On the other hand, in the latter we will prove the existence of solutions belonging to $W_0^{1,1}(\Omega)$ and not belonging to $W_0^{1,q}(\Omega)$, $1 < q < p$. The existence of $W_0^{1,1}(\Omega)$ solutions, instead of $W_0^{1,q}(\Omega)$ or $W_0^{1,p}(\Omega)$ solutions, of the boundary value problem (1.1) is a consequence of the poor summability of the right hand side. We point out that existence results of $W_0^{1,1}(\Omega)$ distributional solutions is not so usual in elliptic problems.

Note that our approach is "direct" and that there are no regularity assumptions w.r.t. $x \in \Omega$.

We have made an effort to keep these lecture notes self-contained, specifically orientated to Master and PhD students. For the basic tools of functional analysis and Sobolev spaces we refer to the book by Brezis [7]. Some similar problems are also studied in the books [1,2].

2 Weak solutions

Theorem 2.1 *If $f \in L^m(\Omega)$ with $m \geq (p^*)' = \frac{Np}{Np+p-N}$, then there exists a weak solution $u \in W_0^{1,p}(\Omega)$ of (1.1), i.e., u satisfies*

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla\varphi = \int_\Omega f\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{2.1}$$

Proof This result is deduced using variational methods. We consider the following functional

$$J(v) = \frac{1}{p}\int_\Omega a(x)|\nabla v|^p - \int_\Omega f\,v, \quad \forall v \in W_0^{1,p}(\Omega).$$

Since $m \geq (p^*)'$, the functional J is well defined. Moreover, using Hölder inequality with exponents $(p^*, (p^*)')$ and (1.4), we obtain

$$J(v) \geq \frac{\alpha}{p} ||v||^p_{W_0^{1,p}(\Omega)} - ||f||_{L^{(p^*)'}(\Omega)} ||v||_{L^{p^*}(\Omega)}.$$

Thus, using Sobolev inequality, we have

$$J(v) \geq \frac{\alpha}{p} ||v||^p_{W_0^{1,p}(\Omega)} - S\, ||f||_{L^{(p^*)'}(\Omega)} ||v||_{W_0^{1,p}(\Omega)},$$

which implies that J is coercive. On the other hand, thanks to the weak lower semicontinuity of the norm $||.||_{W_0^{1,p}(\Omega)}$ in $W_0^{1,p}(\Omega)$, we deduce that the functional J is weakly lower semicontinuous. Then, there exists $u \in W_0^{1,p}(\Omega)$ a minimizer for J and the Euler-Lagrange equation that u satisfies is the equation of (1.1), in the sense of (2.1). □

Theorem 2.2 *If $f \in L^m(\Omega)$ with $m \geq (p^*)' = \frac{Np}{pN+p-N}$, then the weak solution u of* (1.1) *is unique.*

Proof This fact is due to the strict convexity of the functional J defined above. □

2.1 Summability of the weak solutions

We make use of the following functions, defined for $k > 0$ and $s \in I\!R$,

$$T_k(s) := \begin{cases} -k, & s \leq -k \\ s, & |s| \leq k \\ k, & s \geq k \end{cases} \qquad G_k(s) := s - T_k(s). \tag{2.2}$$

Theorem 2.3 *If $f \in L^m(\Omega)$ with $(p^*)' \leq m < \frac{N}{p}$, then the weak solution u of* (1.1) *given by Theorem* 2.1 *belongs to $L^{((p-1)m^*)^*}(\Omega)$.*

Proof The idea is to take a suitable power of the weak solution u as a test function (see [6]). But, it is not possible because the solution is not bounded. In this way, we take as a test function

$$\varphi = |T_k(u)|^{p(\gamma-1)}\, T_k(u), \quad \gamma \geq 1,$$

which is a bounded function. Hence we have,

$$(p\gamma - p + 1) \int_\Omega a(x) |\nabla T_k(u)|^p\, |T_k(u)|^{p(\gamma-1)} \leq ||f||_{L^m(\Omega)} \left(\int_\Omega |T_k(u)|^{(p\gamma-p+1)m'} \right)^{\frac{1}{m'}}.$$

Moreover, using Sobolev inequality and (1.4), we have

$$\int_\Omega a(x) |\nabla T_k(u)|^p\, |T_k(u)|^{p(\gamma-1)} = \frac{1}{\gamma^p} \int_\Omega a(x) |\nabla (T_k(u))^\gamma|^p \geq \frac{\alpha}{(S\,\gamma)^p} \left(\int_\Omega |T_k(u)|^{\gamma\, p*} \right)^{\frac{p}{p^*}}.$$

Summarizing the last inequalities, we deduce that

$$\left(\int_\Omega |T_k(u)|^{\gamma\, p*} \right)^{\frac{p}{p^*}} \leq \frac{(S\,\gamma)^p}{\alpha(p\gamma - p + 1)} ||f||_{L^m(\Omega)} \left(\int_\Omega |T_k(u)|^{(p\gamma-p+1)m'} \right)^{\frac{1}{m'}}.$$

Now, it is sufficient to choose γ such that $\gamma\, p^* = (p\gamma - p + 1)m'$, i.e.,

$$\gamma = \frac{(p-1)m'}{pm' - p^*} = \frac{((p-1)m^*)^*}{p^*}.$$

The fact that $(p*)' \leq m < \frac{N}{p}$ implies that $\gamma \geq 1$ and $\frac{p}{p^*} - \frac{1}{m'} > 0$. To finish, we apply Fatou Lemma (as k tends to infinite) to deduce that

$$\left(\int_\Omega |u|^{\gamma\, p*}\right)^{\frac{p}{p^*} - \frac{1}{m'}} \leq \frac{(S\,\gamma)^p}{\alpha(p\gamma - p + 1)}\, ||f||_{L^m(\Omega)}.$$

That is

$$\left(\int_\Omega |u|^{((p-1)m^*)^*}\right)^{\frac{p}{p^*} - \frac{1}{m'}} \leq \frac{(S\,\gamma)^p}{\alpha(p\gamma - p + 1)}\, ||f||_{L^m(\Omega)},$$

which completes the proof. □

Theorem 2.4 *If $f \in L^m(\Omega)$ with $m > \frac{N}{p}$, then the weak solution u of (1.1) given by Theorem 2.1 belongs to $L^\infty(\Omega)$.*

Proof Following the Stampacchia method (see [10]) for L^∞-estimates, we take $G_k(u)$ as a test function in the weak formulation of (1.1) to obtain, using Hölder inequality and (1.4), that

$$\alpha \int_\Omega |\nabla G_k(u)|^p \leq ||f||_{L^m(\Omega)} \left(\int_{\{|u_n|>k\}} |G_k(u)|^{m'}\right)^{\frac{1}{m'}}. \tag{2.3}$$

Sobolev inequality and Hölder inequality with exponents $\frac{p^*}{m'}$ and its Hölder conjugate imply that

$$\frac{\alpha}{S^p} \left(\int_\Omega |G_k(u)|^{p^*}\right)^{\frac{p}{p^*}} \leq ||f||_{L^m(\Omega)} \left(\int_\Omega |G_k(u)|^{p^*}\right)^{\frac{1}{p^*}} \mu\{|u_n| > k\}^{(1-\frac{m'}{p^*})\frac{1}{m'}},$$

(where μ is the Lebesgue measure) and thus

$$\left(\int_\Omega |G_k(u)|^{p^*}\right)^{\frac{p-1}{p^*}} \leq \frac{S^p}{\alpha}\, ||f||_{L^m(\Omega)}\, \mu\{|u_n| > k\}^{\frac{1}{m'} - \frac{1}{p^*}}.$$

Therefore, using Hölder inequality again (with exponents p^* and its Hölder conjugate) we have

$$\left(\int_\Omega |G_k(u)|\right)^{p-1} \leq \frac{S^p}{\alpha}\, ||f||_{L^m(\Omega)}\, \mu\{|u_n| > k\}^{\frac{1}{m'} - \frac{1}{p^*}}\, \mu\{|u_n| > k\}^{(1-\frac{1}{p^*})(p-1)},$$

and then

$$\int_\Omega |G_k(u)| \leq \left(\frac{S^p}{\alpha}\right)^{\frac{1}{p-1}} ||f||^{1/(p-1)}_{L^m(\Omega)}\, \mu\{|u_n| > k\}^{(\frac{1}{m'} - \frac{1}{p^*})\frac{1}{p-1} + 1 - \frac{1}{p^*}}.$$

The fact that $m > \frac{N}{p}$ implies $(\frac{1}{m'} - \frac{1}{p^*})\frac{1}{p-1} + 1 - \frac{1}{p^*} > 1$ and by Lemma 5.2 (see Appendix A below), we deduce the result.

Remark 2.5 Let f belongs to $L^m(\Omega)$ with $m > \frac{N}{p}$. If a function $u \in W_0^{1,p}(\Omega)$, not necessary a solution of a differential problem, satisfies the inequality (2.3), then u belongs to $L^\infty(\Omega)$.

2.2 Nonlinear b.v.p. with lower order term

We make use of the following well known inequalities.

Lemma 2.6 (See Appendix B below) *Let ξ and η be arbitrary vectors of $\mathbb{R}^N$.*

- *If $2 \le p < N$, then*

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\,(\xi - \eta) \ge \gamma_p\,|\xi - \eta|^p, \tag{2.4}$$

- *If $1 < p < 2$, then*

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\,(\xi - \eta) \ge \gamma_p\,\frac{|\xi - \eta|^2}{(1 + |\xi| + |\eta|)^{2-p}}, \tag{2.5}$$

where γ_p denotes positive constants depending on p. □

Next, we study the Dirichlet problem for the p-Laplace operator with a lower order term. We refer to the paper [8] as a starting point of this type of problems. In particular, we consider the problem

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) + u\,|u|^{r-1} = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{2.6}$$

where $r > 1$ and $f \in L^m(\Omega)$ with $m \ge (p^*)'$.

Theorem 2.7 *Assume that $r > 1$ and $f \in L^m(\Omega)$ with $m \ge (p^*)' = \frac{Np}{Np+p-N}$. Then, there exists a weak solution $u \in W_0^{1,p}(\Omega)$ of (2.6), i.e., $|u|^r \in L^1(\Omega)$ and u satisfies*

$$\int_\Omega a(x)|\nabla u|^{p-2}\,\nabla u\,\nabla\varphi + \int_\Omega |u|^{r-1}\,u\,\varphi = \int_\Omega f\,\varphi, \quad \forall\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Proof We follow a standard approximation procedure. We fix $n \in \mathbb{N}$ and define the function

$$g_n(s) := |T_n(s)|^{r-1}\,T_n(s), \quad \forall s \in \mathbb{R},$$

where the function T_n is given by (2.2). Firstly, using again variational methods, we study existence results for the following approximated problems

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u_n|^{p-2}\nabla u_n) + g_n(u_n) = f, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

To this aim we consider, for each $n \in \mathbb{N}$, the function

$$\phi_n(s) := \int_0^s g_n(t)\,dt, \quad \forall s \in \mathbb{R},$$

and we define the functional

$$J_n(v) = \frac{1}{p}\int_\Omega a(x)|\nabla v|^p + \int_\Omega \phi_n(v) - \int_\Omega f\,v, \quad \forall v \in W_0^{1,p}(\Omega).$$

We observe that J_n is well defined (since the function g_n is bounded and $m \ge (p^*)'$). Moreover, using that ϕ_n is a positive function, we get

$$J_n(v) \ge \frac{1}{p}\int_\Omega a(x)|\nabla v|^p - \int_\Omega f\,v, \quad \forall v \in W_0^{1,p}(\Omega).$$

Thus, recalling the proof of Theorem 2.1, we deduce that J_n is a coercive and weakly lower semicontinuous functional. As a consequence, there exists $u_n \in W_0^{1,p}(\Omega)$ a minimizer for J_n and the Euler–Lagrange equation that u satisfies is the equation of (2.7), in the sense

$$\int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi + \int_\Omega g_n(u_n)\,\varphi = \int_\Omega f\,\varphi, \quad \forall\varphi \in W_0^{1,p}(\Omega). \tag{2.8}$$

Next, we find a solution of (2.6) as a limit (in a sense) of the sequence $\{u_n\}$. Keeping this in mind, we divide the proof into four steps.

STEP 1. *The sequence* $\{u_n\}$ *is bounded in* $W_0^{1,p}(\Omega)$ by a positive constant R. Indeed, using u_n as a test function in (2.8), we obtain

$$\int_\Omega a(x)|\nabla u_n|^p + \int_\Omega g_n(u_n)\,u_n \le \int_\Omega |f||u_n|,$$

which implies, dropping the positive term $\int_\Omega g_n(u_n)\,u_n$ and using (1.4), that

$$\alpha \int_\Omega |\nabla u_n|^p \le \int_\Omega |f||u_n|.$$

Since $m \ge (p^*)'$, using Hölder inequality and next Sobolev inequality, we deduce that

$$\alpha ||u_n||^p_{W_0^{1,p}(\Omega)} \le S\,||f||_{L^{(p^*)'}(\Omega)}\,||u_n||_{W_0^{1,p}(\Omega)}.$$

Therefore, if $R := (\frac{S}{\alpha}\,||f||_{L^{(p^*)'}(\Omega)})^{\frac{1}{p-1}}$, we conclude that

$$\|u_n\|_{W_0^{1,p}(\Omega)} \le R.$$

As a consequence, there exists a subsequence (not relabeled) such that u_n *converges weakly in* $W_0^{1,p}(\Omega)$ *and a.e. in* Ω *to a function* $u \in W_0^{1,p}(\Omega)$.

STEP 2. *Strong convergence in* $L^1(\Omega)$ *of the lower order term.* Using again u_n as a test function in (2.8), we obtain that

$$0 \le \int_\Omega g_n(u_n)\,u_n \le \int_\Omega |f||u_n| \le S\,||f||_{L^{(p^*)'}(\Omega)}\,||u_n||_{W_0^{1,p}(\Omega)} \le C_R, \tag{2.9}$$

where C_R is a positive constant depending on R (which is given by Step 1).

To finish, we want to use Vitali's Theorem to prove that the sequence $\{g_n(u_n)\}$ converges strongly in $L^1(\Omega)$ to $|u|^{r-1}u$. To this aim, recalling that $u_n(x)$ converges a.e. in Ω to u (by Step 1), we only need to prove that, for every subset measurable E, we have

$$\lim_{\text{meas(E)}\to 0} \int_E |g_n(u_n)| = 0, \quad \text{uniformly with respect to } n.$$

Indeed, for every $k > 0$, we have, using (2.9), that

$$\int_E |g_n(u_n)| \le \int_{\{k\le|u_n|\}} |g_n(u_n)| + \int_E |k|^r \le \frac{C_R}{k} + \int_E |k|^r,$$

which implies that

$$\lim_{\text{meas(E)}\to 0} \int_E |g_n(u_n)| \le \frac{C_R}{k}.$$

Therefore, thanks to Vitali's Theorem, we conclude that

$$g_n(u_n) \longrightarrow u\,|u|^{r-1}, \quad \text{strongly in } L^1(\Omega).$$

As a consequence, we have also obtained that $|u|^r \in L^1(\Omega)$.

STEP 3. *Passing to the limit.* In order to pass to the limit in (2.8), we observe that the weak convergence of u_n is not sufficient due to the nonlinearity of the principal part. We need to prove that the sequence $\{\nabla u_n\}$ converges strongly in $L^p(\Omega)$ to ∇u. So, we use $[u_n - T_k(u)]$ as test function in (2.8). Hence,

$$\begin{aligned}
&\int_\Omega (a(x)|\nabla u_n|^{p-2}\nabla u_n - a(x)|\nabla u|^{p-2}\nabla u)\nabla(u_n-u)\\
&\quad+\int_\Omega (g_n(u_n)-g_n(T_k(u)))[u_n-T_k(u)]\\
&\quad= -\int_\Omega a(x)|\nabla u|^{p-2}\nabla u\nabla(u_n-u) - \int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n\nabla[u-T_k(u)]\\
&\qquad -\int_\Omega g_n(T_k(u))[u_n-T_k(u)] + \int_\Omega f[u_n-T_k(u)],
\end{aligned}$$

which implies, using that $\int_\Omega (g_n(u_n)-g_n(T_k(u)))[u_n-T_k(u)] \geq 0$ and (1.4),

$$\begin{aligned}
&\alpha\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n-u)\\
&\quad\leq -\int_\Omega a(x)|\nabla u|^{p-2}\nabla u\nabla(u_n-u) - \int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n\nabla[u-T_k(u)]\\
&\qquad -\int_\Omega g_n(T_k(u))[u_n-T_k(u)] + \int_\Omega f[u_n-T_k(u)].
\end{aligned}\tag{2.10}$$

In order to pass to the limit in the right hand side of (2.10), we observe firstly that

$$\lim_{n\to\infty}\int_\Omega a(x)|\nabla u|^{p-2}\nabla u\nabla(u_n-u)=0.$$

Moreover, (1.4) and the fact that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ (by Step 1) implies that the sequence $\{a(x)|\nabla u_n|^{p-1}\}$ is bounded in $L^{\frac{p}{p-1}}(\Omega)$ and so

$$\left|\int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n\nabla[u-T_k(u)]\right| \leq C_1\left[\int_\Omega |\nabla[u-T_k(u)]|^p\right]^{\frac{1}{p}} = \omega_1(k).$$

On the other hand,

$$\lim_{n\to\infty}\int_\Omega f[u_n-T_k(u)] = \int_\Omega f[u-T_k(u)] = \omega_2(k),$$

and, using that $|g_n(T_k(u))| \leq |T_k(u)|^r \leq k^r$, we also deduce that

$$\lim_{n\to\infty}\int_\Omega g_n(T_k(u))[u_n-T_k(u)] = \int_\Omega g(T_k(u))[u-T_k(u)] = \omega_3(k),$$

where $\omega_i(k)$, $i=1,2,3$, goes to 0 when k tends to infinite. Passing to the limit in (2.10), we obtain that

$$0 \leq \limsup_{n\to\infty}\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n-u) \leq \omega_1(k)+\omega_2(k)+\omega_3(k),$$

which implies, letting k tends to infinite,

$$\lim_{n\to\infty}\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n-u)=0. \tag{2.11}$$

As expected, the cases $p \geq 2$ and $p < 2$ are different. In the case $2 \leq p < N$, recalling (2.4), we deduce from (2.11) that the sequence ∇u_n converges strongly in $W_0^{1,p}(\Omega)$ to ∇u.

On the other hand, if $1 < p < 2$, using (2.5), it follows from (2.11) that

$$\lim_{n\to\infty} \int_\Omega \frac{|\nabla(u_n - u)|^2}{(1 + |\nabla u_n| + |\nabla u|)^{2-p}} \leq 0. \tag{2.12}$$

But Hölder inequality with exponents $(\frac{2}{p}, \frac{2}{2-p})$ and Step 1 imply that

$$\begin{aligned}
\int_\Omega |\nabla(u_n - u)|^p &= \int_\Omega \frac{|\nabla(u_n - u)|^p}{(1 + |\nabla u_n| + |\nabla u|)^{\frac{p(2-p)}{2}}} (1 + |\nabla u_n| + |\nabla u|)^{\frac{p(2-p)}{2}} \\
&\leq \left(\int_\Omega \frac{|\nabla(u_n - u)|^2}{(1 + |\nabla u_n| + |\nabla u|)^{(2-p)}}\right)^{\frac{p}{2}} \left(\int_\Omega (1 + |\nabla u_n| + |\nabla u|)^p\right)^{\frac{(2-p)}{2}} \\
&\leq \tilde{C}_R \left(\int_\Omega \frac{|\nabla(u_n - u)|^2}{(1 + |\nabla u_n| + |\nabla u|)^{(2-p)}}\right)^{\frac{p}{2}},
\end{aligned}$$

that is to say

$$\int_\Omega \frac{|\nabla(u_n - u)|^2}{(1 + |\nabla u_n| + |\nabla u|)^{(2-p)}} \geq \bar{C}_R \left(\int_\Omega |\nabla(u_n - u)|^p\right)^{\frac{2}{p}}. \tag{2.13}$$

Therefore, using (2.12), we deduce that

$$\lim_{n\to\infty} \left(\int_\Omega |\nabla(u_n - u)|^p\right)^{\frac{2}{p}} \leq 0,$$

which implies that the sequence $\{\nabla u_n\}$ converges strongly in $W_0^{1,p}(\Omega)$ to ∇u.

Finally, summarizing all the steps, we can pass to the limit in (2.8) and we conclude that

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla\varphi + \int_\Omega u\,|u|^{r-1}\varphi = \int_\Omega f\varphi, \quad \forall\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

□

Remark 2.8 We observe that, if u is a solution of (2.6) given by Theorem (2.7), then we can use $T_k(u)$ as a test function to deduce

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla T_k(u) + \int_\Omega u\,|u|^{r-1} T_k(u) = \int_\Omega f\, T_k(u).$$

Therefore, Levi Theorem (as k tends to ∞) gets

$$\int_\Omega a(x)|\nabla u|^p + \int_\Omega |u|^{r+1} = \int_\Omega f\, u,$$

which implies that it is possible to use u as test function, despite his unboundedness.

3 Existence results: problems with low summable data

In this section, we study existence results for the problem (1.1) when f belongs to $L^m(\Omega)$ with $1 < m < (p^*)'$. Here we follow [3,4]. Observe that in this case we do not have a variational formulation.

Theorem 3.1 *If* $f \in L^m(\Omega)$ *with* $\max(1, \frac{N}{N(p-1)+1}) < m < (p^*)' = \frac{Np}{pN+p-N}$, *then there exists a distributional solution* $u \in W_0^{1,(p-1)m^*}(\Omega)$ *of* (1.1), *in the sense that u satisfies*

$$\int_\Omega a(x)|\nabla u|^{p-2}\,\nabla u\,\nabla\varphi = \int_\Omega f\,\varphi, \quad \forall\varphi \in C_c^\infty(\Omega).$$

Remark 3.2 Observe that $m > \frac{N}{N(p-1)+1}$ implies that $(p-1)m^* > 1$ and $m < (p^\star)'$ implies that $(p-1)m^* < p$.

Proof We work by approximation to prove the existence of distributional solutions. By Theorem 2.1, there exists $u_n \in W_0^{1,p}(\Omega)$ weak solution of the problem

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u_n|^{p-2}\nabla u_n) = f_n(x), & \text{in } \Omega; \\ u_n = 0, & \text{on } \partial\Omega; \end{cases} \tag{3.1}$$

where f_n is a sequence of function in $L^\infty(\Omega)$ such that $f_n \to f$ in $L^m(\Omega)$ and $|f_n(x)| \le |f(x)|$ a.e. in Ω, (for example $f_n = \frac{f}{1+\frac{1}{n}|f|}$ or $f_n = T_n(f)$, with T_n given by (2.2)). Moreover, by Theorem 2.4, $u_n \in L^\infty(\Omega)$.

Our aim is to pass to the limit. Keeping this in mind, we split the proof into four steps.

STEP 1. *The sequence* $\{u_n\}$ *is bounded in* $L^{((p-1)m^*)^*}(\Omega)$. Following the same ideas of the proof of Theorem 2.3, we define $\theta = \frac{(p-1)m'}{pm'-p^*}$. We observe that $pm' - p^* > 0$, since $m < \frac{N}{p}$. Moreover, the fact that $m < (p^*)'$, implies that $\theta < 1$. Let ϵ be a strictly positive real number. The function $v_\epsilon = [(\epsilon + |u_n|)^{1-p(1-\theta)} - \epsilon^{1-p(1-\theta)}]\mathrm{sign}(u_n)$ is bounded since $1 - p(1-\theta) > 0$ (which is equivalent to $p > 1$). Thus, we can use v_ϵ as a test function in the weak formulation of (3.1) to deduce, using (1.4) and Sobolev embedding, that

$$\begin{aligned} C_{1,p}\left(\int_\Omega \{(\epsilon + |u_n|)^\theta - \epsilon^\theta\}^{p^*}\right)^{\frac{p}{p^*}} &\le C_{2,p}\int_\Omega \frac{a(x)|\nabla u_n|^p}{(\epsilon + |u_n|)^{p(1-\theta)}} \\ &\le \left(\int_\Omega |f|^m\right)^{\frac{1}{m}} \left(\int_\Omega \{(\epsilon + |u_n|)^{1-p(1-\theta)} - \epsilon^{1-p(1-\theta)}\}^{m'}\right)^{\frac{1}{m'}}, \end{aligned} \tag{3.2}$$

where $C_{i,p}$ denotes a strictly positive constant. Since, for every $n \in I\!N$, u_n belongs to $L^\infty(\Omega)$, the limit as ϵ tends to zero yields, thanks to Lebesgue theorem,

$$C_{1,p}\left(\int_\Omega |u_n|^{\theta\,p^*}\right)^{\frac{p}{p^*}} \le \left(\int_\Omega |f|^m\right)^{\frac{1}{m}} \left(\int_\Omega |u_n|^{[1-p(1-\theta)]m'}\right)^{\frac{1}{m'}}. \tag{3.3}$$

The fact that $m < \frac{N}{p}$, implies that $\frac{p}{p^*} > \frac{1}{m'}$. Furthermore, the choice of θ implies that $\theta\,p^* = [1 - p(1-\theta)]m'$ and that $\theta\,p^* = ((p-1)m^*)^*$. As a consequence we have proved that

$$C_{1,p}\left(\int_\Omega |u_n|^{((p-1)m^*)^*}\right)^{\frac{1}{m}-\frac{p}{N}} \le \left(\int_\Omega |f|^m\right)^{\frac{1}{m}}, \tag{3.4}$$

which gives us Step 1.

STEP 2. *The sequence* $\{u_n\}$ *is bounded in* $W_0^{1,(p-1)m^*}(\Omega)$. Firstly, we observe that Step 1, Fatou Lemma, (1.4) and (3.2) implies the boundedness, with respect to n, of

$$\int_\Omega \frac{|\nabla u_n|^p}{|u_n|^{p(1-\theta)}}.$$

Now we can estimate $\int_\Omega |\nabla u_n|^q$ with $q = (p-1)m^*$. Indeed we have

$$\int_\Omega |\nabla u_n|^q = \int_\Omega \frac{|\nabla u_n|^q}{|u_n|^{(1-\theta)\,q}} |u_n|^{(1-\theta)\,q} \leq \left(\int_\Omega \frac{|\nabla u_n|^p}{|u_n|^{p(1-\theta)}}\right)^{\frac{q}{p}} \left(\int_\Omega |u_n|^{(1-\theta)\,\frac{q\,p}{p-q}}\right)^{1-\frac{q}{p}}.$$

We observe that $(1-\theta)\,\frac{q\,p}{p-q} = q^*$, so the right hand side is bounded by Step 1. Then, the sequence $\{u_n\}$ is bounded by a positive constant R in $W_0^{1,(p-1)m^*}(\Omega)$.

As a consequence, there exists $u \in W_0^{1,(p-1)m^*}(\Omega)$ such that, up to a subsequence, u_n converges weakly to u in $W_0^{1,(p-1)m^*}(\Omega)$.

In what follows, C_R denotes (different) positive constants depending only on R, given by Step 2.

STEP 3. *Passing to the limit.* In order to pass to the limit in the weak formulation of (3.1), the weak convergence of u_n is not sufficient due to the nonlinearity of the principal part. We prove that the sequence $\{\nabla u_n\}$ is Cauchy in $L^r(\Omega)$ with a suitable $r > 1$. To this aim, we fix $1 < r < \min\{2, (p-1)m^*\}$ such that $\dfrac{r}{2-r}(2-p) < (p-1)m^*$. Observe that it is possible because, if $1 < p < 2$, then $2 - p < 1 < (p-1)m^*$. Next we take $T_k(u_n - u_m)$ as a test function to obtain, using (1.4),

$$\begin{aligned} &\alpha \int_{\{|u_n-u_m|\leq k\}} \{|\nabla u_n|^{p-2}\,\nabla u_n - |\nabla u_m|^{p-2}\,\nabla u_m\}\,\nabla(u_n-u_m) \\ &\leq \int_{\{|u_n-u_m|\leq k\}} \{\,a(x)|\nabla u_n|^{p-2}\,\nabla u_n - a(x)|\nabla u_m|^{p-2}\,\nabla u_m\}\,\nabla(u_n-u_m) \\ &\leq k \int_\Omega |f_n - f_m|. \end{aligned} \tag{3.5}$$

If $1 < p < 2$, using (2.5), we deduce from (3.5) that

$$\alpha\,\gamma_p \int_{\{|u_n-u_m|\leq k\}} \frac{|\nabla(u_n-u_m)|^2}{(1+|\nabla u_n| + |\nabla u_m|)^{2-p}} \leq k\,||f_n - f_m||_{L^1(\Omega)}.$$

Thanks to Step 2, we have (using Hölder inequality) that

$$\begin{aligned} &\int_{\{|u_n-u_m|\leq k\}} |\nabla(u_n-u_m)|^r \\ &= \int_{\{|u_n-u_m|\leq k\}} \frac{|\nabla(u_n-u_m)|^r}{(1+|\nabla u_n| + |\nabla u_m|)^{\frac{r(2-p)}{2}}}\,(1+|\nabla u_n| + |\nabla u_m|)^{\frac{r(2-p)}{2}} \\ &\leq \left(\int_{\{|u_n-u_m|\leq k\}} \frac{|\nabla(u_n-u_m)|^2}{(1+|\nabla u_n| + |\nabla u_m|)^{(2-p)}}\right)^{\frac{r}{2}} \left(\int_\Omega (1+|\nabla u_n| + |\nabla u_m|)^{\frac{r}{2-r}(2-p)}\right)^{1-\frac{r}{2}} \\ &\leq C_R \left(\frac{k}{\alpha\,\gamma_p}\right)^{r/2} ||f_n - f_m||_{L^1(\Omega)}^{r/2} = \epsilon^1_{n,m}, \end{aligned} \tag{3.6}$$

where $\epsilon^1_{n,m}$ tends to zero as n, m tend to infinite.

On the other hand, i.e., $p > 2$, using (2.4), we deduce from (3.5) that

$$\alpha\,\gamma_p \int_{\{|u_n-u_m|\leq k\}} |\nabla(u_n-u_m)|^p \leq k\,||f_n - f_m||_{L^1(\Omega)}.$$

Then, using Hölder inequality, we have

$$\int_{\{|u_n-u_m|\le k\}} |\nabla(u_n-u_m)|^r \le \left(\int_{\{|u_n-u_m|\le k\}} |\nabla(u_n-u_m)|^p\right)^{\frac{r}{p}} \mu(\Omega)^{1-\frac{r}{p}}$$
$$\le \left(\frac{k}{\alpha\,\gamma_p}\right)^{r/p} ||f_n-f_m||_{L^1(\Omega)}^{r/p}\, \mu(\Omega)^{1-\frac{r}{p}} = \epsilon_{n,m}^2, \tag{3.7}$$

where $\epsilon_{n,m}^2$ tends to zero as n, m tend to infinite.

In every case ($1 < p < 2$ or $p \ge 2$) we deduce, using (3.6) or (3.7), Hölder inequality and Step 2, that

$$\int_\Omega |\nabla(u_n-u_m)|^r = \int_{\{|u_n-u_m|\le k\}} |\nabla(u_n-u_m)|^r + \int_{\{|u_n-u_m|>k\}} |\nabla(u_n-u_m)|^r$$
$$\le \epsilon_{n,m}^i + \left(\int_\Omega |\nabla(u_n-u_m)|^{(p-1)m^*}\right)^{\frac{r}{(p-1)m^*}} \mu(\{|u_n-u_m|>k\})^{1-\frac{r}{(p-1)m^*}}$$
$$\le \epsilon_{n,m}^i + \tilde{C}_R\, \mu(\{|u_n-u_m|>k\})^{1-\frac{r}{(p-1)m^*}}.$$

Using that u_n converges strongly to u in $L^{(p-1)m^*}(\Omega)$, by Step 1 and Sobolev's embedding, we conclude from the last inequality that $\{\nabla u_n\}$ is a Cauchy sequence in $L^r(\Omega)$ ($r > 1$) and consequently, up to a subsequence, converges to ∇u a.e. in Ω. Since, by Step 1 and (1.4), $\{a(x)|\nabla u_n|^{p-1}\}$ is bounded in $L^{m^*}(\Omega)$ we deduce that $a(x)|\nabla u_n|^{p-2}\nabla u_n$ strongly converges to $a(x)|\nabla u|^{p-2}\nabla u$ in $(L^\sigma(\Omega))^N$, $1 \le \sigma < m^*$. Therefore, given $\varphi \in C_c^\infty(\Omega)$, we conclude that

$$\lim_{n\to\infty} \int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi = \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla\varphi.$$

To finish, we pass to the limit in the weak formulation of (3.1) to deduce that

$$\int_\Omega a(x)|\nabla u|^{p-2}\,\nabla u\,\nabla\varphi = \int_\Omega f\,\varphi, \quad \forall\varphi \in C_c^\infty(\Omega),$$

i.e., u is a distributional solution. □

3.1 Regularizing effect of a power lower order term on the summability of solutions

In this section we are going to study the unexpected regularizing effect on the existence of finite energy solutions of the problem:

$$\begin{cases} -\mathrm{div}(\,a(x)|\nabla u|^{p-2}\nabla u) + u\,|u|^{r-1} = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{3.8}$$

where $f \in L^m(\Omega)$ with

$$\frac{N}{N(p-1)+1} < m < (p^*)' = \frac{Np}{Np+p-N}.$$

Specifically we prove the following theorem (see [9]).

Theorem 3.3 *Assume that* $f \in L^m(\Omega)$ *with* $\max(1, \frac{N}{N(p-1)+1}) < m < (p^*)' = \frac{Np}{Np+p-N}$. *If* $r > \frac{1}{m-1}$, *then there exists a distributional solution* $u \in W_0^{1,p}(\Omega)$ *of* (3.8), *i.e.,* $|u|^r \in L^1(\Omega)$ *and* u *satisfies*

$$\int_\Omega a(x)|\nabla u|^{p-2}\,\nabla u\,\nabla\varphi + \int_\Omega |u|^{r-1}\,u\,\varphi = \int_\Omega f\,\varphi, \quad \forall\varphi\in C_c^\infty(\Omega).$$

Remark 3.4 Observe that $p > (p-1)m^*$ and compare with the result of Theorem 3.1 to see the regularizing effect of the lower order term.

Proof We consider the following approximated problem

$$\begin{cases} -\mathrm{div}(\,a(x)|\nabla u_n|^{p-2}\nabla u_n) + u_n\,|u_n|^{r-1} = f_n(x), & \text{in } \Omega;\\ u=0, & \text{on } \partial\Omega; \end{cases} \tag{3.9}$$

where f_n is a sequence of functions in $L^\infty(\Omega)$ such that $f_n \to f$ in $L^m(\Omega)$ and $|f_n(x)| \le |f(x)|$ a.e. in Ω. By Theorem 2.7, there exists $u_n \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega a(x)|\nabla u_n|^{p-2}\,\nabla u_n\,\nabla\varphi + \int_\Omega |u_n|^{r-1}\,u_n\,\varphi = \int_\Omega f_n\,\varphi, \quad \forall\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).$$

Moreover, for each $n\in I\!N$ fixed, we prove that u_n belongs to $L^\infty(\Omega)$. Indeed, consider the real function ψ_k defined in $I\!R$ by

$$\psi_k(s) = \begin{cases} -1, & \text{if } s < -k-1,\\ s+k, & \text{if } -k-1 \le s < -k,\\ 0, & \text{if} -k \le s \le k,\\ s-k, & \text{if } k < s \le k+1,\\ 1, & \text{if } k+1 < s. \end{cases}$$

Fixed $n\in I\!N$, we take $\varphi=\psi_k(u_n)\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ as a test function in (3.9) to deduce, dropping the positive term coming from the principal part, that

$$\int_\Omega |u_n|^r|\psi_k(u_n)| = \int_\Omega u_n|u_n|^{r-1}\,\psi_k(u_n) \le \int_\Omega f_n\psi_k(u_n) \le \int_\Omega |f_n||\psi_k(u_n)|,$$

that is

$$\int_{\{k\le|u_n|\}} [|u_n|^r - |f_n|]|\psi_k(u_n)| = \int_\Omega [|u_n|^r - |f_n|]|\psi_k(u_n)| \le 0.$$

Thus, if we take k such that $k^r = \|f_n\|_{L^\infty(\Omega)}$, then we have

$$0 \le \int_{\{\|f_n\|_{L^\infty(\Omega)}\le|u_n|^r\}} [|u_n|^r - |f_n|]|\psi_k(u_n)| \le 0.$$

Therefore

$$|u_n| \le \|f_n\|_{L^\infty(\Omega)}^{\frac{1}{r}}.$$

Consequently, it is possible to take powers of u_n as test function.

Next, we find a solution of (3.8) as a limit of the sequence $\{u_n\}$. We divide the proof into three steps.

Step 1. *The sequence* $\{u_n\}$ *is bounded in* $W_0^{1,p}(\Omega)$. We use $|u_n|^{\frac{r}{m'-1}}\,sign(u_n)$ as test function in (3.9). Firstly, we observe that $r > \frac{1}{m-1}$ implies that $\frac{r}{m'-1} > 1$ and thus $\frac{r}{m'-1} - 1 > 0$. Hence,

$$\int_\Omega a(x)|\nabla u_n|^p\,|u_n|^{(\frac{r}{m'-1}-1)} + \int_\Omega |u_n|^{rm} \le \|f\|_{L^m(\Omega)} \left(\int_\Omega |u_n|^{rm}\right)^{\frac{1}{m'}},$$

which implies, first of all,

$$\left(\int_\Omega |u_n|^{rm}\right)^{\frac{1}{m}} \leq \|f\|_{L^m(\Omega)} \tag{3.10}$$

and then

$$\alpha \int_\Omega |\nabla u_n|^p\, |u_n|^{(\frac{r}{m'-1}-1)} \leq \int_\Omega a(x)|\nabla u_n|^p\, |u_n|^{(\frac{r}{m'-1}-1)} \leq \|f\|^m_{L^m(\Omega)}. \tag{3.11}$$

Now, we write

$$\int_\Omega |\nabla u_n|^p = \int_{\{|u_n|\leq 1\}} |\nabla u_n|^p + \int_{\{1<|u_n|\}} |\nabla u_n|^p.$$

For the first integral of the right hand side we use the estimate

$$\int_{\{|u_n|\leq 1\}} |\nabla u_n|^p \leq \frac{1}{\alpha}\int_\Omega |f||T_1(u_n)| \leq \frac{1}{\alpha}\int_\Omega |f|.$$

For the second integral of the right hand side we use (3.11) to get

$$\begin{aligned}\int_{\{1<|u_n|\}} |\nabla u_n|^p &\leq \int_{\{1<|u_n|\}} |\nabla u_n|^p\, |u_n|^{(\frac{r}{m'-1}-1)}\\ &\leq \int_\Omega |\nabla u_n|^p\, |u_n|^{(\frac{r}{m'-1}-1)} \leq \frac{\|f\|^m_{L^m(\Omega)}}{\alpha}.\end{aligned}$$

Therefore, summarizing the above two estimates, we conclude

$$\alpha \int_\Omega |\nabla u_n|^p \leq \|f\|_{L^1(\Omega)} + \|f\|^m_{L^m(\Omega)}.$$

As a consequence, there exists $u \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, u_n converges weakly in $W_0^{1,p}(\Omega)$ to u.

STEP 2. *Convergence of the lower order term.* Observe that, by (3.10), the sequence $\{u_n\}$ is bounded in $L^{rm}(\Omega)$. Moreover, by Step 1 and using Sobolev embedding, u_n converges (up to subsequence) to u a.e. in Ω. Then, since $r < rm$, we deduce that the sequence $\{|u_n|^r\}$ converges strongly to $|u|^r$ in $L^\sigma(\Omega)$, $1 \leq \sigma < m$. Furthermore $|u|^r \in L^1(\Omega)$.

STEP 3. *Passing to the limit.* We easily check that we can pass to the limit in the principal part. Indeed, we observe that the use of $T_k(u_n - u_m)$ as a test function implies

$$\begin{aligned}&\int_\Omega (a(x)|\nabla u_n|^{p-2}\nabla u_n - a(x)|\nabla u_m|^{p-2}\nabla u_m)\nabla T_k(u_n-u_m)\\ &\quad+\int_\Omega (|u_n|^{r-1}u_n - |u_m|^{r-1}u_m)T_k(u_n-u_m) \leq k\int_\Omega |f_n - f_m|.\end{aligned}$$

Hence, dropping the positive term,

$$\alpha \int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)\nabla T_k(u_n-u_m) \leq k\int_\Omega |f_n - f_m|,$$

i.e., we have the inequality (3.5). Following the same arguments of Step 3 of the proof of Theorem 3.1, we prove that the sequence $\{\nabla u_n\}$ converges to ∇u a.e. in Ω. By Step 1, the sequence $\{|\nabla u_n|^{p-1}\}$ is bounded in $L^{\frac{p}{p-1}}(\Omega)$ and then using the almost everywhere convergence of the gradient we deduce that $|\nabla u_n|^{p-2}\nabla u_n$ strongly converges to $|\nabla u|^{p-2}\nabla u$ in $(L^1(\Omega))^N$. Therefore,

$$\lim_{n\to+\infty}\int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi=\int_\Omega a(x)|\nabla u|^{p-2}\nabla u\nabla\varphi,$$

for all $\varphi\in C_c^\infty(\Omega)$.

Using Step 2, we pass to the limit in the lower order term to deduce that

$$\lim_{n\to+\infty}\int_\Omega |u_n|^{r-1}u_n\,\varphi=\int_\Omega |u|^{r-1}u\,\varphi,$$

for all $\varphi\in C_c^\infty(\Omega)$ and so $u\in W_0^{1,p}(\Omega)$ satisfies

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u\nabla\varphi+\int_\Omega |u|^{r-1}u\,\varphi=\int_\Omega f\,\varphi,\ \ \forall\varphi\in C_c^\infty(\Omega),$$

which gives us the result. □

4 $W_0^{1,1}$ solutions

In this section we study the problem (1.1) when f belongs to $L^m(\Omega)$ with $1<m<(p^*)'$ and $(p-1)m^*=1$. Recall Theorem 3.1 where it is proved existence results when $(p-1)m^*>1$. The main difficulty of this case is due to the lack of compactness of bounded sequences, since $W_0^{1,1}(\Omega)$ is not reflexive. In this section, we follow [5].

Theorem 4.1 *Assume that $f\in L^m(\Omega)$ with $1<m=\frac{N}{N(p-1)+1}$, and that $1<p<2-\frac{1}{N}$. Then, there exists a distributional solution $u\in W_0^{1,1}(\Omega)$ of (1.1), i.e., u satisfies*

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u\nabla\varphi=\int_\Omega f\,\varphi,\ \ \forall\varphi\in C_c^\infty(\Omega).$$

Remark 4.2 Observe that $m=\frac{N}{N(p-1)+1}$ implies that $(p-1)m^*=1$.

Proof Following the same arguments used in the proof of Theorem 3.1, we consider $u_n\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, solutions of (3.1). Furthermore, we observe that the use of $T_k(u_n)$ as a test function yields, using (1.4), that

$$\alpha\int_\Omega|\nabla T_k(u_n)|^p\le k\int_\Omega|f|,\tag{4.1}$$

i.e., the sequence $\{T_k(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$.

As in the proof of Theorem 3.1, we are going to find a solution of (1.1) as a limit of the sequence $\{u_n\}$. Keeping this in mind, we divide the proof into several steps.

Step 1. *The sequence $\{u_n\}$ is bounded in $L^{\frac{N}{N-1}}(\Omega)$ and in $W_0^{1,1}(\Omega)$.* This is immediately deduced following the same arguments of Step 1 and Step 2 of the proof of Theorem 3.1 in the case $(p-1)m^*=1$.

As a consequence, there exists a subsequence, not relabelled, such that $\{u_n\}$ converges in $L^r(\Omega)$, with $1\le r<\frac{N}{N-1}$, and almost everywhere in Ω to a function u in $L^r(\Omega)$.

Step 2. *There exists Z such that $\{\nabla u_n\}$ converges to Z in measure.*

We define the function

$$g(t) = \frac{t}{1+|t|}, \quad \forall t \in I\!R,$$

and use $g(u_n - u_m)$ as a test function in the weak formulation of (3.1). Hence, we have

$$\int_\Omega (a(x)|\nabla u_n|^{p-2}\nabla u_n - a(x)|\nabla u_m|^{p-2}\nabla u_m)\nabla(u_n-u_m)\, g'(u_n-u_m) \\ \leq \int_\Omega (f_n - f_m) g(u_n-u_m),$$

which implies, using (1.4) and (2.5), that

$$\alpha\,\gamma_p \int_\Omega \frac{|\nabla(u_n-u_m)|^2}{(1+|\nabla u_n|+|\nabla u_m|)^{2-p}}\, g'(u_n-u_m) \leq \int_\Omega |f_n - f_m|.$$

Thus, using Hölder inequality, we have

$$\int_\Omega \frac{|\nabla(u_n-u_m)|}{1+|u_n-u_m|} = \int_\Omega \frac{|\nabla(u_n-u_m)|\sqrt{g'(u_n-u_m)}}{(1+|\nabla u_n|+|\nabla u_m|)^{1-\frac{p}{2}}}\,\frac{(1+|\nabla u_n|+|\nabla u_m|)^{1-\frac{p}{2}}}{(1+|u_n-u_m|)\sqrt{g'(u_n-u_m)}} \\ \leq \left(\int_\Omega \frac{|\nabla(u_n-u_m)|^2 g'(u_n-u_m)}{(1+|\nabla u_n|+|\nabla u_m|)^{2-p}}\right)^{\frac{1}{2}} \left(\int_\Omega \frac{(1+|\nabla u_n|+|\nabla u_m|)^{2-p}}{(1+|u_n-u_m|)^2 g'(u_n-u_m)}\right)^{\frac{1}{2}}$$

which implies that

$$\int_\Omega \frac{|\nabla(u_n-u_m)|}{1+|u_n-u_m|} \leq \left(\frac{1}{\alpha\,\gamma_p}\int_\Omega |f_n-f_m|\right)^{\frac{1}{2}} \left(\int_\Omega (1+|\nabla u_n|+|\nabla u_m|)^{2-p}\right)^{\frac{1}{2}}.$$

Since $\frac{1}{2-p} > 1$, from the a priori estimates given by Step 1, it follows that the last term is bounded. Then, using Hölder inequality again,

$$\int_\Omega |\nabla u_n - \nabla u_m|^{\frac{1}{2}} \leq \int_\Omega \frac{|\nabla(u_n-u_m)|^{\frac{1}{2}}}{(1+|u_n-u_m|)^{\frac{1}{2}}}(1+|u_n-u_m|)^{\frac{1}{2}} \\ \leq C_R \left(\frac{1}{\alpha\,\gamma_p}\int_\Omega |f_n-f_m|\right)^{\frac{1}{4}}.$$

Therefore, since the metric space $(L^{\frac{1}{2}}(\Omega), d(f,g) = \int_\Omega |f-g|^{\frac{1}{2}})$ is complete, there exists Z such that

$$\int_\Omega |\nabla u_n - Z|^{\frac{1}{2}} \to 0$$

which implies that

$$\nabla u_n(x) \text{converges in measure to } Z$$

and Step 2 is proved.

STEP 3. *The sequence* $\{\frac{\partial u_n}{\partial x_i}\}$ *is equi-integrable.* Following the same ideas of Step 1 and Step 2 of the proof of Theorem 3.1, we use $(|u_n|^{1-p(1-\theta)} - k^{1-p(1-\theta)})^+ \mathrm{sign}(u_n)$ as a test function in the weak formulation of (3.1) with $\theta = \frac{(p-1)m'}{pm'-p^*}$. Thanks to (1.4) and Step 1 (see (3.4) too), we have that

$$C_{3,p} \int_{\{k\le|u_n|\}} \frac{|\nabla u_n|^p}{|u_n|^{p(1-\theta)}} \le \left(\int_{\{k\le|u_n|\}} |f|^m\right)^{\frac{1}{m}} \left(\int_{\{k\le|u_n|\}} \{|u_n|^{1-p(1-\theta)} - k^{1-p(1-\theta)}\}^{m'}\right)^{\frac{1}{m'}}$$
$$\le \left(\int_{\{k\le|u_n|\}} |f|^m\right)^{\frac{1}{m}} \left(\int_{\{k\le|u_n|\}} |u_n|^{[1-p(1-\theta)]m'}\right)^{\frac{1}{m'}} \le C_{4,p} \left(\int_{\{k\le|u_n|\}} |f|^m\right)^{\frac{1}{m}}.$$

Consequently, by Hölder's inequality we have (using that $p'(1-\theta) = \frac{N}{N-1}$)

$$\int_{\{k\le|u_n|\}} |\nabla u_n| = \int_{\{k\le|u_n|\}} \frac{|\nabla u_n|}{|u_n|^{(1-\theta)}} |u_n|^{(1-\theta)}$$
$$\le \left(\int_{\{k\le|u_n|\}} \frac{|\nabla u_n|^p}{|u_n|^{p(1-\theta)}}\right)^{\frac{1}{p}} \left(\int_\Omega |u_n|^{p'(1-\theta)}\right)^{\frac{1}{p'}} \le C_{5,p} \left(\int_{\{k\le|u_n|\}} |f|^m\right)^{\frac{1}{m}},$$

where $C_{i,p}$ denotes a strictly positive constant. Thus, for every measurable subset E, thanks to (4.1) and the last inequality we have

$$\int_E \left|\frac{\partial u_n}{\partial x_i}\right| \le \int_E |\nabla u_n| \le \int_E |\nabla T_k(u_n)| + \int_{\{k\le|u_n|\}} |\nabla u_n|$$
$$\le \text{meas}(E)^{\frac{1}{p'}} \left(\frac{k}{\alpha}\|f\|_{L^1(\Omega)}\right)^{\frac{1}{p}} + C_{5,p} \left(\int_{\{k\le|u_n|\}} |f|^m\right)^{\frac{1}{m}}$$

which implies the result.

Step 4. *Passing to the limit.* As a consequence of Step 2 and Step 3 and using Vitali's Theorem, we deduce that

$$\nabla u_n \longrightarrow Z \quad \text{strongly in } (L^1(\Omega))^N.$$

Since $\frac{\partial u_n}{\partial x_i}$ is the distributional partial derivative of u_n, we have, for every $n \in \mathbb{N}$,

$$\int_\Omega \frac{\partial u_n}{\partial x_i} \varphi = -\int_\Omega u_n \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_c^\infty(\Omega).$$

We now pass to the limit in the above identities. We use that $\partial_i u_n$ converges to Z_i in $L^1(\Omega)$ and that, by Step 1, u_n converges to u in $L^1(\Omega)$. We obtain

$$\int_\Omega Z_i \varphi = -\int_\Omega u\, \partial_i \varphi, \quad \forall \varphi \in C_c^\infty(\Omega),$$

which implies that $Z_i = \partial_i u$, and then

$$\nabla u_n \longrightarrow \nabla u \quad \text{strongly in } (L^1(\Omega))^N.$$

Finally, summarizing all the steps, we can pass to the limit in the weak formulation of (3.1) to deduce that u satisfies

$$\int_\Omega a(x)|\nabla u|^{p-2} \nabla u \nabla \varphi = \int_\Omega f \varphi, \quad \forall \varphi \in C_c^\infty(\Omega),$$

which gives us the result. □

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Appendix A

For convenience of the reader we have collected in this section all the necessary prerequisites used (in particular) in Sect. 2.

Given a measurable function $f : \Omega \to I\!R$, we use the following notation

$$g(k) := \int_\Omega |G_k(f)| \quad \text{and} \quad A_k := \{x \in \Omega : |f(x)| > k\}, \quad k > 0.$$

Lemma 5.1 *If $f \in L^1(\Omega)$, then the function g is differentiable a.e. and $g'(k) = -\mu\{|u_n| > k\}$.*

Proof Firstly, we observe that it is sufficient to prove that the function

$$\tilde{g}(k) = \int_{\{f-k>0\}} (f-k), \quad k > 0,$$

is differentiable a.e. with $\tilde{g}'(k) = -\mu(A_{k,+})$, where $A_{k,+} = \{x \in \Omega : f(x) - k > 0\}$.

We observe that the function $\tilde{g}$ is monotone and then $\tilde{g}$ is differentiable a.e. Next we check its derivative. Let h be a positive number, then

$$\begin{aligned}\frac{\tilde{g}(k+h) - \tilde{g}(k)}{h} &= \frac{1}{h}\left(\int_{A_{k+h,+}} (f-k-h) - \int_{A_{k,+}} (f-k)\right)\\ &= \frac{1}{h}\left(\int_{A_{k+h,+}} -h - \int_{\{k<f\le k+h\}} (f-k)\right)\\ &= -\int_\Omega \chi_{\{f>k+h\}} - \frac{1}{h}\int_{\{k<f\le k+h\}} (f-k).\end{aligned}$$

The fact that

$$0 \le \int_{\{k<f\le k+h\}} (f-k) \le \int_{\{k<f\le k+h\}} h,$$

implies

$$0 \le \frac{1}{h}\int_{\{k<f\le k+h\}} (f-k) \le \int_\Omega \chi_{\{k<f\le k+h\}}$$

and then the term $\int_{\{k<f\le k+h\}} (f-k)$ converges to 0 as $h \to 0^+$. As a consequence,

$$\lim_{h\to 0} \frac{\tilde{g}(k+h) - \tilde{g}(k)}{h} = -\lim_{h\to 0}\int_\Omega \chi_{\{f>k+h\}} = -\mu(\{f > k\}) = -\mu(A_{k,+})$$

which gives us the result. □

Lemma 5.2 *Assume that $f \in L^1(\Omega)$. If there exist $\alpha > 1$ and $B > 0$ such that the function g satisfies*

$$g(k) \le B\mu\{|u_n| > k\}^\alpha, \quad \textit{for every } k > 0,$$

then $f \in L^\infty(\Omega)$. Moreover, there exists a positive constant $\gamma = \gamma(\alpha, \Omega)$ such that

$$\|f\|_{L^\infty(\Omega)} \le B\gamma.$$

Proof Using Lemma 5.1 one has

$$g(k) \leq B[-g'(k)]^{\alpha}, \quad \text{for every } k > 0,$$

that is,

$$g'(k)[g(k)]^{-\frac{1}{\alpha}} \leq -\frac{1}{B^{\frac{1}{\alpha}}}, \quad \text{for every } k > 0. \tag{5.1}$$

Integrating this inequality on $(0, k)$ we get

$$-\left(1-\frac{1}{\alpha}\right)\frac{k}{B^{\frac{1}{\alpha}}} \geq g(k)^{1-\frac{1}{\alpha}} - g(0)^{1-\frac{1}{\alpha}} = g(k)^{1-\frac{1}{\alpha}} - \|f\|_{L^1(\Omega)}^{1-\frac{1}{\alpha}}.$$

Consequently,

$$g(k)^{1-\frac{1}{\alpha}} \leq \|f\|_{L^1(\Omega)}^{1-\frac{1}{\alpha}} - \left(1-\frac{1}{\alpha}\right)\frac{k}{B^{\frac{1}{\alpha}}}, \quad \forall k > 0. \tag{5.2}$$

In particular, (5.2) holds true for $k_0 = \frac{B^{\frac{1}{\alpha}}\|f\|_{L^1(\Omega)}^{1-\frac{1}{\alpha}}}{1-\frac{1}{\alpha}}$. This implies that $g(k_0) = 0$ and as a consequence

$$|f(x)| \leq k_0 = \frac{B^{\frac{1}{\alpha}}\|f\|_{L^1(\Omega)}^{1-\frac{1}{\alpha}}}{1-\frac{1}{\alpha}} \leq \frac{B^{\frac{1}{\alpha}}\|f\|_{L^\infty(\Omega)}^{1-\frac{1}{\alpha}}\mu(\Omega)^{1-\frac{1}{\alpha}}}{1-\frac{1}{\alpha}}.$$

Then, we deduce that

$$\|f\|_{L^\infty(\Omega)} \leq \left(1-\frac{1}{\alpha}\right)^{-\alpha}\mu(\Omega)^{\alpha-1}B.$$

□

Appendix B

Here we give just an idea about the proof of (2.4), (2.5): we only work with the simple case $N = 1$; the general case can be found in http://www.uam.es/personal_pdi/ciencias/ireneo/ALMERIA1.

- If $1 < p < 2$, then $\frac{1}{p-1} > 1$; so that the local Lipschitz continuity of the real function $s|s|^{\frac{1}{p-1}-1}$ says

$$\left||a|^{\frac{1}{p-1}-1}a - |b|^{\frac{1}{p-1}-1}b\right| \leq \frac{1}{p-1}|a-b|(|a|+|b|)^{\frac{2-p}{p-1}}$$
$$\leq \frac{1}{p-1}|a-b|2^{\frac{2-p}{p-1}}\left(|a|^{\frac{1}{p-1}}+|b|^{\frac{1}{p-1}}\right)^{2-p}.$$

  Define $x = |a|^{\frac{1}{p-1}-1}a, \ y = |b|^{\frac{1}{p-1}-1}b$. If $a > b$, we have

$$\frac{(x-y)^2}{(1+|x|+|y|)^{2-p}} \leq \frac{2^{\frac{2-p}{p-1}}}{p-1}(|x|^{p-2}x - |y|^{p-2}y)(x-y), \quad x > y.$$

  If $a < b$, the symmetry implies the same inequality.

- If $p > 2$, thanks to the symmetry, we prove the inequality

$$|x - y|^p \leq (|x|^{p-2}x - |y|^{p-2}y)(x - y),$$

in the case

$$(x - y)^p \leq (x^{p-1} - y^{p-1})(x - y), \quad x > y,$$

which is equivalent to the positivity of

$$\psi(x) = (x^{p-1} - y^{p-1}) - (x - y)^{p-1} \geq 0, \quad x > y.$$

The function $\psi(x)$ is positive, since it is increasing and $\psi(y) = 0$.

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