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Geometric problems in PDEs with applications to mathematical physics

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Abstract Many questions that arise naturally in the study of the PDEs of mathematical physics are of a strongly geometric or topological nature. In this paper we will provide a gentle invitation to this vibrant area of research by presenting, in a completely non-technical manner, some recent results of the author in this direction.

Keywords Euler equation · Eigenvalues · Elliptic PDE · Wave equations · Level sets

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1 Introduction

There are many high-profile open problems in the analysis of the PDEs of mathematical physics that ultimately boil down to assertions of a strongly geometric or topological nature. One feature that makes these problems both very difficult and extremely appealing is that, despite having received much attention, there is not a standard set of techniques that one can routinely resort to in order to attack these problems. More often than not, the reason is that they involve extracting topological information from a PDE, which is very hard. Roughly speaking, this is because the "softness" of the techniques of differential topology and dynamical systems makes them unsuitable to deal with PDEs, while controlling topological objects using the quantitative, more rigid methods of analysis is often awkward. The very nature of these problems makes them strongly interdisciplinary, so successful approaches require finely tailored combinations of ideas and techniques coming from different branches of mathematics (analysis, geometry and topology), often interspersed with some physical intuition.

It should come as no surprise that the study of physical problems leads to a rich interaction between different areas of mathematics, for the origin of very diverse mathematical subjects is deeply rooted in physics. Just to name a few, we can mention PDEs, where the Laplace,

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heat, wave and Schrödinger equations and their nonlinear analogs constitute the core of any course on this matter; harmonic analysis, which is born out of the desire to understand the heat equation; dynamical systems, which, just as symplectic and to some extent Riemannian geometry, are closely related to classical mechanics and have spectacular connections with optics and quantum mechanics via high-frequency limits; spectral theory, which underwent an enormous development with the advent of quantum mechanics,…Furthermore, the idea that the rigorous solution of important problems in physics leads to first-rate mathematics is still applicable, as can be inferred from the fact that two of the Millenium Problems established by the Clay Institute (namely, the (in)stability of the Navier–Stokes equation and the existence of a mass gap for Yang–Mills quantum field theories) are based on the interaction between mathematics and physics.

Our goal in this paper is to discuss a few geometric problems in PDEs, which arise as diverse contexts as fluid mechanics, quantum mechanics, electrostatics and cosmology, and discuss some of the ideas that have been developed to solve them. To narrow the scope of the paper, I will concentrate on developments where I have been directly involved. Furthermore, to make the paper accessible to a broader audience, I will strive not to get bogged down in the technical details of the definitions, statements and proofs, which can be found in the references, but to discuss some of the underlying ideas and to present, in a completely non-technical manner, the context in which these problems arise. Hence, this paper should be understood as a gentle if slightly inaccurate invitation to the fascinating area of research at the boundary between analysis and geometry that appears when one analyzes in detail many problems in mathematical physics.

2 Vortex lines and vortex tubes in stationary fluids

Let us consider the Euler equation in $\mathbb{R}^3$,

$$\partial_t u + (u \cdot \nabla)u = -\nabla P, \qquad \operatorname{div} u = 0,$$

which models the behavior of an inviscid incompressible fluid in space. In this equation the unknowns are the velocity field $u(x, t)$, which is a time-dependent vector field, and the pressure function $P(x, t)$.

In this section we will be concerned with the existence of *steady* (that is, time-independent) solutions to the Euler equation in $\mathbb{R}^3$. As is well known, a vector field $u(x)$ is a steady solution of the Euler equation if and only if it satisfies the (quite unmanageable) system of PDEs

$$u \times \omega = \nabla B, \qquad \operatorname{div} u = 0, \tag{2.1}$$

where $\omega := \operatorname{curl} u$ is the *vorticity* and

$$B := P + \frac{1}{2}|u|^2$$

is the Bernoulli function.

Let us recall that a *vortex line* in a steady fluid is just an integral curve of the vorticity, that is, the solution $x(t)$ to the ODE

$$\dot{x} = \omega(x)$$

with some initial condition $x(0) = x_0$. A major conjecture in the field, popularized by the works of Arnold [2] and Moffatt [25] in the 1960s, was that any link (i.e., a set of smooth disjoint curves in space, possibly knotted and linked in a nontrivial fashion) could be realized as a set of vortex lines of a steady solution to the Euler equation in $\mathbb{R}^3$, up to a diffeomorphism. In physics, this is of direct interest because knotted vortex lines play a central role in our understanding of turbulent behavior through the theory of Lagrangian turbulence [28].

An influential argument supporting the validity of this conjecture is due to Moffatt [26], who resorts to an auxiliary magnetohydrodynamics system of PDEs to connect this conjecture with the phenomenon of magnetic relaxation, discovered by the physicists Zakharov and Zeldovich. However, making Moffatt's program precise seems to be way out of reach despite the recent results on magnetic relaxation [5], and remains a fascinating open problem. Attempts at attacking this conjecture and other related problems that do not rely on magnetic relaxation have not been very successful either. In short, the reason is that the various approaches were developed either from a purely topological or from a purely analytic standpoint, as illustrated by the variational approach developed in [24] to obtain some control on axisymmetric solutions or the methods of contact topology employed in [19].

Although the experts had long held a firm belief in the validity of this conjecture because of its physical interpretation, it is not hard to see that the problem is rather subtle. To see this, let us go back to Eq. (2.1). The celebrated structure theorem of Arnold [2,3] asserts that if u is a steady Euler flow for which B is not constant, under mild technical assumptions the vortex lines must be tangent to a family of invariant tori, cylinders and planes that defines a rigid geometric structure closely related to that of an integrable Hamiltonian system with two degrees of freedom. For our purposes, the content of Arnold's theorem is that when the Bernoulli function is not constant (i.e., the velocity and the vorticity are not everywhere collinear), there is not much freedom in choosing how the vortex lines and possible vortex tubes can sit in space, which severely constrains the kind of knots that can be vortex lines in this case. Consequently, Arnold conjectured that the right class of steady Euler flows one should consider to prove the conjecture were the (strong) *Beltrami fields*, which satisfy

$$\operatorname{curl} u = \lambda u \tag{2.2}$$

for some nonzero constant λ and whose associated Bernoulli function is constant. It is remarkable that Beltrami fields, the best known examples of which are the ABC flows [9], have found many other applications in fluid mechanics, notably as tools to attack the Onsager conjecture [7,8]. Notice that Beltrami fields are automatically analytic because they satisfy the elliptic equation $\Delta u + \lambda^2 u = 0$.

Following Arnold, in joint work with Peralta-Salas, we developed an approach using Beltrami fields that enabled us to arrive at the following statement, which proves the above conjecture [16]:

Theorem 2.1 *Let $L \subset \mathbb{R}^3$ be a possibly unbounded, locally finite link. Then for any real constant $\lambda \neq 0$ one can transform L using a C^∞ diffeomorphism Φ of $\mathbb{R}^3$ arbitrarily close to the identity in any C^k norm, so that $\Phi(L)$ is a set of vortex (or stream) lines of a Beltrami field u, which satisfies $\operatorname{curl} u = \lambda u$ in $\mathbb{R}^3$.*

The strategy that leads to the proof of this result is surprisingly versatile and can be applied, mutatis mutandis, in many different contexts. Leaving technicalities aside and assuming the link is connected (i.e., a knot) for simplicity, the proof is carried out in three steps. The first step is the construction of a local Beltrami field. For this, one takes the knot L and perturbs it a little if necessary to make it analytic. One then embeds the perturbed knot into an analytic two-dimensional strip and constructs a vector field on this strip having the latter knot as a stable hyperbolic limit cycle. Next one extends this vector field on the strip to a local Beltrami field v, defined in a tubular neighborhood of the cycle. For this one develops a rather delicate

Cauchy–Kowalewski theorem for the curl operator, which presents a couple of unexpected features reflecting the fact that curl does not possess any non-characteristic surfaces, and uses v as a Cauchy datum. This construction is relatively involved but offers us an important boon: by construction, v has a periodic orbit diffeomorphic to the knot L, which can be shown to be hyperbolic. This shows that the periodic orbit is preserved under small perturbations up a diffeomorphism, thereby taking care of the second step of the proof. To conclude, in the third step one shows that there exists a global Beltrami field u that approximates the local solution v by proving a suitable global approximation theorem for the curl operator. Since the global solution u must then have a vortex line diffeomorphic to L because of the robustness of the limit cycle of v, the theorem then follows after taking care of the necessary technical details.

A well-known geometric problem in steady Euler flows was the existence of steady solutions to the Euler equation having thin vortex tubes of arbitrarily complicated topology, where "thin" means that the width of tube is much smaller than its length. Let us recall that a (closed) *vortex tube* in a steady fluid is domain in $\mathbb{R}^3$ that is the union of vortex lines and whose boundary is an embedded torus. Obviously the boundary of the tube is an invariant torus of the vorticity.

This was one of the oldest conjectures in fluid mechanics, as it was formulated by Lord Kelvin in 1875 [31] and the original evidence for it came from a theoretical argument of Helmholtz and some recorded experimental observations of Maxwell. It should be emphasized that the interest of this conjecture is not merely academic; in fact, recent experimental results have shown [23] that thin vortex tubes knotted and linked in complicated ways can be created in the laboratory using real fluids and cleverly designed hydrofoils. In mathematics, vortex tubes have long played a role in the construction of possible blow-up scenarios for the Euler equation [6,28], since a change in the topology of the tubes under time evolution is known to signal the onset of a singularity.

The theorem that establishes Kelvin's conjecture can be stated as follows [17]:

Theorem 2.2 *Given a collection of possibly knotted and linked closed curves $\gamma_1, \dots, \gamma_N$ in $\mathbb{R}^3$, let us consider the tubes of thickness ϵ defined by these curves,*

$$\mathcal{T}_\epsilon(\gamma_j) := \left\{x \in \mathbb{R}^3 : \operatorname{dist}(x, \gamma_j) = \epsilon\right\}.$$

If the thickness ϵ is small enough, one can transform the collection of pairwise disjoint thin tubes

$$\mathcal{T}_\epsilon(\gamma_1), \dots, \mathcal{T}_\epsilon(\gamma_N)$$

by a diffeomorphism Φ of $\mathbb{R}^3$, arbitrarily close to the identity in any C^m norm, so that

$$\Phi[\mathcal{T}_\epsilon(\gamma_1)], \dots, \Phi[\mathcal{T}_\epsilon(\gamma_N)]$$

are vortex tubes of a Beltrami field u, which satisfies the equation

$$\operatorname{curl} u = \lambda u$$

in $\mathbb{R}^3$ for some nonzero constant λ. Moreover, the field u decays at infinity as

$$|D^j u(x)| < \frac{C_j}{|x|} \quad \textit{for all } j.$$

Ultimately, this is a result on the existence of invariant tori (with controlled geometry) in steady solutions to the Euler equation in $\mathbb{R}^3$, which combines ideas from partial differential equations and dynamical systems. For concreteness, to explain the gist of the proof we will concentrate on constructing a solution where we are prescribing just one vortex tube $\mathcal{T}_\epsilon \equiv \mathcal{T}_\epsilon(\gamma)$. There is no loss of generality in assuming that the curve γ is analytic. The heart of the matter is that we do not have any standard tools to construct solutions of the Eq. (2.2) in $\mathbb{R}^3$ that contain the prescribed invariant torus $\mathcal{T}_\epsilon$. What might be expected to be easier, of course, is to construct a solution v of (2.2) only in the interior of the tube with the condition that u be tangent to $\partial\mathcal{T}_\epsilon$, since this is some kind of boundary value problem; indeed, this is precisely the first step of the proof. The connection between the solution v in the tube and the global problem considered in the theorem is that we will show that, given v, one can always construct a solution u in the whole space $\mathbb{R}^3$ which is close to v in $C^k(\mathcal{T}_\epsilon)$. Hence, in order to ensure that the solution u has a vortex tube close to $\mathcal{T}_\epsilon$ we need to prove a result on the preservation of the invariant torus $\partial\mathcal{T}_\epsilon$ under suitably small perturbations of v. This is an extremely delicate point, since the verification of the (KAM-type) nondegeneracy conditions for the preservation of these tori hinges on a fine analysis of the solutions to the boundary problem. Furthermore, that we can actually satisfy the nondegeneracy conditions is not something we infer from a choice of boundary data, but something we extract from the equation using in a crucial way the geometry of the tube $\mathcal{T}_\epsilon$ (and, in particular, the smallness of ϵ). Unfortunately, making these ideas precise is a very technical matter that would take us too far along the road to topological fluid mechanics.

3 Eigenvalue distribution for spin chains

In the analysis of vortex lines and vortex tubes in fluid mechanics, we have seen how innovative mathematical reasoning can lead to results that are relevant for the analysis of physical problems. In this section we shall see this principle at work in a context that is technically lighter: the study of the distribution of the eigenvalues for spin chains.

More precisely, the problem that we will consider now is the asymptotic distribution of the eigenvalues of Haldane–Shastry spin chains. This is a family of finite-dimensional quantum models with long-range interactions which has received considerable attention because they are the simplest models in condensed matter physics exhibiting fractional statistics. Historically, the Haldane–Shastry chains were introduced as a simplified version of the one-dimensional Hubbard model of superconductivity with long-range hopping. Spin chains of Haldane–Shastry type also appear in many other areas of current interest such as quantum chaos, conformal field theory or the AdS/CFT correspondence. Throughout this section, we refer to [12] and references therein for details.

In the simplest case, a spin chain with N "sites" (or spins) is modeled by a Hamiltonian H_N, which is a self-adjoint matrix acting on the complex vector space $\mathcal{V}_N := (\mathbb{C}^2)^{\otimes N}$ of dimension 2^N. The easiest example of a Haldane–Shastry N-site Hamiltonian is [21,29]

$$H_N := \sum_{1\leq i<j\leq N} \frac{1-S_{ij}}{\sin^2(\frac{\pi(i-j)}{N})},$$

where S_{ij} is the operator on $\mathcal{V}_N$ that acts on the canonical basis of this space by permuting the ith and jth spins. We will not need any specific details about these Hamiltonians here, so again we refer to [12] for details. In general, these Hamiltonians take the form

$$H_N = \sum_{1\leq i\neq j\leq N} c_{ij} T_{ij},$$

where c_{ij} are real constants and T_{ij} are self-adjoint matrices whose action is only nontrivial on the ith and jth factors of the tensor product space $\mathcal{V}_N$.

It is by no means obvious in which sense Haldane–Shastry spin chains can be regarded from a geometric view point. The reason is, however, of fundamental importance, and it is related to the fact that an arbitrary choice of constants c_{ij} and matrices T_{ij} with the aforementioned structure does not qualify as a Haldane–Shastry spin chain. In fact, as was shown by Polychronakos, the key feature of Haldane–Shastry spin chains is that they arise as semiclassical limits of a certain class of matrix-valued Schrödinger operators known as Calogero–Moser–Sutherland (spin) models, which are of the form

$$\tilde{H}_{N,\hbar} := -\hbar^{-2}\Delta + V(x).$$

Intuitively speaking, as $\hbar \searrow 0$, the lowest eigenvalues of these Hamiltonians correspond to quantum-mechanical states in which the particle is "frozen" at the global minimum x_0 of the potential, the eigenvalues being related to the eigenvalues of the matrix $V(x_0)$. Omitting the details, the key property of Haldane–Shastry chains is that they are essentially a rescaling of the matrix $V(x_0)$. In turn, the geometry is now recovered from the fact that Calogero–Sutherland–Moser Hamiltonians closely related to reductions under isometry subgroups of the Laplace–Beltrami operator on symmetric spaces. This is the reason for the existence of a rich algebraic structure that simplifies the analysis of the spectrum of Haldane–Shastry chains, and allows to classify these chains using root systems.

In the study of the eigenvalues of spin chains, the main object of interest is the behavior of the eigenvalues as the number of sites N tends to infinity, which is the so-called thermodynamic limit. The underlying algebraic structure of these Hamiltonians leads to explicit but very unmanageable formulas for the spectrum of these chains, where the eigenvalues are typically presented in terms of complicated objects arising in number theory. A well-known conjecture on spin chains of Haldane–Shastry type stated that their level density becomes Gaussian as the number of sites tends to infinity. Although this conjecture had been numerically verified for all chains of HS type whose spectrum had been computed in closed form, a rigorous proof thereof was lacking.

In joint work with Finkel and González-López, we settled the conjecture in the affirmative for spin chains associated with the root system of type A [10,11], which is both the simplest and the most studied case. The proof essentially relies on two key properties of these chains, namely the characterization of the eigenvalues in terms of motifs, which can be seen as a reflection of the underlying geometric structure of these models, and estimates for Fourier series associated with the dispersion relation of the models.

Our result has implications in connection with two fundamental conjectures in the theory of quantum chaos that we shall now discuss. The first of these conjectures, due to Berry and Tabor, asserts that the probability density of spacings between consecutive levels in the spectrum of a quantum system whose classical analog is integrable follows Poisson's law. The second conjecture, formulated by Bohigas, Giannoni and Schmidt, posits that for a chaotic quantum system this density is instead given by Wigner's surmise. Using that the asymptotic distribution of the eigenvalues is Gaussian one readily obtains that the spacing density follows a remarkable distribution which is neither of Poisson's nor of Wigner's type.

Moreover, in addition to its intrinsic mathematical interest, the Gaussian asymptotics have direct physical applicability, as they imply that one does not need to work one's way through pages of algebraic minutiae using the awkward closed formulas for the eigenvalues. This is particularly relevant in the thermodynamic (or large-N) limit, which is not readily extracted from the exact expression for the eigenvalues. Fortunately, the methods we used enabled us to carry out a general and very detailed study of the thermodynamics of Haldane–Shastry chains, which generalizes and puts on a mathematically rigorous footing many previous works on this subject [12].

4 Wave equations in cosmology

The study of wave equations in geometric contexts is a many-faceted subject with many connections with physics. A particularly interesting problem is the analysis of the nonlinear wave equation

$$\Box_g \phi - \mu\phi = F(\phi, \nabla\phi), \tag{4.1}$$

in spaces whose geometry at infinity is close to that of the n-dimensional anti-de Sitter (AdS) space. These spaces are called asymptotically AdS and roughly correspond to Lorentzian manifolds whose metric is asymptotic (in a precise sense) as $r \to \infty$ to the AdS metric

$$g_{\mathrm{AdS}} = -K^{-2}\cosh^2(Kr)\,dt^2 + dr^2 + K^{-2}\sinh^2(Kr)\,g_{\mathbb{S}^{n-2}},$$

where $-K$ is the curvature of AdS. Here $t \in \mathbb{R}$ is the time variable, $r \in \mathbb{R}^+$ is the radial coordinate and $g_{\mathbb{S}^{n-2}}$ is the canonical metric on the $(n-2)$-sphere. As is well known, AdS is the foremost solution of the vacuum Einstein equation with negative cosmological constant.

To analyze the wave equation in AdS, it is convenient to replace the radial coordinate r by a certain function thereof, x, which takes values in $(0, 1]$ and is defined through the relation

$$x := \left(2\cosh\frac{r}{2} - 1\right)^{-\frac{1}{2}}.$$

In terms of this variable, the wave equation (4.1) with $F = 0$ takes the form

$$-\partial_t^2\phi + \partial_x^2\phi - \frac{n-2}{x}\,\partial_x\phi + \Delta_\theta\phi - \frac{\mu}{x^2}\,\phi + \cdots = 0,$$

where Δ_θ is the angular Laplacian (in $\mathbb{S}^{n-2}$) and the dots stand for terms that are smaller, in a certain sense, for x close to zero. The coefficients of the equation are obviously singular at $x = 0$, which is directly related to the fact that AdS is not globally hyperbolic because null geodesics escape to infinity for finite values of their affine parameter. This has the effect that the evolution under the wave equation in AdS of smooth, compactly supported initial data does not remain compactly supported in space for all values of time. From a practical point of view, this key feature of the equation forces us to consider not only initial conditions, but also asymptotic boundary conditions of the form

$$x^{\alpha - \frac{n-1}{2}}\phi|_{x=0} = f(t, \theta), \tag{4.2}$$

with

$$\alpha := \left[\left(\frac{n-1}{2}\right)^2 + \mu\right]^{1/2}.$$

The motivation to consider the nonlinear wave equation in an asymptotically AdS background with initial and asymptotic boundary conditions comes from the AdS/CFT correspondence in string theory [32]. This is a conjectural relation which posits that a gravitational field on a space endowed with an Einstein metric close to the AdS metric at infinity can be recovered from a gauge field living on the conformal boundary of the manifold. The gravitational field is modeled using some PDE of hyperbolic character in the manifold (typically, the Einstein equation) and the gauge field at conformal infinity plays the role of boundary datum through a relation analogous to (4.2). Since, in harmonic coordinates, the Einstein equation reduces to a nonlinear wave equation, at the heart of the AdS/CFT correspondence lies a boundary-value problem closely related to the one we have just stated. Indeed, the *holographic principle* asserts that the boundary data (which here, in the context of scalar wave equations, would be the function f), defined on defined on an $(n-1)$-dimensional boundary, propagates through a suitable n-manifold (which is referred to as the *bulk*) to determine the field (here ϕ) via a locally well-posed problem.

It is important at this stage to note that most rigorous results related to the holographic principle have been obtained for the elliptic counterparts of these equations, where there are important contributions due to Sullivan [30], Anderson–Schoen [1] and Graham–Lee [20], among others. Linear wave equations in asymptotically AdS backgrounds have also been considered by various authors.

In joint work with Kamran, we have recently proved that the initial-boundary value problem at infinity for the nonlinear wave equation in an asymptotically AdS manifold is well posed in a certain scale of Sobolev spaces adapted to the geometry of the space-time. In the language of physics, this can be rephrased as saying that the holographic prescription problem is well posed for scalar fields. The prescription has the physically critical properties of being *fully holographic* and *causal*, which essentially means that, for trivial initial conditions and compactly supported datum f on the conformal boundary, the field ϕ is purely controlled by f and is identically zero for all times below the support of this function [13].

The proof is based on suitable energy estimates to prove that, given (compatible) initial conditions and a datum on the conformal boundary f, there is a unique solution. In the case of the linear wave equation, this solution is defined for all time. Estimates are given in terms of the norms of the data in suitable Sobolev spaces that are defined in a way that compensates for the singular behavior of the metric at the conformal boundary. An interesting feature is that, due to the form of the energy, one does not simply get the usual weighted Sobolev spaces, but rather a twisted version thereof, involving both a weight vanishing at the conformal boundary of the manifold and twisted derivatives, where the twist factor conjugating the derivative is directly related to the geometry of the asymptotically AdS space at infinity through the "renormalized energy" considered by Breitenlohner and Freedman [4]. One also shows that these Sobolev-type estimates imply the pointwise decay of the solutions by proving suitable Sobolev embedding theorems. An important consequence of this result is that it paves the way towards a Lorentzian analog of the celebrated Graham–Lee theorem [20].

5 Critical points and level sets in potential theory

Let Ω be a bounded open subset of $\mathbb{R}^n$ $(n \geq 3)$ with a smooth connected boundary and consider the following boundary value problem:

$$\Delta u = 0 \qquad \text{in } \mathbb{R}^n \backslash \overline{\Omega}\,, \tag{5.1a}$$

$$u|_{\partial\Omega} = 1\,, \qquad \lim_{|x|\to\infty} u(x) = 0\,. \tag{5.1b}$$

This is a classical topic in mathematical physics and potential theory, where $\partial\Omega$ is interpreted as a grounded conducting surface and u represents the electric potential.

The study of the qualitative properties of the solutions to this problem was actually pioneered by Faraday and Maxwell in the XIX century. In physics, the level sets of the function u are equipotential surfaces, whereas its critical points correspond to the equilibrium positions for the motion of a charged particle in presence of the conducting surface $\partial\Omega$. The relevance of Faraday's lines of force, which are simply the integral curves of the gradient vector field

$-\nabla u$, was unveiled by Maxwell, which used them to obtain the first hints of what is nowadays known as Morse theory.

It is easy to prove that if the domain Ω is star-shaped, the function u does not have any critical points (that is, the gradient ∇u is always nonzero) and all the level sets of u are diffeomorphic to a sphere $\mathbb{S}^{n-1}$. It is also classical that the solutions to the analogous problem in $\mathbb{R}^2$, where the solutions are not required to tend to zero at infinity but to be upper bounded, have totally analogous properties whenever the domain Ω is diffeomorphic to a disk. A well-known question in this direction, raised by Kawohl in 1988 [22], was whether u could have any critical points when Ω is diffeomorphic to an n-ball.

In joint work with Peralta-Salas, we answered this question in the negative [14]. More precisely, we proved that for any N there is a domain $\Omega \subset \mathbb{R}^n$ diffeomorphic to the ball such that the solution u has at least N nondegenerate critical points. One can also have degenerate critical points, with a critical set of codimension at most 3.

The proof of this result relies on an unusual combination of classical potential theory, transversality techniques and the geometry of real analytic sets. Let us describe the underlying idea in $\mathbb{R}^3$, where things are easier to visualize. The basic observation is that, if $N =: 2g$ is even and we take as our domain a torus of genus g which is invariant under the reflections

$$x_2 \mapsto -x_2 \quad \text{and} \quad x_3 \mapsto -x_3\,,$$

elementary arguments enable us to prove that the corresponding solution u must have at least N critical points. Of course, what makes the problem nontrivial is that this domain is not diffeomorphic to a sphere. The key idea here is that one can choose as the domain Ω the above torus of genus g with a smooth small-measure set removed (one can think of this small set as a thin slab) so that the resulting domain is contractible: this yields a domain which is topologically a sphere but, geometrically, is very close to being a torus of genus g. The theorem is then proved by deriving suitable estimates which show that the geometry of the domain, and not its topology, is what governs the behavior of the function u.

An important feature of the strategy of the proof is that the underlying principle of "simulating topological effects with geometry" has a very wide applicability range. In particular, complemented with further technology from geometric analysis and the calculus of variations, this idea was very useful to prove that there are Riemannian metrics in $\mathbb{R}^n$ whose minimal Green's function have level sets of arbitrarily complicated topology [15].

6 Conclusions and prospective work

Through examples, we have seen how ideas of PDEs, dynamical systems and geometric analysis can be combined to solve a number of geometric questions that appear in mathematical physics. Needless to say, the potential applications of these methods are not restricted to the problems presented in this paper, so we will conclude with a sample of the problems that we will pursue in the near future.

In fluid mechanics there is a plethora of high-profile problems that are of a strongly geometric nature, including evolution free boundary problems such as the water wave equation or the connection between solutions of the Euler equation and the orbits of the group of volume-preserving diffeomorphism studied by Arnold. A concrete problem that we plan to study is the helical flow paradox of Morgulis et al. [27]: how come that Beltrami fields with a nonconstant proportionality factor $\lambda(x)$ are generically "laminar", in the sense that the function $\lambda(x)$ is a first integral of the fluid flow, and fluids are believed to be generically "turbulent"? This is meaningful because Beltrami fields are expected to capture the generic behavior of a smooth stationary fluid flow in $\mathbb{R}^3$. The key to this paradox is that there are local, very stringent obstructions to the existence of Beltrami fields for "most" factors $\lambda(x)$ [18]. We will also consider evolution problems, both in fluid mechanics and continuing our work on wave and dispersive equations (in asymptotically AdS spaces and in more classical settings).

We have also surveyed several results about the closely related topics of elliptic PDEs, spectral problems and quantum mechanics. This is a subject with a very rich interaction between geometric quantities and analytic estimates. One of the directions in which we intend to continue our research here is the study of nodal (i.e., zero) sets of eigenfunctions [33], both in the context of bounded domains with Dirichlet boundary conditions and for compact manifolds without boundary. In the first case, a well-known problem is Payne's conjecture about the structure of the nodal set of the second eigenfunction for a simply connected planar domain. In the second case, the high-energy limit of the nodal set has been thoroughly studied from different points of view, as it is closely related to quantum chaos.

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References

1. Anderson, M.T., Schoen, R.: Positive harmonic functions on complete manifolds of negative curvature. Ann. Math. **121**, 429–461 (1985)
2. Arnold, V.I.: Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. Ann. Inst. Fourier **16**, 319–361 (1966)
3. Arnold, V.I., Khesin, B.: Topological Methods in Hydrodynamics. Springer, New York (1999)
4. Breitenlohner, P., Freedman, D.Z.: Stability in gauged extended supergravity. Ann. Phys. **144**, 249–281 (1982)
5. Brenier, Y.: Topology-preserving diffusion of divergence-free vector fields and magnetic relaxation. Commun. Math. Phys. (2014, in press)
6. Córdoba, D., Fefferman, C.: On the collapse of tubes carried by 3D incompressible flows. Commun. Math. Phys. **222**, 293–298 (2001)
7. De Lellis, C., Székelyhidi, L.: The Euler equations as a differential inclusion. Ann. Math. **170**, 1417–1436 (2009)
8. De Lellis, C., Székelyhidi, L.: Dissipative continuous Euler flows. Invent. Math. **193**, 377–407 (2013)
9. Dombre, T., Frisch, U., Greene, J.M., Hénon, M., Mehr, A., Soward, A.M.: Chaotic streamlines in the ABC flows. J. Fluid Mech. **167**, 353–391 (1986)
10. Enciso, A., Finkel, F., González-López, A.: Spin chains of Haldane–Shastry type and a generalized central limit theorem. Phys. Rev. E **79**, 060105R(4) (2009)
11. Enciso, A., Finkel, F., González-López, A.: Level density of spin chains of Haldane–Shastry type. Phys. Rev. E **82**, 051117(6) (2010)
12. Enciso, A., Finkel, F., González-López, A.: Thermodynamics of spin chains of Haldane–Shastry type and one-dimensional vertex models. Ann. Phys. **327**, 2627–2665 (2012)
13. Enciso, A., Kamran, N.: Causality and the conformal boundary of AdS in real-time holography. Phys. Rev. D **85**, 106016 (2012)
14. Enciso, A., Peralta-Salas, D.: Critical points and level sets in exterior boundary problems. Indiana Univ. Math. J. **58**, 1947–1969 (2009)
15. Enciso, A., Peralta-Salas, D.: Critical points of Green's functions on complete manifolds. J. Differ. Geom. **92**, 1–29 (2012)
16. Enciso, A., Peralta-Salas, D.: Knots and links in steady solutions of the Euler equation. Ann. Math. **175**, 345–367 (2012)
17. Enciso, A., Peralta-Salas, D.: Existence of knotted vortex tubes in steady Euler flows. arXiv:1210.6271
18. Enciso, A., Peralta-Salas, D.: Beltrami fields with a nonconstant proportionality factor are rare. arXiv:1402.6825
19. Etnyre, J., Ghrist, R.: Contact topology and hydrodynamics III. Knotted orbits. Trans. Am. Math. Soc. **352**, 5781–5794 (2000)

20. Graham, C.R., Lee, J.M.: Einstein metrics with prescribed conformal infinity on the ball. Adv. Math. **87**, 186–225 (1991)
21. Haldane, F.D.M.: Exact Jastrow–Gutzwiller resonating-valence-bond ground state of the spin-$1/2$ antiferromagnetic Heisenberg chain with $1/r^2$ exchange. Phys. Rev. Lett. **60**, 635–638 (1988)
22. Kawohl, B.: Open problems connected with level sets of harmonic functions. In: Kral, J., et al. (eds.) Potential Theory, Surveys and Problems, Lecture Notes in Mathematics, vol. 1344. Springer, Berlin (1988)
23. Kleckner, D., Irvine, W.T.M.: Creation and dynamics of knotted vortices. Nat. Phys. **9**, 253–258 (2013)
24. Laurence, P., Stredulinsky, E.W.: Two-dimensional magnetohydrodynamic equilibria with prescribed topology. Commun. Pure Appl. Math. **53**, 1177–1200 (2000)
25. Moffatt, K.: The degree of knottedness of tangled vortex lines. J. Fluid Mech. **35**, 117–129 (1969)
26. Moffatt, K.: Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology I. J. Fluid Mech. **159**, 359–378 (1985)
27. Morgulis, A., Yudovich, V.I., Zaslavsky, G.M.: Compressible helical flows. Commun. Pure Appl. Math. **48**, 571–582 (1995)
28. Pelz, R.B., Yakhot, V., Orszag, S.A., Shtilman, L., Levich, E.: Velocity–vorticity patterns in turbulent flow. Phys. Rev. Lett. **54**, 2505–2508 (1985)
29. Shastry, B.S.: Exact solution of an $S = 1/2$ Heisenberg antiferromagnetic chain with long-ranged interactions. Phys. Rev. Lett. **60**, 639–642 (1988)
30. Sullivan, D.: The Dirichlet problem at infinity for a negatively curved manifold. J. Differ. Geom. **18**, 723–732 (1983)
31. Thomson, W. (Lord Kelvin): Vortex statics. Proc. R. Soc. Edinb. **9**, 59–73 (1875)
32. Witten, E.: Anti de Sitter space and holography. Adv. Theor. Math. Phys. **2**, 253–291 (1998)
33. Yau, S.T.: Open problems in geometry. Proc. Sympos. Pure Math., vol. 54, pp. 1–28, Amer. Math. Soc., Providence (1993)