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A *p*-adic interpolation of generalized Heegner cycles and integral Perrin-Riou twist I

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Abstract
In this paper, we develop an integral refinement of the Perrin-Riou theory of exponential maps. We also formulate the Perrin-Riou theory for anticyclotomic deformation of modular forms in terms of the theory of the Serre–Tate local moduli and interpolate generalized Heegner cycles p-adically.

Résumé
Dans cet article, nous développons un raffinement entier de la théorie des applications exponentielles de Perrin- Riou. Nous formulons également la théorie de Perrin-Riou pour les déformations anticyclotomiques de formes modulaires en utilisant la théorie des modules locaux de Serre- Tate et nous interpolons p-adiquement les cycles de Heegner généralisés.

Mathematics Subject Classification Primary 11R23; Secondary 11G40 · 11F11 · 11G15 · 11F67 · 11F85

1 Introduction

The Perrin-Riou theory of the big exponential map is the fundamental theory in the local Iwasawa theory for the cyclotomic deformation of Galois representations, and it continues to be a source of development of new p-adic theories beyond the Iwasawa theory. The purpose of the paper is twofold. First, we give a generalization of the Soulé twist on Galois cohomology groups inspired by the Perrin-Riou theory and Amice-Velu–Vishik theory of the p-adic distribution. Though the original Perrin-Riou theory is a local theory, our twist theory works in fairly general Galois representations even for global fields with torsion coefficients similar to the Soulé twist. Such a theory is essential when we twist Euler systems that are not norm-compatible in the p-power direction. Second, we describe a geometric interaction between the Perrin-Riou theory and the theory of the Serre–Tate local moduli of ordinary elliptic curves. Since Katz, the fruitful relationship between the local moduli and Iwasawa

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theory is known. Our description is also not essentially new, and it is a reformulation of known results such as [2] in terms of the Perrin-Riou theory. However, we think that the theory is naturally described in terms of the Perrin-Riou theory, which is also crucial for applications in the non-ordinary case.

Our motivation comes from the Euler system of generalized Heegner cycles defined by Bertolini–Darmon–Prasanna [2], for which the Soulé twist does not work because it is not norm compatible in the p-power direction, and torsion coefficients are also crucial. In the sequel [24] of this paper, we give an example of the Coates–Wiles–Kolyvagin type result for elliptic cusp forms twisted by anticyclotomic Hecke characters of imaginary quadratic fields satisfying the classical Heegner hypothesis as application of our theory. More precisely, if the special value of the associated L-function does not vanish at a critical value, then we show that the corresponding p-primary Selmer group is finite for almost all p. (There are similar results proved by different methods. cf. [3, 19, 28].) The key to our result is our twist theory and a p-adic interpolation of generalized Heegner cycles by a power series (Theorem 6.11), which is considered as a Coleman power series interpolating "zeta elements". Based on the theory developed in this paper, we also prove a one-side divisibility of the Iwasawa main conjecture in this setting in [25].

The organization of the paper is as follows. In Sect. 2, we develop our theory of the integral Perrin-Riou twist. In Sect. 3, we recall generalized Heegner cycles by Bertolini–Darmon–Prasanna, and in Sect. 4, we prove a certain horizontal congruence for generalized Heegner cycles which is the key ingredient of the application of our twist theory. In Sect. 5, we explain the relation between the Serre–Tate local moduli and anticyclotomic extensions. In Sect. 6, we construct the logarithmic Coleman power series interpolating generalized Heegner cycles. See also the beginning of Sect. 6 as an introduction to our formulation. In the appendix, we summarize the theory of the Perrin-Riou exponential map for crystalline representations over the division tower of a relative Lubin–Tate group of height 1.

2 The integral Perrin-Riou twist

In this section, we give a generalization of the Soulé twist. The idea goes back to the work of Amice-Vélu and Vishik for the construction of the cyclotomic p-adic L-function of higher weight elliptic modular forms at non-ordinary primes. The same idea has been already used in Perrin-Riou [33, 35] and see also [8, 26, 28]. Our generalization is integral and works even in torsion coefficients.

Lemma 2.1 *Let R be a commutative ring and M an R-module. Let $(a_n)_{n=0,1,\dots}$ be a sequence in M and put*

$$b_n := \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_{n-i} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a_i .$$

Then for $u \in R$, we have

$$\sum_{i=1}^{n} (-1)^i \binom{n}{i} (u^i - 1) a_{n-i} = \sum_{i=1}^{n} (-1)^i \binom{n}{i} (u-1)^i b_{n-i} . \tag{2.1}$$

Proof By considering the universal case, it suffices to show this in the case $R = \mathbb{Z}[Y]$, $M = R[X, Y_1, \dots, Y_i, \dots]$, $u = X$, $a_i = Y_i$ for indeterminate X and $(Y_i)_i$. Let $R\langle\langle t \rangle\rangle$ be the formal power series ring consisting of elements $\sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$ for $c_n \in R$. If we put

$f(t)=\sum_{n=0}^{\infty}a_n\frac{t^n}{n!}$ and $g(t)=\sum_{n=0}^{\infty}b_n\frac{t^n}{n!}$, we have $g(t)=e^{-t}f(t)$ in $R\langle\langle t\rangle\rangle$. Then we have the identity by looking the coefficients of t^n of both sides of

$$e^{-ut}f(t)-e^{-t}f(t)=e^{(1-u)t}g(t)-g(t).$$

We consider a triple (Γ,γ,ρ) where Γ is a profinite group isomorphic to $\mathbb{Z}_p$ with a topological generator γ of Γ and ρ is an embedding of topological groups $\Gamma\hookrightarrow 1+p\mathbb{Z}_p$. Let Γ_n be the open subgroup of Γ of index p^n generated by $\gamma_n:=\gamma^{p^n}$. Let $\mathbb{Z}_p(\rho)$ be a rank one $\mathbb{Z}_p$-representation of Γ with a basis e_ρ with the action $ge_\rho=\rho(g)e_\rho$. For a continuous p-adic representation T of Γ and an integer i, we let $T(i)_\rho$ be the representations of Γ defined by the tensor product of T and $\mathbb{Z}_p(\rho)^{\otimes i}$. We put $\mathrm{Tr}_{m+n/n}:=\sum_{a=0}^{p^m-1}\gamma^{p^n a}\in\mathbb{Z}[\Gamma]$. Note that $\mathrm{Tr}_{m+n/n}=\mathrm{Tr}_{n+1/n}\mathrm{Tr}_{m+n/n+1}$.

Theorem 2.2 *Let h be a natural number and let α be an element of $\mathbb{C}_p$ such that $|p^h/\alpha|_p<1$. Let M be a p-adically complete $\mathbb{Z}_p[\alpha]$-module with a continuous action of Γ. For $0\le i\le h-1$, suppose that we have a sequence $(c_n^{(i)})_{n\in\mathbb{N}}$ in $M(i)_\rho$ and $(r_{n,i})_{n\in\mathbb{N}}$ in M satisfying the following conditions:*

(a) The projection of $c_n^{(i)}$ to the free part M/M_{tor} is fixed by Γ_n.
(b) $\mathrm{Tr}_{n+1/n}c_{n+1}^{(i)}=\alpha c_n^{(i)}$.
(c) For an element $d_n^{(i)}:=c_n^{(i)}\otimes e_\rho^{\otimes -i}\in M$, we have

$$\sum_{j=0}^{i}(-1)^j\binom{i}{j}d_n^{(j)}=p^{i(n-1)}r_{n,i}.$$

Then there exists a functorial way to extend $c_n^{(i)}$ for arbitrary integer i and extend $r_{n,i}$ to $r_{n,i,k}\in M$ ($k\in\mathbb{Z}$, $i\in\mathbb{Z}^{\ge 0}$) with $r_{n,i,0}=r_{n,i}$, so that they satisfy (a), (b) and

$$\sum_{j=0}^{i}(-1)^j\binom{i}{j}d_n^{(j+k)}=p^{i(n-1)}r_{n,i,k} \tag{2.2}$$

for any non-negative integer i. Here, a functorial way means the compatibility with morphisms $(\Gamma,\gamma,\rho)\to(\Gamma',\gamma',\rho')$ and $M'\to M$ in the obvious sense. Furthermore, if M is torsion-free, the extensions $c_n^{(i)}$ and $r_{n,i,k}$ are characterized by (a), (b) and (2.2), and independent of γ.

Proof First, we construct $(c_n^{(h)})_n$. For $x\in M$, we put $x(h):=x\otimes e_\rho^{\otimes h}\in M(h)$. We let

$$\tilde{c}_n^{(h)}:=-\sum_{i=1}^{h}(-1)^i\binom{h}{i}d_n^{(h-i)}(h).$$

For $g\in G$, applying Lemma 2.1 for $a_n=d_{k+1}^{(n)}(h)$, we have

$$\begin{aligned}g\tilde{c}_{k+1}^{(h)}&=-\sum_{i=1}^{h}(-1)^i\binom{h}{i}gd_{k+1}^{(h-i)}(h)\\&=-\sum_{i=1}^{h}(-1)^i\binom{h}{i}\rho(g)^{-i}gd_{k+1}^{(h-i)}(h)+\sum_{i=1}^{h}(-1)^i\binom{h}{i}(\rho(g)^{-1}-1)^i p^{k(h-i)}g\cdot r_{k+1,h-i}(h)\\&=-\sum_{i=1}^{h}(-1)^i\binom{h}{i}(gc_{k+1}^{(h-i)})\otimes e_\rho^{(i)}+\sum_{i=1}^{h}(-1)^i\binom{h}{i}(\rho(g)^{-1}-1)^i p^{k(h-i)}g\cdot r_{k+1,h-i}(h).\end{aligned} \tag{2.3}$$

Hence by the condition (b), we have

$$\mathrm{Tr}_{k+1/k}\tilde{c}_{k+1}^{(h)} - \alpha\tilde{c}_k^{(h)} = \sum_{a=0}^{p-1}\sum_{i=1}^{h}(-1)^i\binom{h}{i}(\rho(\gamma^{-p^k a})-1)^i p^{k(h-i)}\gamma^{p^k a}\cdot r_{k+1,h-i}(h).$$

We put

$$x_k := \sum_{a=0}^{p-1}\sum_{i=1}^{h}(-1)^i\binom{h}{i}\frac{(\rho(\gamma^{-p^k a})-1)^i}{p^{ki}}\cdot\gamma^{p^k a}\cdot r_{k+1,h-i}(h).$$

and $y_k := \left(\frac{p^h}{\alpha}\right)^k x_k$. Then we define

$$c_n^{(h)} := \tilde{c}_n^{(h)} + \alpha^{n-1}\sum_{i=0}^{\infty}\mathrm{Tr}_{n+i/n}y_{n+i}.$$

(Note that $\lim_{k\to\infty} y_k = 0$ by our assumption on α.) Since $\alpha^n y_n = \mathrm{Tr}_{n+1/n}\tilde{c}_{n+1}^{(h)} - \alpha\tilde{c}_n^{(h)}$, we have $\mathrm{Tr}_{n+1/n}c_{n+1}^{(h)} = \alpha c_n^{(h)}$. By construction,

$$\sum_{i=0}^{h}(-1)^{h-i}\binom{h}{i}d_n^{(i)}(h) = c_n^{(h)} - \tilde{c}_n^{(h)} = p^{(n-1)h}\sum_{i=0}^{\infty}\left(\frac{p^h}{\alpha}\right)^{i+1}\mathrm{Tr}_{n+i/n}x_{n+i}.$$

Hence define $r_{n,h,0}$ by

$$r_{n,h,0} := (-1)^h\sum_{i=0}^{\infty}\left(\frac{p^h}{\alpha}\right)^{i+1}\mathrm{Tr}_{n+i/n}x_{n+i}\otimes e_\rho^{\otimes -h}.$$

To show the property (a), we may assume M is torsion-free. By (2.3), we have $\gamma_k\tilde{c}_k^{(h)} \equiv \tilde{c}_k^{(h)} \mod p^{kh}M$. Hence

$$\begin{aligned}\gamma_n\mathrm{Tr}_{n+m/n}\tilde{c}_{m+n}^{(h)} &= \sum_{a=0}^{p^m-1}\gamma^{(a+1)p^n}\tilde{c}_{m+n}^{(h)} = \gamma^{p^{n+m}}\tilde{c}_{m+n}^{(h)} + \sum_{a=1}^{p^m-1}\gamma^{ap^n}\tilde{c}_{m+n}^{(h)}\\ &\equiv \mathrm{Tr}_{n+m/n}\tilde{c}_{m+n}^{(h)} \mod p^{(n+m)h}M.\end{aligned}$$

The property $a)$ then follows from

$$c_n^{(h)} = \lim_{m\to\infty}\alpha^{-m}\mathrm{Tr}_{n+m/n}\tilde{c}_{n+m}^{(h)}.$$

By induction for h, we have $(c_n^{(i)})_n$ and $r_{n,i,0}$ for any non-negative i satisfies (a), (b) and (c). Since $\binom{i+1}{j+1} = \binom{i}{j+1} + \binom{i}{j}$, we have

$$\sum_{j=0}^{i}(-1)^j\binom{i}{j}d_n^{(j+k+1)} = -\sum_{j=0}^{i+1}(-1)^j\binom{i+1}{j}d_n^{(j+k)} + \sum_{j=0}^{i}(-1)^j\binom{i}{j}d_n^{(j+k)}.$$

Using this, $r_{n,i,k}$ is defined inductively for $k \geq 1$. In the negative direction, we put ${}^\iota\rho = \rho^{-1}$ and let ${}^\iota M$ be the module M but the action of G is by ${}^\iota\rho$. We define ${}^\iota c_n^{(i)} := c_n^{(h-1-i)}$ and apply our theorem for $i, k \geq 0$ and $({}^\iota M)(1-h)$ as M. Then

$${}^\iota d_n^{(i)} := {}^\iota c_n^{(i)} \otimes e_{{}^\iota\rho}^{(-i)} = c_n^{(h-i-1)} \otimes e_\rho^{(i)} \in M(h-1).$$

We have

$$\begin{aligned}\sum_{j=0}^{i}(-1)^j\binom{i}{j}{}^{\iota}d_n^{(j)} &= \sum_{j=0}^{i}(-1)^j\binom{i}{j}c_n^{(h-j-1)}\otimes e_\rho^{(j)}\\ &= (-1)^i\sum_{j=0}^{i}(-1)^j\binom{i}{j}c_n^{(h-i+j-1)}\otimes e_\rho^{(i-j)}\\ &= (-1)^i\sum_{j=0}^{i}(-1)^j\binom{i}{j}d_n^{(h-i+j-1)}\otimes e_\rho^{(h-1)}\\ &= (-1)^i p^{i(n-1)}r_{n,i,h-i-1}\otimes e_\rho^{(h-1)}.\end{aligned}$$

Hence by putting ${}^{\iota}r_{n,i} = (-1)^i r_{n,i,h-i-1}\otimes e_\rho^{(h-1)}$, and our theorem can be applied for this system. Then ${}^{\iota}c_n^{(i)}$ and ${}^{\iota}r_{n,i,k}$ are extended to any $i, k \geq 1$.

For the last assertion, suppose that $M_{\mathrm{tor}} = \{0\}$. First, by the condition (a), the trace of $c_n^{(i)}$ in condition (b) is independent of the choice of γ. Hence, the independence of γ follows if we show the uniqueness of the extensions of $c_n^{(i)}$ and $r_{n,i,k}$. Consider $(c_n^{(i)})_{i\in\mathbb{Z},n\in\mathbb{N}}$ satisfying (a), (b), (2.2) and $c_n^{(i)} = 0$ for i such that $0 \leq i < h$. Then $c_n^{(h)} \equiv 0 \mod p^{h(n-1)}$ by (2.2) for all n. By b), we have $c_n^{(h)} = \alpha^{-m}\mathrm{Tr}_{m+n}c_{m+n}^{(h)}$, and hence $c_n^{(h)} = 0$. Inductively, we have $c_n^{(i)} = 0$ for all $i \geq 0$. For a negative i, the proof is similar. Thus, the extension of $c_n^{(i)}$ is unique. Since $M_{\mathrm{tor}} = \{0\}$, the relation (2.2) and $(c_n^{(i)})_i$ determine $r_{n,i,k}$ uniquely.

We consider a 4-tuple $(G, G_\infty, \rho, \gamma)$ where G is a profinite group with a normal subgroup G_∞ such that $\Gamma := G/G_\infty$ is isomorphic to $\mathbb{Z}_p$, γ is a topological generator of Γ, and ρ is an embedding of topological groups $\Gamma \hookrightarrow 1 + p\mathbb{Z}_p$. Let Γ_n be the open subgroup of Γ of index p^n and G_n the inverse image of Γ_n by ρ in G. Let $\mathbb{Z}_p(\rho)$ be a rank one $\mathbb{Z}_p$-representation of G with a basis e_ρ with the action $ge_\rho = \rho(g)e_\rho$. For a continuous p-adic representation T of G and an integer i, we denote by $T(i)_\rho$ the tensor product of T and $\mathbb{Z}_p(\rho)^{\otimes i}$ as representations of G.

Corollary 2.3 *Let h be a natural number and let α be an element of $\mathbb{C}_p$ such that $|p^h/\alpha|_p < 1$. Let T be a finitely generated $\mathbb{Z}_p[\alpha]$-module with continuous action of G. Assume that $H^0(G_\infty, T) = \{0\}$ and $p^{n_0}H^1(G_\infty, T)_{\mathrm{tor}} = \{0\}$. Suppose that for $0 \leq i \leq h-1$, we have a system $(c_n^{(i)})_n \in \prod_{n\in\mathbb{N}} H^1(G_n, T(i)_\rho)$ satisfying the following two conditions:*

(a) $\mathrm{Cor}_{n+1/n}c_{n+1}^{(i)} = \alpha c_n^{(i)}$

(b) We identify $c_n^{(i)}$ and its image by the natural inclusion $H^1(G_n, T(i)_\rho) \to H^1(G_\infty, T(i)_\rho)$. Elements $d_n^{(i)} := c_n^{(i)} \otimes e_\rho^{\otimes -i} \in H^1(G_\infty, T)$ satisfy the congruence relation

$$\sum_{j=0}^{i}(-1)^j\binom{i}{j}d_n^{(j)} \equiv 0 \mod p^{in}H^1(G_\infty, T).$$

Then $d_n^{(j)}$ can be extended for any integer j such that

$$\sum_{j=0}^{i}(-1)^j\binom{i}{j}d_n^{(j+k)} \equiv 0 \mod p^{in}H^1(G_\infty, T) \tag{2.4}$$

for any natural number i and any integer k, and $p^{n_0}c_n^{(i)} = p^{n_0}d_n^{(i)} \otimes e_\rho^{\otimes i}$ *satisfies the* α*-norm compatible relation a). Furthermore,* $p^{n_0}c_n^{(i)}$ *does not depend on the choice of an extension of* $d_n^{(j)}$.

Remark 2.4

(i) If $\varprojlim_n H^1(G_n, T)$ is a finitely generated Λ-module, as in [35, Proposition 1.8], we have an element z of $\mathscr{H}_\infty(\Gamma) \otimes_{\mathbb{Z}_p[\![\Gamma]\!]} \varprojlim_n H^1(G_n, T)$ interpolating $\alpha^{-n}c_n^{(i)}$ $(0 \le i \le h-1)$. By the projection of z to $H^1(G_n, V(i))$ for any integer i, we have a system $(P_{n,i})_n$. Then the image of our twist $(c_n^{(i)})$ after inverting p is equal to $(\alpha^n P_{n,i})_n$ by the characterization of ours and that in [35, Proposition 1.8].

(ii) In [25], we give a slightly different generalization of the Perrin-Riou twist that works not only for algebraic twists as in this paper but also for any continuous characters. However, in [25], we do not directly consider the twist on l-local cohomology groups.

3 Generalized Heegner cycles and the *p*-adic Abel–Jacobi map

In this section, following [2, §2], we introduce generalized Heegner cycles and their p-adic Abel–Jacobi images.

3.1 Kuga–Sato variety

Let N be a natural number. For the moment, we assume that $N > 4$ and consider the universal generalized elliptic curve $\overline{\pi} : \overline{\mathscr{E}} \to X_1(N)$ over $\mathbb{Z}[1/N]$ with $\Gamma_1(N)$-level structure (a point of order N) and the universal elliptic curve $\pi : \mathscr{E} \to Y_1(N)$. For a non-negative integer m, let W_m be the Kuga–Sato variety with $\Gamma_1(N)$-level structure, that is, W_m is the canonical desingularization of

$$\overline{\mathscr{E}}^{(m)} := \overline{\mathscr{E}} \times_{X_1(N)} \cdots \times_{X_1(N)} \overline{\mathscr{E}} \qquad (m\text{-times}).$$

By the construction of W_m, we have naturally

$$\mathscr{E}^{(m)} := \mathscr{E} \times_{Y_1(N)} \cdots \times_{Y_1(N)} \mathscr{E} \subset W_m.$$

The group $((\mathbb{Z}/N\mathbb{Z}) \rtimes \{\pm 1\})^m \rtimes \mathfrak{S}_m$ ($\mathfrak{S}_m$: the m-th symmetric group) acts on $\mathscr{E}^{(m)}$ by the translation by the level structure, ± 1-multiplication of each component and by the permutation of components. This action is canonically extended on W_r by a property of the canonical desingularization. Let ϵ_{W_m} be the idempotent in the group algebra of $((\mathbb{Z}/N\mathbb{Z}) \rtimes \{\pm 1\})^m \rtimes \mathfrak{S}_m$ with coefficients in $\mathbb{Z}[1/2(m!)]$ corresponding the character that sends $\mathbb{Z}/N\mathbb{Z}$ to the identity, $\{\pm 1\}$ identically to $\{\pm 1\}$ and $\mathfrak{S}_m$ to $\{\pm 1\}$ as the sign character. We also regard ϵ_{W_m} as an element of $\mathbb{Q}[\mathrm{Aut}(W_m)]$. For details, see [2, Appendix].

3.2 Generalized Heegner cycles

Let E be an elliptic curve with a $\Gamma_1(N)$-structure Lv defined over a number field F. We put $X_{E,m} := E^m \times W_m$. Following [2], we define a cycle on $X_{E,m}$ for an isogeny $\iota : E \to E'$ with $(\mathrm{Ker}\, \iota) \cap E[N] = \{0\}$. By our condition on ι, there exists a natural level structure on E' compatible with ι and Lv. If E' and ι is defined a finite extension F' of

F, it defines a point $z_{E'} \in X_1(N)(F')$ and E' is the fiber of $\mathscr{E}$ over $z_{E'}$. In particular, $E^m \times (E')^m \subset E^m \times W_m = X_{E,m}$. Hence the m-th power of the graph $\Gamma_\iota \subset E \times E'$ of ι may be regarded as a cycle on $X_{E,m}$. Let ϵ_{E^m} be an idempotent in $\mathbb{Z}[1/2(m!)][\{\pm 1\} \rtimes \mathfrak{S}_m]$ similarly defined as ϵ_{W_m}. We also regard ϵ_{E^m} as an element of $\mathbb{Q}[\mathrm{Aut}(E^m)]$, and put $\epsilon_X = \epsilon_{E^m}\epsilon_{W_m} \in \mathbb{Q}[\mathrm{Aut}(X_{E,m})]$. Then we let $\Delta_\iota := \epsilon_X \Gamma_\iota^m$, which may be regarded as an element of the Chow group $\mathrm{CH}^{m+1}(X_{E,m}) \otimes \mathbb{Q}$. Let $\mathrm{CH}^{m+1}(X_{E,m})_0 \otimes \mathbb{Q}$ be the subset of $\mathrm{CH}^{m+1}(X_{E,m}) \otimes \mathbb{Q}$ consisting of homologically trivial elements. The cycle Δ_ι is homologically trivial if $m > 0$ because then $\epsilon_X H_B^{2m+2}(X_{E,m}, \mathbb{Q}) = 0$. ([2, Proposition 2.7]) If E has complex multiplication, Δ_ι is called a generalized Heegner cycle associated with ι and Lv.

We fix an embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let K be an imaginary quadratic field with discriminant D_K such that $\mathcal{O}_K^\times = \{\pm 1\}$. Let A be a CM elliptic curve defined over the Hilbert class field H of K, and let ψ_H be the Grössencharacter $H^\times \backslash \mathbb{A}_H^\times \to \mathbb{C}^\times$ associated to A/H. We denote $X_{A,m}$ by X_m if we fixed A. (Sect. 5.13, we will choose A more precisely.) We fix an invariant differential ω_A of A and an embedding $[\ ] : K \to \mathrm{End}\, A \otimes \mathbb{Q}$ so that $[a]^*\omega_A = \iota_\infty(a)\omega_A$. Then consider the complex uniformization $\pi_A : \mathbb{C}/\Lambda_A \cong A(\mathbb{C})$ such that $\pi_A^*(\omega_A) = dz$. By replacing A by its conjugate by $\mathrm{Gal}(H/K)$ if necessary, we may assume that the lattice Λ_A is written in the form $\mathcal{O}_K\Omega_K$ for a complex number Ω_K. For a natural number c, we let $\mathcal{O}_c$ be the order of K of conductor c, that is, $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K$. Then $\frac{1}{c}\mathcal{O}_c\Omega_K$ defines a subgroup C of $A[c]$ and the theory of complex multiplication shows that C is defined over the ring class field $H_c = K(j(\mathcal{O}_c))$ of conductor c. We let A_c be the quotient A/C and $\pi_c : A \to A_c$ the canonical projection. Then the complex uniformization of A_c with respect to $(\pi_c)_*\omega_A$ is $\mathbb{C}/\Lambda_c$ for the lattice $\Lambda_c = \mathcal{O}_c\Omega_K$ and π_c is identified with the isogeny $\mathbb{C}/\mathcal{O}_K\Omega_K \to \mathbb{C}/\Lambda_c, z \mapsto cz$.

Now we assume the classical Heegner hypothesis, that is

$$\text{(Heeg)} \qquad \text{all prime factors of } N \text{ splits in } K.$$

Choose an ideal $\mathfrak{N}$ of $\mathcal{O}_K$ such that $\mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$. Then the pair $(A, A[\mathfrak{N}])$ defines a point z_A in $X_0(N)(H)$ so-called a Heegner point (of conductor 1). Take a point z'_A of $X_1(N)$ above z_A with respect to the natural projection $\pi_1 : X_1(N) \to X_0(N)$. Suppose that $(c, N) = 1$. Then $(\mathrm{Ker}\, \pi_c) \cap A[N] = \{0\}$ and hence we have the generalized Heegner cycle $\Delta_{\pi_c, z'_A} := \Delta_{\pi_c}$ associated with z'_A and π_c. Let Δ_c be the cycle $\frac{1}{\deg \pi_1}\sum \Delta_{\pi_c, z'_A}$ where the sum runs through all points z'_A (counting multiplicity) over z_A by π_1. Then Δ_c is defined over H_c. We call it the generalized Heegner cycle of conductor c.

3.3 The p-adic Abel–Jacobi map

Let p be a prime number such that $p \nmid D_K N$. Suppose that p splits in K and write $(p) = \mathfrak{p}\mathfrak{p}^*$ as an ideal of $\mathcal{O}_K$. We fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ which is compatible with $\mathfrak{p}$. We denote $K_\mathfrak{q}$ by the completion of K at a prime ideal $\mathfrak{q}$ of $\mathcal{O}_K$. For $\mathfrak{q}|p$, we regard as $K_\mathfrak{q} = \mathbb{Q}_p$ by the natural inclusion map $\mathbb{Q}_p \hookrightarrow K_\mathfrak{q}$. Define the Serre–Tate character $\tilde{\psi}_H : \mathbb{A}_H^\times \to K^\times$ by $\tilde{\psi}_H(x) := \psi_H(x_f)$ where x_f is the finite part of $x \in \mathbb{A}_H$, namely, it is obtained from x by replacing the component of the archimedean places by 1. Let $\mathbf{N}_p$ be the norm map $(H \otimes_\mathbb{Q} \mathbb{Q}_p)^\times \to (K \otimes_\mathbb{Q} \mathbb{Q}_p)^\times$, and consider the homomorphism

$$\psi_{H,p} : \mathbf{A}_H^\times \to (K \otimes_\mathbb{Q} \mathbb{Q}_p)^\times, \quad x \mapsto (\tilde{\psi}_H(x) \otimes 1)\mathbf{N}_p(x_p)^{-1}$$

where x_p is the component of x in $(H \otimes_\mathbb{Q} \mathbb{Q}_p)^\times$. Then $\psi_{H,p}$ is trivial on $H^\times$ and kills the connected component of idèle class group, it induces a Galois representation $\mathrm{Gal}(\overline{H}/H) \to (K \otimes_\mathbb{Q} \mathbb{Q}_p)^\times$. This is equal to the Galois representation $\mathrm{Gal}(\overline{H}/H) \to \mathrm{Aut}_{K\otimes_\mathbb{Q}\mathbb{Q}_p}(V_p A) =$

$(K \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$ on the Tate module of A. Composing with the projections on $(K \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times = (K_{\mathfrak{p}} \times K_{\mathfrak{p}^*})^\times$ to each factors and with the natural identification $\mathbb{Q}_p \cong K_{\mathfrak{p}}$ and $\mathbb{Q}_p \cong K_{\mathfrak{p}^*}$, we obtain characters $\psi_{A,\mathfrak{p}}, \psi_{A,\mathfrak{p}^*} : \mathrm{Gal}(\overline{H}/H) \to \mathbb{Q}_p^\times$. The Galois action on $V_{\mathfrak{p}}A$ and $V_{\mathfrak{p}^*}A$ are given by $\psi_{A,\mathfrak{p}}, \psi_{A,\mathfrak{p}^*}$, which are also denoted by $\psi_{\mathfrak{p}}, \psi_{\mathfrak{p}^*}$ for simplicity.

Let f be a normalized eigen newform for $\Gamma_0(N)$ of weight $k \geq 2$. We consider the map to Bloch–Kato Selmer groups

$$\begin{aligned} \mathrm{CH}^{k-1}(X_{k-2} \otimes H_c)_0 &\longrightarrow H^1_{\mathrm{f}}(H_c, H^{2k-3}_{\text{ét}}((X_{k-2})_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(k-1))) \\ &\longrightarrow H^1_{\mathrm{f}}(H_c, V_f \otimes [\mathrm{Sym}^{k-2} H^1_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)](k-1)) \\ &= \prod_{i=1}^{k-1} H^1_{\mathrm{f}}(H_c, V_f(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i})). \end{aligned}$$

Here the first map is the p-adic Abel–Jacobi map and the second is obtained by the isomorphism

$$\begin{aligned} \epsilon_X H^{2k-3}_{\text{ét}}((X_{k-2})_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) &\cong \epsilon_{W_{k-2}} H^{k-1}_{\text{ét}}((W_{k-2})_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) \otimes \epsilon_{A^{k-2}} H^{k-2}_{\text{ét}}(A^{k-2}_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) \\ &\cong H^1_{\text{ét}}(X_1(N)_{\overline{\mathbb{Q}}}, j_* R^1\pi \mathbb{Q}_p) \otimes \mathrm{Sym}^{k-2} H^1_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) \end{aligned} \tag{3.1}$$

(j is the inclusion $Y_1(N) \hookrightarrow X_1(N)$. cf. [39, 1.2.1]) and the projection to the f-part. We consider the image of the homologically trivial cycle Δ_c for $k > 2$ or $\Delta_c - (\infty)$ for $k = 2$ by this map. We denote it by $(z_c^{(i)})_i$. For the Euler system argument later, it is important that the denominator of $z_c^{(i)}$ is bounded independent of c. In fact, the Abel–Jacobi map is defined integrally and the denominator comes from ϵ_X, $\deg \pi_1$, the isomorphism (3.1), the projector of taking f-part and the order of $H^{2k-2}_{\text{ét}}((X_{k-2})_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathrm{tor}}$ to be Δ_c homologically trivial. These are all independent of c. Note that the isomorphism (3.1) is also defined integrally. (cf. [29, Proposition 2.1].)

We call $z_c^{(i)}$ the i-th Abel–Jacobi image of generalized Heegner cycle of conductor c, or just for simplicity, the generalized Heegner cycle of conductor c. It is known that the system $(z_c^{(i)})_c$ forms an Euler system (cf. [6, Chapter 4, 7]).

So far, we have assumed $N > 4$, but as in [31, Chapter II, (3.7), (3.8)], we can eliminate this assumption to define $z_c^{(i)}$.

4 Congruences on generalized Heegner cycles

In this section, we prove a key congruence relation for applying Theorem 2.2 and Corollary 2.3 with $h = k - 1$. (k is the weight of a modular form we apply for.) If $k = 2$ ($h = 1$), the congruence relation is trivial. Hence we assume $k > 2$ in the below. As before, first, we assume $N > 4$. Let Δ_ι be the generalized Heegner cycle associated to an isogeny $\iota : (A, \mathrm{Lv}) \to (A', \mathrm{Lv}')$ defined over a number field F with compatible $\Gamma_1(N)$-level structures. Let P be the point of $X_1(N)$ corresponding to (A', Lv'). Let X be the generalized Kuga–Sato variety X_{k-2} associated to (A, Lv) and let $\pi : X \to X_1(N)$ be the canonical map. Then the fiber $X_P := \pi^{-1}(P)$ is the product $(A \times A')^{k-2}$. Then by the Künneth formula and the definition of ϵ_X, we have

$$\begin{aligned} V_P &:= \epsilon_X H^{2(k-2)}((X_P)_{\overline{F}}, \mathbb{Q}_p)(k-2) \\ &= [\mathrm{Sym}^{k-2} H^1(A_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{Sym}^{k-2} H^1(A'_{\overline{F}}, \mathbb{Q}_p)](k-2) \\ &= \mathrm{Hom}_{\mathbb{Q}_p}(\mathrm{Sym}^{k-2} H^1(A'_{\overline{F}}, \mathbb{Q}_p), \mathrm{Sym}^{k-2} H^1(A_{\overline{F}}, \mathbb{Q}_p)) \\ &= \mathrm{Hom}_{\mathbb{Q}_p}(\mathrm{Sym}^{k-2} V_p(A), \mathrm{Sym}^{k-2} V_p(A')). \end{aligned}$$

As a cycle on X_P, the image of Δ_ι by the étale cycle map

$$\mathrm{cl}_P : \mathrm{CH}^{k-2}(X_P) \longrightarrow V_P^{G_F}$$

is the natural map $\mathrm{Sym}^{k-2} V_p(A) \to \mathrm{Sym}^{k-2} V_p(A')$ induced by ι.

We interpret the Abel–Jacobi image of the generalized Heegner cycle as an extension of Galois representations. Put

$$V_X = \epsilon_X H^{2k-3}((X_{k-2})_{\overline{F}}, \mathbb{Q}_p(k-1)), \quad W_X = \epsilon_X H^{2k-3}((X_{k-2} - X_P)_{\overline{F}}, \mathbb{Q}_p(k-1)).$$

Then we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_X & \longrightarrow & W_{\Delta_\iota} & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{cl}_P(\Delta_\iota)} & & \\ 0 & \longrightarrow & V_X & \longrightarrow & W_X & \longrightarrow & V_P & \longrightarrow & 0. \end{array} \tag{4.1}$$

The bottom exact sequence is obtained by the localization sequence and purity, and the surjectivity comes from $k > 2$ (Δ_ι is homologically trivial). The upper exact sequence is the pull-back of the bottom one by the Galois equivariant map $\mathbb{Q}_p \to V_P$ sending 1 to $\mathrm{cl}_P(\Delta_\iota)$. Then the cohomology class of the upper extension gives the Abel–Jacobi image of Δ_ι in $H^1(F, V_X)$.

Now we assume A is the CM elliptic curve defined over H in Sect. 3.2 and consider a base change of A to a field F where $H \subset F \subset H_{cp^\infty}$ for a natural number c. (Later, we will take ι as π_{cp^n} and $F = H_{cp^n}$.) Let $\mathbb{Q}_p(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i}) := (V_{\mathfrak{p}} A)^{\otimes i} \otimes_{\mathbb{Q}_p} (V_{\mathfrak{p}^*} A)^{\otimes k-i}$. It is a 1-dimensional Galois representation of G_F over $\mathbb{Q}_p$ with the character $\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i}$. For a Galois representation U of G_F, we write $U \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i})$ simply by $U(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i})$. We have

$$\mathrm{Sym}^{k-2} H^1(A_{\overline{F}}, \mathbb{Q}_p)(k-2) = \prod_{i=0}^{k-2} \mathbb{Q}_p(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-2-i}).$$

By pushing the upper sequence in (4.1) by the canonical projection

$$V_X \longrightarrow V_f \otimes [\mathrm{Sym}^{k-2} H^1(A_{\overline{F}}, \mathbb{Q}_p)](k-1) = \prod_{i=1}^{k-1} V_f(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i}) \to V_f(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i}),$$

we obtain an extension

$$0 \longrightarrow V_f(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i}) \longrightarrow W_{f,i} \longrightarrow \mathbb{Q}_p \longrightarrow 0. \tag{4.2}$$

This extension class corresponds to the element $z_f^{(i)}[\iota] \in H^1(F, V_f(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i}))$ defined by the generalized Heegner cycle Δ_ι. We may also construct (4.2) as follows. Put that

$$\tilde{V}_X = H^1(X_1(N)_{\overline{F}}, j_* \mathrm{Sym}^{k-2} R^1 \pi_* \mathbb{Q}_p), \quad \tilde{W}_X = H^1((X_1(N) - P)_{\overline{F}}, j_* \mathrm{Sym}^{k-2} R^1 \pi_* \mathbb{Q}_p)$$

where $\pi : \mathcal{E} \to Y_1(N)$ is the universal elliptic curve and j is the inclusion map $Y_1(N) \to X_1(N)$. Then as in [31, II, Proposition 2.4], we have an exact sequence

$$0 \longrightarrow \tilde{V}_X \longrightarrow \tilde{W}_X \longrightarrow [\mathrm{Sym}^{k-2} H^1(A'_{\overline{F}}, \mathbb{Q}_p)](-1) \longrightarrow 0 \tag{4.3}$$

By taking the tensor product with $\mathrm{Sym}^{k-2} H^1(A_{\overline{F}}, \mathbb{Q}_p)(k-1)$, we have the bottom sequence of (4.1). Similarly, for $1 \le i \le k-1$, by taking the tensor product with $\mathbb{Q}_p(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i})$, we have

$$0 \longrightarrow \tilde{V}_X(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i}) \longrightarrow \tilde{W}_X(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i}) \longrightarrow V_{P,i} \longrightarrow 0 \tag{4.4}$$

where

$$V_{P,i} := \mathrm{Hom}_{\mathbb{Q}_p}(\mathbb{Q}_p(\psi_{\mathfrak{p}}^{k-1-i} \psi_{\mathfrak{p}^*}^{i-1}), \mathrm{Sym}^{k-2} V_p(A')).$$

The natural Galois equivariant map $(T_{\mathfrak{p}} A)^{\otimes i} \otimes_{\mathbb{Q}_p} (T_{\mathfrak{p}^*} A)^{\otimes k-i} \to \mathrm{Sym}^{k-2} V_p(A')$ induced by ι corresponds to a map of Galois representations $\mathrm{cl}_P(\Delta_\iota)_i : \mathbb{Q}_p \to V_{P,i}^{G_F} \subset V_{P,i}$. Then the pull-back of (4.4) by $\mathrm{cl}_P(\Delta_\iota)_i$ gives an extension

$$0 \longrightarrow \tilde{V}_X(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i}) \longrightarrow W_i \longrightarrow \mathbb{Q}_p \longrightarrow 0 \tag{4.5}$$

and its push-forward induced by the quotient $\tilde{V}_X \to V_f$ gives (4.2). There is also the integral version of (4.4). We put

$$\tilde{T}_X = H^1(X_1(N)_{\overline{F}}, j_* \mathrm{Sym}^{k-2} R^1 \pi_* \mathbb{Z}_p), \quad \tilde{U}_X = H^1((X_1(N) - P)_{\overline{F}}, j_* \mathrm{Sym}^{k-2} R^1 \pi_* \mathbb{Z}_p)$$

and

$$T_{P,i} := \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(\psi_{\mathfrak{p}}^{k-1-i} \psi_{\mathfrak{p}^*}^{i-1}), \mathrm{Sym}^{k-2} T_p(A')).$$

Then, we have an exact sequence

$$0 \longrightarrow \tilde{T}_X(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i}) \longrightarrow \tilde{U}_X(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i}) \longrightarrow T_{P,i}$$

and hence as the push-forward,

$$0 \longrightarrow T_f(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i}) \longrightarrow U_X(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i}) \longrightarrow T_{P,i}. \tag{4.6}$$

Here the cokernel of the last map is finite whose order is bounded independent of P. (cf. [31, II. (1.10), (5.5)].) Therefore, there exists a natural integer C depending only on N, k, p such that p^C kills the cokernel.

Let w be a basis of $\mathcal{O}_K$-module $H_1(A(\mathbb{C}), \mathbb{Z})$ of rank 1. We take a basis $u \in T_{\mathfrak{p}} A$, $v \in T_{\mathfrak{p}^*} A$ so that

$$u - v \in \mathbb{Z}_p w \subset T_p A = H_1(A(\mathbb{C}), \mathbb{Z}_p).$$

Put that $e = u \otimes v^{\otimes -1} \in T_{\mathfrak{p}} A \otimes (T_{\mathfrak{p}^*} A)^{\otimes -1}$. Note that e does not depend on the choice of u, v, and w. In Sect. 5 below, we see that for $H_{cp^\infty} = \cup_n H_{cp^n}$, the Galois group $G_{H_{cp^\infty}}$ fixes the element e. (cf. (5.1) and Proposition 5.8.) We write $z_f^{(i)}[\iota]$ by $z^{(i)}[cp^n]$ and let $w^{(i)}[cp^n]$ be the image of $z_f^{(i)}[cp^n]$ by the morphism

$$H^1(H_{cp^n}, V_f(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i})) \to H^1(H_{cp^\infty}, V_f(\psi_{\mathfrak{p}}^i \psi_{\mathfrak{p}^*}^{k-i})) \cong H^1(H_{cp^\infty}, V_f(\psi_{\mathfrak{p}} \psi_{\mathfrak{p}^*}^{k-1}))$$

where the last map is given by tensoring $e^{\otimes -(i-1)}$.

Proposition 4.1 *Suppose that $\iota : A \to A_{cp^n}$ is the natural projection over H_{cp^n} whose complex uniformizations with respect to ω_A and $\iota_*\omega_A$ give*

$$\mathbb{C}/\mathcal{O}_K\Omega_K \longrightarrow \mathbb{C}/\mathcal{O}_{cp^n}\Omega_K, \quad z \longmapsto cp^n z.$$

(i) *We have $\iota(u-v) \in p^n T_p A_{cp^n}$.*
(ii) *We put*

$$g_i : \mathbb{Q}_p \longrightarrow T_{P,1}, \quad 1 \longmapsto \left(u^{\otimes k-2} \mapsto \iota(u)^{\otimes k-2-i} \left(\frac{\iota(u-v)}{p^n} \right)^{\otimes i} \right).$$

Then by (4.6), the map $p^C g_i$ defines an element $r_i[cp^n] \in H^1(H_{cp^\infty}, T_f(\psi_{\mathfrak{p}}\psi_{\mathfrak{p}^*}^{k-1}))$ such that

$$p^C \sum_{j=0}^{i} (-1)^j \binom{i}{j} w^{(j)}[cp^n] = p^{ni} r_i[cp^n]. \tag{4.7}$$

Proof For simplicity, we put $F_\infty := H_{cp^\infty}$. Let f_i be the $\mathbb{Q}_p$-linear map

$$f_i : \mathbb{Q}_p \longrightarrow V_{P,1}, \quad 1 \longmapsto \left(u^{\otimes k-2} \mapsto \iota(u)^{\otimes k-2-i} \iota(v)^{\otimes i} \right).$$

Since $u^{\otimes k-2-i} \otimes v^{\otimes i} = u^{\otimes k-2} \otimes e^{\otimes -i}$, this is a morphism of G_{F_∞}-modules. Then the extension corresponding to $w_f^{(i)}[cp^n]$ is the pull-back of

$$0 \longrightarrow \tilde{V}_X(\psi_{\mathfrak{p}}\psi_{\mathfrak{p}^*}^{k-1}) \longrightarrow \tilde{W}_X(\psi_{\mathfrak{p}}\psi_{\mathfrak{p}^*}^{k-1}) \longrightarrow V_{P,1} \longrightarrow 0$$

by f_i. (cf. (4.4).) Hence the element

$$\sum_{j=0}^{i} (-1)^j \binom{i}{j} w_f^{(j)}[cp^n] \in H^1(F_\infty, V_f(\psi_{\mathfrak{p}}\psi_{\mathfrak{p}^*}^{k-1}))$$

corresponds to the extension by the pull-back of (4.4) by $\sum_{j=0}^{i}(-1)^j \binom{i}{j} f_j$.

The image of $\mathbb{Z}\Omega_K \subset H_1(A(\mathbb{C}), \mathbb{Z})$ by ι in $H_1(A_{cp^n}(\mathbb{C}), \mathbb{Z})$ is divisible by cp^n. Hence by our choice of u, v, we have $\iota(u-v) \in p^n T_p A_{cp^n}$. Therefore,

$$\sum_{j=0}^{i} (-1)^j \binom{i}{j} \iota(u)^{\otimes k-2-j} \iota(v)^{\otimes j} = \iota(u)^{\otimes k-2-i} \iota(u-v)^{\otimes i} \in p^{ni} \mathrm{Sym}^{k-2} T_p A_{cp^n}.$$

Hence, by (4.6), the map $p^C g_i$ defines the element $r_i[cp^n] \in H^1(H_{cp^\infty}, T_f(\psi_{\mathfrak{p}}\psi_{\mathfrak{p}^*}^{k-1}))$, and (4.7) holds.

Next, we prove a Frobenius relation for $r_i[cp^n]$, which we use in the proof of the main theorem of the sequel [24].

Let ℓ be an inert prime of K prime to $cD_K Np$ and let λ_ℓ be a place of $H_{c\ell}$ over ℓ. The elliptic curve A_c has good reduction at λ_ℓ and $A_{\ell c}$, too since it is isogenous to A_c. By the Néron mapping property, the natural isogeny $A_c \to A_{c\ell}$ of degree ℓ reduces to an isogeny $\tilde{A}_c \to \tilde{A}_{c\ell}$ of degree ℓ over the residue field $\mathbb{F}_{\ell^2}$. (Note that ℓ splits completely on H_c and totally ramified for $H_{c\ell}/H_c$.) Since $\tilde{A}_c$ is supersingular, there exists an isomorphism $\tilde{A}_{c\ell} \cong \tilde{A}_c^{\mathrm{Fr}_\ell}$ and the isogeny $\tilde{A}_c \to \tilde{A}_{c\ell} \cong \tilde{A}_c^{\mathrm{Fr}_\ell}$ must be the ℓ-th Frobenius map.

Let P_c (resp. $P_{c\ell}$) be a point of $X_1(N)$ corresponding to A_c (resp. $A_{c\ell}$) with a level structure. The representation $T_{P_c,i}$ is unramified at ℓ and we may identify it with

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(\psi_{\mathfrak{p}}^{k-1-i}\psi_{\mathfrak{p}^*}^{i-1}), \mathrm{Sym}^{k-2}T_p(\tilde{A}_c))$$

as a $G_{\mathbb{F}_{\ell^2}}$-representation. For a $G_{\mathbb{F}_{\ell^2}}$-equivariant map $h : \mathbb{Z}_p \to T_{P_c,i}$, let c_h be the cohomology class corresponding to the extension class obtained by the pull-back of (4.6) by $p^C h$. By composing $p^C h$ and the natural map $T_p\tilde{A}_c \to T_p\tilde{A}_{c\ell}$, we have a map $G_{\mathbb{F}_{\ell^2}}$-equivariant map $\mathbb{Z}_p \to T_{P_{c\ell},i}$. By the identification $\tilde{A}_{c\ell} \cong \tilde{A}_c^{\mathrm{Fr}_\ell}$, this map gives the cohomology class equal to $c_h^{\mathrm{Fr}_\ell}$. Applying this for cp^n as c, we have

$$\mathrm{Fr}_\ell(p^C \mathrm{loc}_\ell(z^{(i)}[cp^n])) = p^C \mathrm{loc}_\ell(z^{(i)}[\ell cp^n]), \quad \mathrm{Fr}_\ell(\mathrm{loc}_\ell(r_i[cp^n])) = \mathrm{loc}_\ell(r_i[\ell cp^n]) \tag{4.8}$$

for $1 \le i \le k-1$.

So far, we assumed $N > 4$ but as in [31, Chapter II, (3.7), (3.8)], we can eliminate this assumption.

5 Serre–Tate local moduli and anticyclotomic extension

In this section, first, we review the theory of Serre–Tate local moduli. We follow Katz's article [21] but slightly modify it to work on a finite residue field. (In [21], the residue field of deformations is assumed to be algebraically closed.) In most proofs, the finite residue field case is reduced to the algebraically closed residue field case by flat descent arguments. However, we work directly on finite residue fields for two reasons. First, in the finite residue field case, the formal group representing the Serre–Tate local moduli of an ordinary elliptic curve is not isomorphic to the formal multiplicative group. We see that it is the formal group whose division points produce anticyclotomic extensions. Second, at least in the classical setting, the Perrin-Riou theory is developed only for a finite unramified extension of $\mathbb{Q}_p$ because of the local duality pairing. (cf. [35, p.221, ERRATA].)

5.1 Relative Lubin–Tate formal groups of height 1

We recall relative Lubin–Tate formal groups of height 1. ([12, Chapter 1].) Let k be the finite field $\mathbb{F}_q$ of q-elements. Let W be the ring of Witt vectors $W(k)$ and $L = W[1/p]$ with the Frobenius σ. For an element $\xi \in \mathbb{Z}_p$ with $v_p(\xi) = v_p(q)$, we consider a power series $\varphi(T) \in W[[T]]$ satisfying the following two conditions:

- $\varphi(T) \equiv \pi T \mod \deg 2$ for an element $\pi \in W$ such that $N_{L/\mathbb{Q}_p}(\pi) = \xi$.
- $\varphi(T) \equiv T^p \mod p$.

Then there exists a unique one-dimensional formal group $\mathcal{G}_\xi$ over W that has a "Frobenius" $\mathcal{G}_\xi \to \mathcal{G}_\xi^\sigma$ induced by $\varphi(T)$. The isomorphism class of $\mathcal{G}_\xi$ over W depends only on ξ, and it is called the relative Lubin–Tate formal group corresponding to ξ. The parameter ξ is characterized from $\mathcal{G}_\xi$ as the eigenvalue of the q-th Frobenius on the Dieudonné module of the special fiber of $\mathcal{G}_\xi$. The isomorphism class of a formal group over W is determined by the isomorphism class associated with weakly admissible filtered φ-module. Hence, the relative Lubin–Tate formal group with parameter ξ is characterized as the formal group over W associated with the filtered φ-module D over L of rank 1 satisfying $\mathrm{Fil}^1 D = D$, $\mathrm{Fil}^2 D = \{0\}$

and the q-th power Frobenius acts by ξ. (If one chooses π as above, one may define the σ-semi-linear Frobenius action φ on D by putting $\varphi\omega = \pi\omega$ on a fixed generator ω of D. The isomorphism class of the filtered φ-module D does not depend on the choice of π and ω.)

5.2 The canonical lift

Let $\overline{E}$ be an *ordinary* elliptic curve defined over k. We denote the set-theoretical Tate module by $T_p\overline{E}$, which is a free $\mathbb{Z}_p$-module of rank 1. Let $u_q \in \mathbb{Z}_p^\times$ be the eigenvalue of the q-th Frobenius φ_q on $T_p\overline{E}$. We put $\xi = qu_q^{-1}$ and we denote $\mathcal{G}_\xi$ by $\mathcal{G}_{\overline{E}}$. The special fiber $\overline{\mathcal{G}}_{\overline{E}}$ of $\mathcal{G}_{\overline{E}}$ is isomorphic to the formal group of $\overline{E}$ since their filtered φ-modules are isomorphic. We regard naturally $\overline{E}[p^\infty]$ as an étale p-divisible group over W. We fix an isomorphism ε between $\overline{E}[p^\infty] \times \overline{\mathcal{G}}_{\overline{E}}[p^\infty]$ and the p-divisible group associated with $\overline{E}$. Then by [21, Theorem 1.2.1], there is an elliptic curve E over W whose p-divisible group is isomorphic to the p-divisible group $\overline{E}[p^\infty] \times \mathcal{G}_{\overline{E}}[p^\infty]$ over W. Since $\operatorname{End}\mathcal{G}_{\overline{E}}[p^\infty] \cong \operatorname{End}\overline{\mathcal{G}}_{\overline{E}}[p^\infty]$ by the natural map, the isomorphism class of the triple $(\overline{E}, \mathcal{G}_{\overline{E}}[p^\infty], \varepsilon)$ does not depend on the choice of ε, and $\operatorname{End} E \cong \operatorname{End}\overline{E}$ by the natural map. In particular, E is a CM elliptic curve. The elliptic curve E/W is called the canonical lift of $\overline{E}/k$. The n-times composition of Frobenius on the p-divisible group induces the Frobenius lift $F_{p^n} : E \to E^{\sigma^n}$ over W. Note that for an invariant differential ω_E of E, we have $F_q^*\omega_E = \xi\omega_E$.

5.3 The local moduli functor and Frobenii

We consider the moduli functor $\hat{\mathscr{M}}_{\overline{E}/k}$ that corresponds an artin local ring R with residue field k to the set of isomorphism classes of lifts of $\overline{E}/k$ over R. This functor is pro-representable by a formal scheme $\hat{\mathcal{M}}_{\overline{E}} := \operatorname{Spf}\mathfrak{R}_{\overline{E}}$ and let $\mathbb{E}/\mathfrak{R}_{\overline{E}}$ be the universal lift of $\overline{E}/k$. For simplicity, we sometimes write $\hat{\mathcal{M}}_{\overline{E}}$, $\mathfrak{R}_{\overline{E}}$ by $\hat{\mathcal{M}}$, $\mathfrak{R}$. The ring $\mathfrak{R}$ is non-canonically isomorphic to the one-variable formal power series ring over W. For any W-scheme X, we denote by $X^{(\sigma^n)}$ the W-scheme obtained from X/W by $\sigma^n : W \to W$ as fiber product:

$$\begin{array}{ccc} X^{(\sigma^n)} & \xrightarrow{\Sigma_n} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} W & \xrightarrow{\operatorname{Spec}(\sigma^n)} & \operatorname{Spec} W. \end{array}$$

Similarly, for formal schemes. By corresponding a deformation $\mathcal{E}/R$ of $\overline{E}/k$ to $\mathcal{E}^{(\sigma)}/R^{(\sigma)}$ of $\overline{E}^{(\sigma)}/k$, we have a bijection

$$\hat{\mathscr{M}}_{\overline{E}/k}(R) \cong \hat{\mathscr{M}}_{\overline{E}^{(\sigma)}/k}(R^{(\sigma)}),$$

and hence by taking $R = \mathfrak{R}_{\overline{E}}$ in the above, the tautological section gives an isomorphism $\hat{\mathcal{M}}_{\overline{E}/k}^{(\sigma)} \cong \hat{\mathcal{M}}_{\overline{E}^{(\sigma)}/k}$. For a deformation $\mathcal{E}/R$, we denote by $\mathcal{E}^{(n)}$ the quotient of $\mathcal{E}$ by the canonical subgroup $\hat{\mathcal{E}}[p^n]$, and by F_{p^n} the projection $\mathcal{E} \to \mathcal{E}^{(n)}$, which is a lift of the Frobenius $F_{p^n} : \overline{E} \to \overline{E}^{(\sigma^n)}$. (Note that the notation is compatible with that in Sect. 5.2.) Since $\mathcal{E}^{(n)}$ is a deformation of $\overline{E}^{(\sigma^n)}$, this defines a morphism

$$\varphi_{p^n} : \hat{\mathcal{M}}_{\overline{E}/k} \to \hat{\mathcal{M}}_{\overline{E}^{(\sigma^n)}/k} \cong \hat{\mathcal{M}}_{\overline{E}/k}^{(\sigma^n)}$$

and $\mathbb{E}^{(n)}$ is the pull-back of $\mathbb{E}^{(\sigma^n)}$ by φ_{p^n}. Hence we have a morphism of W-algebra $\mathfrak{R}^{(\sigma^n)} \to \mathfrak{R}$, which is also denoted by φ_{p^n} by abuse of notation. We also write φ_{p^n} by φ if $n = 1$.

For $\omega \in \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}})$, the invariant differential form $\varphi^*_{p^n}\Sigma^*_n\omega$ may be regarded as an invariant form on $\mathbb{E}^{(n)}$ by the identification $\mathbb{E}^{(n)} \cong \varphi^*_{p^n}\mathbb{E}^{(\sigma^n)}$. We denote $F^*_{p^n}\varphi^*_{p^n}\Sigma^*_n\omega \in \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}})$ simply by $\Phi^*_n\omega$ and by $\Phi^*\omega$ if $n = 1$. (Note that in [21], the letter Φ is used for φ in the above.)

5.4 The group structure on the local moduli

By the (relative) Lubin–Tate theory, the formal group law on $\mathfrak{R}$ over W is defined in a unique way so that the Frobenius φ is a group homomorphism. However, for our later purpose, we define it more geometrically. First, we construct a free W-submodule M of rank 1 in $\hat{\Omega}^1_{\mathfrak{R}/W}$, and then construct the group law so that M is the space of invariant differential forms.

Let ∇ be the Gauss–Manin connection

$$\nabla : H^1_{\mathrm{dR}}(\mathbb{E}/\mathfrak{R}) \longrightarrow \hat{\Omega}^1_{\mathfrak{R}/W}\hat{\otimes}_{\mathfrak{R}} H^1_{\mathrm{dR}}(\mathbb{E}/\mathfrak{R}).$$

By the principal polarization, we regard $\overline{E} \cong \overline{E}^\vee$ and $\mathbb{E} \cong \mathbb{E}^\vee$. Then the Kodaira–Spencer map

$$\mathrm{KS} : \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}})^{\otimes 2} \longrightarrow \hat{\Omega}^1_{\mathfrak{R}/W}$$

is given by $\mathrm{KS}(\omega \otimes \eta) := \langle \nabla(\omega), \eta\rangle$ where $\langle\ ,\ \rangle$ is the Poincaré pairing on $H^1_{\mathrm{dR}}(\mathbb{E}/\mathfrak{R})$.

Proposition 5.1 *There exists a functorial map of W-modules*

$$s_{\mathbb{E}} : \Gamma(E, \Omega^1_{E/W}) \longrightarrow \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}}) \subset H^1_{\mathrm{dR}}(\mathbb{E}/\mathfrak{R}), \qquad \omega_E \longmapsto \omega_{\mathbb{E}}$$

characterized by the following properties. It is a section of the specialization map $\Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}}) \to \Gamma(E, \Omega^1_{E/W})$ at the origin, and Frobenius compatible, that is, the differential $\omega_{\mathbb{E}}$ satisfies

$$\Phi^*_q\omega_{\mathbb{E}} = \xi\omega_{\mathbb{E}}.$$

The map $s_{\mathbb{E}}$ induces isomorphisms of W-modules

$$\Gamma(E, \Omega^1_{E/W}) \cong \{\omega \in \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}}) \mid \Phi^*_q\omega = \xi\omega\}$$

and

$$\Gamma(E, \Omega^1_{E/W}) \otimes_W \mathfrak{R} \cong \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}}).$$

Proof This is a modification of Corollary 4.1.5 of [21] and the proof is essentially the same. Let ω_E be an invariant differential of E/W. We take a basis v of $T_p\overline{E}^\vee$. Then v induces isomorphism $\iota_E : \hat{E} \cong \hat{\mathbb{G}}_m$ over $W(\overline{k})$ and $\iota_{\mathbb{E}} : \hat{\mathbb{E}} \cong \hat{\mathbb{G}}_m$ over $\mathfrak{R}\hat{\otimes}_W W(\overline{k})$. Then we have the p-adic period $\Omega_{E,v} \in W(\overline{k})^\times$ satisfying $\iota^*_E(dt/(1+t)) = \Omega_{E,v}\omega_E$ and $\sigma_q\Omega_{E,v} = u_q\Omega_{E,v}$. We put

$$\omega_{\mathbb{E}} = \Omega^{-1}_{E,v}\iota^*_{\mathbb{E}}(dt/(1+t)) \in \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}})\hat{\otimes}_W W(\overline{k}).$$

Since the action of σ_q on $T_p\overline{E}^\vee \cong \mathrm{Hom}(\hat{\mathbb{E}}, \hat{\mathbb{G}}_m)$ is given by u_q, the differential $\omega_{\mathbb{E}}$ is fixed by σ_q. Hence it is defined over W. Note also that $\omega_{\mathbb{E}}$ does not depend on the choice of v. Since $\Phi^*_p\iota^*_{\mathbb{E}}(dt/(1+t)) = p\iota^*_{\mathbb{E}}(dt/(1+t))$, the Frobenius compatibility holds. Suppose that

$\omega \in \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}})$ satisfies $\Phi_q^*\omega = \xi\omega$. Write as $\omega = g\omega_{\mathbb{E}}$ with $g \in \mathfrak{R}$. Then the Frobenius compatibility implies $\Phi_q^* g = g$. Hence g is an element of W and $\{\omega \in \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}}) \mid \Phi_q^*\omega = \xi\omega\}$ is a free W-module rank 1 generated by $\omega_{\mathbb{E}}$.

Let M be the image of $\Gamma(E, \Omega^1_{E/W})^{\otimes 2}$ in $\hat{\Omega}^1_{\mathfrak{R}/W}$ by the composition of $s_{\mathbb{E}}$ and the Kodaira–Spencer map. Let ω_E be a generator of W-module $\Gamma(E, \Omega^1_{E/W})$ and let $u_p \in W^\times$ be such that $F_p^*\Sigma^*\omega_E = pu_p^{-1}\omega_E$. Let $\omega_{\mathbb{E}}$ be the lift of ω_E by Proposition 5.1. Since the Frobenius action is compatible with Gauss–Manin connection and Poincaré pairing (with Tate twist “(-1)" on the target), we have

$$\varphi^*(\langle \nabla(\omega_{\mathbb{E}}), \omega_{\mathbb{E}} \rangle) = p^{-1}\langle \nabla(\Phi^*\omega_{\mathbb{E}}), \Phi^*\omega_{\mathbb{E}} \rangle = pu_p^{-2}\langle \nabla(\omega_{\mathbb{E}}), \omega_{\mathbb{E}} \rangle.$$

Hence we naturally have a structure of strongly divisible filtered module on $M \subset \Omega^1_{\mathfrak{R}/W}$ by $\mathrm{Fil}^1 M = M$, $\mathrm{Fil}^2 M = \{0\}$ and $\varphi\omega_{\hat{\mathcal{M}}} = pu_p^{-2}\omega_{\hat{\mathcal{M}}}$ where $\omega_{\hat{\mathcal{M}}} := \mathrm{KS}(\omega_{\mathbb{E}}^{\otimes 2}) = \langle \nabla(\omega_{\mathbb{E}}), \omega_{\mathbb{E}} \rangle$. If we choose a non-canonical isomorphism $\mathfrak{R} \cong W[\![X]\!]$ such that $\omega_{\hat{\mathcal{M}}}/dX|_{X=0} = 1$, we can associate the formal group structure on $\mathrm{Spf}\,\mathfrak{R}$ that makes $\omega_{\hat{\mathcal{M}}}$ is an invariant differential. (The formal group law is given by $F(X, Y) := f^{-1}(f(X) + f(Y))$ where f is the formal primitive of $\omega_{\hat{\mathcal{M}}}$ with $f(0) = 0$. Note that we have $F(X, Y) \in W[\![X, Y]\!]$ by the Honda theory (cf. [18]).) Since $\varphi(X) = pu_p^{-2}X \mod \deg 2$, the formal group is the relative Lubin–Tate corresponding to $\xi = qu_q^{-2}$ with $u_q := N_{L/\mathbb{Q}_p}u_p$. The group structure is also characterized by the property that φ becomes a group homomorphism $\hat{\mathcal{M}} \to \hat{\mathcal{M}}^\sigma$. In particular, the group structure does not depend on the choice of ω_E and the isomorphism $\mathfrak{R} \cong W[\![X]\!]$.

5.5 The Tate module of $\hat{\mathcal{M}}$ and the Serre–Tate coordinate

By definition, $\hat{E}$ is the relative Lubin–Tate group corresponding to the parameter $\xi = qu_q^{-1}$ with the strongly divisible lattice $D(\hat{E}) := \Gamma(E, \Omega^1_{E/W})$. By construction, the Kodaira–Spencer map (composed with the map $s_{\mathbb{E}}$) gives an isomorphism of filtered φ-modules,

$$D(\hat{E})^{\otimes 2}(1) \cong M = D(\hat{\mathcal{M}}).$$

Hence it induces the functorial isomorphism of Galois representations of G_L,

$$T_p\hat{\mathcal{M}} \cong T_p\hat{E}^{\otimes 2}(-1) = \mathrm{Hom}_{\mathbb{Z}_p}(T_p\overline{E}^{\otimes 2}, \mathbb{Z}_p(1)). \tag{5.1}$$

Let $\kappa_{\hat{E}}$ (resp. κ_{cyc}) be the relative Lubin–Tate character for $\hat{E}$ (resp. the cyclotomic character). Then the relative Lubin–Tate character $\kappa_{\hat{\mathcal{M}}}$ for $\hat{\mathcal{M}}$ is $\kappa_{\hat{E}}^2\kappa_{cyc}^{-1}$.

For a generator $\overline{v} \in T_p\overline{E}$, the map (5.1) induces an isomorphism $T_p\hat{\mathcal{M}} \to \mathbb{Z}_p(1)$ of Galois representations over $W(\overline{k})[1/p]$. Hence we have an isomorphism $\iota : \hat{\mathcal{M}} \to \hat{\mathbb{G}}_m$ as formal groups over $W(\overline{k})$.

We explain that the isomorphism ι coincides with the Serre–Tate coordinate, essentially the main theorem of [21]. For the reader's convenience, first, we recall the Serre–Tate coordinate. Take a generator $(\overline{v}, \overline{v}^\vee) \in T_p\overline{E} \times T_p\overline{E}^\vee$. Suppose that a lift $\mathcal{E}/R$ of $\overline{E}/\overline{k}$ is given. Here R is an artin local ring with the residue field $\overline{k}$. The natural projection $T_p\mathcal{E}^\vee \to T_p\overline{E}^\vee$ is surjective, and the Weil pairing for $\mathcal{E}$ gives the isomorphism $T_p\overline{E}^\vee \cong \mathrm{Hom}(T_p\hat{\mathcal{E}}, \mathbb{Z}_p(1)) = \mathrm{Hom}_R(\hat{\mathcal{E}}, \hat{\mathbb{G}}_m)$. Let ρ be the isomorphism $\hat{\mathcal{E}} \to \hat{\mathbb{G}}_m$ associated to $\overline{v}^\vee$. Write that $\overline{v} = (\overline{v}_n)_n$ with $\overline{v}_n \in \overline{E}[p^n]$. Since $\mathcal{E}(R) \to \overline{E}(\overline{k})$ is surjective, we can take a lift $v_n \in \mathcal{E}(R)$ of $\overline{v}_n$. Then

$p^n v_n \in \hat{\mathcal{E}}(R)$. Since $\cap_m [p^m]\hat{\mathcal{E}}(R) = \{0\}$, the limit $x := \lim_{n\to\infty} p^n v_n$ exists and it is easy to see the well-definedness. Then $\rho(x) \in \hat{\mathbb{G}}_m(R)$ is the Serre–Tate coordinate of $\mathcal{E}/R$ with respect to $(\overline{v}, \overline{v}^\vee)$, which is denoted by $q(\mathcal{E}/R, \overline{v}, \overline{v}^\vee)$.

As before, we identify $\overline{E}^\vee$ and E by the principal polarization. Then, for a generator $\overline{v} \in T_p\overline{E}$, the Serre–Tate coordinate

$$q(\mathbb{E}/\mathfrak{R}^{sh}, \overline{v}, \overline{v}) \in 1 + \mathfrak{m}_{\mathfrak{R}^{sh}}$$

defines a map of formal schemes $\hat{\mathcal{M}} \to \hat{\mathbb{G}}_m$ over $W(\overline{k})$. Here $\mathfrak{R}^{sh} := \mathfrak{R}\hat{\otimes}_W W(\overline{k})$. (i.e. the completion of the strict henselizaion.) We call this map the Serre–Tate map. The following is a reformulation of the main theorem of [21].

Proposition 5.2 *The isomorphism ι coincides with the Serre–Tate map, that is, ι sends the tautological section of $\hat{\mathcal{M}}(\mathfrak{R})$ to the Serre–Tate parameter. In particular, the Serre–Tate map is a group isomorphism, and $\mathfrak{R}^{sh} = W(\overline{k})[\![T]\!]$ with $T = q(\mathbb{E}/\mathfrak{R}^{sh}, \overline{v}, \overline{v}) - 1$. The group structure of $\hat{\mathcal{M}}$ is the unique structure that makes the Serre–Tate map a group homomorphism.*

Proof The element

$$\overline{v} \in T_p\overline{E}^\vee = \mathrm{Hom}_{W(\overline{k})}(\hat{E}, \hat{\mathbb{G}}_m) = \mathrm{Hom}_{W(\overline{k})}(\hat{\mathbb{E}}, \hat{\mathbb{G}}_m)$$

defines invariant differential forms $\omega_{\mathbb{E}}$ on $\mathbb{E}$ and ω_E on E by the pull-back by $\overline{v}$ of the invariant differential $\omega_{\hat{\mathbb{G}}_m}$ on $\hat{\mathbb{G}}_m$. By the extension of scalars, the map $s_{\mathbb{E}}$ is extended over $W(\overline{k})$. Then $s_{\mathbb{E}}$ sends ω_E to $\omega_{\mathbb{E}}$. The main theorem of [21] is that

$$\mathrm{KS}(\omega_{\mathbb{E}}^{\otimes 2}) = \frac{dT}{1+T}. \tag{5.2}$$

Hence the map

$$\iota^* : \quad D(\hat{\mathbb{G}}_m) \xrightarrow{(\overline{v}^{\otimes 2})^*} D(\hat{E})^{\otimes 2}(1) \xrightarrow{\mathrm{KS}\circ s_{\mathbb{E}}} D(\hat{\mathcal{M}})$$

coincides with the one induced by the Serre–Tate map.

Remark 5.3 In [21], Katz first defined the Serre–Tate coordinate, then computed the Kodaira–Spencer map in terms of the Serre–Tate coordinate. We took the reverse order, that is, we first defined the formal group $\hat{\mathcal{M}}$ via the Kodaira–Spencer map and then related it to the Serre–Tate coordinate. The advantage of our approach is that the formal group structure on $\hat{\mathcal{M}}$ is directly defined over W and the relation between $\hat{\mathcal{M}}$ and the anticyclotomic extension becomes apparent in the following.

5.6 A moduli interpretation

The isomorphism (5.1) recovers the moduli property of $\hat{\mathcal{M}}$ as follows. Let $\mathcal{E}/R$ be a deformation of $\overline{E}/k$. Then there exists an exact sequence of fppf sheaves on R,

$$0 \longrightarrow \widehat{\mathcal{E}}[p^n] \longrightarrow \mathcal{E}[p^n] \longrightarrow \overline{E}[p^n] \longrightarrow 0.$$

From this, we have

$$0 \longrightarrow \underline{\mathrm{Hom}}(\overline{E}[p^n]^{\otimes 2}, \mu_{p^n}) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{E}[p^n]\underline{\otimes}\overline{E}[p^n], \mu_{p^n}) \longrightarrow \underline{\mathrm{Hom}}(\widehat{\mathcal{E}}[p^n]\underline{\otimes}\overline{E}[p^n], \mu_{p^n}) \longrightarrow 0. \tag{5.3}$$

Here the tensor product $\underline{\otimes}$ and the $\underline{\mathrm{Hom}}$ are taken as fppf sheaves. By sending 1 to the Weil pairing $\widehat{\mathcal{E}}[p^n] \times \overline{E}[p^n] \to \mu_{p^n}$, we have a morphism of fppf sheaves

$$\mathbb{Z}/p^n\mathbb{Z} \longrightarrow \underline{\mathrm{Hom}}(\widehat{\mathcal{E}}[p^n] \otimes \overline{E}[p^n], \mu_{p^n}).$$

By the pull-back of (5.3) by this morphism, we obtain an extension of fppf sheaves

$$0 \longrightarrow \underline{\mathrm{Hom}}(\overline{E}[p^n]^{\otimes 2}, \mu_{p^n}) \longrightarrow \mathscr{E} \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow 0.$$

By (5.1), the sheaf $\underline{\mathrm{Hom}}(\overline{E}[p^n]^{\otimes 2}, \mu_{p^n})$ is representable by $\hat{\mathcal{M}}[p^n]$. This extension defines an element in the flat cohomology group $H^1_{\mathrm{fl}}(\mathrm{Spf}\ R,\ \hat{\mathcal{M}}[p^n])$, which is isomorphic to $\hat{\mathcal{M}}(R)/p^n$ by the Kummer map and the Hilbert theorem 90. (Note that since $H^i(k, \hat{\mathcal{M}}(R^{s\hbar})) = 0$ for $i > 0$ (cf. [5, Proposition 3.9]), the proof of Hilbert 90 in our case is reduced to the case of the formal multiplicative group.) Hence by taking limit for n, we have an element $x(\mathcal{E}/R) \in \hat{\mathcal{M}}(R)$.

Proposition 5.4 *The element $-x(\mathcal{E}/R) \in \hat{\mathcal{M}}(R)$ corresponds to the deformation $\mathcal{E}/R$.*

Proof To show this, we may work over $W(\overline{k})$ by scalar extensions. We fix a generator $\overline{v} = (\overline{v}_n)_n$ of $T_p\overline{E}$. It suffices to show that $-x(\mathcal{E}/R)$ is sent to the Serre–Tate coordinate of $\mathcal{E}/R$ by the Serre–Tate map $\hat{\mathcal{M}} \to \hat{\mathbb{G}}_m$. Considering the universal deformation case and then by (infinitely many) specializations, we may reduce to the case that $R = W(\overline{k})$. We identify $T_p\overline{E} \cong \mathbb{Z}_p$ and $\hat{\mathcal{E}}[p^n] \cong \mu_{p^n}$ by $\overline{v}$. Then the extension class of

$$0 \longrightarrow \widehat{\mathcal{E}}[p^n] \longrightarrow \mathcal{E}[p^n] \longrightarrow \overline{E}[p^n] \longrightarrow 0$$

defines an element of $H^1_{\mathrm{fl}}(\mathrm{Spf}\ R,\ \hat{\mathcal{E}}[p^n]) \cong H^1_{\mathrm{fl}}(\mathrm{Spf}\ R,\ \mu_{p^n}) = (1+\mathfrak{m}_R)/(1+\mathfrak{m}_R)^{p^n}$. This is the image of $x = x(\mathcal{E}/R)$ by the Serre–Tate map. We compute it in

$$H^1_{\mathrm{fl}}(\mathrm{Spf}\ R,\ \hat{\mathcal{E}}[p^n]) \hookrightarrow H^1_{\mathrm{fl}}(\mathrm{Spec}\ R[1/p],\ \hat{\mathcal{E}}[p^n]) = H^1_{\mathrm{\acute{e}t}}(R[1/p],\ \hat{\mathcal{E}}[p^n])$$

where the last cohomology is the Galois (étale) cohomology of the field $R[1/p]$. (cf. [14, Lemme 3.6].) Let $v_n \in \mathcal{E}(R)$ be a lift of $\overline{v}_n$. Then the Serre–Tate coordinate of $\mathcal{E}/R$ is the limit of $p^n v_n \in \hat{\mathcal{E}}(R) \cong \hat{\mathbb{G}}_m(R)$. By the definition of the Kummer map, the image of $p^n v_n$ in $H^1_{\mathrm{\acute{e}t}}(R[1/p],\ \hat{\mathcal{E}}[p^n])$ is the cocycle class $\sigma \longmapsto \sigma w_n - w_n$ where $w_n \in \hat{\mathcal{E}}(R')$ such that $p^n w_n = p^n v_n \in \hat{\mathcal{E}}(R)$ for some finite flat extension of R' of R. We put $z_n = v_n - w_n$, then z_n is a p^n-torsion point of $\mathcal{E}$ and a lift of $\overline{v}_n$. Hence the element $x \in H^1_{\mathrm{\acute{e}t}}(R[1/p],\ \hat{\mathcal{E}}[p^n])$ may be represented by the cocycle

$$\sigma \longmapsto \sigma z_n - z_n = -\sigma w_n + w_n.$$

(Note that $\sigma v_n = v_n$ since $v_n \in \mathcal{E}(R)$.)

5.7 The moduli interpretation of translation by torsion points

We describe the moduli theoretic description of the addition $x \oplus y$ on $\hat{\mathcal{M}}$ when x or y is a torsion point.

Let $\mathcal{E}/R$ be a deformation of E/k. In the following, for $\overline{P} \in \overline{E}[p^n]$, we consider two kinds of lifts of $\overline{P}$. One is a lift in $\mathcal{E}[p^n]$ over a finite flat extension of R by using $\mathcal{E}[p^n]/\hat{\mathcal{E}}[p^n] \cong \overline{E}[p^n]$, which we call a finite order lift and denote by P^f though there are several choices of P^f. The other is a lift in $\mathcal{E}(R^{s\hbar})$ which we call an unramified lift and denote by P^u though there are several choices of P^u.

Let x and y be elements of $\hat{\mathcal{M}}(R)$, and suppose that x is torsion of order p^n and y corresponds to the elliptic curve $\mathcal{E}/R$. By the isomorphism (5.1), the element x defines an element of

$$\mathrm{Hom}_{\mathbb{Z}_p}(\overline{E}[p^n]^{\otimes 2}, \mu_{p^n}) = \mathrm{Hom}_{\mathbb{Z}_p}(\overline{E}[p^n], \mathrm{Hom}(\overline{E}[p^n], \mu_{p^n})) = \mathrm{Hom}_{\mathbb{Z}_p}(\overline{E}[p^n], \hat{\mathcal{E}}[p^n]),$$

which is also denoted by x by abuse of notation. We put $\mathcal{E}_0 = \mathcal{E}/\hat{\mathcal{E}}[p^n]$. For $\overline{P} \in \overline{E}[p^n]$, let $x'(\overline{P})$ be an element of $\hat{\mathcal{E}}[p^{2n}]$ such that $[p^n]x'(\overline{P}) = x(\overline{P})$. We denote the image of the subgroup of $\mathcal{E}_0$ generated by

$$P^f - x'(\overline{P}) \mod \hat{\mathcal{E}}[p^n] \qquad (\overline{P} \in \overline{E}[p^n])$$

by $C_{x,y}$. Note that it does not depend on the choices of $x'(\overline{P})$ and P_f, and $C_{x,y}$ can be defined over R. The reduction of $C_{x,y}$ is the unique subgroup of order p of $\overline{E}^{\sigma^n}$. We put $\mathcal{E}_{x,y} := \mathcal{E}_0/C_{x,y}$ and consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{E} & \longrightarrow & \mathcal{E}_0 & \longrightarrow & \mathcal{E}_{x,y} \\ \downarrow{\scriptstyle \mathrm{red}} & & \downarrow{\scriptstyle \mathrm{red}} & & \downarrow{\scriptstyle \mathrm{red}} \\ \overline{E} & \xrightarrow{F_{p^n}} & \overline{\mathcal{E}_0} \cong \overline{E}^{\sigma^n} & \xrightarrow{F_{p^n}^{\vee}} & \overline{\mathcal{E}_{x,y}} \cong \overline{E}. \end{array} \tag{5.4}$$

The vertical arrows are the reduction maps. Hence the elliptic curve $\mathcal{E}_{x,y}$ is also a deformation of $\overline{E}/k$.

Proposition 5.5 (i) *Let x and y be as above. The deformation corresponding to the point $x \oplus y \in \hat{\mathcal{M}}$ is $\mathcal{E}_{x,y}$.*

(ii) *Let ω_E be an invariant differential for E/W and $\omega_{\mathbb{E}} = s_{\mathbb{E}}(\omega_E)$ the lift to the universal deformation. Let $\pi_{x,y}$ be the isogeny $\mathcal{E} \to \mathcal{E}_{x,y}$ in (5.4). Let $\omega_{\mathcal{E}}$ and $\omega_{\mathcal{E}_{x,y}}$ be the pull-back of $\omega_{\mathbb{E}}$ to $\mathcal{E}$ and $\mathcal{E}_{x,y}$ by the universality. Then we have $\pi_{x,y}^*\omega_{\mathcal{E}_{x,y}} = p^n\omega_{\mathcal{E}}$, or equivalently, $\omega_{\mathcal{E}_{x,y}} = p^{-n}(\pi_{x,y})_*\omega_{\mathcal{E}}$.*

Proof For $\overline{v} = (\overline{v}_n)_n \in T_p\overline{E}$, we consider the following commutative diagram

$$\begin{array}{ccccc} \hat{\mathcal{E}} & \longrightarrow & \hat{\mathcal{E}}_0 & \longrightarrow & \hat{\mathcal{E}}_{x,y} \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota_0} & & \downarrow{\scriptstyle \iota_{x,y}} \\ \hat{\mathbb{G}}_m & \xrightarrow{p^n} & \hat{\mathbb{G}}_m & = & \hat{\mathbb{G}}_m \end{array} \tag{5.5}$$

where ι, $\iota_{x,y}$ are trivializations induced by $\overline{v}$ and ι_0 by $\sigma^n(\overline{v}) \in \overline{E}^{\sigma^n}[p^n]$. Let R^{sh} be the strict henselization of R. By (5.4), as a lift of $\overline{v}_m \in \overline{\mathcal{E}_{x,y}}[p^m]$ to $\mathcal{E}_{x,y}(R^{sh})$, we may take $\pi_x(v^u_{m+n}) \in E_x$. Then the Serre–Tate coordinate of $\mathcal{E}_{x,y}$ is given by $\lim_{m\to\infty} \iota_{x,y}(p^m\pi_{x,y}(v^u_{m+n}))$. On the other hand, by the definition of $C_{x,y}$, the element $\pi_{x,y}(v^f_n)$ is equal to $\pi_{x,y}(x'(\overline{v}_n))$. Then by $(v^u_{m+n} - v^f_{m+n}), x'(\overline{v}_n) \in \hat{\mathcal{E}}$ and (5.5), we have

$$\begin{aligned} \iota_{x,y}(p^m\pi_{x,y}(v^u_{m+n})) &= \iota_{x,y}(p^m\pi_{x,y}(v^u_{m+n} - v^f_{m+n})) \cdot \iota_{x,y}(p^m\pi_{x,y}(v^f_{m+n})) \\ &= \iota(p^{m+n}v^u_{m+n}) \cdot \iota_{x,y}(\pi_{x,y}(x'(\overline{v}_n))) = \iota(p^{m+n}v^u_{m+n}) \cdot \iota(x(\overline{v}_n)). \end{aligned}$$

The Serre–Tate coordinate of $\mathcal{E}$ is $\lim_{m\to\infty} \iota(p^{m+n}v^u_{m+n})$ and by Proposition 5.2, $\iota(x(\overline{v}_n))$ is also the Serre–Tate coordinate of the deformation associated with x. The assertion follows.

For (ii), we may assume that $\omega_{\mathcal{E}} = \iota^*(\omega_{\hat{\mathbb{G}}_m})$ and $\omega_{\mathcal{E}_{x,y}} = \iota_x^*(\omega_{\hat{\mathbb{G}}_m})$. Then by (5.5), we have $\pi_x^*\omega_{\mathcal{E}_{x,y}} = p^n\omega_{\mathcal{E}}$. (ii) follows from this. (Note that $\deg \pi_x = p^{2n}$.)

Let $x = (x_n)_n$ be an element of $T_p\hat{\mathcal{M}}$. We have an isomorphism of formal groups $\varphi_n^\vee : \hat{\mathcal{M}} \to \hat{\mathcal{M}}^{\sigma^{-n}}$ such that $\varphi_n \circ \varphi_n^\vee = [p^n]_{\hat{\mathcal{M}}}$. (cf. [12, Chapter I, Proposition 1.5].) We put $\varpi_n = \varphi_n^\vee(x_n) \in \hat{\mathcal{M}}^{\sigma^{-n}}[p^n]$. The system $(\varpi_n)_n$ satisfies that $\varphi(\varpi_{n+1}) = \varpi_n$, and we call it the system of φ-power torsion points associated to the system of p-power torsion points $x \in T_p\hat{\mathcal{M}}$. We give the interpretation of ϖ_n as a deformation.

Since E is the canonical lift, we have the splitting $s : \overline{E}[p^n] \to E[p^n]$ defined by the decomposition $T_pE = T_p\overline{E} \times T_p\hat{E}$, which is compatible with the Galois action over $L = W[1/p]$. It may also be described as follows. For $\overline{P} \in \overline{E}[p^n]$, take $\overline{P}_{n+m} \in \overline{E}[p^{n+m}]$ such that $p^m\overline{P}_{n+m} = \overline{P}$. Then consider a lift $P_{n+m} \in E(W(\overline{k}))$ of $\overline{P}_{n+m}$. The section $s(\overline{P})$ is defined to be the limit of p^mP_{n+m} when $m \to \infty$. (Note that $p^{n+m}P_{n+m} \to 0$ since the Serre–Tate coordinate of the canonical lift is 1, and the limit depends only on $\overline{P}_n$ since $\hat{E}(W(\overline{k})) \cap E[p^\infty] = \{O\}$.)

The element x_n is regarded as a map

$$x_n \in \mathrm{Hom}_{\mathbb{Z}_p}(\overline{E}[p^n], \hat{E}[p^n]).$$

Let C_n be the étale subgroup

$$C_n := \{s(P_n) - x_n(P_n) \mid P_n \in \overline{E}[p^n]\} \subset E[p^n].$$

Clearly, $pC_{n+1} = C_n$ and E/C_n is a deformation of $\overline{E}^{\sigma^{-n}}$. (The isomorphism on the fiber is given by $F_{p^n}^\vee : \overline{E}/\overline{E}[p^n] \cong \overline{E}^{\sigma^{-n}}$.) For an invariant differential form ω_E of E over W, let $\omega_{E^{\sigma^{-n}}} := \Sigma_{-n}^*\omega_E$ be the twist of ω_E by σ^{-n} on $E^{\sigma^{-n}}$ and let $\omega_{\mathbb{E}^{\sigma^{-n}}}$ be the Frobenius compatible lift of $\omega_{E^{\sigma^{-n}}}$ on the universal deformation.

Proposition 5.6 *Let E' be the deformation of $\overline{E}^{\sigma^{-n}}$ corresponding to $\varpi_n \in \hat{\mathcal{M}}^{\sigma^{-n}}[p^n]$, that is, $E' = (E^{\sigma^{-n}})_0/C_{\varpi_n}$ (cf. Proposition 5.5* (i)). *Let ω' be the pull-back of $\omega_{\mathbb{E}^{\sigma^{-n}}}$ on E' by the universal property. Then there exists a unique isomorphism $\iota : E' \to E/C_n$ such that the following diagram commutative:*

$$\begin{array}{ccc} E^{\sigma^{-n}} & \xrightarrow{F_{p^n}} & E \\ \Big\downarrow{\scriptstyle \pi_n} & & \Big\downarrow{\scriptstyle \pi} \\ E' & \xrightarrow{\iota} & E/C_n \end{array}$$

where π_n and π are the canonical projections. In particular, E/C_n is the deformation of $\overline{E}^{\sigma^{-n}}$ corresponding to ϖ_n. If $F_{p^n}^*\omega_E = \varrho_n\omega_{E^{\sigma^{-n}}}$, then $\pi_*\omega_E = \varrho_n\iota_*\omega'$.

Proof The map $\varphi_n : T_p\hat{\mathcal{M}} \to T_p\hat{\mathcal{M}}^{\sigma^n}$ is given by

$$\varphi_n : \mathrm{Hom}(T_p\overline{E}, T_p\hat{E}) \to \mathrm{Hom}(T_p\overline{E}^{\sigma^n}, T_p\hat{E}^{\sigma^n}), \quad x \mapsto \left(P \mapsto F_{p^n}x(F_{p^n}^{-1}P)\right).$$

Then since $\varphi_n \circ \varphi_n^\vee = [p^n]_{\hat{\mathcal{M}}}$, the map corresponding to ϖ_n is given by

$$\overline{E}^{\sigma^{-n}}[p^n] \longrightarrow \hat{E}^{\sigma^{-n}}[p^n], \quad Q \longmapsto F_{p^n}^\vee(x_n(F_{p^n}Q)).$$

Hence the image of C_{ϖ_n} by F_{p^n} is

$$\{s(F_{p^n}\overline{Q}) - x_n(F_{p^n}\overline{Q}) \mid \overline{Q} \in \overline{E}^{\sigma^{-n}}[p^n]\},$$

that is, C_n. The assertion follows from this and Proposition 5.5 (ii).

Let c be a natural number such that $(c, p) = 1$ and let A_c be the elliptic curve defined in Sect. 3.2. Let k be the residue field of H_c, and we take $\overline{E}$ as the reduction of A_c. Then A_c is the canonical lift of $\overline{E}$, and we may take E as A_c. As in Sect. 4, let w be a generator of the $\mathcal{O}_c$-module $H_1(A_c, \mathbb{Z})$ and consider generators of $\mathbb{Z}_p$-modules $u \in T_\mathfrak{p} A_c$ and $v \in T_{\mathfrak{p}^*} A_c$ so that $u - v \in \mathbb{Z}_p w \subset T_p A_c = H_1(A_c, \mathbb{Z}_p)$, and put that

$$e = u \otimes v^{\otimes -1} \in T_\mathfrak{p} A_c \otimes (T_{\mathfrak{p}^*} A_c)^{\otimes -1} = T_p \hat{\mathcal{M}}. \tag{5.6}$$

Let $(\varpi_n)_n$ be the φ-power system associated to e.

Proposition 5.7 *The deformation corresponding to ϖ_n is A_{cp^n}.*

Proof The torsion point ϵ_n of $\hat{\mathcal{M}}[p^n] = \mathrm{Hom}(\overline{A}_c[p^n], \hat{A}_c[p^n])$ corresponds to the map

$$x_n \in \mathrm{Hom}_{\mathbb{Z}_p}(\overline{A}_c[p^n], \hat{A}_c[p^n]), \quad v_n \longmapsto u_n.$$

Hence by Proposition 5.6, the deformation corresponding to ϖ_n is A_c/C_n with

$$C_n := \{s(P_n) - x_n(P_n) \mid P_n \in \overline{A}_c[p^n]\} = \mathbb{Z}w_n.$$

If we identify A_c with $\mathbb{C}/\mathcal{O}_c$, then we may take $w_n = \frac{1}{p^n} \in \frac{1}{p^n}\mathcal{O}_c/\mathcal{O}_c$. Therefore, $A_c/C_n \cong A_{cp^n}$.

5.8 Local moduli and anticyclotomic extensions

We explain the relation between the localization of anticyclotomic extensions and the torsion tower for the local moduli.

As before, let K be an imaginary quadratic field, and p splits in $\mathcal{O}_K$. We fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and let $\mathfrak{p}$ be a prime over p compatible with ι_p. We denote the closure of a field $F \subset \overline{\mathbb{Q}}$ in $\mathbb{C}_p$ by $\hat{F}$. For $p \nmid c$, let A_c be the elliptic curve with $\mathrm{End}\,\overline{A}_c \cong \mathcal{O}_c$ as before. For simplicity, we denote $\hat{\mathcal{M}}_{\overline{A}_c}$ by $\hat{\mathcal{M}}_c$. Let K_∞ be the anticyclotomic $\mathbb{Z}_p$-extension of K. Since p splits, $\hat{K}_\infty$ is a ramified $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$. We show that this extension is contained in the field obtained by adjoining torsion points of $\hat{\mathcal{M}}_c$ to $\hat{H}_c$.

Proposition 5.8 *The j-invariant $j(\mathcal{O}_{cp^n})$ is contained in $\hat{H}_c(\hat{\mathcal{M}}_c[p^n])$. In particular, we have $\hat{H}_{cp^n} \subset \hat{H}_c(\hat{\mathcal{M}}_c[p^n])$. Furthermore,*

$$[\hat{H}_c(\hat{\mathcal{M}}_c[p^n]) : \hat{H}_{cp^n}] = [\mathcal{O}_c^\times : \mathcal{O}_{cp^n}^\times] \le \sharp\mathcal{O}_K^\times/2.$$

In particular, if $c \neq 1$ or $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, the ring class field tower locally coincides with the torsion tower of the local moduli $\hat{\mathcal{M}}_c$, that is, $\hat{H}_{cp^n} = \hat{H}_c(\hat{\mathcal{M}}_c[p^n])$.

Proof The first assertion follows from Proposition 5.7. (Note that $\hat{\mathcal{M}}_c^{\sigma^{-n}} \cong \hat{\mathcal{M}}_c$ by $\varphi_n^\vee$.) It is known that

$$[H_f : H] = f[\mathcal{O}_K^\times : \mathcal{O}_f^\times] \prod_{\ell \mid f} \left(1 - \left(\frac{d_K}{\ell}\right)\frac{1}{\ell}\right)$$

for the order $\mathcal{O}_f$ of conductor f. (cf. [10, Theorem 7.24].) We have $[\hat{H}_{cp^n} : \hat{H}_c] = [H_{cp^n} : H_c] = p^{n-1}(p-1)[\mathcal{O}_c^\times : \mathcal{O}_{cp^n}^\times]$. On the other hand, by the theory of formal groups, we have $[\hat{H}_c(\hat{\mathcal{M}}_c[p^n]) : \hat{H}_c] = p^{n-1}(p-1)$. The assertion follows from these.

5.9 Theta operator on local moduli

As before, let $\overline{E}/k$ be an ordinary elliptic curve over a finite field $k = \mathbb{F}_q$ with $\mathrm{End}_{\overline{k}}(\overline{E}) \cong \mathcal{O}_c$ for a natural number c prime to p. Let E be the canonical lift of $\overline{E}$ over $W := W(k)$ and let $\mathbb{E}/\mathfrak{R}$ be the universal deformation of $\overline{E}/k$.

We fix an invariant differential ω_E of E and let $\omega_{\mathbb{E}}$ be the lift of ω_E on $\mathbb{E}$ by $s_{\mathbb{E}}$. Let $u_p \in W^\times$ be such that $F_p^*\omega_{E^\sigma} = pu_p^{-1}\omega_E$. As before, let $\omega_{\hat{\mathcal{M}}}$ be the invariant differential of $\hat{\mathcal{M}}$ defined by

$$\mathrm{KS}(\omega_{\mathbb{E}}^{\otimes 2}) = \omega_{\hat{\mathcal{M}}}$$

and $\partial_{\hat{\mathcal{M}}}$ be the differential operator on $\mathfrak{R}$ associated to $\omega_{\hat{\mathcal{M}}}$. That is, $dg = \partial_{\hat{\mathcal{M}}}(g)\omega_{\hat{\mathcal{M}}}$. We define $\xi_{\mathbb{E}} \in H^1_{\mathrm{dR}}(\mathbb{E}/\mathfrak{R})$ by

$$\nabla(\omega_{\mathbb{E}}) = \xi_{\mathbb{E}} \otimes \omega_{\hat{\mathcal{M}}} \in H^1_{\mathrm{dR}}(\mathbb{E}/\mathfrak{R}) \otimes_{\mathfrak{R}} \hat{\Omega}^1_{\mathfrak{R}/W}.$$

Lemma 5.9 $\xi_{\mathbb{E}}$ *lies in the unit root space, that is,* $\Phi_p^*\xi_{\mathbb{E}} = u_p\xi_{\mathbb{E}}$, *and* $\langle \omega_{\mathbb{E}}, \xi_{\mathbb{E}} \rangle = 1$. *These property characterizes* $\xi_{\mathbb{E}}$. *In particular,* $\omega_{\mathbb{E}}, \xi_{\mathbb{E}}$ *become a basis of the* $\mathfrak{R}$*-module* $H^1_{\mathrm{dR}}(\mathbb{E}/\mathfrak{R})$. *We also have* $\nabla(\xi_{\mathbb{E}}) = 0$.

Proof Since $\mathrm{KS}(\omega_{\mathbb{E}}^{\otimes 2}) = \langle \omega_{\mathbb{E}}, \nabla(\omega_{\mathbb{E}}) \rangle$, we have $\langle \omega_{\mathbb{E}}, \xi_{\mathbb{E}} \rangle = 1$. Since ∇ is compatible with the Frobenius structure and $\varphi_p^*\omega_{\hat{\mathcal{M}}^\sigma} = pu_p^{-2}\omega_{\hat{\mathcal{M}}}$, we have $\Phi_p^*\xi_{\mathbb{E}} = u_p\xi_{\mathbb{E}}$. Since the image of $\nabla \circ (\Phi_q^*)^m$ is divisible by q^m, we have $\nabla(\xi_{\mathbb{E}}) = 0$.

The quotient by the unit root space generated by $\xi_{\mathbb{E}}$ defines a splitting

$$s_u : H^1_{\mathrm{dR}}(\mathbb{E}/\mathfrak{R}) \to \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}})$$

of the Hodge filtration as $\mathfrak{R}$-module. This is also obtained by the pull-back map on the formal group $H^1_{\mathrm{dR}}(\mathbb{E}/\mathfrak{R}) \to H^1_{\mathrm{dR}}(\hat{\mathbb{E}}/\mathfrak{R}) \cong \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}})$. For a natural number n, we put $\mathbb{L}_n = \mathrm{Sym}^n_{\mathfrak{R}} H^1_{\mathrm{dR}}(\mathbb{E}/\mathfrak{R})$ and naturally extend the connection ∇ to $\mathbb{L}_n \to \mathbb{L}_n \otimes_{\mathfrak{R}} \hat{\Omega}^1_{\mathfrak{R}/W}$. The splitting s_u also naturally defines $\mathbb{L}_n \to \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}})^{\otimes n}$, which is also denoted by s_u by abuse of notation. Then the theta operator ϑ is defined by the composition

$$\vartheta : \quad \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}})^{\otimes n} \longrightarrow \mathbb{L}_n \xrightarrow{\nabla} \mathbb{L}_n \otimes_{\mathfrak{R}} \hat{\Omega}^1_{\mathfrak{R}/W} \xrightarrow{\mathrm{id}\otimes \mathrm{KS}^{-1}} \mathbb{L}_{n+2} \xrightarrow{s_u} \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}})^{\otimes (n+2)}.$$

Lemma 5.10 *For* $g \in \mathfrak{R}$, *we have* $\vartheta(g\omega_{\mathbb{E}}^{\otimes k}) = \partial_{\hat{\mathcal{M}}}(g)\omega_{\mathbb{E}}^{\otimes (k+2)}$.

Proof The assertion follows from $\nabla(g\omega_{\mathbb{E}}^{\otimes k}) = dg \otimes \omega_{\mathbb{E}}^{\otimes k} + kg\omega_{\mathbb{E}}^{\otimes (k-1)} \otimes \xi_{\mathbb{E}} \otimes \omega_{\hat{\mathcal{M}}}$.

Hence if we identify $\Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}})^{\otimes n}$ with $\mathfrak{R}$ by the basis $\omega_{\mathbb{E}}^{\otimes n}$, the operator ϑ is $\partial_{\hat{\mathcal{M}}}$ on $\mathfrak{R}$.

5.10 The $\boldsymbol{\psi}$-operator on $\mathfrak{R}$

Let ψ be the unique σ^{-1}-semilinear map on $\mathfrak{R}$ satisfying $\psi \circ \varphi = 1$ and

$$\varphi \circ \psi(g) = p^{-1} \sum_{P \in \hat{\mathcal{M}}[\varphi]} t_P^* g$$

for $g \in \mathfrak{R}$, where t_P^* is the pull-back by translation by P with respect to the addition on $\hat{\mathcal{M}}$. (Note that $\hat{\mathcal{M}}[\varphi] = \hat{\mathcal{M}}[p]$.)

We give the moduli theoretic interpretation of ψ. To give a σ^{-1}-semilinear map on $\mathfrak{R}$ is equivalent to give a map from $\mathrm{Mor}(\hat{\mathcal{M}}^{\sigma^{-1}}, \hat{\mathbb{G}}_a)$ to $\mathrm{Mor}(\hat{\mathcal{M}}, \hat{\mathbb{G}}_a)$ where Mor means morphisms of formal schemes (not group homomorphisms). For $f \in \mathrm{Mor}(\hat{\mathcal{M}}^{\sigma^{-1}}, \hat{\mathbb{G}}_a)$ and for $x \in \hat{\mathcal{M}}$ corresponding to a deformation $\mathcal{E}/R$, we put

$$\tilde{\psi}(f)(x) := \sum_C f(\mathcal{E}/C) \ \in R'$$

where C runs through étale subgroups of $\mathcal{E}$ of order p, and R' is a finite flat extension of R. Then by Proposition 5.5, we have

$$(\varphi^*(\tilde{\psi})(f))(x) = \sum_{C'} f(\mathcal{E}_0/C') = \sum_{y \in \hat{\mathcal{M}}[\varphi]} f(x \oplus y)$$

where C' runs through étale subgroups of $\mathcal{E}_0$ of order p. By the general theory of formal groups, the right-hand side is of the form $p\varphi^*(g)(x)$ for $g \in \mathrm{Mor}(\hat{\mathcal{M}}^{\sigma^{-1}}(R), \hat{\mathbb{G}}_a(R))$. Hence we have $\tilde{\psi}(f)(x) \in pR$. The above argument also shows that $p^{-1}\tilde{\psi}$ has the characterization property of ψ. Hence

Proposition 5.11 *The operator ψ is the map that associates $f \in \mathrm{Mor}(\hat{\mathcal{M}}^{\sigma^{-1}}, \hat{\mathbb{G}}_a)$ to the map*

$$\hat{\mathcal{M}} \to \hat{\mathbb{G}}_a, \quad \mathcal{E} \mapsto \frac{1}{p}\sum_C f(\mathcal{E}/C)$$

where C runs through étale subgroups of $\mathcal{E}$ of order p.

Lemma 5.12 $\partial_{\hat{\mathcal{M}}} : \mathfrak{R}^{\psi=0} \to \mathfrak{R}^{\psi=0}$ *is bijective.*

Proof The kernel is $W \cap \mathfrak{R}^{\psi=0} = \{0\}$. For the surjectivity, it is sufficient to show it after the scalar extension to $W(\bar{k})$ and we may use the Serre–Tate coordinate. We fix a basis $\overline{v} \in T_p\overline{E}$ and consider the Serre–Tate coordinate $q(\mathbb{E}/\mathfrak{R}, \overline{v}) = 1 + t$. Then $\mathfrak{R}^{s\hbar} = W(\bar{k})[\![t]\!]$. Let ι be the Serre–Tate map $\hat{\mathcal{M}} \to \hat{\mathbb{G}}_m$ over $W(\bar{k})$ associated to $\overline{v}$. Take the p-adic period $\Omega_{\hat{\mathcal{M}},p} \in W(\bar{k})^\times$ such that $\iota^*(\frac{dt}{1+t}) = \Omega_{\hat{\mathcal{M}},p}\omega_{\hat{\mathcal{M}}}$. As operators on $W(\bar{k})[\![t]\!]$, we have

$$\partial_{\hat{\mathcal{M}}} = \Omega_{\hat{\mathcal{M}},p}(1+t)\frac{d}{dt}, \qquad \varphi(t) = (1+t)^p - 1.$$

Hence the assertion is reduced to the well-known case.

For $g \in \mathfrak{R}$, we let

$$g^\flat = (\psi \circ \varphi - \varphi \circ \psi)g = (1 - \varphi \circ \psi)g.$$

Then clearly, we have $g^\flat \in \mathfrak{R}^{\psi=0}$. If we regard $g \in W(\bar{k})[\![t]\!]$ with the Serre–Tate coordinate and μ is the corresponding measure on $\mathbb{Z}_p$, then $g^\flat$ is the power series corresponding to the measure on $\mathbb{Z}_p$ obtained by the extension of $\mu|_{\mathbb{Z}_p^\times}$ by zero.

5.11 The Galois action on the local moduli

First, we recall the following general fact.

Lemma 5.13 *Let ι be an isogeny of elliptic curves $\overline{A} \to \overline{B}$ over k of degree c prime to p. It induces an isomorphism $\iota_{\hat{\mathcal{M}}} : \hat{\mathcal{M}}_{\overline{A}} \to \hat{\mathcal{M}}_{\overline{B}}$ as formal groups characterized by the following equivalent properties* (1), (2)*:*

(1) *Let $\mathcal{A}/R$ be a deformation of $\overline{A}/k$ and let $\mathcal{B}/R$ be the corresponding deformation of $\overline{B}/k$ by $\iota_{\hat{\mathcal{M}}}$. Then there exists a unique isogeny $\mathcal{A} \to \mathcal{B}$ over R of degree c compatible with ι on the special fiber.*
(2) *Let $(\overline{v}, \overline{w})$ be an element of $T_p\overline{A} \times T_p\overline{B}^{\vee}$. Then*

$$q(\mathbb{A}/\mathfrak{R}^{sh}, \overline{v}, {}^t\iota(\overline{w})) = \iota_{\mathfrak{R}}(q(\mathbb{B}/\mathfrak{R}^{sh}, \iota(\overline{v}), \overline{w}))$$

where $\iota_{\mathfrak{R}}$ is the morphism on universal deformation rings $\mathfrak{R}^{sh}_{\overline{B}} \to \mathfrak{R}^{sh}_{\overline{A}}$ naturally induced by $\iota_{\hat{\mathcal{M}}}$, and ${}^t\iota$ is the dual isogeny of ι.

Proof Since c is prime to p, for a deformation $\mathcal{A}/R$ of $\overline{A}/k$, the reduction map induces an isomorphism $\mathcal{A}[c] \cong \overline{A}[c]$. Hence the kernel C of ι is uniquely lifted to a subgroup of $\mathcal{A}[c]$. We associate $\mathcal{A}/R$ to the deformation $\mathcal{B}$ as the quotient of $\mathcal{A}$ by C and the isomorphism on the special fiber by $\overline{\mathcal{B}} = \overline{A}/C \cong \overline{B}$ induced by ι. The equivalence (1) and (2) follows from the following diagram and [21, Theorem 2.1, 4)].

$$\begin{array}{ccccc} \mathbb{A} & \longrightarrow & \mathbb{A}/C & \longrightarrow & \mathbb{B} \\ \downarrow & & \downarrow & \square & \downarrow \\ \operatorname{Spf}\mathfrak{R}_{\overline{A}} & = & \operatorname{Spf}\mathfrak{R}_{\overline{A}} & \longrightarrow & \operatorname{Spf}\mathfrak{R}_{\overline{B}}. \end{array} \qquad (5.7)$$

By (2), $\iota_{\hat{\mathcal{M}}}$ is compatible with the Serre–Tate map, hence it is a homomorphism. By considering the dual isogeny of ι, it is straightforward to show that $\iota_{\hat{\mathcal{M}}}$ is an isomorphism.

Let c be an integer prime to pd_K. We let $\mathcal{G}_{cp^n} := \mathrm{Gal}(H_{cp^n}/K)$ and $\Gamma_n := \mathrm{Gal}(H_{cp^n}/H_c)$. ($n$ may be ∞.) Now we consider a Galois action of $\mathcal{G}_{cp^n}$ on the local moduli. For this purpose, as in [6], we assume that the discriminant D_K is odd or $8 \mid D_K$, and take the CM elliptic curve A_c more precisely. (We can also consider the case $4 \parallel D_K$ in the below if we assume the existence of a CM elliptic curve A that is $\mathbb{Q}$-curve satisfying the Shimura condition and good at all places over p. However, in the sequel [24], we only consider the case that D_K is odd or $8 \mid D_K$ because we use results of [6].) Then by [36], there is a canonical Hecke character $\varphi_K : I_K(\mathfrak{f}) \to \mathbb{C}^\times$ of conductor $\mathfrak{f}$ satisfying

(1) $\varphi_K(\overline{\mathfrak{a}}) = \overline{\varphi_K(\mathfrak{a})}$ for all $\mathfrak{a} \in I_K(\mathfrak{f})$.
(2) $\varphi_K(\alpha\mathcal{O}_K) = \pm\alpha$ for every $\alpha \in K^\times$ prime to $\mathfrak{f}$.
(3) The conductor $\mathfrak{f}$ is divisible only by primes ramified in $K/\mathbb{Q}$.

Up to an ideal class character of K, there is a unique canonical Hecke character if D_K is odd and there are two if $8 \mid D_K$. (If $4 \parallel D_K$, there is no Hecke character satisfying the above conditions (especially, (2)) but a variant is considered in [41].) We define the Hecke character ψ_H of H by $\psi_H = \varphi_K \circ N_{H/K}$. We also denote the Grössencharacter $H^\times\backslash\mathbb{A}_H^\times \to \mathbb{C}^\times$ associated to ψ_H by the same letter. Then the Serre–Tate character $\tilde{\psi}_H$ has the open kernel and $\tilde{\psi}_H(\alpha) = N_{H/K}\alpha$ for any principal idelé $\alpha \in H^\times$. Hence there is an elliptic curve A with $\mathrm{End}_{\mathbb{C}}(A) = \mathcal{O}_K$ defined over $H^+ := \mathbb{Q}(j(\mathcal{O}_K))$ such that $j(A) = j(\mathcal{O}_K)$ and its Serre–Tate character is $\tilde{\psi}_H$. (cf. [17, Theorem 9.1.3, Theorem 10.1.3].) The elliptic curve A is a $\mathbb{Q}$-curve in the sense of [17], and by construction, it satisfies the Shimura condition, i.e. $H(A_{\mathrm{tor}})/K$ is abelian. (cf. [11, Condition (S)].) In particular, the Weil restriction $B := \mathrm{Res}_{H/K} A$ is a CM abelian variety. (cf. [15, §4])

Let $\mathfrak{P}$ be the prime of H_c compatible with the fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and let k be the residue field of H_c at $\mathfrak{P}$. For $\tau \in \mathcal{G}_c$, let $\mathfrak{R}_\tau$ be the universal deformation ring for $\overline{A}_c^\tau/k$. We let $\tilde{\mathfrak{R}} = \prod_{\tau \in \mathcal{G}_c} \mathfrak{R}_\tau$. We define an action of $\mathcal{G}_{cp^\infty}$ on $\tilde{\mathfrak{R}}$.

Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ prime to d_K and let $\sigma_{\mathfrak{a}}$ be the element of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ associated to $\mathfrak{a}$ by the Artin reciprocity map. Since A satisfies the Shimura condition, we have an isogeny $\lambda(\mathfrak{a}) : A \to A^{\sigma_{\mathfrak{a}}}$ such that $\sigma_{\mathfrak{a}}(P) = \lambda(\mathfrak{a})(P)$ for $P \in A[\mathfrak{b}]$ with $(\mathfrak{a}, \mathfrak{b}) = 1$. Suppose further that $(\mathfrak{a}, c) = 1$. Then $\lambda(\mathfrak{a})(\ker(\pi_c)) = \sigma_{\mathfrak{a}}(\ker(\pi_c))$ and hence $\lambda(\mathfrak{a})$ induces an isogeny $A_c \to A_c^{\sigma_{\mathfrak{a}}}$ over H_c, which is also denoted by $\lambda(\mathfrak{a})$. Let $\lambda_{\mathfrak{a}}$ be the composition $\lambda(\mathfrak{a}) \circ \pi_c : A \to A_c^{\sigma_{\mathfrak{a}}}$.

Suppose that $\sigma \in \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is represented by an integral ideal $\mathfrak{a}$ prime to pcD_K as $\sigma = \sigma_{\mathfrak{a}}$. Then by Lemma 5.13, the isogeny $\lambda^\tau(\mathfrak{a}) : A_c^\tau \to A_c^{\sigma\tau}$ induces a morphism of formal groups $\hat{\mathcal{M}}_{\overline{A}_c^\tau} \to \hat{\mathcal{M}}_{\overline{A}_c^{\sigma\tau}}$ over $W(k)$ or in other words, a ring homomorphism $[\sigma_{\mathfrak{a}}] : \mathfrak{R}_{\sigma\tau} \to \mathfrak{R}_\tau$. If $\sigma_{\mathfrak{a}}$ fixes elements of H_c, the action $[\sigma_{\mathfrak{a}}] : \mathfrak{R}_\tau \to \mathfrak{R}_\tau$ coincides with the relative Lubin–Tate action of $\hat{\mathcal{M}}_{\overline{A}_c^\tau}$, that is, $[\sigma_{\mathfrak{a}}] = [\kappa_\tau(\sigma_{\mathfrak{a}})]$ with the Lubin–Tate character κ_τ of $\hat{\mathcal{M}}_{\overline{A}_c^\tau}$. We define the action of $\mathcal{G}_{cp^\infty}$ on $\tilde{\mathfrak{R}}$ as the unique continuous action extending that of $\sigma_{\mathfrak{a}}$ for integral ideals $\mathfrak{a}$, which are dense in $\mathcal{G}_{cp^\infty}$. Then we have a Galois-compatible ring isomorphism

$$\tilde{\mathfrak{R}} \cong \mathfrak{R} \otimes_{\mathbb{Z}_p[\![\Gamma_\infty]\!]} \mathbb{Z}_p[\![\mathcal{G}_{cp^\infty}]\!], \quad (x_\tau)_\tau \mapsto \sum_\tau [\tilde{\tau}] x_\tau \otimes \tilde{\tau}^{-1} \tag{5.8}$$

where $\tilde{\tau}$ is any extension of τ to $\mathcal{G}_{cp^\infty}$. We define the action of φ, ψ diagonally on $\tilde{\mathfrak{R}}$, which corresponds to $\varphi \otimes 1$, $\psi \otimes 1$ on $\mathfrak{R} \otimes_{\mathbb{Z}_p[\![\Gamma_\infty]\!]} \mathbb{Z}_p[\![\mathcal{G}_{cp^\infty}]\!]$.

We fix a generator $\overline{\boldsymbol{v}} = (\overline{v}, \overline{v}^\vee)$ of $T_p\overline{A} \oplus T_p\overline{A}^\vee$, and put

$$t := q(\mathbb{A}/\mathfrak{R}^{s\hbar}, \overline{v}, \overline{v}^\vee) - 1.$$

For a non-zero integral ideal $\mathfrak{a}$, we also put

$$t_{\mathfrak{a}} := q(\mathbb{A}^{\sigma_{\mathfrak{a}}}/\mathfrak{R}^{s\hbar}_{\sigma_{\mathfrak{a}}}, \lambda_{\mathfrak{a}}(\overline{v}), {}^t\lambda_{\mathfrak{a}}^{-1}(\overline{v}^\vee)) - 1. \tag{5.9}$$

We regard $t_{\mathfrak{a}}$ as an element of $\tilde{\mathfrak{R}}^{s\hbar} = \prod_\tau \mathfrak{R}^{s\hbar}_\tau$ by putting it in the component $\mathfrak{R}^{s\hbar}_{\sigma_{\mathfrak{a}}}$ and 0 in other components.

Lemma 5.14 (i) *Under the ring isomorphism in (5.8), the component $\mathfrak{R}^{s\hbar}_{\sigma_{\mathfrak{a}}} \subset \tilde{\mathfrak{R}}^{s\hbar}$ corresponds to the submodule*

$$\mathfrak{R}^{s\hbar} \otimes \sigma_{\mathfrak{a}}^{-1} \subset \mathfrak{R}^{s\hbar} \otimes_{\mathbb{Z}_p[\![\Gamma_\infty]\!]} \mathbb{Z}_p[\![\mathcal{G}_{cp^\infty}]\!].$$

Furthermore, the element $t_{\mathfrak{a}} \in \tilde{\mathfrak{R}}^{s\hbar}$ corresponds to $t \otimes \sigma_{\mathfrak{a}}^{-1}$.

(ii) *If $\sigma_{\mathfrak{a}}$ fixes elements of H_c, we have*

$$[\sigma_{\mathfrak{a}}]\left(q(\mathbb{A}/\mathfrak{R}^{s\hbar}, \overline{v}, \overline{w})\right) = q(\mathbb{A}/\mathfrak{R}^{s\hbar}, \overline{v}, \overline{w})^{\kappa_r(\sigma_{\mathfrak{a}})} \tag{5.10}$$

where $\kappa_r = \kappa_{\hat{\mathcal{M}}}^{-1}$. (Note that κ_r is the local reciprocity map $\mathrm{Gal}(H_{cp^\infty}/H_c) \cong \mathbb{Z}_p^\times$.*)*

Proof By Lemma 5.13, we have

$$[\sigma_{\mathfrak{a}}]\left(q(\mathbb{A}^{\sigma_{\mathfrak{a}}}/\mathfrak{R}^{s\hbar}_{\sigma_{\mathfrak{a}}}, \lambda_{\mathfrak{a}}(\overline{v}), \overline{w})\right) = q(\mathbb{A}/\mathfrak{R}^{s\hbar}, \overline{v}, {}^{t}\lambda_{\mathfrak{a}}(\overline{w})) \tag{5.11}$$

for $(\overline{v}, \overline{w}) \in T_p\overline{A} \times T_p(\overline{A}^{\vee})^{\sigma_{\mathfrak{a}}}$. Thus, if $\sigma_{\mathfrak{a}}$ fixes elements of H_c, we have

$$[\sigma_{\mathfrak{a}}]\left(q(\mathbb{A}/\mathfrak{R}^{s\hbar}, \overline{v}, \overline{w})^{\kappa_A(\sigma_{\mathfrak{a}})}\right) = q(\mathbb{A}/\mathfrak{R}^{s\hbar}, \overline{v}, \overline{w})^{\kappa_{\mathrm{cyc}}(\sigma_{\mathfrak{a}})\kappa_A(\sigma_{\mathfrak{a}})^{-1}}. \tag{5.12}$$

The assertion follows from these.

As in the proof of Lemma 5.12, we have $\varphi(t) = (1+t)^p - 1$. Hence $1 + t \in (\mathfrak{R}^{st})^{\psi=0}$. We identify

$$(1+t) \otimes 1 \ \in \ (\mathfrak{R}^{s\hbar})^{\psi=0} \otimes_{\mathbb{Z}_p[\![\Gamma_\infty]\!]} \mathbb{Z}_p[\![\mathcal{G}_{cp^\infty}]\!]$$

with $1 + t \in (\mathfrak{R}^{s\hbar})^{\psi=0} \subset (\tilde{\mathfrak{R}}^{s\hbar})^{\psi=0}$.

Proposition 5.15 *Suppose that $c > 1$. Then $\tilde{\mathfrak{R}}^{\psi=0}$ is a free $W[\![\mathcal{G}_{cp^\infty}]\!]$-module of rank 1. As a $W(\overline{k})[\![\mathcal{G}_{cp^\infty}]\!]$-module, $(\tilde{\mathfrak{R}}^{s\hbar})^{\psi=0}$ is free of rank 1 generated by $1 + t$.*

Proof By descent theory, it suffices to show the last assertion. We have

$$W(\overline{k})[\![t]\!]^{\psi=0} = W(\overline{k})[\![\mathrm{Gal}(\hat{K}^{ur}_\infty/\hat{K}^{ur})]\!](1+t)$$

where $\hat{K}^{ur}_\infty = \hat{K}^{ur}(\hat{\mathcal{M}}_c[p^\infty])$. (cf. [7, Theorem 3], [16, Theorem 2.6], [23, Proposition 3.11]). Then the assertion follows from Proposition 5.8 and (5.10).

Remark 5.16 The action on $\mathfrak{R}^{s\hbar}$ with the Serre–Tate coordinate is the inverse of the Lubin–Tate character, which is the opposite of classical normalization. For example, in the cyclotomic setting, γ acts on $(1 + t)$ by the cyclotomic character. In the appendix, we also use the classical normalization following Perrin-Riou. We write $\mathfrak{R}^{s\hbar}$ with our action by ${}^{\iota}\mathfrak{R}^{s\hbar}$ if we use the classical normalization.

6 Logarithmic Coleman power series interpolating generalized Heegner cycles

In this section, we construct the logarithmic Coleman power series interpolating generalized Heegner cycles.

First, we recall the classical Coleman power series theory and Perrin-Riou theory to compare them with our theory (cf. [7, 33]). Let $\mathbb{Q}_{p,n}$ be the cyclotomic field $\mathbb{Q}_p(\zeta_{p^{n+1}})$ and $\mathbb{Q}_{p,\infty} := \cup_n \mathbb{Q}_{p,n}$. We put $G^{cyc}_\infty := \mathrm{Gal}(\mathbb{Q}_{p,\infty}/\mathbb{Q}_p)$. Fix a basis $\xi = (\zeta_{p^{n+1}})_n$ of $\mathbb{Z}_p(1)$. Let U_n be the group of the principal units in $\mathbb{Q}_{p,n}$, and $U_\infty = \varprojlim_n U_n$. Then for $u = (u_n)_n \in U_\infty$, there exists a power series, $f_{\xi,u} \in 1 + p\mathbb{Z}_p[\![t]\!]$, called the Coleman power series associated to u, such that $f_{\xi,u}(\zeta_{p^{n+1}} - 1) = u_n$. We have $\left(1 - \frac{\varphi}{p}\right) \log f_{\xi,u} \in \mathbb{Z}_p[\![t]\!]^{\psi=0}$. Here φ is the operator defined by $\varphi(t) = (1+t)^p - 1$ and ψ is the left inverse of φ. Then there is an exact sequence of G_∞-modules

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow U_\infty \xrightarrow{\log^\flat \circ \mathrm{Col}} \mathbb{Z}_p[\![t]\!]^{\psi=0} \longrightarrow \mathbb{Z}_p(1) \longrightarrow 0 \tag{6.1}$$

where $\log^\flat \circ \mathrm{Col}$ is the map sending u to $\left(1-\frac{\varphi}{p}\right)\log f_{\xi,u}$. It is known that

$$\mathbb{Z}_p[\![t]\!]^{\psi=0} = \mathbb{Z}_p[\![G_\infty^{cyc}]\!](1+t)$$

where the action of $g \in G_\infty^{cyc}$ is given by $g \cdot (1+t) = (1+t)^{\kappa_{cyc}(g)}$ with the cyclotomic character κ_{cyc}. Let η be a non-trivial tame even character of G_∞^{cyc} and let e_η be the idempotent of $\mathbb{Z}_p[\![G_\infty^{cyc}]\!]$ associated to η. Then the image of a system of the η-part of cyclotomic units by $\log^\flat \circ \mathrm{Col}$ is $\mathscr{L}_{p,\eta}^{cyc} \cdot (1+t)$ where $\mathscr{L}_{p,\eta}^{cyc}$ is the Kubota–Leopoldt p-adic L-function in $e_\eta \mathbb{Z}_p[\![G_\infty^{cyc}]\!]$.

In [33], Perrin-Riou developed a certain integral exponential theory interpolating Bloch–Kato exponential maps for the cyclotomic deformation of crystalline representations of $G_{\mathbb{Q}_p}$. (The base field may also be taken as a finite unramified extension of $\mathbb{Q}_p$.) Her exponential is a generalization of the inverse of $\log^\flat \circ \mathrm{Col}$ in (6.1). More precisely, for a crystalline representation V of $G_{\mathbb{Q}_p}$, let $D_p(V)$ be the filtered φ-module associated to V. Let $h \geq 1$ be a natural number such that $\mathrm{Fil}^{-h} D_p(V) = D_p(V)$ and for simplicity, we assume that $V^{G_{\mathbb{Q}_p(\zeta_{p^\infty})}} = \{0\}$ and $D_p(V)^{\varphi=p^{-j}} = \{0\}$ for $j \geq 0$. Then for an element $g \in D_p(V) \otimes \mathbb{Z}_p[\![t]\!]^{\psi=0}$, she constructed a family of local points $c_{h,n}(g) \in H^1_{\mathrm{f}}(\mathbb{Q}_{p,n}, V)$ $(n = 0, 1, \dots)$ with bounded denominators for n. More precisely, first, take a (unique by our assumption) solution of $(1-\varphi)G = g$ in $D_p(V) \otimes \mathscr{H}_\infty$ where $\mathscr{H}_\infty \subset \mathbb{Q}_p[\![t]\!]$ is a certain convergent power series ring on the open unit disc. (cf. (6.4)). Then for a suitable Galois stable lattice T, it can be shown that $p^{(n+1)(h-1)} G(\zeta_{p^{n+1}} - 1)$ is in the image of $H^1_{\mathrm{f}}(\mathbb{Q}_{p,n}, T)$ for all n under the Bloch–Kato logarithm. Hence there is an element $c_{h,n}(g)$ such that $\log_V c_{h,n}(g) = p^{(n+1)(h-1)} G(\zeta_{p^{n+1}} - 1)$, and G (resp. g) is an analogue of $\log f_{\xi,u}$ (resp. $\left(1-\frac{\varphi}{p}\right)\log f_{\xi,u}$). (In the appendix, we write $c_{h,n}(g)$ by $c_{h,n}(G)$.) The system $(c_{h,n}(g))_n$ satisfies a certain norm relation related to the characteristic polynomial of φ on $D_p(V)$. Hence the system $(c_{h,n}(g))_n$ is not norm compatible in general, however, by modifying $c_{h,n}(g)$ possibly admitting denominators (in non-ordinary cases), she constructed an element

$$\Omega^{\xi}_{V,h}(g) \in \mathscr{H}_\infty(G_\infty^{cyc}) \otimes_{\mathbb{Z}_p[\![G_\infty^{cyc}]\!]} \varprojlim_n H^1_{\mathrm{f}}(\mathbb{Q}_{p,n}, T)$$

where $\mathscr{H}_\infty(G_\infty^{cyc})$ is a power series ring containing $\mathbb{Z}_p[\![G_\infty^{cyc}]\!]$ with huge denominators (cf. (6.4)). She also defined the map $\Omega^{\xi}_{V(j),h}$ for $j \in \mathbb{Z}$. It is not difficult to generalize her theory not only for the p-power cyclotomic tower but also for the p-power torsion tower of a relative Lubin–Tate group of height 1. We summarized it in the Appendix. (cf. [43] for Lubin–Tate groups of height 1 but not for the "relative" Lubin–Tate groups.)

Our purpose is to construct the logarithmic Coleman power series interpolating generalized Heegner cycles in the following sense. For a natural number c prime to p and $-r < i < r$ with $r = k/2$, the localization of the Abel–Jacobi image of the generalized Heegner cycles of conductor cp^n gives a system of local points

$$z^{(i+r)}_{cp^n} \in H^1_{\mathrm{f}}(\hat{H}_{cp^n}, V_f(\psi_{\mathfrak{p}}^{i+r}\psi_{\mathfrak{p}^*}^{r-i})) \quad (n = 0, 1, \dots)$$

and this satisfies the norm relation

$$\mathrm{Cor}_{n+1/n}\, z^{(i+r)}_{n+1} - a_p(f) z^{(i+r)}_n + p^{k-2}\mathrm{Res}_{n/n-1}\, z^{(i+r)}_{n-1} = 0.$$

(cf. [6, Proposition 4.4].) This is precisely the relation in the Perrin-Riou theory for $c_{h,n}(g)$ in this context. Hence, it is natural to expect that there is a vector-valued power series $g_i \in D_p(V_f(\psi_{\mathfrak{p}}^{i+r}\psi_{\mathfrak{p}^*}^{r-i})) \otimes \mathfrak{R}^{\psi=0}$ such that $c_{r,n}(g_i) = z^{(i+r)}_{cp^n}$. In fact, we show that such g_i is given in terms of the t-expansion of the Coleman primitive $F_{f^\flat}$ of the p-depletion $f^\flat$ of the modular form f, and the solution G_i of $(1-\varphi)G_i = g_i$ is also given in terms of the t-expansion of the Coleman primitive F_f of f. This strongly connects the Coleman integration theory and the Perrin-Riou theory. We shall see that identifying

$$D_p(V_f(r)) \otimes \mathfrak{R}^{\psi=0} \cong D_p(V_f(r)) \otimes \mathcal{O}[\![\mathrm{Gal}(\hat{H}_{cp^\infty}/\hat{H}_c)]\!],$$

the element g_0 gives a vector-valued Bertolini–Darmon–Prasanna (BDP) p-adic L-function in $D_p(V_f(r)) \otimes \mathcal{O}[\![\mathrm{Gal}(\hat{H}_{cp^\infty}/\hat{H}_c)]\!]$. (Precisely, we need a semi-local version of the above argument when the class number of K is greater than 1.) Note that in [2, §3.8], only the primitive of the p-depleted modular form $f^\flat$ is calculated. Perrin-Riou theory enables us to calculate the primitive of the original modular form f.

6.1 Coleman primitives of modular forms

Assume that $N > 4$. Let $\underline{\omega}$ be the invertible sheaf $\pi_*\Omega^1_{\mathscr{E}/Y_1(N)}$ for $\pi : \mathscr{E} \to Y_1(N)$ and let $\mathcal{L}_1$ be the relative de Rham cohomology group $H^1_{\mathrm{dR}}(\mathscr{E}/Y_1(N))$. Then we have an exact sequence

$$0 \longrightarrow \underline{\omega} \longrightarrow \mathcal{L}_1 \longrightarrow \underline{\omega}^\vee \longrightarrow 0 \tag{6.2}$$

where $^\vee$ is the dual of $\mathcal{O}_{Y_1(N)}$-modules. We extend them on $X_1(N)$ using the canonical differential form and the Gauss–Manin connection on the Tate curve around cusps. (cf. [2, §1].) For a natural number n, we put $\mathcal{L}_n := \mathrm{Sym}^n\mathcal{L}_1$. The Hodge filtration on $\mathcal{L}_n$ is defined naturally from (6.2) and the Poincaré duality defines a pairing $\langle\, ,\, \rangle : \mathcal{L}_n \times \mathcal{L}_n \to \mathcal{O}_{X_1(N)}$. By construction, we also have the Gauss–Manin connection $\nabla : \mathcal{L}_n \to \mathcal{L}_n \otimes \Omega^1_{X_1(N)}(\mathrm{cusps})$.

Let p be a prime not dividing N. Let $\overline{S}$ be the subset of $X_1(N)(\overline{\mathbb{F}}_p)$ consisting of all cusps and all supersingular points. Let $\mathcal{X}$ be the rigid analytic space over $\mathbb{Q}_p$ associated with $X_1(N)$ and $\mathcal{L}_n^{\mathrm{rig}}$ denote the rigid analytic coherent sheaf associated with $\mathcal{L}_n$. Let $\mathcal{Y}_{\mathrm{ord}}$ be the affinoid obtained by subtracting all residue discs over the points in $\overline{S}$ and let $\mathcal{W}$ be a wide-open neighborhood of $\mathcal{Y}_{\mathrm{ord}}$. By using the Gauss–Manin connection of $\mathcal{L}_n$ on $\mathcal{W}$, we let

$$H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{L}_n^{\mathrm{rig}}, \nabla) := \frac{\Gamma(\mathcal{W}, \mathcal{L}_n^{\mathrm{rig}} \otimes \Omega^1_{\mathcal{W}})}{\nabla\Gamma(\mathcal{W}, \mathcal{L}_n^{\mathrm{rig}})},$$

which is known to be independent of the choice of $\mathcal{W}$. By the theory of the canonical subgroup, the Frobenius $\varphi : \mathcal{Y}_{\mathrm{ord}} \to \mathcal{Y}_{\mathrm{ord}}$ is overconvergent, that is, there is a wide-open neighborhood $\mathcal{W}'$ such that $\mathcal{W} \supset \mathcal{W}' \supset \mathcal{Y}_{\mathrm{ord}}$ and φ is extended to $\varphi : \mathcal{W}' \to \mathcal{W}$. We also have the Frobenius structure on the relative de Rham cohomology $\mathcal{L}_1$ compatible with the Gauss–Manin connection and it induces a horizontal morphism $\mathrm{Fr} : \varphi^*\mathcal{L}_n^{\mathrm{rig}} \to \mathcal{L}_n^{\mathrm{rig}}|_{\mathcal{W}'}$. By composing these, we have a map

$$\Gamma(\mathcal{W}, \mathcal{L}_n^{\mathrm{rig}} \otimes \Omega^1_{\mathcal{W}}) \to \Gamma(\mathcal{W}', \varphi^*(\mathcal{L}_n^{\mathrm{rig}} \otimes \Omega^1_{\mathcal{W}})) \to \Gamma(\mathcal{W}', \mathcal{L}_n^{\mathrm{rig}} \otimes \Omega^1_{\mathcal{W}'}).$$

In particular, this induces a map on the space of overconvergent modular forms and actions on $H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{L}_n^{\mathrm{rig}}, \nabla)$. By abuse of notation, we denote all of these by φ. For details, see [2, §3.5].

Let f be a primitive normalized eigenform for $\Gamma_1(N)$ of weight $k \geq 2$ with Neben character ε. Let $\mathcal{O}$ be a finite flat $\mathbb{Z}_p$-algebra in $\mathbb{C}_p$ containing the coefficients of f, roots α, β of $P_f(t) := t^2 - a_p(f)t + \varepsilon(p)p^{k-1}$ and a primitive N-th root of unity by the fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. If p is ordinary for f, we take α as the unit root. For a $\mathbb{Z}_p$-module M, we write $M \otimes_{\mathbb{Z}_p} \mathcal{O}$ by $M_{\mathcal{O}}$. If there is no fear of confusion, we sometimes write $M_{\mathcal{O}}$ simply by M by abuse of notation. Let $\omega_f \in \Gamma(X_1(N), \underline{\omega}^{\otimes k})_{\mathcal{O}}$ be the section corresponding to f. We may also regard ω_f as a section of $\Gamma(X_1(N), \underline{\omega}^{\otimes(k-2)} \otimes \Omega^1_{X_1(N)}(\text{cusps}))_{\mathcal{O}}$ by the Kodaira–Spencer isomorphism. Let M_f be the $\mathcal{O}[\varphi]$-submodule in $H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{L}_k^{\mathrm{rig}}, \nabla)_{\mathcal{O}}$ generated by the image of ω_f. (The action of φ on $H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{L}_k^{\mathrm{rig}}, \nabla) \otimes_{\mathbb{Z}_p} \mathcal{O}$ is $\varphi \otimes 1$.) Then we have $P_f(\varphi)\omega_f = 0$ in M_f. Hence $\mathcal{O}$-module M_f is at most of rank 2 and it is of rank 1 if and only if $\mathcal{O}\omega_f$ is closed by the action of φ. This happens only when the p-adic Galois representation associated with f is ordinary and decomposable at all places over p. (cf. [13, Proposition 4].) There are operators U, V on the space of p-adic modular forms compatible with ∇ such that

$$U\left(\sum b_n q^n\right) = \sum b_{np} q^n, \quad V\left(\sum b_n q^n\right) = \sum b_n q^{pn}$$

on q-expansions. The moduli interpolation of U is that it associates a triple $(A, \omega_A, \mathrm{Lv})$ of ordinary elliptic curves to the cycle $\frac{1}{p}\sum_C (A/C, \omega_{A/C}, \mathrm{Lv})$ where C runs through étale subgroup of A of order p, $\omega_{A/C}$ is the invariant differential form on A/C whose pull-back to A is ω_A and the level structure is the natural one. (Note $p \nmid N$.) Similarly, V associates it to the triple $(A/\hat{A}[p], p\omega_A, \frac{1}{p}\mathrm{Lv})$. (cf. [2, p. 1085].) Note also that the Frobenius map Fr_p associates it to $(A/\hat{A}[p], p\omega_A, \mathrm{Lv})$. Hence V and φ differ by the diamond operator $\langle p\rangle$.

Following [2], for p-adic modular form g, we let

$$g^\flat := (UV - VU)g = (1 - VU)g.$$

(Note that [2] uses the right action for U and V.) Let $\omega_{f^\flat}$ be the section in $\Gamma(\mathcal{W}, \underline{\omega}^{\otimes k})$ associated to $f^\flat$ for a wide open $\mathcal{W}$ of $\mathcal{Y}_{\mathrm{ord}}$. In this subsection, we consider (Coleman) primitive functions of ω_f and $\omega_{f^\flat}$ with respect to ∇. By general theory, primitives are determined up to horizontal sections of ∇. We eliminate the ambiguity in the following lemmas.

First, consider the primitive of $\omega_{f^\flat}$. Since $P_f(\varphi)\omega_f = 0$ in the rigid cohomology $H^1_{\mathrm{dR}}(\mathcal{W}, \mathcal{L}_n^{\mathrm{rig}}, \nabla)$, there is a rigid analytic function $F_{f^\flat}$ (a section in $\Gamma(\mathcal{W}, \mathcal{L}_{k-2}^{\mathrm{rig}})$) such that $P_f(\varphi)\omega_f = P(0)\nabla F_{f^\flat}$. The following lemma shows that $F_{f^\flat}$ is a primitive of $\omega_{f^\flat}$.

Lemma 6.1 (i) *The q-expansion of $f^\flat$ is $\sum_{p\nmid n} a_n(f)q^n$.*
(ii) *We have $P_f(\varphi)\omega_f = P(0)\omega_{f^\flat}$ as p-adic modular forms. In particular, $F_{f^\flat}$ is a primitive of $\omega_{f^\flat}$.*

Proof i) follows from the direct calculation of the action of U, V on the q-expansion. The q-expansion of ω_f is given $\omega_f = f(q)\frac{dq}{q}\omega_{\mathrm{can}}^{\otimes k-2}$ where ω_{can} is the canonical invariant differential form of the Tate curve. On the q-expansion of f, the Frobenius acts as $\epsilon(p)V$. (cf. [20, §1.3, (1.3.2)]). Hence

$$\begin{aligned}P_f(\varphi)\omega_f &= (\epsilon(p)p^{k-1} - a_p(f)\epsilon(p)p^{k-1}V + \epsilon(p)^2p^{2k-2}V^2)f(q)\cdot\frac{dq}{q}\,\omega_{\mathrm{can}}^{\otimes k-2}\\ &= \epsilon(p)p^{k-1}(1 - a_p(f)V + \epsilon(p)p^{k-1}V)f(q)\cdot\frac{dq}{q}\,\omega_{\mathrm{can}}^{\otimes k-2} = P(0)\omega_{f^\flat}.\end{aligned}$$

Lemma 6.2 *There is a unique rigid analytic primitive $F_{f^\flat}$ of $\omega_{f^\flat}$ such that $U(F_{f^\flat}) = 0$.*

Proof Take a rigid analytic primitive and put $g = VU(F_{f^\flat})$. Then g is a horizontal section of ∇. Then by replacing $F_{f^\flat}$ by $F_{f^\flat} - g$, we have such a primitive. The space of horizontal sections is a finite-dimensional vector space, and U defines a linear transform on it having the right inverse V. Hence U is invertible on it. Hence the condition $U(F_{f^\flat}) = 0$ characterize the primitive uniquely.

Actually, there is no algebraic horizontal section when $k > 2$ and the above lemma has a meaning only when $k = 2$. However, such a characterization is important in general. In fact, the same idea has already been used in [27].

From now on, we denote by $F_{f^\flat}$ the primitive in the above lemma.

Let $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ be the sheaf of locally analytic functions on $\mathcal{X}$ with values in the fraction field of $\mathcal{O}$ and $\mathcal{O}_{\mathcal{X}}^{\mathrm{col}}$ the subsheaf consisting of Coleman functions. We put $\mathcal{L}_{k-2}^{\mathrm{col}} = \mathcal{L}_{k-2} \otimes_{\mathcal{O}_{X_1(N)}} \mathcal{O}_{\mathcal{X}}^{\mathrm{col}}$. By the theory of Coleman integration, we have $F_f \in \mathcal{L}_{k-2}^{\mathrm{col}}$ such that $\nabla F_f = \omega_f$. Note that F_f is determined uniquely up to a horizontal section of $\mathcal{L}_{k-2}$.

Lemma 6.3 *There is a unique Coleman primitive function F_f of ω_f such that $P_f(\varphi)F_f = P_f(0)F_{f^\flat}$.*

Proof Since $P_f(1) \neq 0$, $P_f(\varphi)$ is invertible on the space of horizontal sections of $\mathcal{L}_{k-2}$. The assertion follows from Lemma 6.1 (ii).

We fix F_f as the primitive in the above lemma.

6.2 Expansion at the Heegner point

We use the same setting and notations in Sect. 5.9. We fix a $\Gamma_1(N)$-level structure $\mathrm{Lv} : \mathbb{Z}/N\mathbb{Z} \hookrightarrow \overline{E}$. Since $(N, p) = 1$, it is canonically extended to the level structure on E and $\mathbb{E}$. Then the residue disc of $X_1(N)$ at $(\overline{E}/k, \mathrm{Lv})$ is identified with the rigid analytic disc associated with the formal group $\mathcal{M}_{\overline{E}/k}$. Considering the formal completion of $X_1(N)$ over W at the closed point corresponding to the isomorphism class of $(\overline{E}/k, \mathrm{Lv})$, the completion of the universal elliptic curve on $X_1(N)$ is identified with $\mathbb{E}$. Then we may regard $\omega_f \in \Gamma(\mathbb{E}, \Omega^1_{\mathbb{E}/\mathfrak{R}})^{\otimes k}_{\mathcal{O}}$ and write $\omega_f = f(\mathbb{E}/\mathfrak{R}, \omega_{\mathbb{E}}, \mathrm{Lv})\omega_{\mathbb{E}}^{\otimes k}$ where $f(\mathbb{E}/\mathfrak{R}, \omega_{\mathbb{E}}, \mathrm{Lv}) \in \mathfrak{R}_{\mathcal{O}}$ is the value at $(\mathbb{E}/\mathfrak{R}, \omega_{\mathbb{E}}, \mathrm{Lv})$ as the Katz modular form associated to f (with coefficients in $\mathcal{O}$). The operator VU associates a point in the residue disc at $(\overline{E}/k, \mathrm{Lv})$ to a cycle whose support is in the same residue disc. By Proposition 5.11, its moduli interpolation coincides with that of $\varphi \circ \psi$. Hence we have $VU = \varphi \circ \psi$ on the residue disc. In particular,

$$f^\flat(\mathbb{E}/\mathfrak{R}, \omega_{\mathbb{E}}, \mathrm{Lv}) = (f(\mathbb{E}/\mathfrak{R}, \omega_{\mathbb{E}}, \mathrm{Lv}))^\flat.$$

Let $\omega_{\mathbb{E}}^\vee, \xi_{\mathbb{E}}^\vee$ be the dual basis of $\omega_{\mathbb{E}}, \xi_{\mathbb{E}}$. Then by the identification of the de Rham pairing, we have $\xi_{\mathbb{E}}^\vee = -\omega_{\mathbb{E}}$ and $\omega_{\mathbb{E}}^\vee = \xi_{\mathbb{E}}$. Hence $\nabla(\xi_{\mathbb{E}}^\vee) = -\omega_{\mathbb{E}}^\vee \otimes \omega_{\hat{\mathcal{M}}}$ and $\nabla(\omega_{\mathbb{E}}^\vee) = 0$.

Proposition 6.4 *Let $F_{f^\flat}(\mathbb{E}/\mathfrak{R}, \mathrm{Lv}) \in \mathbb{L}_{k-2}$ be the formal expansion of $F_{f^\flat}$ at $(\overline{E}, \mathrm{Lv})$. We have*

$$\langle F_{f^\flat}(\mathbb{E}/\mathfrak{R}, \mathrm{Lv}), (\omega_{\mathbb{E}}^\vee)^{k-2-j}(\xi_{\mathbb{E}}^\vee)^j\rangle_{\mathbb{E}} = (-1)^j j!\partial_{\hat{\mathcal{M}}}^{-1-j} f^\flat(\mathbb{E}/\mathfrak{R}, \omega_{\mathbb{E}}, \mathrm{Lv}) \in \mathfrak{R}_{\mathcal{O}}^{\psi=0}.$$

Proof The proof is similar to [2, Proposition 3.24]. First, note that by acting $VU = \varphi \circ \psi$, we have

$$\langle F_{f^\flat}(\mathbb{E}/\mathfrak{R}, \mathrm{Lv}), (\omega_{\mathbb{E}}^\vee)^{k-2-j}(\xi_{\mathbb{E}}^\vee)^j\rangle_{\mathbb{E}} \in \mathfrak{R}_{\mathcal{O}}^{\psi=0}.$$

Since $\nabla(\omega_{\mathbb{E}}^\vee) = 0$, we have

$$\partial_{\hat{\mathcal{M}}}\langle F_{f^\flat}(\mathbb{E}/\mathfrak{R}, \mathrm{Lv}), (\omega_{\mathbb{E}}^\vee)^{k-2}\rangle_{\mathbb{E}} = \langle f^\flat(\mathbb{E}/\mathfrak{R}, \mathrm{Lv}), (\omega_{\mathbb{E}}^\vee)^{k-2}\rangle_{\mathbb{E}} = f^\flat(\mathbb{E}/\mathfrak{R}, \omega_{\mathbb{E}}, \mathrm{Lv}).$$

Hence we show the equality in the assertion by operating $\partial_{\hat{\mathcal{M}}}^{j+1}$. For $j \geq 1$, we also have

$$\begin{aligned}
&\partial_{\hat{\mathcal{M}}} \langle F_{f^\flat}(\mathbb{E}/\mathfrak{R}, \mathrm{Lv}), (\omega_{\mathbb{E}}^\vee)^{k-2-j}(\xi_{\mathbb{E}}^\vee)^j\rangle_{\mathbb{E}} \\
&= \langle \nabla F_{f^\flat}(\mathbb{E}/\mathfrak{R}, \mathrm{Lv}), (\omega_{\mathbb{E}}^\vee)^{k-2-j}(\xi_{\mathbb{E}}^\vee)^j\rangle_{\mathbb{E}} + \langle F_{f^\flat}(\mathbb{E}/\mathfrak{R}, \mathrm{Lv}), \nabla((\omega_{\mathbb{E}}^\vee)^{k-2-j}(\xi_{\mathbb{E}}^\vee)^j)\rangle_{\mathbb{E}} \\
&= -j\langle F_{f^\flat}(\mathbb{E}/\mathfrak{R}, \mathrm{Lv}), (\omega_{\mathbb{E}}^\vee)^{k-2-j}(\xi_{\mathbb{E}}^\vee)^{j-1}\rangle_{\mathbb{E}}.
\end{aligned}$$

Hence the assertion follows by induction.

6.3 The construction of the logarithmic Coleman power series

As before, let f be a normalized cusp form for $\Gamma_0(N)$ of weight k. Assume that $p \nmid N$ and let α, β be roots of $x^2 - a_p(f)x + p^{k-1}$ in the fixed algebraic closure $\overline{\mathbb{Q}}$ and the embedding into $\mathbb{C}_p$. Let $\mathcal{O}$ be the integer ring of a finite extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$ including the Hecke field of f and α. We take α as a unit root if p is ordinary, and any if p is non-ordinary.

The strategy for the construction of $g_i \in M_f \otimes \mathfrak{R}^{\psi=0}$ is that by choosing an appropriate splitting N_f of the Hodge filtration of M_f, we construct a map $M_f^\vee \to \mathfrak{R}^{\psi=0}$ with the identification

$$M_f \otimes \mathfrak{R}^{\psi=0} = \mathrm{Hom}(M_f^\vee, \ \mathfrak{R}^{\psi=0}).$$

We put

$$N_f := \begin{cases} \mathcal{O}\varphi\omega_f, & \text{if } \quad M_f \quad \text{is of rank 2} \\ \{0\}, & \text{otherwise,} \end{cases}$$

and let $\mathcal{I}^\flat : M_f \to \Gamma(\mathcal{W}, \mathcal{L}_{k-2}^{\mathrm{rig}})_{\mathcal{O}}$ be the map defined as the composition

$$M_f \longrightarrow M_f/N_f \cong \mathcal{O}\,\omega_f \longrightarrow \Gamma(\mathcal{W}, \mathcal{L}_{k-2}^{\mathrm{rig}})_{\mathcal{O}}, \qquad \omega_f \longmapsto F_{f^\flat}.$$

If M_f is of rank 2, define an "integration" map $\mathcal{I} \in \mathrm{Hom}_{\mathcal{O}}(M_f, \Gamma(\mathcal{W}, \mathcal{L}_{k-2}^{\mathrm{col}})_{\mathcal{O}})$ by

$$\mathcal{I}(\omega_f) = F_f, \qquad \mathcal{I}(\varphi\omega_f) = \varphi F_f.$$

If M_f is of rank 1, define $\mathcal{I}$ by $\mathcal{I}(\omega_f) = (1 - \alpha^{-1}\varphi)F_f$.

Now we consider the α-stabilized version. We let $\omega_{f_\alpha} = (1 - \beta^{-1}\varphi)\omega_f$. Then $\varphi\omega_{f_\alpha} = \alpha\omega_{f_\alpha}$ in M_f. We put $N_\alpha := \mathcal{O}[1/p]\omega_{f_\alpha}$. Then we have $M_f[1/p]/N_\alpha \cong \mathcal{O}[1/p]\,\omega_f$. (Note that if M_f is of rank 1, we have $\omega_{f_\alpha} = 0$ since we choose α to be the unit root.) Then we define $\mathcal{I}_\alpha^\flat : M_f \to \Gamma(\mathcal{W}, \mathcal{L}_{k-2}^{\mathrm{rig}})_{\mathcal{O}}$ by

$$M_f \longrightarrow M_f[1/p]/N_\alpha \cong \mathcal{O}[1/p]\,\omega_f \longrightarrow \Gamma(\mathcal{W}, \mathcal{L}_{k-2}^{\mathrm{rig}})_{\mathcal{O}}, \qquad \omega_f \longmapsto F_{f^\flat}.$$

Similarly, define $\mathcal{I}_\alpha \in \mathrm{Hom}_{\mathcal{O}}(M_f, \Gamma(\mathcal{W}, \mathcal{L}_{k-2}^{\mathrm{col}})_{\mathcal{O}})$ by

$$M_f \longrightarrow M_f[1/p]/N_\alpha \cong \mathcal{O}[1/p]\,\omega_f \longrightarrow \Gamma(\mathcal{W}, \mathcal{L}_{k-2}^{\mathrm{rig}})_{\mathcal{O}}, \qquad \omega_f \longmapsto (1 - \alpha^{-1}\varphi)F_f.$$

Proposition 6.5 *We have* $(1 - \varphi)\mathcal{I} = \mathcal{I}^\flat$ *and* $(1 - \varphi)\mathcal{I}_\alpha = \mathcal{I}_\alpha^\flat$.

Proof The relation $(1 - \varphi)\mathcal{I} = \mathcal{I}^\flat$ follows from Lemma 6.6 below. Since $\mathcal{I}_\alpha(\varphi\omega_f) = \beta\mathcal{I}_\alpha(\omega_f)$ and $\varphi^{-1}\omega_f = (\alpha\beta)^{-1}(a_p(f) - \varphi)\omega_f$, we have

$$\mathcal{I}_\alpha(\varphi^{-1}\omega_f) = \beta^{-1}(1 - \alpha^{-1}\varphi)F_f.$$

Then by Lemma 6.3, we have

$$(1 - \varphi)\mathcal{I}_\alpha(\omega_f) = \mathcal{I}_\alpha(\omega_f) - \varphi\mathcal{I}_\alpha(\varphi^{-1}\omega_f) = P_f(0)^{-1}P_f(\varphi)F_f = F_{f^\flat}.$$

Let R be a commutative ring with an automorphism σ. Let M, N be R-modules with σ-semilinear endomorphism φ. We assume that $\varphi : M \to M$ is bijective and consider the natural action of φ on $\mathrm{Hom}_R(M, N)$ by $\varphi(f)(m) = \varphi f(\varphi^{-1}m)$. Let $R[t]_\sigma$ be the non-commutative ring such that the underlying set is the polynomials over R with variable t and the multiplication is twisted by the rule $t \cdot a = \sigma(a) \cdot t$ for $a \in R$. Suppose that $M = R[\varphi]_\sigma \omega$ and $P(\varphi)\omega = 0$ for $\omega \in M$ and a monic polynomial P of degree h.

Lemma 6.6 *Assume $h \geq 2$ and let $g : M \to N$ be a morphism of R-modules such that the kernel contains $\sum_{i=1}^{h-1} R\varphi^i\omega$. Suppose that there exists $G \in \mathrm{Hom}_R(M, N)$ such that $(1-\varphi)G = g$. Then G satisfies $P(\varphi)(G(\omega)) = P(0)g(\omega)$ and $G(\varphi^i\omega) = \varphi^i G(\omega)$ for $i = 1, \ldots, h-1$. Conversely, if G satisfies these relations, G is a solution of $(1-\varphi)G = g$.*

Proof For $\eta \in M$, the condition $(1-\varphi)G = g$ implies that

$$G(\eta) - \varphi G(\varphi^{-1}\eta) = g(\eta).$$

Since $g(\varphi^i\omega) = 0$ for $i = 1, \ldots, h-1$, we inductively have $G(\varphi^i\omega) = \varphi^i G(\omega)$ for such i. We also have

$$G(\varphi^h\omega) - \varphi^h G(\omega) = G(\varphi^h\omega) - \varphi G(\varphi^{h-1}\omega) = g(\varphi^h\omega).$$

Hence

$$P(\varphi)G(\omega) = P(\varphi)G(\omega) - G(P(\varphi)\omega) = -g(\varphi^h\omega) = P(0)g(\omega).$$

The converse is also clear.

We let $D_E = H^1_{\mathrm{dR}}(E/W)$, which has the structure of a strongly divisible module. Since E is the canonical lift, we have the decomposition $D_E = D_{E,\mathfrak{p}} \oplus D_{E,\mathfrak{p}^*}$ where $D_{E,\mathfrak{p}}$ is the filtered φ-module associated to the formal group $\hat{E}$ (or the p-adic representation $(T_\mathfrak{p}E)^\vee$, the $\mathbb{Z}_p$-dual of $T_\mathfrak{p}E$) and $D_{E,\mathfrak{p}^*}$ is the filtered φ-module associated to the p-adic representation $(T_{\mathfrak{p}^*}E)^\vee = (T_p\overline{E})^\vee$. If there is no fear of confusion, we omit E in the notation of $D_{E,\mathfrak{p}}$ and $D_{E,\mathfrak{p}^*}$. The module $D_\mathfrak{p}$ is a W-module of rank 1 generated by an invariant differential form and $\mathrm{Fil}^1 D_\mathfrak{p} = D_\mathfrak{p}$, $\mathrm{Fil}^2 D_\mathfrak{p} = \{0\}$. The module $D_{\mathfrak{p}^*}$ is a W-module of rank 1 generated by a unit root vector of φ in D_E and $\mathrm{Fil}^0 D_\mathfrak{p} = D_\mathfrak{p}$, $\mathrm{Fil}^1 D_\mathfrak{p} = \{0\}$. Then we let

$$L_{E,n} := \mathrm{Sym}^n D_E = \bigoplus_{i=0}^{n} D_\mathfrak{p}^{\otimes i} \otimes_W D_{\mathfrak{p}^*}^{\otimes(n-i)}.$$

Now we define the formal completion of $\mathcal{I}^\flat, \mathcal{I}$ at $(\overline{E}/k, \mathrm{Lv})$. By the formal completion at $(\overline{E}/k, \mathrm{Lv})$, we have the map $\Gamma(\mathcal{W}, \mathcal{L}^{\mathrm{rig}}_{k-2}) \to \mathbb{L}_{k-2} \otimes \mathbb{Q}_p$. Then we extend the pairing $\langle\ ,\ \rangle_\mathbb{E} : \mathbb{L}_{k-2} \times \mathbb{L}^\vee_{k-2} \to \mathfrak{R}$ to

$$\Gamma(\mathcal{W}, \mathcal{L}^{\mathrm{rig}}_{k-2}) \times \mathbb{L}^\vee_{k-2} \longrightarrow \mathfrak{R} \otimes \mathbb{Q}_p.$$

The lift $s_\mathbb{E}$ in Proposition 5.1 naturally defines a lift $L^\vee_{E,k-2} \to \mathbb{L}^\vee_{k-2}$, which is also denoted by $s_\mathbb{E}$. Then we have a map

$$\mathcal{I}^\flat_E : M_f \otimes L^\vee_{E,k-2} \longrightarrow \mathfrak{R}^{\psi=0}_\mathcal{O}, \quad \xi \otimes \eta \longmapsto \langle \mathcal{I}^\flat(\xi), s_\mathbb{E}(\eta)\rangle_\mathbb{E}.$$

(By Proposition 6.4, we do not need denominators here.) By definition, it satisfies that

$$\mathcal{I}^\flat_E(\omega_f \otimes \eta^\vee_E) = \langle F_{f^\flat}(\mathbb{E}/\mathfrak{R}, \mathrm{Lv}), \eta^\vee_\mathbb{E}\rangle_\mathbb{E}, \quad \mathcal{I}^\flat_E(\varphi\omega_f \otimes \eta^\vee_E) = 0 \tag{6.3}$$

where $\eta_{\mathbb{E}}^{\vee}$ is the Frobenius compatible lift of $\eta_E^{\vee}$.

We let

$$\mathscr{H}_h(\mathfrak{R}) := \left\{ \sum_{n=0}^{\infty} a_n t^n \mid a_n \in W[1/p],\ |a_n|_p n^{-h} \to 0 \right\} \tag{6.4}$$

where t is a generator of W-module $\mathfrak{m}/\mathfrak{m}^2$ for the ideal $\mathfrak{m}$ of $\mathfrak{R}$ corresponding to the canonical lift E. We put $\mathscr{H}_\infty(\mathfrak{R}) = \cup_{h=1}^{\infty} \mathscr{H}_h(\mathfrak{R})$.

Lemma 6.7 *The formal completion of* $\Gamma(\mathcal{W}, \mathcal{L}_k^{\mathrm{col}})$ *at* $(\overline{E}/k, \mathrm{Lv})$ *lies in* $\mathbb{L}_k \otimes_{\mathfrak{R}} \mathscr{H}_\infty(\mathfrak{R})$.

Proof After multiplying by a polynomial of φ, Coleman functions become rigid analytic functions on the closed disc at $(\overline{E}/k, \mathrm{Lv})$. The formal completion of a rigid analytic function on the closed disc has bounded denominators. In particular, it is an element of $\mathscr{H}_\infty$. Hence the assertion follows from the general fact that for a given $g \in \mathscr{H}_\infty$, a solution G of $P(\varphi)G = P(0)g$ where P is a monic polynomial with coefficients in W satisfying $P(1) \neq 0$ lives in $\mathscr{H}_\infty$. This is shown by the same argument in the proof of Proposition (ii) of 2.2.1 of [33]. Note that we may change the equation $P(\varphi)G = g$ into the form $(1-\varphi)G = g$ as in [33] by using Lemma 6.6. In fact, let N be a $W[\varphi]_\sigma$-module containing g and $M = W[\varphi]_\sigma/(P(\varphi))$. (Write $1 \in M$ formally as ω.) Define $\tilde{g} : M \to N$ by $\tilde{g}(1) := g$ and $\tilde{g}(\varphi^i) := 0$ for $i = 1, \dots, h-1$. Then by Lemma 6.6, the equation $P(\varphi)G = g$ is equivalent to the equation $(1-\varphi)\tilde{G} = \tilde{g}$. See also Proposition 7.2 in the appendix.

Then by the formal completion, we define the map by

$$\mathcal{I}_E : M_f \otimes L_{E,k-2}^{\vee} \longrightarrow \mathscr{H}_\infty(\mathfrak{R})_{\mathcal{O}}, \quad \xi \otimes \eta \longmapsto \langle \mathcal{I}(\xi), s_{\mathbb{E}}(\eta) \rangle_{\mathbb{E}}.$$

Similarly, by using $\mathcal{I}_\alpha$ and $\mathcal{I}_\alpha^\flat$ instead of $\mathcal{I}$ and $\mathcal{I}^\flat$, we define $\mathcal{I}_{\alpha,E}$ and $\mathcal{I}_{\alpha,E}^\flat$.

Proposition 6.8 *We have* $(1-\varphi)\mathcal{I}_E = \mathcal{I}_E^\flat$ *and* $(1-\varphi)\mathcal{I}_{\alpha,E} = \mathcal{I}_{\alpha,E}^\flat$.

Proof This follows from Proposition 6.5 and that $s_{\mathbb{E}}$ is Frobenius compatible.

We apply the Perrin-Riou theory in our appendix for $\mathcal{G} = \hat{\mathcal{M}}$ and use the same notations. We identify $D_{\mathfrak{p}} \otimes D_{\mathfrak{p}^*}^{\otimes -1}$ with $D_{\hat{\mathcal{M}}}$ by the Kodaira–Spencer map as before. By definition, we have

$$\omega_{\hat{\mathcal{M}}} = \omega_E \otimes \xi_E^{\vee}. \tag{6.5}$$

Let V be $V_f(r)$ and put $D := D_p(V)$ (resp. $D_{\mathbb{Q}_p}$) the filtered φ-module associated to representation V of $G_{W[1/p]}$ (resp. $G_{\mathbb{Q}_p}$) with coefficients in $\mathcal{O}$. Let D_f be the filtered φ-module associated to f with coefficients in $\mathcal{O}$. Note that $D_f = M_f[1/p]$ if M_f is of rank 2, and M_f is the quotient by the unit root space if M_f is of rank 1. The module $D_{E,\mathfrak{p}}^{\vee} \otimes D_{E,\mathfrak{p}^*}^{\vee}$ has a canonical basis $\omega_E^{\vee} \otimes \xi_E^{\vee}$, which is independent of the choice of ω_E and we denote it symbolically by $\omega^{\vee}\xi^{\vee}$. We identify

$$D(\mathbb{Q}_p(1)) \cong D_{E,\mathfrak{p}}^{\vee} \otimes D_{E,\mathfrak{p}^*}^{\vee}$$

by $\omega^{\vee}\xi^{\vee}$. Then we have canonically

$$D = D_f(r) = D_f \otimes_{\mathbb{Q}_p} (D_{E,\mathfrak{p}}^{\vee} \otimes_W D_{E,\mathfrak{p}^*}^{\vee})^{\otimes r}.$$

We consider the twist associated with $\hat{\mathcal{M}}$:

$$D\langle i \rangle := D \otimes D_{\mathfrak{p}}^{\otimes -i} \otimes D_{\mathfrak{p}^*}^{\otimes i} = D_f \otimes D_{\mathfrak{p}}^{\otimes -r-i} \otimes D_{\mathfrak{p}^*}^{\otimes -r+i}.$$

Now we construct the desired element $g_{E,0} \in D \otimes \mathfrak{R}^{\psi=0}$ and put

$$g_{E,i} := d^i g_{E,0} \in D\langle -i\rangle \otimes \mathfrak{R}^{\psi=0}$$

such that the associated local points by the Perrin-Riou exponential map are twists of the Abel–Jacobi image of generalized Heegner cycles. Here d is the canonical derivation

$$d : \mathfrak{R} \longrightarrow \hat{\Omega}^1_{\mathfrak{R}/W} = D_{\hat{\mathcal{M}}} \otimes_W \mathfrak{R} = \mathfrak{R}\langle -1\rangle$$

and it induces an invertible map

$$d : D\langle -i\rangle \otimes \mathfrak{R}^{\psi=0} \longrightarrow D\langle -i-1\rangle \otimes \mathfrak{R}^{\psi=0}.$$

First we define $g_{E,i}$ for $-r < i < r$ directly. We identify

$$D\langle -i\rangle \otimes_W \mathfrak{R}^{\psi=0} = \mathrm{Hom}_{W\otimes\mathcal{O}}(D^\vee\langle i\rangle,\ \mathfrak{R}^{\psi=0}_{\mathcal{O}}).$$

If $-r < i < r$, we have

$$D^\vee\langle i\rangle = D(-1)\langle i\rangle = D_f \otimes_{\mathbb{Q}_p} (D^\vee_{E,\mathfrak{p}})^{\otimes r-1+i} \otimes (D^\vee_{E,\mathfrak{p}^*})^{\otimes r-1-i} \subset D_f \otimes L^\vee_{E,k-2}. \quad (6.6)$$

There is a natural projection $D_f \to M_f \otimes W[1/p]$. In fact, $D_f = M_f \otimes W[1/p]$ if M_f is of rank 2, and M_f is the quotient by the unit root space if M_f is of rank 1. By composing it with $\mathcal{I}^\flat_E$, and restricting it on $D^\vee\langle i\rangle$, we have an element

$$\mathcal{I}^\flat_{E,i} \in D\langle -i\rangle \otimes \mathfrak{R}^{\psi=0}.$$

Similarly, we define $\mathcal{I}^\flat_{E,\alpha,i} \in D\langle -i\rangle \otimes \mathfrak{R}^{\psi=0}$. Note that $\mathcal{I}^\flat_{E,i} = \frac{\alpha}{\alpha-\beta}\mathcal{I}^\flat_{E,\alpha,i} + \frac{\beta}{\beta-\alpha}\mathcal{I}^\flat_{E,\beta,i}$. Then we have

$$\mathcal{I}^\flat_{E,\alpha,i} \in \mathcal{N}_\alpha\langle -i\rangle \otimes \mathfrak{R}^{\psi=0} = \mathrm{Hom}_{W\otimes\mathcal{O}}(D/\mathcal{N}_\alpha\langle i\rangle, \mathfrak{R}^{\psi=0}_{\mathcal{O}}) \quad (6.7)$$

where $\mathcal{N}_\alpha = D_\alpha \otimes_{\mathbb{Z}_p} D^{\otimes -r}_{E,\mathfrak{p}} \otimes D^{\otimes -r}_{E,\mathfrak{p}^*} \subset D$ for the α-eigen space D_α.

Proposition 6.9 *For an integer i such that $-r < i < r-1$, we have*

$$d\mathcal{I}^\flat_{E,i} = -(r-i-1)\mathcal{I}^\flat_{E,i+1}, \quad d\mathcal{I}^\flat_{E,\alpha,i} = -(r-i-1)\mathcal{I}^\flat_{E,\alpha,i+1}.$$

Proof This follows from (6.3) and Proposition 6.4.

For $-r < i < r$, we put

$$g_{E,i} := \frac{(-1)^{r-i-1}}{(r-i-1)!}\mathcal{I}^\flat_{E,i}$$

and then we put

$$g_{E,i} := d^i g_{E,0} \in D\langle -i\rangle \otimes \mathfrak{R}^{\psi=0}$$

for a general integer i. (The notation is consistent for $-r < i < r-1$ by Proposition 6.9.) Similarly, for $i \in \mathbb{Z}$, we define

$$g_{E,\alpha,i} := \frac{(-1)^{r-i-1}}{(r-i-1)!}\mathcal{I}^\flat_{E,\alpha,i} \in \mathcal{N}_\alpha\langle -i\rangle \otimes \mathfrak{R}^{\psi=0}. \quad (6.8)$$

Lemma 6.10 *The operator $1-\varphi$ is bijective on $D\langle -i\rangle$.*

Proof Let n be a natural number such that φ^n is $W[1/p]$-linear. Then it suffices to show that φ^n does not have an eigenvalue 1. Eigenvalues must be of the form $\alpha'(u/v)^m$ where α' is an eigenvalue of φ^n on D, m is an integer, and u, v are the distinct roots of the p-Euler factor of E/W. This is a product of Weil numbers, and the complex absolute value is $|\alpha'(u/v)^m| = |\alpha'| \neq 1$.

By the above lemma, we have a solution $G_{E,i} \in D\langle -i\rangle \otimes \mathscr{H}_\infty(\mathfrak{R})$ of $(1-\varphi)G_{E,i} = g_{E,i}$. (cf. Proposition 7.2 in the appendix.)

Now we consider the case $E = A_c$ with $(c, p) = 1$ and g_{A_c}. By the Perrin-Riou exponential map, for $i > -r$, we have a local point

$$c_{r,n}^{(i)}(G_{A_c}) = \exp_{V\langle i\rangle}(\xi_{r,n}^{(i)}(G_{A_c,-i})) \in H^1_{\mathrm{f}}(\hat{H}_c(\varpi_n), V\langle i\rangle)$$

associated to g_{A_c} with the orientation ϵ. (cf. 5.6.) Here $V\langle i\rangle = V \otimes T_p\hat{\mathcal{M}}_{\overline{A}_c}^{\otimes i}$. Since $(c, p) = 1$, the projection $\pi_c : A \to A_c$ induces an isomorphism $T_p\hat{\mathcal{M}}_{\overline{A}} \cong T_p\hat{\mathcal{M}}_{\overline{A}_c}$. We identify $V\langle i\rangle$ for A_c with that for A.

Theorem 6.11 (i) *For an integer i such that $-r < i < r$, the local point $c_{r,n}^{(i)}(G_{A_c})$ is the Abel–Jacobi image of the generalized Heegner cycle $z_{cp^n}^{(i+r)}$.*

(ii) *The local point $c_{r,n}^{(i)}(G_{A_c,\alpha})$ is*

$$z_{cp^n,\alpha}^{(i+r)} = z_{cp^n}^{(i+r)} - p^{k-2}\alpha^{-1}\mathrm{Res}_{n/n-1} z_{cp^{n-1}}^{(i+r)}.$$

In particular, $C_{r,n}^{(i)}(G_{A_c,\alpha}) = \alpha^{-n} z_{cp^n,\alpha}^{(i+r)}$.

Proof (i) By definition and Proposition 6.8,

$$\xi_{r,n}^{(i)}(G_{A_c,\alpha}) = p^{(r-1)n}(1 \otimes \varphi^{-n} \otimes 1)\mathcal{I}_{A_c,-i}^{\sigma^{-n}}(\varpi_n).$$

It suffices to show this after taking the Bloch–Kato logarithm. Let ω_{A_c} be an invariant differential form on A_c and put $\omega_A = \pi_c^*\omega_{A_c}$. Then $(\pi_c)_*\omega_A^\vee = \omega_{A_c}^\vee$. We put

$$\omega_{A_c,i}^\vee := (\omega_{A_c}^\vee)^{\otimes r-i-1}(\xi_{A_c}^\vee)^{\otimes r+i-1} = (\omega_{A_c}^\vee \xi_{A_c}^\vee)^{\otimes r-1}\omega_{\hat{\mathcal{M}}_{\overline{A}_c}}^{\otimes i}.$$

(cf. (6.5).) Then we have $\omega_f(r-1) \otimes \omega_{\hat{\mathcal{M}}_{\overline{A}_c}}^{\otimes i} = \omega_f \otimes \omega_{A_c,i}^\vee$, and this is an element in the first de Rham filtration of $D^\vee\langle -i\rangle \subset D_f \otimes L_{E,k-2}^\vee$ (cf. (6.6)). Then it suffices to show that the evaluation of

$$\begin{aligned}\xi_{r,n}^{(i)}(G_{A_c,\alpha}) &\in D\langle i\rangle/\mathrm{Fil}^0 D\langle i\rangle \\ &= \mathrm{Hom}_{W\otimes\mathcal{O}}(\mathrm{Fil}^1 D^\vee\langle -i\rangle,\ W[1/p]\otimes\mathcal{O}) \\ &= \mathrm{Hom}_{W\otimes\mathcal{O}}((\mathrm{Fil}^1 D_f)(r-1) \otimes L_{\hat{\mathcal{M}}_{\overline{A}_c},i},\ W[1/p]\otimes\mathcal{O})\end{aligned}$$

at $\omega_f(r-1) \otimes \omega_{\hat{\mathcal{M}}_{\overline{A}_c}}^{\otimes i} = \omega_f \otimes \omega_{A_c,i}^\vee$ is the Abel–Jacobi image of the generalized Heegner cycle. Let A' be the deformation of $\overline{A}_c^{\sigma^{-n}}$ associated to ϖ_n as in Sect. 5.7. Let ϱ_n be such that $\varphi^n\omega_{A_c} = \varrho_n\omega_{A_c^{\sigma^{-n}}}$. Then we have $\varphi^n\xi_{A_c} = p^n\varrho_n^{-1}\xi_{A_c^{\sigma^{-n}}}$ and hence $\varphi^n\omega_{\hat{\mathcal{M}}_{\overline{A}_c}} = \varrho_n^2 p^{-n}\omega_{\hat{\mathcal{M}}_{\overline{A_c^{\sigma^{-n}}}}}$. By Proposition 5.6, for the isomorphism $\iota : A' \to A_{cp^n}$ and the projection $\pi : A_c \to A_{cp^n}$, we have

$$\iota_*\omega_{A'} = \varrho_n^{-1}\pi_*\omega_{A_c}, \quad \iota_*\xi_{A'} = p^{-n}\varrho_n\pi_*\omega_{A_c}.$$

Then by Proposition 5.7 and Proposition 6.8, we have

$$\begin{aligned} & p^{(r-1)n}(1\otimes\varphi^{-n}\otimes 1)\mathcal{I}^{\sigma^{-n}}_{A_c,-i}(\varpi_n)\left(\omega_f(r-1)\otimes\omega^{\otimes i}_{\hat{\mathcal{M}}_{\overline{A}_c}}\right) \\ & = p^{(r-1)n}\mathcal{I}_{A_c^{\sigma^{-n}},-i}(\varpi_n)\left(\omega_f(r-1)\otimes\varphi^n\omega^{\otimes i}_{\hat{\mathcal{M}}_{\overline{A}_c}}\right) \\ & = p^{(r-i-1)n}\varrho_n^{2i}\mathcal{I}_{A_c^{\sigma^{-n}},-i}(\varpi_n)\left(\omega_f(r-1)\otimes\omega^{\otimes i}_{\hat{\mathcal{M}}_{\overline{A}_c^{\sigma^{-n}}}}\right) \\ & = p^{(r-i-1)n}\varrho_n^{2i}\langle F_f(A'/H_{cp^n},\mathrm{Lv}),(\omega^\vee_{A'})^{\otimes r-i-1}(\xi^\vee_{A'})^{\otimes r+i-1}\rangle_{A'} \\ & = \langle F_f(A_{cp^n}/H_{cp^n},\mathrm{Lv}),\pi_*\omega^\vee_{A_c,i}\rangle_{A_c} \\ & = \langle \pi^*_{cp^n}F_f(A_{cp^n}/H_{cp^n},\mathrm{Lv}),\omega^\vee_{A,i}\rangle_{A}. \end{aligned}$$

Hence the assertion follows from [2, Proposition 3.21].

(ii) follows from $\mathcal{I}_\alpha(\omega_f)=(1-\alpha^{-1}\varphi)F_f$ and i). The last assertion follows from that the difference between (7.2) and (7.3) is $p^{rn}\phi_n^{-1}$ on $\mathcal{N}_\alpha\subset D_p(V)$.

Remark 6.12 By using results in Sects. 2 and 4, we can extend $z^{(i+r)}_{cp^n}$ for an arbitrary integer $i>-r$ (even as a global cohomology class controlling denominators). On the other hand, $c^{(i)}_{r,n}(G_{A_c})$ is also defined for $i>-r$ by the Perrin-Riou theory. They coincide with each other since they do for $-r<i<r$ by Theorem 6.11 and satisfy the same congruence relation by Proposition 7.9 in the Appendix and the uniqueness of our twist theory.

7 Appendix: Perrin-Riou theory for a relative Lubin–Tate extension

We explain the Perrin-Riou theory for a relative Lubin–Tate extension following [33] and [35].

7.1 Notations and setting

Let k be a finite field and $W=W(k)$ the Witt vector with Frobenius σ. In this appendix, let H be the fraction field of W and $\mathbb{C}_p$ the completion of the algebraic closure of H. Let $\mathcal{G}=\mathrm{Spf}\,R_{\mathcal{G}}$ be a relative Lubin–Tate formal group over $\mathbf{W}$ of height 1. Though $R_{\mathcal{G}}$ is isomorphic to the one-variable formal power series ring $W[\![X]\!]$, we prefer a coordinate-free description for our application to the local moduli. We sometimes write $R_{\mathcal{G}}$ simply by R. For an automorphism $\tau\in\mathrm{Gal}(H/\mathbb{Q}_p)$, let $\mathcal{G}^\tau$ be the base change of $\mathcal{G}$ by τ. Hence $R_{\mathcal{G}^\tau}=R$ as a ring but $a\in W$ acts by $\tau^{-1}(a)$. For simplicity, we denote $R_{\mathcal{G}^\tau}$ by $R^{(n)}$ if $\tau=\sigma^{-n}$.

We denote the space of invariant differentials of $\mathcal{G}$ by $D_{\mathcal{G}}$, which is a W-module of rank 1, and put $L_{\mathcal{G},i}=D_{\mathcal{G}}^{\otimes i}$ for $i\geq 0$ and $L_{\mathcal{G},i}=(D_{\mathcal{G}}^\vee)^{\otimes -i}$ for $i<0$. We let $\mathcal{L}_{\mathcal{G},i}=L_{\mathcal{G},i}\otimes_W R$. Let $\mathscr{H}_\infty(\mathcal{G})=\cup_h\mathscr{H}_h(\mathcal{G})$ be the Perrin-Riou ring associated to $\mathcal{G}$ where

$$\mathscr{H}_h(\mathcal{G}):=\left\{\sum_{n=0}^\infty a_nX^n \mid a_n\in W[1/p],\ |a_n|_p n^{-h}\to 0\right\}$$

and X is a coordinate of R. We also write $\mathscr{H}_\infty(\mathcal{G})$ simply by $\mathscr{H}_\infty$ if there is no fear of confusion. We put $\mathscr{L}_{\mathcal{G},i}=L_{\mathcal{G},i}\otimes_W\mathscr{H}_\infty(\mathcal{G})$. We also put $\mathcal{L}_{\mathcal{G}}=\prod_{i\in\mathbb{Z}}\mathcal{L}_{\mathcal{G},i}$ and $\mathscr{L}_{\mathcal{G}}=\prod_{i\in\mathbb{Z}}\mathscr{L}_{\mathcal{G},i}$.

Let ϕ be the Frobenius map $\phi : \mathcal{G} \to \mathcal{G}^\sigma$. We denote by φ the σ-semilinear ring homomorphism $\varphi : R \to R$ associated to ϕ, and by ψ the σ^{-1}-semilinear map $R \to R$ such that $\psi \circ \varphi = 1$ and

$$\varphi \circ \psi(g) = p^{-1} \sum_{x \in \mathcal{G}[\phi]} t_x^* g$$

where t_x is the translation on $\mathcal{G}$ by x. Since the height of $\mathcal{G}$ is 1, the space $L_{\mathcal{G},1}$ is the Dieudonné module of the special fiber of $\mathcal{G}$ and hence the Frobenius act on it, which is denoted by φ. Then the action of φ on $\mathscr{L}_{\mathcal{G},i} = L_{\mathcal{G},i} \otimes_W \mathscr{H}_\infty(\mathcal{G})$ are defined diagonally.

Let d be the formal derivation $R \to \hat{\Omega}^1_{R/W}$, $g \mapsto dg$. Since $\hat{\Omega}^1_{R/W} = R \otimes_W L_{\mathcal{G},1}$, the derivation d is extended to $\mathcal{L}_{\mathcal{G},i} \to \mathcal{L}_{\mathcal{G},i+1}$, and its horizontal section is $L_{\mathcal{G},i}$. The derivation d is also compatible with φ. We denote by $\mathcal{L}_{\mathcal{G}}^{d=1}$ the set of elements in $\mathcal{L}_{\mathcal{G}}$ fixed by d, and similarly for $\mathscr{L}_{\mathcal{G}}^{d=1}$. We fix an invariant differential form $\omega_{\mathcal{G}}$ of $\mathcal{G}$, and let $\partial : R \to R$ be the differential operator such that $d(g) = \partial(g)\omega_{\mathcal{G}}$. Let $\lambda_{\mathcal{G}} \in \mathscr{H}_\infty(\mathcal{G})$ be the logarithm associated with $\omega_{\mathcal{G}}$. Let ϖ be a uniformizer of W such that $\phi^*\omega_{\mathcal{G}^\sigma} = \varpi\omega_{\mathcal{G}}$ or in other words, $\varphi\omega_{\mathcal{G}} = \varpi\omega_{\mathcal{G}}$. In particular, $\varphi\lambda_{\mathcal{G}} = \varpi\lambda_{\mathcal{G}}$. Note that ϖ depends on the choice of $\omega_{\mathcal{G}}$. By the compatibility of d and ϕ, we have $\varpi(\varphi \circ \partial) = \partial \circ \varphi$ on R.

7.2 The *p*-adic period of $\mathcal{G}$

Let x be a point of $\mathcal{G}(\mathcal{O}_{\mathbb{C}_p})$, which corresponds to a continuous ring homomorphism $R \to \mathcal{O}_{\mathbb{C}_p}$ over W. This morphism is uniquely extended to $\mathscr{H}_\infty(\mathcal{G}) \to \mathbb{C}_p$. For $f \in \mathscr{H}_\infty(\mathcal{G})$, we write the image of f by this morphism by $f(x)$. In particular, if x corresponds to the origin of $\mathcal{G}$, we denote it by $f(0)$. Similarly, for a point $\tilde{x} \in \mathcal{G}(\mathbb{A}_{\rm inf})$, we can define $f(\tilde{x}) \in B^+_{\rm cris}$. Let $\theta : \mathbb{A}_{\rm inf} \to \mathcal{O}_{\mathbb{C}_p}$ be the canonical map defined by Fontaine.

Lemma 7.1 *For an element $w = (w_n) \in \varprojlim_n \mathcal{G}(\mathcal{O}_{\mathbb{C}_p})$, there is a unique lift $\tilde{w} = (\tilde{w}_n) \in \varprojlim_n \mathcal{G}(\mathbb{A}_{\rm inf})$ of w with respect to $\theta : \mathcal{G}(\mathbb{A}_{\rm inf}) \to \mathcal{G}(\mathcal{O}_{\mathbb{C}_p})$. Here the transition map is the multiplication $[p]_{\mathcal{G}}$ of $\mathcal{G}$. We have $\varphi(\tilde{w}) = \widetilde{\varphi(w)} = \varphi(X)|_{X=\tilde{w}}$ where φ on the left-hand side is the Frobenius on $\mathbb{A}_{\rm inf}$ and φ on the middle and right-hand side is that of $\mathcal{G}$.*

Proof Suppose that all $\tilde{w}_n$ are in $\mathrm{Ker}\,\theta$. Then $\tilde{w}_n \in \cap_m [p^m]_{\mathcal{G}}(\mathrm{Ker}\,\theta)$. Hence the uniqueness follows from the fact that $\mathbb{A}_{\rm inf}$ is separated with respect to the topology induced by the ideal $(p) + \mathrm{Ker}\,\theta$. We fix n. For a natural number m, we take any lift $\tilde{w}'_{n+m}$ of w_{n+m} to $\mathbb{A}_{\rm inf}$ by θ. Then put $\tilde{w}_n := \lim_{k\to\infty}[p^m]_{\mathcal{G}}\tilde{w}'_{n+m}$. It is straightforward to check that it is well-defined and has the desired property. (The last property follows from the uniqueness. Note also that $\varprojlim_n \mathcal{G}(\mathcal{O}_{\mathbb{C}_p}) = \varprojlim_n \mathcal{G}(\mathcal{O}_{\mathbb{C}_p}/p)$ by the canonical projection.)

For a generator $\epsilon = (\epsilon_n)_n \in T_p\mathcal{G}$, we take the lift $\tilde{\epsilon} = (\tilde{\epsilon}_n)_n \in \varprojlim_n \mathcal{G}(\mathbb{A}_{\rm inf})$ in the above lemma, and define the p-adic period of $\mathcal{G}$ by

$$t_\epsilon = \lambda_{\mathcal{G}}(\tilde{\epsilon}_0) \in B^+_{\rm cris}.$$

By Lemma 7.1, we have

$$\varphi(t_\epsilon) = (\varphi\lambda_{\mathcal{G}})(X)|_{X=\tilde{\epsilon}_0} = \varpi\lambda_{\mathcal{G}}(\tilde{\epsilon}_0) = \varpi t_\epsilon.$$

For a Galois representation V of G_H, we let $D_p(V) = (B_{\rm cris} \otimes_{\mathbb{Q}_p} V)^{G_H}$. Then we have

$$e := \epsilon \otimes t_\epsilon^{-1} \in (V_p\mathcal{G} \otimes_{\mathbb{Q}_p} B_{\rm cris})^{G_H} = D_p(V_p\mathcal{G}).$$

The element e depends only on the choice of $\omega_{\mathcal{G}}$. We denote $e^{\otimes k}$ by e_k. We also denote by φ the Frobenius on $D_p(V_p\mathcal{G})$ coming from the Frobenius on B_{cris} and the identity on $V_p\mathcal{G}$. Then $\varphi e = \varpi^{-1}e$ and the map

$$H^1_{\mathrm{dR}}(\mathcal{G}/W) = D_{\mathcal{G}} \longrightarrow D_p(V_p\mathcal{G})^{\otimes -1} = D_p(H^1_{et}(\mathcal{G})), \qquad \omega_{\mathcal{G}} \longmapsto e^{\otimes -1}$$

is an isomorphism of filtered φ-modules. We sometimes identify $\omega_{\mathcal{G}}$ and $e^{\otimes -1}$. We put $V\langle k\rangle = V \otimes_{\mathbb{Q}_p} V_p\mathcal{G}^{\otimes k}$. Suppose V is crystalline. Then the morphism

$$D_p(V\langle k\rangle) = D_p(V)\otimes_H He_k \longrightarrow D_p(V)\otimes_H B_{\mathrm{cris}}, \qquad d\otimes e_k \longmapsto d\otimes t_\epsilon^{-k} \tag{7.1}$$

is an embedding of filtered φ-modules, and we regard

$$D_p(V\langle k\rangle) = D_p(V)\otimes L_{\mathcal{G},-k} \subset D_p(V)\otimes_H B_{\mathrm{cris}}.$$

7.3 Solution of $(1-\Phi)G = g$

Let D be a finite-dimensional vector space over H with a semi-linear action of φ. Let Φ be the action $\varphi\otimes\varphi$ on $D\otimes_W \mathcal{L}_{\mathcal{G}}$. The derivation d is extended on $D\otimes_W \mathscr{L}_{\mathcal{G}}$ by $1\otimes d$. Let M be W-lattice of D and suppose that there exist a natural number h and a non-negative integer c_0 satisfying $p^{nh}\varphi^n M \subset p^{-c_0}M$ for all n.

Proposition 7.2 *Assume that $1-\Phi$ is invertible on $D\otimes L_{\mathcal{G}}$. For $g\in D\otimes\mathscr{L}_{\mathcal{G}}^{d=1}$, there exists a unique solution $G\in D\otimes\mathscr{L}_{\mathcal{G}}^{d=1}$ of $(1-\Phi)G = g$.*

Proof This is similarly proven in [35, §2].

Proposition 7.3 *There exists an integer c such that*

$$p^{n(h+i-1)}(\tau-1)G_{-i}(\tilde{\epsilon}_n) \in p^{-c}M\otimes L^{d=1}_{\mathcal{G},-i}\otimes A_{\mathrm{cris}}$$

for all $\tau\in G_{H(\epsilon_n)}$, all i such that $h+i-1\geq 0$ and a solution of $(1-\Phi)G = g$ for $g\in\mathcal{L}_{\mathcal{G}}^{d=1}\otimes M$.

Proof This is similarly proven in [33, §2.2.1].

7.4 Evaluation at φ-torsion points

As before, let $\epsilon = (\epsilon_n)\in T_p\mathcal{G}$ be a generator. Let $\phi_n : \mathcal{G}^{(n)}\to\mathcal{G}$ be the p^n-th Frobenius map. We have a morphism of formal groups $\phi_n^\vee : \mathcal{G}\to\mathcal{G}^{(n)}$ such that $\phi_n\circ\phi_n^\vee = [p^n]_{\mathcal{G}}$. We put $\varpi_n = \phi_n^\vee(\epsilon_n)\in\mathcal{G}^{(n)}[p^n]$. The system $(\varpi_n)_n$ satisfies that $\phi(\varpi_{n+1}) = \varpi_n$ and $\varpi_1\neq 0$. For an element $G\in D\otimes\mathscr{L}_{\mathcal{G}}$, we define the value

$$G^{(n)}(\varpi_n)\in D^{(n)}\otimes L_{\mathcal{G}^{(n)}}\otimes H(\varpi_n)$$

as follows. Here, for a W-module M, we write by $M^{(n)}$ the abelian group M and W-structure is twisted by σ^n, that is $a\in W$ acts by $\sigma^n(a)$ on M. As $\mathbb{Z}_p$-modules, we have $D\otimes\mathscr{L}_{\mathcal{G}} = D^{(n)}\otimes\mathscr{L}_{\mathcal{G}}^{(n)} = D^{(n)}\otimes\mathscr{L}_{\mathcal{G}^{(n)}}$ and identify G with $G^{(n)}\in D^{(n)}\otimes\mathscr{L}_{\mathcal{G}^{(n)}}$. Then $G^{(n)}(\varpi_n)$ is the image of G by the morphism

$$D^{(n)}\otimes\mathscr{L}_{\mathcal{G}^{(n)}} \longrightarrow D^{(n)}\otimes L_{\mathcal{G}^{(n)}}\otimes H(\varpi_n)$$

induced by ϖ_n. Suppose that φ is bijective on D. We denote by ϕ_n the morphism $D \to D^{(n)}$ of W-modules induced by the semi-linear map φ^n on D. By considering the image of $G^{(n)}(\varpi_n)$ by

$$\phi_n^{-1} \otimes \phi_n^{-1} : D^{(n)} \otimes L_{\mathcal{G}^{(n)}} \longrightarrow D \otimes L_{\mathcal{G}},$$

we have $(\phi_n^{-1} \otimes \phi_n^{-1} \otimes 1)G^{(n)}(\varpi_n) \in D \otimes L_{\mathcal{G}} \otimes H(\varpi_n)$.

Suppose that D is defined over $\mathbb{Q}_p$, that is, there is a filtered φ-module $D_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$ and $D = H \otimes_{\mathbb{Q}_p} D_{\mathbb{Q}_p}$. Then we have a map

$$1 \otimes \phi_n^{-1} : D^{(n)} \otimes L_{\mathcal{G}^{(n)}} = D_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L_{\mathcal{G}^{(n)}} \longrightarrow D_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L_{\mathcal{G}} = D \otimes_H L_{\mathcal{G}}.$$

Hence we have

$$(1 \otimes \phi_n^{-1} \otimes 1)G^{(n)}(\varpi_n) \in D \otimes L_{\mathcal{G}} \otimes H(\varpi_n).$$

7.5 Norm compatible family of local points

Let V be a crystalline representation of G_H. Let h be a natural number such that $\mathrm{Fil}^{-h} D_p(V) = D_p(V)$. For simplicity, we assume that $1 - \Phi$ is invertible on $D \otimes L_{\mathcal{G}}$. According to [35, 3.2.1], for an element $G = (G_j)_j \in D_p(V) \otimes \mathscr{L}_{\mathcal{G}}^{d=1}$ and an integer i such that $h + i - 1 \geq 0$, we put

$$\Xi_{h,n}^{(i)}(G) = (-1)^{h+i-1}(h+i-1)!\,p^{-n}(\phi_n^{-1} \otimes \phi_n^{-1} \otimes 1)G_{-i}^{(n)}(\varpi_n) \in D_p(V\langle i \rangle) \otimes H(\varpi_n). \tag{7.2}$$

(Note that in [35], $\Xi_{h,n}^{(i)}(G)$ is denoted by $\Xi_{n,i}^{(h)}(G)$. We change it because of the compatibility with the notation in Sect. 2.) Then we consider the image of the Bloch–Kato exponential map

$$C_{h,n}^{(i)}(G) = \exp_{V\langle i \rangle}(\Xi_{h,n}^{(i)}(G)) \in H^1_{\mathrm{f}}(H(\varpi_n), V\langle i \rangle).$$

Sometimes, we omit the index h or G from $C_{h,n}^{(i)}(G)$ for simplicity.

Proposition 7.4 *Suppose that G is a solution of $(1-\varphi)G = g$ for $g \in D_p(V) \otimes (\mathscr{L}_{\mathcal{G}}^{d=1})^{\psi=0}$. For $i \geq 1-h$, the system $(C_{h,n}^{(i)}(G))_n$ is norm compatible for $n \geq 1$, that is,*

$$\mathrm{Cor}_{n+1/n} C_{h,n+1}^{(i)}(G) = C_{h,n}^{(i)}(G).$$

Proof We write that $g = (g_i)_i$ with $g_i \in D_p(V) \otimes L_{\mathcal{G},i} \otimes R_{\mathcal{G}}^{\psi=0}$ and $G = (G_i)_i$ with $G_i \in D_p(V) \otimes L_{\mathcal{G},i} \otimes \mathscr{H}_\infty(\mathcal{G})$. Then applying $1 \otimes 1 \otimes (\varphi \circ \psi)$ to $(1-\Phi)G_i = g_i$, we have

$$(1 \otimes 1 \otimes \varphi \circ \psi)G_i = (\varphi \otimes \varphi \otimes \varphi)G_i.$$

Hence we have

$$\begin{aligned}
&\mathrm{Tr}_{n+1/n}(\phi_{n+1}^{-1} \otimes \phi_{n+1}^{-1} \otimes 1)G_i^{(n+1)}(\varpi_{n+1}) \\
&\quad = p(\phi_{n+1}^{-1} \otimes \phi_{n+1}^{-1} \otimes 1)[(1 \otimes 1 \otimes \varphi \circ \psi)G_i^{(n+1)}](\varpi_{n+1}) \\
&\quad = p[(\phi_{n+1}^{-1} \otimes \phi_{n+1}^{-1} \otimes 1)(\varphi \otimes \varphi \otimes \varphi)G_i^{\sigma^{-n-1}}](\varpi_{n+1}) = p(\phi_n^{-1} \otimes \phi_n^{-1} \otimes 1)G_i^{(n)}(\varpi_n).
\end{aligned}$$

The assertion follows from this.

7.6 Integral family of local points

In this subsection, we assume that V is a crystalline representation of $G_{\mathbb{Q}_p}$ and denote the associated filtered φ-module by $D_p(V)_{\mathbb{Q}_p} = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_{\mathbb{Q}_p}}$, and put $D_p(V) = D_p(V)_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} H$. Then for an element $G = (G_i)_i \in D_p(V) \otimes \mathscr{L}_{\mathcal{G}}^{d=1}$, and an integer i such that $h + i - 1 \geq 0$, we put

$$\begin{aligned}\xi_{h,n}^{(i)}(G) &= (-1)^{h+i-1}(h+i-1)!p^{(h-1)n}(1 \otimes \phi_n^{-1} \otimes 1)G_{-i}^{(n)}(\varpi_n) \\ &\in D_p(V)_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L_{\mathcal{G},-i} \otimes H(\varpi_n) = D_p(V\langle i\rangle) \otimes H(\varpi_n). \qquad (7.3)\end{aligned}$$

We consider the image of the Bloch–Kato exponential map

$$c_{h,n}^{(i)}(G) = \exp_{V\langle i\rangle}(\xi_{h,n}^{(i)}(G)) \ \in \ H_{\mathrm{f}}^1(H(\varpi_n), V\langle i\rangle).$$

Sometimes, we omit the index h or G from $c_{h,n}^{(i)}(G)$ if there is no fear of confusion.

To show the integral property of $c_{h,n}^{(i)}(G)$, we recall an explicit cocycle representation of it. The Bloch–Kato exponential map is the connecting map of the fundamental exact sequence

$$0 \longrightarrow V\langle i\rangle \longrightarrow (V\langle i\rangle \otimes B_{\mathrm{cris}}) \oplus (V\langle i\rangle \otimes B_{\mathrm{dR}}^+) \longrightarrow (V\langle i\rangle \otimes B_{\mathrm{cris}}) \oplus (V\langle i\rangle \otimes B_{\mathrm{dR}}) \longrightarrow 0$$

where the second map is diagonal and the third map is given by $(x, y) \mapsto ((1-\varphi)x, x-y)$. First, take a lift $\tilde{z} \in D_p(V\langle i\rangle) \otimes B_{\mathrm{cris}}$ of

$$z \in D_p(V\langle i\rangle) \otimes H(\varpi_n) \subset V\langle i\rangle \otimes B_{\mathrm{dR}}$$

such that $\tilde{z} - z \in \mathrm{Fil}^0(D_p(V\langle i\rangle) \otimes B_{\mathrm{dR}})$. Since

$$1 - \varphi : \mathrm{Fil}^0(D_p(V\langle i\rangle) \otimes B_{\mathrm{cris}}) \ \longrightarrow \ D_p(V\langle i\rangle) \otimes B_{\mathrm{cris}}$$

is bijective, we find $\tilde{z}_0 \in \mathrm{Fil}^0(D_p(V\langle i\rangle) \otimes B_{\mathrm{cris}})$ such that $(1-\varphi)\tilde{z} = (1-\varphi)\tilde{z}_0$. Then the cocycle is given by

$$\tau \in G_{H(\varpi_n)} \ \longrightarrow \ (\tau - 1)(\tilde{z} - \tilde{z}_0) \in \mathrm{Fil}^0(D_p(V\langle i\rangle) \otimes B_{\mathrm{cris}})^{\varphi=1} = V\langle i\rangle. \qquad (7.4)$$

(Note that since $(\tau-1)z = 0$, we have $(\tau-1)\tilde{z} = (\tau-1)(\tilde{z}-z) \in \mathrm{Fil}^0(D_p(V\langle i\rangle) \otimes B_{\mathrm{dR}})$.)

Now we investigate the integral properties of the local points. Let M be a W-lattice of $D_p(V)$ and let T be the Galois stable lattice

$$T = \mathrm{Fil}^0(M \otimes t^{-b}A_{\mathrm{cris}})^{\varphi=1} \subset D_p(V) \otimes t^{-b}A_{\mathrm{cris}}.$$

of V where b is a natural number such that $\mathrm{Fil}^b D_p(V) = 0$ and t is "$2\pi i$" in B_{dR}. Then we also have

$$T\langle i\rangle = \mathrm{Fil}^0(M \otimes L_{\mathcal{G},-i} \otimes t^{-b}A_{\mathrm{cris}})^{\varphi=1}.$$

Note that by the identification (7.1) as filtered φ-modules,

$$\begin{aligned}T\langle i\rangle &= \mathrm{Fil}^0(M \otimes L_{\mathcal{G},-i} \otimes t^{-b}A_{\mathrm{cris}})^{\varphi=1} \\ &= \mathrm{Fil}^0(M \otimes t^{-b+i}A_{\mathrm{cris}})^{\varphi=1} \subset D_p(V) \otimes t^{-b+i}A_{\mathrm{cris}}.\end{aligned}$$

Then we have $T\langle i\rangle = T \otimes \mathbb{Z}_p t_\epsilon^i$ in $D_p(V) \otimes B_{\mathrm{cris}}$.

The key of the Perrin-Riou theory is an explicit construction of the lift $\tilde{z}$. For simplicity, we fix a (non-canonical) isomorphism $R \cong W[\![X]\!]$ and an invariant differential $\omega_{\mathcal{G}}$ such that $\omega_{\mathcal{G}}/dX|_{X=0} = 1$. We regard the logarithm $\lambda_{\mathcal{G}}$ as a convergent power series in $H[\![X]\!]$

and $\exp_{\mathcal{G}}$ the formal power series in $H[\![X]\!]$ such that $\exp_{\mathcal{G}} \circ \lambda_{\mathcal{G}}(X) = X$. For an integer m, we put $\epsilon_n^{(m)} = \psi_m(\epsilon_n) \in \mathcal{G}^{(m)}[p^n]$ and $\epsilon^{(m)} = (\epsilon_n^{(m)})_n \in T_p\mathcal{G}^{(m)}$. We take the lift $(\tilde{\epsilon}_n^{(m)})_n \in \varprojlim_n \mathcal{G}^{(m)}(\mathbb{A}_{\mathrm{inf}})$ by Lemma 7.1 and consider the p-adic period $t_{\epsilon^{(n)}} = \lambda_{\mathcal{G}^{(n)}}(\tilde{\epsilon}_0^{(n)})$. Note that if $\varphi^n \omega_{\mathcal{G}^{(n)}} = \varpi^{(n)}\omega_{\mathcal{G}}$, then we have $p^n t_\epsilon = \varpi^{(n)} t_{\epsilon^{(n)}}$. The following is the key formula that describes an explicit lift of ϖ_n in B_{dR}.

Lemma 7.5 *In B_{dR}, we have*

$$\varpi_n = \tilde{\epsilon}_n^{(n)} \oplus_{\mathcal{G}^{(n)}} \exp_{\mathcal{G}^{(n)}} \left(-\frac{t_{\epsilon^{(n)}}}{p^n} \right).$$

Proof The right hand side is an element of $\mathcal{G}^{(n)}[p^n]$, and its projection to $\mathbb{C}_p$ is ϖ_n. The assertion follows from these.

For simplicity, we put $t_n := -\frac{t_{\epsilon^{(n)}}}{p^n}$. We let

$$\tilde{G}(Z) := G^{(n)}(\tilde{\epsilon}_n^{(n)} \oplus_{\mathcal{G}^{(n)}} \exp_{\mathcal{G}^{(n)}} Z) \in D_p(V)^{(n)} \otimes_W L_{\mathcal{G}^{(n)}} \otimes_W \mathbb{A}_{\mathrm{inf}}\langle\!\langle Z \rangle\!\rangle.$$

where $\mathbb{A}_{\mathrm{inf}}\langle\!\langle Z \rangle\!\rangle = \{\sum_{n=0}^{\infty} a_n \frac{Z^n}{n!} | a_n \in \mathbb{A}_{\mathrm{inf}}\}$. Suppose that $\tilde{G}(Z) = (\tilde{G}_i(Z))_i$ where

$$\tilde{G}_i(Z) \in D_p(V)^{(n)} \otimes_W L_{\mathcal{G}^{(n)},i} \otimes_W \mathbb{A}_{\mathrm{inf}}\langle\!\langle Z \rangle\!\rangle.$$

Let $P(Z)$ be a polynomial in $D_p(V)^{(n)} \otimes L_{\mathcal{G}^{(n)}} \otimes \mathbb{A}_{\mathrm{inf}}[Z]$ such that the i-th component $P_i(Z)$ is the polynomial part of the power series $\tilde{G}_i(Z)$ of degree $\leq h-1-i$. (If $h-1-i < 0$, we put $P_i(Z) = 0$.) We denote $P(Z)$ by $P(G)(Z)$ if we emphasize the dependence of G. Since $\mathrm{Fil}^i L_{\mathcal{G}^{(n)},i} = L_{\mathcal{G}^{(n)},i}$ and $\mathrm{Fil}^{-h} D_p(V) = D_p(V)$, we have

$$\begin{aligned} &G^{(n)}(\varpi_n) - P(t_n) = \tilde{G}(t_n) - P(t_n) \\ &\in \prod_{i \in \mathbb{Z}} L_{\mathcal{G}^{(n)},i} \otimes D_p(V)^{(n)} \otimes \mathrm{Fil}^{h-i} B_{\mathrm{dR}} \subset \mathrm{Fil}^0 \left(D_p(V)^{(n)} \otimes L_{\mathcal{G}^{(n)}} \otimes B_{\mathrm{dR}} \right). \end{aligned} \tag{7.5}$$

Lemma 7.6 *Suppose that $i \geq 1-h$. We have*

$$P_{-i}(G)(Z) = \sum_{k=0}^{h+i-1} (-1)^k \frac{\partial^k G_{-i}^{(n)}(\tilde{\epsilon}_n^{(n)})}{k!} Z^k$$

where $\partial = \lambda'_{\mathcal{G}}(X)^{-1} \frac{d}{dX}$. (Note that $G_{-i}^{(n)} \in D_p(V\langle i \rangle)^{(n)} \otimes \mathscr{H}_\infty(G) \subset D_p(V\langle i \rangle)^{(n)} \otimes H[\![Z]\!]$.)

Proof Put $Z = \lambda_{\mathcal{G}}(X)$. Then $\frac{d}{dZ} = \partial$. Since $\lambda'_{\mathcal{G}}(X)dX$ is an invariant differential, we have

$$\partial^k(G^{(n)}(\tilde{\epsilon}_n^{(n)} \oplus_{\mathcal{G}^{(n)}} X)) = (\partial^k G^{(n)})(\tilde{\epsilon}_n^{(n)} \oplus_{\mathcal{G}^{(n)}} X).$$

The assertion follows from this.

Proposition 7.7 *Assume that $1 - \Phi$ is invertible on $D \otimes L_{\mathcal{G}}$. Suppose that $g \in M \otimes \mathcal{L}_{\mathcal{G}}^{d=1}$ and let G be the solution of $(1-\varphi)G = g$. Then there is an integer c independent of g and n such that*

$$(h+i-1)!p^{(h+i-1)n}(\tau - 1)P_{-i}(t_n) \in p^{-c} M \otimes L_{\mathcal{G},-i} \otimes A_{\mathrm{cris}}$$

for $i \geq 1-h$ and any $\tau \in G_{H(\varpi_n)}$.

Proof For a non-negative integer k, we have

$$\partial^k G_{-i}^{(n)} \cdot \omega_{\mathcal{G}}^{\otimes k} = d^k G_{-i}^{(n)} = G_{k-i}^{(n)}.$$

Hence by Proposition 7.3, we have a constant c such that

$$p^{(h+i-k-1)n}(\tau - 1)\partial^k G_{-i}^{(n)}(\tilde{\epsilon}_n^{(n)}) \in p^{-c} M^{(n)} \otimes L_{\mathcal{G}^{(n)},-i} \otimes A_{\mathrm{cris}}.$$

The assertion follows from these and Lemma 7.6.

Theorem 7.8 *Suppose that V is a crystalline representation of $G_{\mathbb{Q}_p}$. With the same notations and assumptions as Proposition 7.7, there exists an integer c such that*

$$p^c\, c_n^{(i)}(G) \in H^1(H(\varpi_n), T\langle i\rangle)$$

for all natural number n and all integers i such that $i \geq 1-h$.

Proof By (7.5), as a lift $\tilde{z}$ of $z = \xi_{h,n}^{(i)}(G)$ in (7.4), we may take

$$(-1)^{h+i-1}(h+i-1)!p^{(h-1)n}(1 \otimes \phi_n^{-1} \otimes 1)P_{-i}(G)(t_n) \;\in\; D_p(V)_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L_{\mathcal{G},-i} \otimes B_{\mathrm{cris}}.$$

Note that $p^i\varphi$ is invertible on $L_{\mathcal{G},-i}$. Then by Proposition 7.7, we have that

$$(h+i-1)!p^{(h-1)n}(\tau - 1)(1 \otimes \phi_n^{-1} \otimes 1)P_{-i}(G)(t_n) \in p^{-c} M_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L_{\mathcal{G},-i} \otimes A_{\mathrm{cris}}. \quad (7.6)$$

This gives the desired estimate for $(\tau - 1)\tilde{z}$. For the estimate of $\tilde{z}_0$, first, note that

$$(1-\varphi)P_{-i}(G)(t_n) = P_{-i}(g)(t_n) \in D_p(V)^{(n)} \otimes L_{\mathcal{G}^{(n)},-i} \otimes A_{\mathrm{cris}}. \quad (7.7)$$

(Here, φ is the σ-semi-linear Frobenius, which acts diagonally on $D_p(V)^{(n)} \otimes L_{\mathcal{G}^{(n)},-i} \otimes A_{\mathrm{cris}}$.) In fact, we have

$$(\partial^k G_{-i}^{(n)})(\tilde{\epsilon}_n^{(n)}) \cdot t_{\epsilon^{(n)}}^k = (d^k G_{-i}^{(n)})(\tilde{\epsilon}_n^{(n)}) \cdot \omega_{\mathcal{G}^{(n)}}^{\otimes -k} \cdot t_{\epsilon^{(n)}}^k = G_{k-i}^{(n)}(\tilde{\epsilon}_n^{(n)})(\omega_{\mathcal{G}^{(n)}}^{\otimes -1} \cdot t_{\epsilon^{(n)}})^k.$$

Then $\omega_{\mathcal{G}^{(n)}}^{\otimes -1} \cdot t_{\epsilon^{(n)}}$ is fixed by φ, and by Lemma 7.1, we have

$$(1-\varphi)G_{k-i}^{(n)}(\tilde{\epsilon}_n^{(n)}) = (1-\varphi)G_{k-i}^{(n)}(X)|_{X=\tilde{\epsilon}_n^{(n)}} = g_{k-i}^{(n)}(\tilde{\epsilon}_n^{(n)}).$$

(The Frobenius φ of A_{cris} in the left-hand side is replaced by φ on $R = W[\![X]\!]$ of $\mathcal{G}$ in the middle one.) Hence (7.7) follows from Lemma 7.6. Then we have

$$\begin{aligned}(1-\varphi)\tilde{z} &= (-1)^{h+i-1}(h+i-1)!p^{(h-1)n}(1 \otimes \phi_n^{-1} \otimes 1)P_{-i}(g)(t_n) \\ &= (-1)^{h+i-1}(h+i-1)!\left(1 \otimes (p^{-in}\phi_n^{-1}) \otimes 1\right)\left(p^{(h+i-1)n}P_{-i}(g)(t_n)\right) \\ &\in p^c M \otimes L_{\mathcal{G},-i} \otimes A_{\mathrm{cris}}.\end{aligned}$$

Then we have the desired estimate for $\tilde{z}_0$ such that $(1-\varphi)\tilde{z} = (1-\varphi)\tilde{z}_0$ by using the fact that there exists an integer s such that the image of

$$1-\varphi : \mathrm{Fil}^0(M \otimes L_{\mathcal{G},-i} \otimes t^{-b}A_{\mathrm{cris}}) \longrightarrow D_p(V\langle i\rangle) \otimes B_{\mathrm{cris}}$$

contains $p^s M \otimes L_{\mathcal{G},-i} \otimes t^{-b}A_{\mathrm{cris}}$ for all i. (cf. [33, Lemme 2.3.4].)

Proposition 7.9 *Suppose that G is a solution of* $(1-\varphi)G = g$ *for* $g \in D_p(V) \otimes (\mathcal{L}_{\mathcal{G}}^{d=1})^{\psi=0}$. *Then there is a constant c such that*

$$p^{-ni+c} \sum_{j=0}^{i} (-1)^j \binom{i}{j} d_n^{(j+1-h)} \in H^1(H_\infty, T) \tag{7.8}$$

for all $n, i \geq 0$. *Here* $d_n^{(j)}$ *is the image of* $c_{h,n}^{(j)}(G)$ *by the restriction on* H_∞ *twisted by* $\epsilon^{\otimes -j}$.

Proof We use the cocycle representation for $z_j = \xi_{h,n}^{(j+1-h)}(G)$ in the proof of Theorem 7.8. If $\tau \in G_{H_\infty}$, then $(\tau - 1)\tilde{z}_j = 0$ since the formula (7.6) is identically equal to zero. We calculate the part coming from $(1-\varphi)\tilde{z}_j$. Suppose that $\varphi^n \omega_{\mathcal{G}^{(n)}} = \varpi^{(n)} \omega_{\mathcal{G}}$. Then $p^n t_\epsilon = \varpi^{(n)} t_{\epsilon^{(n)}}$. We have

$$\begin{aligned}
&\sum_{j=0}^{i} \binom{i}{j} j! [(1 \otimes \phi_n^{-1} \otimes 1) P_{h-1-j}(g)(t_n)] \otimes \omega_{\mathcal{G}}^{\otimes h+1-j} t_\epsilon^{h-1-j} \\
&= \sum_{j=0}^{i} \binom{i}{j} (i-j)! [(1 \otimes \phi_n^{-1} \otimes 1) P_{h-1-i+j}(g)(t_n)] \otimes \omega_{\mathcal{G}}^{\otimes i-j+1-h} t_\epsilon^{h-1-i+j} \\
&= i! [(1 \otimes \phi_n^{-1} \otimes 1) \sum_{j=0}^{i} (-1)^j \partial^j P_{h-1-i}(g)(t_n) \frac{t_n^j}{j!}] \otimes \omega_{\mathcal{G}}^{\otimes i+1-h} t_\epsilon^{h-1-i} \\
&= i! [(1 \otimes \phi_n^{-1} \otimes 1) P_{h-1-i}(g)(0)] \otimes \omega_{\mathcal{G}}^{\otimes i+1-h} t_\epsilon^{h-1-i}.
\end{aligned}$$

In the final equation above, we use the Taylor expansion and the fact that the degree of P_{h-1-i} is i. We have $P_{h-1-i}(g)(0) = g_{h-1-i}^{(n)}(\tilde{\epsilon}_n^{(n)}) \in \mathbb{A}_{\text{inf}}$ (cf. Lemma 7.6), and

$$p^{-ni} \sum_{j=0}^{i} (-1)^j \binom{i}{j} (1-\varphi)\tilde{z}_j = (1 \otimes p^{n(h-1-i)} \phi_n^{-1} \otimes 1) P_{h-1-i}(g)(0)$$

is bounded. Hence the assertion follows.

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