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Isometries of $CAT(0)$ cube complexes are semi-simple

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Abstract
We consider an automorphism of an arbitrary $CAT(0)$ cube complex. We study its combinatorial displacement and we show that either the automorphism has a fixed point or it preserves some combinatorial axis. It follows that when a f.g. group contains a distorted cyclic subgroup, it admits no proper action on a discrete space with walls. As an application Baumslag-Solitar groups and Heisenberg groups provide examples of groups having a proper action on measured spaces with walls, but no proper action on a discrete space with wall.

Résumé
Nous considérons un automorphisme d'un complexe cubique $CAT(0)$ général. Nous étudions son déplacement combinatoire, et nous établissons une dichotomie: ou bien l'automorphisme fixe un point, ou bien il préserve un axe combinatoire. Il en résulte qu'un groupe de type fini contenant un sous-groupe cyclique distordu n'agit pas proprement sur un espace à murs discret. Ainsi les groupes de Baumslag–Solitar ou de Heisenberg fournissent des exemples de groupes agissant proprement sur un espace à murs mesurés, mais pas sur un espace à murs discret.

Keywords CAT(0) cube complexes · Spaces with walls · Classification of isometries · Baumslag–Solitar Groups

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1 Introduction

A *space with walls* is a set V together with a collection $\mathcal{H}$ of "separating" objects: the walls. It is required that two points are separated by finitely many walls (see [7]). Spaces with walls appear in various classical combinatorial structures, like the Davis-Moussong complex of a Coxeter group (see [4,8]), simply-connected polygonal complexes all of whose polygons have an even number of sides or $CAT(0)$ cube complexes.

This notion of "discrete" space with walls was generalized in [3] to that of a *space with measured walls*. Here there is a measure on the set of walls such that the measure of the set of walls separating two given points is finite - this finite measure is called the *wall-distance*, it is indeed a (pseudo-)distance. The case of the counting measure on the set of walls corresponds to a discrete space with walls. At the end of their paper the authors of [3] asked :

Question 1.1 Does a discrete group acting properly on a space with measured walls necessarily admit a proper action on a *discrete* space with walls ?

The Baumslag-Solitar group $BS(m,n)$ with parameters $m,n \in \mathbb{N}^*$ has the following presentation:

$$BS(m,n) := \langle\, a, b \mid ba^m b^{-1} = a^n \,\rangle.$$

The groups $BS(m,n)$ all act properly on a space with measured walls. We briefly recall such an action. First we have the action on the (locally finite) Bass-Serre tree $T_{m,n}$ of $BS(m,n)$ seen as the HNN-extension of the cyclic group generated by a by the isomorphism b sending the subgroup $\langle a^m \rangle$ onto the subgroup $\langle a^n \rangle$. This gives an action on a simplicial tree, thus a discrete space with walls. But this action is not proper since a fixes a vertex. We can also see a as a unit translation of the real line $\mathbb{R}$, and b as a well-choosen non trivial homothety of $\mathbb{R}$. For example using matrices we may let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{pmatrix}$ with $(e^{\beta})^2 = \frac{n}{m}$, then $a \mapsto A, b \mapsto B$ induces a representation of $BS(m,n)$ into $PSL(2,\mathbb{R})$, the isometry group of $\mathbb{H}^2$. Thus $BS(m,n)$ is represented as a parabolic group of isometries of $\mathbb{H}^2$, in such a way that the subgroup $\langle a \rangle$ acts properly. Now $\mathbb{H}^2$ has a natural $PSL(2,\mathbb{R})$-invariant structure of space with measured walls (see section 3 in [3]). The associated wall-distance is the standard hyperbolic metric. The element a is an horocyclic translation and it is readily seen that the action of $BS(m,n)$ on the product space with measured walls $T_{m,n} \times \mathbb{H}^2$ is proper (private communication by A. Valette, see also [5])

In this paper we answer in the negative to Question 1.1 by proving the following:

Theorem 1.2 *For $m \neq n$ the group $BS(m,n)$ has no proper action on a (discrete) space with walls.*

We want to show that for every action of $BS(m,n)$, $m \neq n$ on a space with walls $(V,\mathcal{H})$, all orbits of the infinite cyclic group generated by a are bounded. To do so we consider the $CAT(0)$ cube complex X naturally associated to $(V,\mathcal{H})$ and the induced action of $BS(m,n)$ on it. This complex was introduced simultaneously in [2,9], thus generalizing the previous constructions of [10,12].

Let us review some elementary properties of the $CAT(0)$ cube complex X (see [2,9] for details). There is an embedding $V \to X^0$ which is an isometry when V is equipped with the wall-distance (so $d(v,v')$ is the number of walls separating $\{v,v'\}$) and X^0 is equipped with the combinatorial distance of the 1-skeleton X^1. An automorphism f of the space with walls $(V,\mathcal{H})$ extends to an automorphism $\bar{f}$ of X. The property of having bounded orbits is equivalent for f or for $\bar{f}$. And the map $f \mapsto \bar{f}$ is an injective extension morphism $\mathrm{Aut}(V, \mathcal{H}) \to \mathrm{Aut}(X)$. It follows that Theorem 1.2 is a consequence of its analogue for $CAT(0)$ cube complexes:

Theorem 1.3 *For $m \neq n$ the group $BS(m, n)$ has no proper action on a $CAT(0)$ cube complex.*

Here is a short argument establishing Theorem 1.3 when the cube complex is finite dimensional. In that case the element a either fixes a point or preserves a $CAT(0)$ geodesic on which it has a positive translation length L. But this latter case is impossible if $m \neq n$ because the Baumslag-Solitar relation imposes $mL = nL$.

In order to handle the general case we establish a classification of automorphisms of arbitrary $CAT(0)$ cube complexes. We say an automorphism of a cube complex is *combinatorially elliptic* if it fixes a vertex. We say an automorphism f of a cube complex is *combinatorially hyperbolic* if f preserves a combinatorial geodesic on which it has a positive translation length δ, and for any vertex v in the cube complex we have $d(v, f(v)) \geq \delta$. Recall the *cubical subdivision* of a cube complex is obtained by decomposing each n-dimensional cube in the 2^n subcubes determined by the n (centered) hypercubes. Our main result is:

Theorem 1.4 *Every automorphism of a $CAT(0)$ cube complex is either combinatorially elliptic or combinatorially hyperbolic (on the cubical subdivision).*

An automorphism acts *stably without inversion* if every power of the automorphism acts without inversion. Any automorphism acts stably without inversion on the cubical subdivision (see Lemma 4.2). Thus we deduce Theorem 1.4 from the following more precise result:

Theorem 1.5 *Every automorphism of a $CAT(0)$ cube complex acting stably without inversion is either combinatorially elliptic or combinatorially hyperbolic.*

As was suggested to us by C. Druţu, using the fact that $a^{m^k} = ba^{n^k}b^{-1}$ in $BS(m, n)$ we see that the subgroup $\langle a \rangle$ is distorted (for $m \neq n$). So we can deduce Theorem 1.3 from the following:

Theorem 1.6 *Let Γ denote a finitely generated group containing an element a such that the subgroup $\langle a \rangle$ is distorted in Γ. Then the infinite subgroup $\langle a \rangle$ has a fixed point in every action of Γ on a $CAT(0)$ cube complex. Consequently Γ has no proper action on a discrete space with walls.*

For example we deduce:

Corollary 1.7 *Let $H = \langle a, b, c \mid [a, c] = [a, b] = 1, [b, c] = a \rangle$ denote the (discrete) Heisenberg group and let $H \to G$ denote a morphism which is injective on the distorted subgroup $\langle a \rangle$. Then G has no proper action on a discrete space with walls.*

Note that since the Heisenberg group H is nilpotent it is amenable and thus acts properly on a space with measured walls (see [3, Theorem 1 (5)]). So H itself is an other negative answer to Question 1.1.

In Sect. 2 we review some classical facts about the geometry of $CAT(0)$ cube complex, first considered as combinatorial objects by Gromov in [6]. Thus people familiar with $CAT(0)$ cube complexes may skip it. We concentrate on the combinatorial distance between vertices and insist on the role of hyperplanes. All results here are classical, except that no assumption of finite dimension is made. For this reason we use *cubing* instead of $CAT(0)$ *cube complex* in the core of the text.

In Sect. 3 we define the translation lengths of an automorphism of a $CAT(0)$ cube complex, and we give the two (combinatorial) types we are interested in: (combinatorially) elliptic or (combinatorially) hyperbolic.

In Sect. 4 we generalize to arbitrary $CAT(0)$ cube complexes the notion of action without inversion that occurs in the study of groups acting on trees (see [11]). In particular we show that a group acting on a $CAT(0)$ cube complex has no inversion on the cubical subdivision (see Lemma 4.2).

In Sect. 5 we prove that if an automorphism of a $CAT(0)$ cube complex preserves a combinatorial geodesic then its minimal (combinatorial) displacement in the complex is the same as on the geodesic.

In Sect. 6 we prove Theorem 1.5. So we show that an automorphism acting stably without inversion and without fixed point preserves a combinatorial geodesic (Theorem 6.3 in the text). We then deduce Theorem 1.6.

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2 Geometry of $CAT(0)$ cube complexes

2.1 Cube complexes and non-positive curvature conditions

The following notion of *cube complexes* is equivalent to the notion of *cubical complexes* introduced in [1, Definition 7.32 p 112].

Definition 2.1 (*Cube complexes*) Let X denote some set. *A parametrized cube of* X is an injective map $f : C \to X$, where C is some euclidean cube. A *face* of a parametrized cube $f : C \to X$ is a restriction of f to one of the faces of C. Two parametrized cubes $f : C \to X$, $f' : C' \to X$ are *isometric* whenever there is an isometry $\varphi : C \to C'$ such that $f\varphi = f'$.

A cube complex is a set X together with a family $(f_i)_{i\in I}$ of parametrized cubes $f_i : C_i \to X$ such that

1. (Covering) Each point of X belongs to the range of some f_i
2. (Compatibility) For any two maps f_i, f_j either $f_i(C_i) \cap f_j(C_j) = \emptyset$ or f_i, f_j have isometric faces $f_{ij} : C_{ij} \subset C_i \to X$, $f_{ji} : C_{ji} \subset C_j \to X$ such that $f_i(C_{ij}) = f_j(C_{ji}) = f_i(C_i) \cap f_j(C_j)$

(We will require that the edges of all cubes C_i have the same length, for example unit length.)

The *cube* associated to the parametrized cube $f_i : C_i \to X$ is the image $f_i(C_i)$. Note that if two parametrized cubes $f_i : C_i \to X$, $f_j : C_j \to X$ have the same image, then by the compatibility condition above the cubes C_i, C_j are isometric, in particular they have the same dimension. This dimension, say k, will be called the *dimension* of the cube $f_i(C_i) = f_j(C_j)$: we will say for short that the cube is *a k-cube*. The 0-cubes are the *vertices*, the 1-cubes are the *edges*, the 2-cubes are the *squares* ... We say X has dimension $\leq k$ when every cube has dimension $\leq k$. If there are cubes of arbitrarily dimension we say X has infinite dimension.

The interior of a cube $f_i(C_i)$ of X is the image under f_i of the interior of C_i (independant of the parametrized cube $f_j : C_j \to X$ such that $f_j(C_j) = f_i(C_i)$ again by the compatibility condition). Note that by the covering condition each point p of X is contained in the interior of some cube, and by the compatibility this cube is unique.

When only one family $(f_i)_{i\in I}$ is considered on X we will say by abuse of langage that *X is a cube complex*, and we will denote by X^k the union of all k-cubes of X. This is the *k-skeleton of X*. A *subcomplex of X* is an arbitrary union of cubes of X. Each subcomplex inherits a natural structure of cube complex.

Let X, Y be two cube complexes. A map $f : X \to Y$ is said to be *combinatorial* whenever for each parametrized cube $f_i : C_i \to X$, the composite ff_i is isometric to a parametrized cube of Y. In particular such a map sends vertices to vertices, edges to edges ... Observe that the natural inclusions of subcomplexes are combinatorial. Note also that a combinatorial map $f : X \to Y$ sends the interior of a cube $C \subset X$ bijectively onto the interior of $f(C)$.

The automorphism group of a cube complex X is the set of combinatorial bijections $X \to X$ (this is indeed a subgroup of the permutation group of X).

Example 2.2 Assume that $\mathcal{H}$ is a Hilbert space and that $\mathcal{C}$ is a collection of unit euclidean cubes of $\mathcal{H}$, such that for two cubes of $\mathcal{C}$, their intersection is either empty or a cube of $\mathcal{C}$. Then the union X of all cubes $C \in \mathcal{C}$ has a natural structure of cube complex where the parametrized cubes are the inclusion $C \to X$, $C \in \mathcal{C}$.

Definition 2.3 (*Subdivisions*) Let C denote some euclidean cube. For short we will call *barycenter of C* the barycenter of the set of vertices of C (with equal unit weights). *A vertex of the barycentric subdivision of C* is the barycenter b_F of some face F of C. *A simplex of the barycentric subdivision of C* is the simplex affinely generated by vertices $b_{F_0}, \ldots, b_{F_k}$, where the faces F_i satisfy $F_0 \subset \cdots \subset F_k$. This defines a simplicial complex C'_{simpl} called *the simplicial barycentric subdivision of C*, whose geometric realization is identical with C. Note that any isometry $C \to D$ of unit euclidean cubes induces an isomorphism $C'_{\text{simpl}} \to D'_{\text{simpl}}$.

Let X denote a cube complex. The *simplicial barycentric subdivision of X* is the simplicial complex X'_{simpl} whose cells are restrictions of the $f_i : C_i \to X$ to the simplices of ${C_i}'_{\text{simpl}}$.

Given two comparable faces $F \subset G \subset C$ of a eulidean cube, the union of simplices $\{b_{F_0}, \ldots, b_{F_k}\}$ of C' satisfying $F \subset F_0$, $F_k \subset G$ is in fact a eulidean cube, which we call *a cube of the barycentric subdivision of C*. This decomposition of the cube C into smaller cubes endows C with a structure of cube complex, which we call *the cubical subdivision of C*, denoted by C'. Observe the edges of C are twice as long as the edges of C'. An isometry $C \to D$ induces an isomorphism $C' \to D'$.

Let X denote a cube complex. The *cubical subdivision of X* is the cube complex X' whose parametrized cubes are restrictions of the $f_i : C_i \to X$ to the cubes of the cubical subdivision of C_i.

Definition 2.4 (*Links of vertices*) Let X denote a cube complex, and let v denote some vertex. The collection of cubes of X properly containing v is an abstract simplicial complex, whose set of vertices is the set of edges of X containing v. We will denote this simplicial complex by $\text{link}(v, X)$ (the *link of v in X*).

The cube complex X is *combinatorially non positively curved* if each vertex link is *flag* (that is each complete subgraph is the 1-skeleton of a simplex). We say that X is combinatorially $CAT(0)$ whenever X is combinatorially non positively curved and simply-connected. Following Sageev we will rather use the word *cubing* instead of the expression: combinatorially $CAT(0)$ cube complex.

Definition 2.5 (*The pseudometric of a cube complex, see* [1] *7.38 p 114*) Let X denote a cube complex. *A piecewise geodesic of X with endpoints x, y* is a map $c : [a, b] \to X$

such that $c(a) = x, c(b) = y$, there is a subdivision $a = t_0 \leq \cdots \leq t_n = b$, a sequence of parametrized cubes $f_1 : C_1 \to X, \ldots, f_n : C_n \to X$ and a sequence of isometries $(c_i : [t_{i-1}, t_i] \to C_i)_{1 \leq i \leq n}$ satisfying $f_i c_i = c$ on $[t_{i-1}, t_i]$ for $1 \leq i \leq n$.

The *length* of the piecewise geodesic is $|b - a|$.

We define a pseudometric d on X by setting $d(x, y) =$ the infimum of the lengths of piecewise geodesics of X with endpoints x, y.

Here are now two results linking the metric and the combinatorial viewpoint on cube complexes.

Lemma 2.6 *The pseudometric d on a cube complex X is a geodesic length metric.*

Lemma 2.7 [6] *Let X denote some cube complex. Then X is combinatorially non positively curved if and only if the length metric d on X is locally* $CAT(0)$.

In particular a cube complex is metrically $CAT(0)$ *if and only if it is combinatorially* $CAT(0)$.

The two previous lemmas have been established for cube complexes whose cubes have all dimension $\leq n$; see for example [1]. This result is so classical for people working on cube complexes that they usually identify the metric condition $CAT(0)$ with its combinatorial analogue. For instance, when Chatterji-Niblo or Nica achieve the geometrization of spaces with walls [2,9], turning these to $CAT(0)$ cube complexes, they in fact check the combinatorial condition, and do not tell us anything about the length metric - although Gromov's hypothesis of finite dimensionality usually fails.

We do not insist since in the sequel we will be only concerned with the *combinatorial* geometry of $CAT(0)$ cube complexes (or cubings), which we recall in the next section. Our slogan is: in cubings the combinatorial geometry is as nice as the $CAT(0)$ geometry. In fact the result of this paper shows that, in some context, it is even nicer.

In the sequel we do not make any restriction on the cubing X: in particular we do not assume $\dim X < \infty$.

2.2 Hyperplanes, combinatorial distance and convex subcomplexes

Definition 2.8 A *combinatorial path* of a cube complex X is a sequence $\gamma = (v_0, v_1, \ldots, v_n)$ of vertices of X such that for each $i = 0, \ldots, n-1$ either $v_{i+1} = v_i$ or v_{i+1}, v_i are the two (distinct) endpoints of some edge of X. The *initial point of* γ is v_0, *the terminal point of* γ is v_n and *the length of* γ is n. If for each $i = 0, 1, \ldots, n-1$ we have $v_{i+1} \neq v_i$ we say γ is *non stuttering*.

When $\gamma = (v_0, v_1, \ldots, v_n), \gamma' = (w_0, w_1, \ldots, w_m)$ are two combinatorial paths such that the terminal point of γ is the initial point of γ' we define as usual the product $\gamma.\gamma'$ to be the path $(v_0, v_1, \ldots, v_{n-1}, v_n = w_0, w_1, \ldots, w_m)$.

The *combinatorial distance between two vertices* x, y of a connected cube complex is the minimal length of a combinatorial path joining x to y. It will be denoted by $d(x, y)$, and a path of length $d(x, y)$ will be called *a (combinatorial) geodesic*. We note that $d(x, y)$ is also the minimal length of a non stuttering combinatorial path joining x to y (in other words geodesics are non stuttering).

Sequences $(p_n)_{n\in\mathbb{Z}}$ of vertices of the cube complex such that $d(p_n, p_m) = |m - n|$ will also be called *(infinite) geodesics*.

Definition 2.9 (*Convex subcomplexes*) Let X, Y denote cube complexes and let $f : X \to Y$ denote a combinatorial map. We say f is a *local isometry* if for each vertex v of X the induced simplicial map $f_v : \text{link}(v, X) \to \text{link}(f(v), Y)$ is injective and has full image (recall that a subcomplex $L \subset K$ of a simplicial complex is *full* whenever each simplex of K whose vertices are in L in fact belongs to L).

A subcomplex of a cube complex is *locally convex* if the inclusion map is a local isometry.

A subcomplex Y of a cube complex X is *(combinatorially) convex* if it is connected, and any (combinatorial) geodesic between two vertices of Y has all of its vertices inside Y.

Remark 2.10 (*relations between the various convexity notions*) Let X be a cubing and let $Y \subset X$ denote a connected subcomplex. The following belong to folklore :

Y is locally convex in the sense of the above definition iff it is geodesically convex for the $CAT(0)$ metric.

If Y is locally convex then Y is combinatorially convex.

If Y is combinatorially convex and *full* (in the sense that it contains a cube of X if and only if it contains its vertices), then Y is locally convex.

Definition 2.11 (*Walls, hyperplanes*) Let X denote any cube complex. Two edges a, b of X are said to be *elementary parallel* whenever they are disjoint but contained in some (necessarily unique) square of X. We call *parallelism* the equivalence relation on the set of edges of X which is generated by elementary parallelisms. A *wall of X* is an equivalence relation for the parallelism relation. When an edge e belongs to some wall W we say that *W passes through* e, or that *W is dual to e*. We also say that *a cube C of X is dual to the wall W* when C contains an edge e to which W is dual.

Let C denote some euclidean cube of dimension n, and let $\mathbb{E}$ denote the ambient euclidean space. For every edge e of C with endpoints p, q we consider the hyperplane of $\mathbb{E}$ consisting in points which are at the same distance of p and q. Then the intersection of this hyperplane with C is a euclidean cube, whose cubical subdivision is a subcomplex of C'. We denote this subcomplex of C' by h_e, and call it *the hyperplane of C dual to e*. Observe that $h_e = h_{e'}$ iff e and e' are parallel. Note also that the hyperplane of a segment consist in its midpoint.

Now let e denote some edge of a cube complex X, and let W denote the wall through e. For each parametrized cube $f : C \to X$ and each edge a of C such that $e \parallel f(a)$, we consider the image under the induced combinatorial map $f : C' \to X'$ of the hyperplane h_a of C dual to a. The union of all these $f(h_a)$ is called *a hyperplane of X*, it will be denoted by H_e, and we will say that H_e is *dual to e*. (Note that a hyperplane of X is a subcomplex of the subdivision X'.) Clearly $H_e = H_{e'}$ iff e and e' are parallel in X. In other words the set of edges to which a given hyperplane is dual consists in a wall. Thus walls and hyperplanes are in one-to-one correspondance. We will say that a cube C of X is *dual* to some hyperplane H when C contains an edge e to which H is dual.

Let H denote some hyperplane of a cube complex X. The *neighbourhood of H* is the union of all cubes dual to H. We will denote it by N_H.

Combining results of Sageev we get the following description of hyperplane neighbourhoods:

Theorem 2.12 (see [10]) *Let X be a cubing and let H denote some hyperplane of X with neighbourhood N_H.*

1. *H separates X into two connected components, called the* (open) half-spaces delimited by H

2. *N_H is (combinatorially) convex in X*
3. *N_H admits an automorphism σ_H that fixes pointwise H and exchanges the endpoint of each edge dual to H*

Proof 1. This is Theorem 4.10 in [10].

2. By the separation property above, it follows that the union of all cubes of N_H disjoint of H consists in the disjoint union of two connected subcomplexes, which we call the boundary components of N_H. Then Theorem 4.13 of [10] tells us that each boundary component of N_H is a combinatorially convex subcomplex. The combinatorial convexity of N_H itself follows immediately, using the fact that H disconnects X.
3. We first claim that for each vertex x of N_H there exists a unique edge e_x dual to H and containing x. The existence is by definition of N_H, and we just have to check uniqueness. Assume by contradiction that there are two distinct edges e, e' containing x and dual to H. Then the endpoints y, y' of e, e' distinct from x are contained in the same connected boundary component of N_H. By Theorem 4.13 of [10] this boundary component is convex. Thus (y, x, y') is not a geodesic, so that $d(y, y') \leq 1$. The complex X is simply-connected and its 2-faces are polygons with even length: it follows that the length mod. 2 of paths in X depends only on the endpoints. Thus $d(y, y')$ is even, and we deduce that $y = y'$, so that $e = e'$, contradiction.

Let Q denote any cube of X dual to H. Then by the previous remark any two edges of Q dual to H in X are in fact parallel inside Q. Let σ_Q denote the reflection of Q preserving each edge of Q dual to H, and exchanging the endpoints of these edges. For $Q_1 \subset Q_2$ the restriction of σ_{Q_2} to Q_1 is σ_{Q_1}. Thus the collection of reflections $(\sigma_Q)_{Q \text{ dual to } H}$ defines a reflection $\sigma_H : N_H \to N_H$ with the desired properties. □

Definition 2.13 (*hyperplane crossing a non-stuttering path*) Let X be a cube complex, let $\gamma = (x_0, x_1, \ldots, x_n)$ denote a non stuttering path, let $(e_1, \ldots, e_n)$ denote the sequence of edges of X such that the vertices of e_i are v_{i-1}, v_i. A hyperplane *crosses* γ iff it is dual to one of the edges e_i, and *the sequence of hyperplanes that* γ *crosses* is the sequence $(H_1, \ldots, H_n)$ where H_i is the hyperplane of X dual to e_i.

The last part of Theorem 4.13 in [10] gives:

Theorem 2.14 *Let X denote a cubing. Then a non stuttering path is a combinatorial geodesic if and only if the sequence of hyperplanes it crosses has no repetition.*

In particular the combinatorial distance between two vertices x, y is equal to the number of hyperplanes of X that separates x and y.

3 Combinatorial translation length

An automorphism of a cubing is an isometry for the $CAT(0)$ distance, and also for the combinatorial distance on the set of vertices.

Definition 3.1 Let X be a cubing and let $f \in \mathrm{Aut}(X)$. For every point $x \in X$ we denote by $\delta^0(f, x)$ the $CAT(0)$ distance between x and $f(x)$. And for every vertex $p \in X^0$ we denote by $\delta(f, p)$ the combinatorial distance between p and $f(p)$. We then set $\delta^0(f) = \inf_{x\in X} \delta^0(f, x)$ and $\delta(f) = \inf_{p\in X^0} \delta(f, p)$. We call $\delta^0(f)$ the $CAT(0)$ *translation length of* f, and $\delta(f)$ the *combinatorial translation length of* f. Clearly translation lengths are conjugation invariants.

Lemma 3.2 (comparison between the two translation lengths) *Let X be a cubing (where all edges have unit length). Let $f : X \to X$ be an automorphism.*

1. *We have $\delta^0(f) \leq \delta(f)$.*
2. *(the finite dimensional case) Assume the cubing X has dimension $\leq d$. Then $\delta(f) \leq d\,\delta^0(f)$.*

sketch of proof For any vertex $v \in X^0$ the number $\delta(f, v)$ is the length of some combinatorial path from v to $f(v)$. Since edges have length 1, we get $\delta^0(f, v) \leq \delta(f, v)$. The inequality $\delta^0(f) \leq \delta(f)$ follows.

Assume now X has dimension $\leq d$. For vertices $v, w \in X^0$ let $\{H_1, \ldots, H_n\}$ be the family of hyperplanes separating v from w. It can be shown that there exists a subfamily of cardinality $k \geq \frac{n}{d}$ of pairwise disjoint separating hyperplanes. Vertices are at $CAT(0)$-distance $\geq \frac{1}{2}$ of any hyperplane, and two disjoint hyperplanes are at $CAT(0)$-distance at least 1. Consequently the $CAT(0)$-distance between p, q is at least k. The inequality $\delta^0(f) \geq \frac{1}{d}\delta(f)$ follows. □

Estelle Souche told me the following (it also appears in an exercise of [1]):

Example 3.3 Consider the Hilbert space $\ell^2(\mathbb{Z})$ consisting of maps $u : \mathbb{Z} \to \mathbb{R}$ s.t. $\sum_{n\in\mathbb{Z}} |u(n)|^2 < +\infty$.

Let $(e_k)_{k\in\mathbb{Z}}$ denote the Hilbert basis such that $e_k(n) = 1$ if $n = k$ and $e_k(n) = 0$ otherwise. The set $V = \mathbb{Z}^{(\mathbb{Z})}$ of maps $\mathbb{Z} \to \mathbb{Z}$ with finite support is a subset of $\ell^2(\mathbb{Z})$. We consider the unit cubes of $\ell^2(\mathbb{Z})$ whose vertices are elements of V, and whose edges are parallel to one of the e_k's: the union of all these cubes is a cubing X of infinite dimension.

Define an isometry σ of $\ell^2(\mathbb{Z})$ on the Hilbert basis by $\sigma(e_k) = e_{k+1}$. The map $f : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by $f(u) = e_0 + \sigma(u)$ is an affine isometry of $\ell^2(\mathbb{Z})$. Note that $f(V) = V$ and σ preserves $(e_k)_{k\in\mathbb{Z}}$, thus f induces an isometry of X.

For any $p \in \mathbb{N}$ we have $f^p(0) = e_1 + \cdots + e_p$, whose norm tends to ∞ with p. Thus f has no fixed point in $\ell^2(\mathbb{Z})$.

For each $k \in \mathbb{N}$ define a vector $u_k \in X$ as follows: $u_k(n) = 0$ if $n < 0$ or $n > k$ and $u_k(n) = 1 - \frac{n}{k}$ if $0 \leq n \leq k$. Then $\delta^0(f, u_k) = \sqrt{\frac{1}{k}}$, thus $\delta^0(f) = 0$.

In the terminology of [1] the isometry f of the $CAT(0)$ space X is *parabolic*.

Note that $\delta(f) > 0$ since f has no fixed points. Since $f(0) = e_0$ we have $\delta(f) = \delta(f, 0) = 1$. If we set $p_n = f^n(0)$ then $(p_n)_{n\in\mathbb{Z}}$ is a combinatorial infinite geodesic preserved by f, on which f acts as a unit combinatorial translation length.

Definition 3.4 (*Elliptic, hyperbolic*) Let $f \in \mathrm{Aut}(X)$. We say that f is *combinatorially elliptic* if f has a fixed point in X^0. We say that f is *combinatorially hyperbolic* if f is not elliptic and f preserves some infinite combinatorial geodesic γ on which it acts as a non trivial translation. Any such geodesic γ will be called an *axis for f*.

Example 3.5 Let X denote a single edge. Then the automorphism of X exchanging the end-points of the edge is neither elliptic nor hyperbolic.

4 Actions without inversion

In order to get rid of the trouble caused by automorphisms similar to the one described in Example 3.5 we first introduce the corresponding notion:

Definition 4.1 (*Inversions*) Let f denote an automorphism of a cubing X. Let H denote a hyperplane of X, and let X^+, X^- denote the two half-spaces delimited by H. We say that f *has an inversion along* H whenever $f(X^+) = X^-$ (and thus $f(X^-) = X^+$, $f(H) = H$). We say that f *acts without inversion* if there is no hyperplane H such that f has an inversion along H.

We say that the automorphism f acts *stably without inversion* when f and each power of f act without inversion.

And we say that *a group G of automorphisms of X acts without inversion* if all of its elements act without inversion. Note that if a group acts without inversion, then any of its elements act stably without inversion.

Just as in the tree case we have:

Lemma 4.2 *Let f denote an automorphism of some cubing X. Then f acts without inversion on the cubical subdivision X'.*

Proof Every edge e of X' joins the center of a cube $Q(e)$ of X to the center of one of its codimension 1 face. For each edge e of X', denote by $X'^+(e)$ the half-space of X' delimited by the hyperplane dual to e, so that $X'^+(e)$ contains the center of $Q(e)$ but not the center of the codimension 1 face that is perpendicular to e.

For any automorphism f of X and any edge e of X' we have $f(X'^+(e)) = X'^+(f(e))$.

If e_1, e_2 are opposite edges of a square of X' then $X'^+(e_1) = X'^+(e_2)$. The Lemma follows. □

Question 4.3 Is there a finitely generated group G acting on a locally compact cubing all of whose finite index subgroups have an inversion ?

5 Automorphisms preserving a geodesic

The following was suggested by Tomasz Elsner:

Proposition 5.1 *Let $\mathcal{G}$ denote any graph, and let γ denote an infinite combinatorial geodesic of $\mathcal{G}$. If an automorphism f of $\mathcal{G}$ preserves γ then*

1. *either f has a fixed point in γ*
2. *or f exchanges two consecutive vertices of γ*
3. *or there is a number $d \in \mathbb{Z}$, $d \neq 0$ such that*

$$\text{for every } n \in \mathbb{Z}, \text{ we have } f(p_n) = p_{n+d}$$

and furthermore in that case for every $n \in \mathbb{Z}$ we have $\delta(f) = \delta(f, p_n) = |d|$.

Note that the last property shows that for any other f-invariant geodesic γ', the translation length of f on γ' is $\delta(f)$ too. Note also that when $\mathcal{G}$ is the 1-skeleton of a cubing, the second possibility in the Lemma above corresponds to an inversion. We thus get:

Corollary 5.2 *Let X denote a cubing. Assume that $f \in \mathrm{Aut}(X)$ is combinatorially hyperbolic and acts without inversion. Then f has the same translation length d on each axis, and in fact $d = \delta(f)$. Furthermore for any integer $n > 0$ the automorphim f^n is hyperbolic, each axis for f being an axis for f^n, and we have $\delta(f^n) = n\delta(f)$.*

Proof of Proposition 5.1 There is a bijection $\phi : \mathbb{Z} \to \mathbb{Z}$ such that $f(p_n) = p_{\phi(n)}$. Since f is an automorphism we have $|\phi(n+1) - \phi(n)| = 1$. Thus $\phi(n) = d + \varepsilon n$, with $\varepsilon \in \{-1, 1\}$.

Assume first $\varepsilon = -1$. Either d is even: then $f(p_{\frac{d}{2}}) = p_{\frac{d}{2}}$. Or d is odd: then f exchanges the adjacent vertices $p_{\frac{d-1}{2}}, p_{\frac{d+1}{2}}$.

Assume now $\varepsilon = 1$. If $d = 0$ then f fixes each point of γ. Else for every $n \in \mathbb{Z}$ we have $f(p_n) = p_{n+d}$. Let us prove in this case that for every $n \in \mathbb{Z}$ we have $\delta(f) = \delta(f, p_n) = |d|$.

Up to replacing $(p_n)_{n\in\mathbb{Z}}$ by $(p_{-n})_{n\in\mathbb{Z}}$, we may and will assume that $d > 0$.

So f acts on γ as a translation of length d. Let x denote any vertex of $\mathcal{G}$. Then $d(p_0, f^n(p_0)) = nd$ and by the triangle inequality

$$\begin{aligned} nd(x, f(x)) &\geq d(x, f^n(x)) \geq d(p_0, f^n(p_0)) - d(p_0, x) - d(f^n(p_0), f^n(x)) \\ &= nd - 2d(p_0, x). \end{aligned}$$

Divide by n and let n tend to infinity to obtain the desired inequality $d(x, f(x)) \geq d$. □

6 Classification of automorphisms acting stably without inversion

Our main technical result is the following:

Lemma 6.1 *Let f denote an automorphism of some cubing X. Let p denote a vertex of X such that $\delta(f) = \delta(f, p)$. If f and each power of f act without inversion then for any integer $n \geq 0$ we have $d(p, f^n(p)) = n\delta(f)$.*

The condition "without inversion" is necessary in view of Example 3.5. The condition "stably without inversion" is also necessary: consider an order four rotation of a square.

Proof We may assume $\delta(f) > 0$. We then argue by contradiction. So assume that there is a vertex p and a positive integer n such that $\delta(f) = d(p, f(p))$ and $d(p, f^n(p)) \neq n\delta(f)$. Consider a pair (p, n) with the smallest possible integer n. Observe that $n \geq 2$.

Fix a combinatorial geodesic $\gamma_1 = (p_0 = p, p_1, \ldots, p_{\delta(f)} = f(p))$. Set $\gamma_i = f^i(\gamma_1)$. The path $\gamma = \gamma_1\gamma_2 \ldots \gamma_n$ joins p to $f^n(p)$ and has length $n\delta(f)$. By assumption γ is not a geodesic. Thus $d(p, f^n(p)) < n\delta(f)$, and by Theorem 2.14 there exists a hyperplane H of X crossing at least twice the path γ. For the argument we refer to Fig. 1.

By minimality of n the subpaths $\gamma_1\gamma_1 \ldots \gamma_{n-1}$ and $\gamma_2\gamma_3 \ldots \gamma_n$ are geodesic. Thus by Theorem 2.14 the hyperplane H crosses these subpaths of γ only once. This implies that H crosses γ_1 once, H crosses γ_n once and H does not cross at all the subpath $\gamma_2\gamma_3 \ldots \gamma_{n-1}$.

The path γ contains three maximal subpaths not crossed by H. Two of these subpaths contain the endpoints of γ and we let γ_H denote the third one (in the middle). We now choose the hyperplane H crossing twice γ such that the length of γ_H is minimal. Then we claim that the corresponding subpath γ_H is a geodesic: for else there would be a hyperplane H' crossing twice γ_H, hence also γ, and we would have $\gamma_{H'} \subset \gamma_H, \gamma_{H'} \neq \gamma_H$, contradicting the minimality of the length of γ_H.

Note that by maximality of the subpath γ_H the endpoints of γ_H are in N_H (and are not separated by H). By Theorem 2.122. the neighbourhood N_H is convex, thus we have $\gamma_H \subset N_H$. Let σ_H denote the path of N_H symmetric to γ_H with respect to H. The initial vertex of σ_H is one of the vertices p_i of γ_0. Let q' denote the vertex of σ_H adjacent to $f(p)$; the edge joining q' to $f(p)$ is dual to H. We let q denote the vertex adjacent to p such that $f(q) = q'$, and we denote by K the hyperplane dual to the edge a joining q and p.

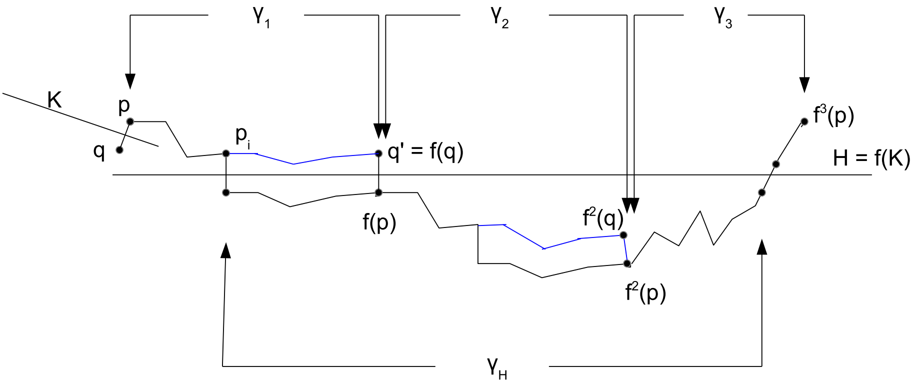

Fig. 1 The vertex p has minimal displacement for the isometry f and for f^2, but not for f^3. We depict a geodesic γ_1 from p to $f(p)$, and its images $\gamma_2 = f(\gamma_1), \gamma_3 = f(\gamma_2)$. We illustrate a hyperplane H that cuts twice the product path $\gamma = \gamma_1\gamma_2\gamma_3$. The preimage hyperplane $K = f^{-1}(H)$ is dual to an edge a from p to q. We have colored in blue a geodesic path symmetric to the maximal subpath of $\gamma_1$that travels in the neighborhood of H, on the same side as $f(p)$ (the f image of this blue path is again in blue)

By symmetry inside N_H (see Theorem 2.123.) we see that the vertex $f(q)$ is on a geodesic from p to $f(p)$. Let γ' denote the part of this geodesic from p to $f(q)$: it has length $\delta(f)-1$. Consider now the path $\gamma'' = (q, p)\gamma'$. The length of this path is $\delta(f)$, and it joins q to $f(q)$. Thus in fact $d(q, f(q)) = \delta(f)$ and $(q, p)\gamma'$ is a geodesic. In particular the hyperplane K separates $\{q, f(q)\}$.

Consider now the product path $\gamma'' f(\gamma'') \dots f^{n-2}(\gamma'')$ joining q to $f^{n-1}(q)$, and of length $(n-1)\delta(f)$. Since $d(q, f(q)) = \delta(f)$, by minimality of n we see that $\gamma'' f(\gamma'') \dots f^{n-2}(\gamma'')$ has to be a geodesic. In particular the hyperplane K separates $\{q, f^{n-1}(q)\}$.

We claim that in fact K separates $\{q, f^{n-1}(p)\}$. Otherwise K separates $f^{n-1}(p)$ and $f^{n-1}(q)$. Thus K is dual to the edge $f^{n-1}(a)$. Since K is also dual to a we get $f^{n-1}(K) = K$. But since K separates $\{q, f^{n-1}(q)\}$ we see that f^{n-1} has an inversion along K, contradiction. Since K separates $\{q, f^{n-1}(p)\}$, when we apply f we see that H separates $\{f(q), f^n(p)\}$. Thus H does not separate $\{f(p), f^n(p)\}$. This is a contradiction, because, as we already noticed, the path $\gamma_2\gamma_3 \dots \gamma_n$ is a geodesic, and it is crossed exactly once by H. □

Corollary 6.2 *Assume that $f \in \mathrm{Aut}(X)$ acts stably without inversion and has no fixed point. Then f is combinatorially hyperbolic. More precisely f has an axis through each vertex p minimizing $d(p, f(p))$. For any integer $n \geq 0$, each axis for f is an axis for f^n and $\delta(f^n) = n\delta(f)$.*

Proof Let p denote any of the vertices of X such that $d(p, f(p)) = \delta(f)$. For brevity we write $\delta(f) = d$.

Let $\gamma_0 = (x_0, x_1, \dots, x_d)$ denote any combinatorial geodesic from p to $f(p)$ (so that in particular $x_d = f(x_0)$). For any integer $k \in \mathbb{Z}$ we define $p_k = f^q(x_r)$ with q, r uniquely defined by $0 \leq r < d$ and $k = r + qd$. Note that for $k = 0, 1, \dots, d$ we have $p_k = x_k$, and for an arbitrary k we have $f(p_k) = p_{k+d}$. Thus $\gamma = (p_k)_{k\in\mathbb{Z}}$ is an infinite path. The map $x \mapsto d(x, f(x))$ achieves its minimal value at $p = p_0$, and thus at each vertex $p_{kd}, k \in \mathbb{Z}$. By Lemma 6.1 it follows that the finite subpath $(p_k)_{k_1 d \leq k \leq k_2 d}$ is a geodesic (for any pair $(k_1, k_2) \in \mathbb{Z}^2$ with $k_1 \leq k_2$). Thus γ is an infinite geodesic, and by construction γ is invariant under f.

We conclude by applying Corollary 5.2. □

We have proved:

Theorem 6.3 *Every automorphism of a cubing acting stably without inversion is either combinatorially elliptic or combinatorially hyperbolic.*

Remark 6.4 Let f denote an automorphism of a cubing X. Assume that f has an inversion along a hyperplane H of X. Then either f is elliptic on the cubical subdivision X' and the set of its fixed point is contained in the subcomplex $H \subset X'$, or f is hyperbolic on X' and all the axes of f are inside H.

We now prove Theorem 1.6 of the Introduction :

Proof Since $a \in \Gamma$ is an infinite order distorted element we have $\frac{|a^n|}{n} \to 0$. Assume Γ acts on a cubing X. Up to passing to the cubical subdivision we may assume Γ acts without inversion on X. We claim that a has a fixed point in X, so that the action of Γ is not proper.

By Corollary 6.2 we have $\delta(a^n) = n\delta(a)$.

Now for any decomposition $g = s_1 \ldots s_k$ we clearly have $\delta(g) \leq \sum_{i=1}^{i=k} \delta(s_i)$. Consider a word-geodesic decomposition $a^n = s_1 \ldots s_{k_n}$ on the finite set S of generators of Γ. We deduce $\delta(a^n) \leq k_n \max_{s \in S} \delta(s)$. Since $\lim \frac{k_n}{n} = 0$ and $\delta(a^n) = n\delta(a)$ we have $\delta(a) = 0$, which concludes the proof. □

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