Annales mathématiques du Québec (2022) 46:195–206
https://doi.org/10.1007/s40316-021-00178-2

Various formulations and approximations of incompressible fluid motions in porous media

Yann Brenier[1]

Received: 31 May 2021 / Accepted: 10 August 2021 / Published online: 17 January 2022

Abstract
We first recall various formulations and approximations for the motion of an incompressible fluid, in the well-known setting of the Euler equations. Then, we address incompressible motions in porous media, through the Muskat system, which is a friction dominated first order analog of the Euler equations for inhomogeneous incompressible fluids subject to an external potential.

Résumé
On commence par rappeler plusieurs formulations et approximations décrivant le mouvement d'un fluide incompressible dans le cadre bien connu des équations d'Euler. On s'intéresse ensuite aux mouvements incompressibles en milieux poreux, au travers du système de Muskat, qui est un analogue du premier ordre, dominé par la friction, des équations d'Euler pour des fluides incompressibles inhomogènes soumis à un potentiel extérieur.

Keywords Partial differential equations · Calculus of variation · Euler's equations · Fluid mechanics · Flows in porous media · Multiphasic flows

1 Geometric formulation of the Euler equations, relaxation and approximations

According to V.I. Arnold 1966 [3,4], the motion of an incompressible fluid, confined in a compact Riemannian manifold D and moving according to the Euler equations, can be described as a (constant speed) geodesic curve along the Lie group $\mathrm{SDiff}(D)$ of all possible orientation and volume preserving maps of D, according to the L^2 metric on its (formal) Lie Algebra [23]. In the simple situation of the flat torus $D = (\mathbb{R}/\mathbb{Z})^d$ or, alternately, a compact domain D of the Euclidian space $\mathbb{R}^d$, we may identify $\mathrm{SDiff}(D)$ as a subset of the Hilbert space $H = L^2(D; \mathbb{R}^d)$. Then, a solution of the Euler equations is just a curve

$$t \in \mathbb{R} \to X_t \in \mathrm{SDiff}(D)$$

that minimizes

$$\int_{t_0}^{t_1} ||\frac{dX_t}{dt}||_H^2 dt$$

✉ Yann Brenier
yann.brenier@ens.fr

1 CNRS, DMA-ENS, 45 rue d'Ulm, 75005 Paris, France

on each short enough time interval $[t_0, t_1]$, once X_{t_0} and X_{t_1} are fixed, which turns out to mean that there is a function

$$(t, x) \in \mathbb{R} \times D \to p(t, x) \in \mathbb{R}$$

such that

$$\frac{d^2 X_t}{dt^2}(a) = -(\nabla p)(t, X_t(a)), \quad \forall (t, a) \in \mathbb{R} \times D.$$

Ebin and Marsden proved that the corresponding geodesic flow is well defined in a small Sobolev neighborhood of the identity map [20]. When D is a contractile domain of $\mathbb{R}^d$, with $d \geq 3$, Shnirelman [30] proved that, as $d \geq 3$, the completion of SDiff(D) with respect to the geodesic distance is the much wider set VPM(D) of *all* Lebesgue measure preserving maps T, in the measure-theoretic sense, i.e.

$$\int_D f(T(a))da = \int_D f(a)da, \quad \forall f \in C(\mathbb{R}^d).$$

He also proved the existence of many pairs of maps (X_0, X_1) in SDiff(D) which admit no minimizing geodesic between them in SDiff(D) and for which sequences of approximate minimizing geodesics generate unlimited micro-structures and cannot converge in any strong sense. It is therefore natural to look for minimizing geodesics in the completion VPM(D), but this also fails as explained in [10]. The cases $d = 2$ and $d = 1$ are also of interest. As $d = 2$, VPM(D) is the right completion of SDiff(D) for the L^2 metric but, surprisingly enough, not for the geodesic distance, as a consequence of a theorem of Eliashberg and Ratiu [21], as also shown by Shnirelman [30]. As $d = 1$, SDiff(D) is reduced to the identity map, while VPM(D) is a very large space on which the concept of minimizing geodesics is not trivial. Anyway, a "better" completion of SDiff(D) for $d \geq 2$ and VPM(D) for $d \geq 1$, is definitely obtained through the embedding

$$T \in \mathrm{VPM}(D) \to \mu_T(dxda) = \delta(x - T(a))da \in DS(D)$$

where $DS(D)$ denotes the set of all doubly stochastic Borel measures μ on the product $D \times D$, doubly stochastic meaning that each projection of μ is just the Lebesgue measure on D. The completion, through the weak convergence of measures, has been proven by Neretin in [26] (see also [15]).

1.1 Discrete incompressible flows

From a more concrete (and computational!) viewpoint, it is worth considering the discrete version of a minimizing geodesic in VPM(D), in the simplest case when D is just the unit cube $[0, 1]^d$. We perform a dyadic subdivision of D in $N = 2^{nd}$ subcubes D_α, $\alpha = 1, \ldots, N$ of equal volume, we denote by x_α their center of mass, and we define a discrete incompressible flow as a succession of M permutations σ^m, $m = 1, \ldots, M$, in $\mathfrak{S}_N$. This type of approximation is crucial in the analysis done by Shnirelman [30]. We may also define a discrete minimizing geodesic as a finite sequence of permutations that minimize

$$\sum_{m=1}^{M-1} \sum_{\alpha=1}^{N} |x_{\sigma_\alpha^{m+1}} - x_{\sigma_\alpha^m}|^2,$$

where $|\cdot|$ denoted the Euclidian norm in $\mathbb{R}^d$ and permutations σ^1 and σ^M are fixed. At the numerical level, this idea was already considered in [9]. A natural iteration scheme for this difficult combinatorial optimization problem amounts in updating, for each $m = 2, \dots, M-1$, σ^m by $\tilde{\sigma}^m$ where $\sigma = \tilde{\sigma}^m$ minimizes

$$\sum_{\alpha=1}^{N} \Big| x_{\sigma_\alpha} - x_{\sigma_\alpha^{m-1}}|^2 + |x_{\sigma_\alpha^{m+1}} - x_{\sigma_\alpha}\Big|^2,$$

or, equivalently,

$$\sum_{\alpha=1}^{N} \left| x_{\sigma_\alpha} - \frac{x_{\sigma_\alpha^{m-1}} + x_{\sigma_\alpha^{m+1}}}{2} \right|^2,$$

which is much easier to do, in particular as $d = 1$, when it just amounts to sorting the sequence

$$\alpha \to x_{\sigma_\alpha^{m-1}} + x_{\sigma_\alpha^{m+1}}$$

in increasing order! Numerical experiments can therefore be easily performed in the case $d = 1$. Let us show on Fig. 1 the example of

$$\sigma_\alpha^1 = \alpha, \quad \sigma_\alpha^M = \inf(2\alpha, 2N - 2\alpha + 1).$$

As $N \uparrow \infty$, we can guess from Fig. 1 how the sequence of permutations converge to a minimizing geodesic along $DS([0, 1])$, with end points μ_0 and μ_1 respective given by

$$\mu_0(dxda) = \delta(x - a)da, \quad \mu_1(dxda) = \delta(x - T(a))da,$$

where

$$T(a) = \inf(2a, 1 - 2a)$$

is the well-known "triangle" (a.k.a "Matterhorn") map in VPM([0, 1]).

1.2 Relaxation of the Euler equation

The combinatorial numerical scheme we just described suggests the following relaxed formulation of the Euler equations:

Proposition 1 *let (X, p) be a smooth solution to the Euler equations, written in material coordinates:*

$$\frac{d^2 X_t(a)}{dt^2} = -(\nabla p)(t, X_t(a)),$$

where each X_t belongs to SDiff(D). *Then, the pair of measures (c, q), respectively nonnegative and valued in $\mathbb{R}^d$,*

$$c(t, x, a) = \delta(x - X_t(a)), \quad q(t, x, a) = \frac{dX_t}{dt}(a)\delta(x - X_t(a)),$$

together with the scalar field p, obey the following self-consistent (pseudo-differential) system of evolution equations:

$$\partial_t c(t, x, a) + \nabla_x \cdot q(t, x, a) = 0,$$
$$(\partial_t q + \nabla_x \cdot \frac{q \otimes q}{c})(t, x, a) = -c(t, x, a)\nabla_x p(t, x).$$

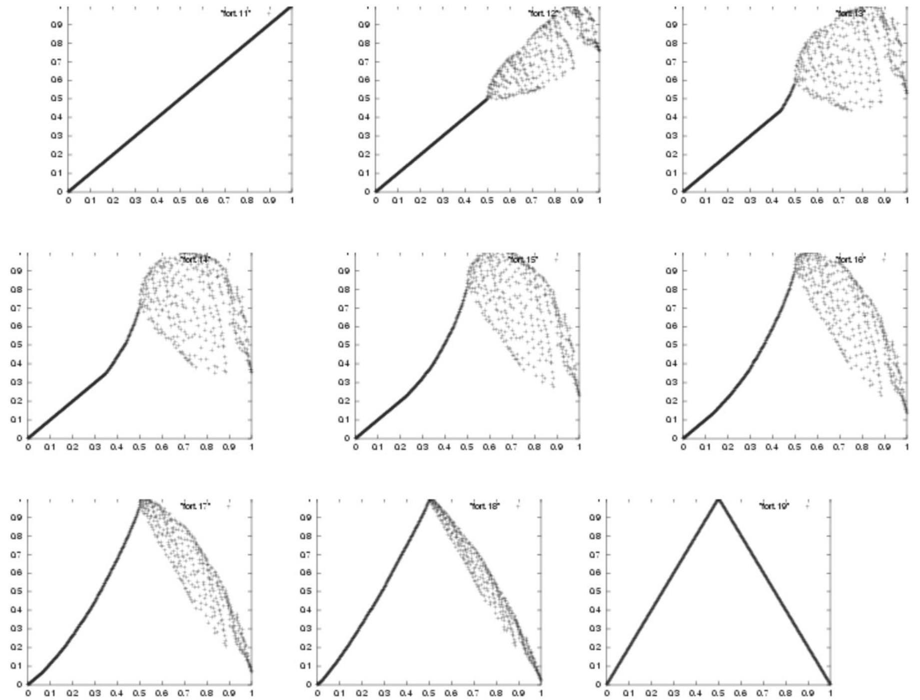

Fig. 1 Example of a minimizing geodesic along DS([0, 1]) The initial and final maps are given in VPM([0, 1] and drawn in box 1 and box 9. For the discretization, we use 16 time steps and 4000 cells. Intermediate permutations are drawn every two time steps in box 2 through box 8 and approximate generalized maps that obviously belong to DS([0, 1]) but not to VPM(D)

$$-\Delta p(t,x) = \nabla_x \otimes \nabla_x \cdot \int_a \frac{q \otimes q}{c}(t,x,a),$$

where we use notation

$$\frac{q \otimes q}{c}(t,x,a) = \frac{dX_t(a)}{dt} \otimes \frac{dX_t(a)}{dt} \delta(x - X_t(a)).$$

Proof

Notice first that

$$\int_a c(t,x,a) = 1,$$

directly follows from the fact that X_t belongs to SDiff(D).

Next, for every test-function f, we get

$$\frac{d}{dt} \int_{(x,a)} f(x,a) c(t,x,a) = \frac{d}{dt} \int f(X_t(a),a) da = \int (\nabla_x f)(X_t(a),a) \cdot \frac{dX_t}{dt}(a) da$$

$= \int_{(x,a)} \nabla_x f(x,a) \cdot q)(t,x,a)$. Similarly:

$$\frac{d}{dt} \int_{(x,a)} f(x,a) q(t,x,a) = \frac{d}{dt} \int f(X_t(a),a) \frac{dX_t}{dt}(a) da$$

$$= \int (\nabla_x f)(X_t(a), a) \cdot \left(\frac{dX_t}{dt} \otimes \frac{dX_t}{dt}\right)(a)da - \int f(X_t(a), a)(\nabla_x p)(t, X_t(a))da$$

$$= \int_{(x,a)} \nabla_x f(x, a) \cdot \frac{q \otimes q}{c}(t, x, a) - f(x, a)c(t, x, a)\nabla_x p(t, x),$$

Finally:

$$-\Delta p(t, x) = \nabla_x \otimes \nabla_x \cdot \int_a \frac{q \otimes q}{c}(t, x, a)$$

follows from

$$\int_a c(t, x, a) = 1.$$

End of proof.

1.3 Variational origin of the relaxed Euler equations

The relaxed Euler equations turn out to be the optimality conditions for the *convex* minimization problem

$$\inf \left\{ \int_{t,x,a} \frac{|q(t, x, a)|^2}{2c(t, x, a)} \; ; \quad (\partial_t c + \nabla_x \cdot q)(t, x, a) = 0, \quad \int_a c(t, x, a) = 1 \right\},$$

where ($c \geq 0, q \in \mathbb{R}^d$) is a pair of Borel measures over $[t_0, t_1] \times D^2$, c being prescribed at $t = t_0$ and $t = t_1$. Note that, in this formulation,

$$\int_{t,x,a} \frac{|q|^2}{2c}$$

should be more precisely understood as

$$\sup \int_{t,x,a} A(t, x, a)c(t, x, a) + B(t, x, a) \cdot q(t, x, a)$$

for all continuous functions $A \in \mathbb{R}$, $B \in \mathbb{R}^d$ subject to

$$2A(t, x, a) + |B(t, x, a)|^2 \leq 0, \quad \forall (t, x, a) \in [t_0, t_1] \times D^2.$$

Also notice that it makes sense to prescribe c at times t_0 and t_1 because of the "continuity equation" $\partial_t c + \nabla_x \cdot q = 0$ and the finiteness of the energy to be minimized.

Let us emphasize that this convex minimization problem, studied in [12], is just the correct relaxation, as $d \geq 3$, of the Minimizing Geodesic Problem in Arnold's geometric formulation of the Euler equations [12]. The need of a relaxed framework is clearly justified by Shnirelman's negative result [30] that the minimizing geodesic problem may admit no solution in the classical framework of SDiff(D) (and not even in VPM(D) as mentioned in [10]). Moreover, another crucial result of Shnirelman [31] fully justifies this convex relaxation, through the following *dynamical* approximation result of *generalized* solutions by classical curves along SDiff(D):

Proposition 2 *Let* $(c(t, x, a) \geq 0, q(t, x, a) \in \mathbb{R}^d)$ *be a pair of Borel measures on* $[t_0, t_1] \times D^2$*, such that*

$$\partial_t c(t, x, a) + \nabla_x \cdot q(t, x, a) = 0, \quad \int_a c(t, x, a) = 1, \quad \int_{t,x,a} \frac{|q(t, x, a)|^2}{c(t, x, a)} < +\infty.$$

Then, as long as $d \geq 2$, there is a sequence of smooth curves

$$t \in [t_0, t_1] \to X_t^n \in \mathrm{SDiff}(D),$$

such that the corresponding measures (c^n, q^n) defined by

$$c^n(t, x, a) = \delta(x - X_t^n(a)), \quad q^n(t, x, a) = \frac{dX_t^n(a)}{dt}\delta(x - X_t^n(a)),$$

weakly converge to (c, q) with no energy gap in the sense that

$$\int_{t,x,a} \frac{|q(t, x, a)|^2}{c(t, x, a)} = \lim_{n\uparrow\infty} \int_{t_0}^{t_1} \int_D \left|\frac{dX_t^n(a)}{dt}\right|^2 dadt.$$

1.4 Connection with permutation valued processes

In the particular case $t_0 = 0$, $t_1 = 1$, $D = [0, 1]$

$$c(0, x, a) = \delta(x - a), \quad c(1, x, a) = \delta(x - T(a)), \quad T(a) = 1 - a,$$

the relaxed minimization problem admits an exact solution (already described in [10]). It turns out that this solution plays a crucial rose in the analysis of permutation valued processes, for which we refer to [17,18,29] (where no connection seems to be made with the Euler model of incompressible flows before [17]). The author is very grateful to Gérard Ben–Arous for pointing out this remarkable connection after a lecture delivered by the author in Toulouse in 2017. From that observation, we make the conjecture that the Euler model, through its Arnold geometric interpretation, can be just interpreted as the macroscopic limit of the large deviation principle for the process of random exchanges of subcubes D_α with common interfaces, as $N \uparrow \infty$. In our opinion, this would be a particularly exciting and quite fundamental interpretation of the Euler equations!

1.5 Some properties of the relaxed equations

Due to the convexity of the relaxed minimization problem, it is shown in [12] that there is a unique pressure gradient ∇p, entirely determined by the data of c at the end points t_0, t_1, such that for all solutions (c, q), we get

$$\nabla p(t, x) = -\int_a \partial_t q(t, x, a) + \nabla_x \cdot \left(\frac{q \otimes q}{c}\right)(t, x, a).$$

For this unique pressure gradient, we have partial regularity and continuous dependence results for which we refer to [1,2,5,12,14]. In addition, the "Boltzmann entropy" $\int_{a,x}(c \log c - c)(t, x, a) \geq 0$ turns out to be a convex function of $t \in [t_0, t_1]$. This had been conjectured in [13] and proven in [7,24]. We conjecture that this property might lead to a proof that the group of volume preserving diffeomorphisms $\mathrm{SDiff}(D)$ enjoys, in a suitable sense, a nonnegative Ricci curvature, à la Lott–Sturm–Villani.

1.6 Frequent ill-posedness of the Cauchy problem

As just seen, the relaxed Euler equations:

$$\partial_t c(t, x, a) + \nabla_x \cdot q(t, x, a) = 0, \quad \int_a c(t, x, a) = 1,$$

$$(\partial_t q + \nabla_x \cdot \frac{q \otimes q}{c}(t, x, a) = -c(t, x, a)\nabla_x p(t, x),$$

are well suited for the "minimizing geodesic problem". However, the Cauchy problem is rarely well-posed, only under severe conditions (à la Rayleigh–Penrose) on the initial data $(c, q)(t = 0, x, a)$. There have been a lot of related works, in the last 20 years, from numerous authors such as A. Baradat, C. Bardos, N. Besse, Y. Brenier, E. Grenier, D. Han-Kwan, M. Iacobelli, F. Rousset. (See [6] as a very recent reference on these issues.)

2 Incompressible flow in porous media

The "Muskat system", also known as "incompressible porous media equations", rules the motion of an inhomogeneous incompressible fluid driven by a potential Φ through a porous medium. The traditional writing, in "Eulerian coordinates", of this model reads

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \quad v = \mathbb{P}(-\rho \nabla \Phi),$$

where $\mathbb{P}$ is the Helmholtz L^2 projection onto divergence-free fields. This was a starting point for Otto's gradient flow theory [22,27,28]. It is also of high interest [33] for the application of convex integration in mathematical Fluid Mechanics à la De Lellis and Székelyhidi [19] (which followed another very influential work of Shnirelman [32]!).

The Muskat system on a compact domain $D \subset \mathbb{R}^d$ can be easily written in material (a.k.a. Lagrangian) coordinates:

$$\frac{dX_t}{dt}(a) = -\rho_0(a)(\nabla\Phi)(X_t(a)) - (\nabla p)(t, X_t(a)), \ a \in D,$$

where X_t is a volume-preserving map of D, $\forall t \geq 0$.

So the Muskat system is just a friction dominated first order analog of the Euler equations of an incompressible inhomogeneous fluid accelerated by $\nabla\Phi$, with Boussinesq approximation, written in material (or Lagrangian) coordinates:

$$\frac{d^2X_t}{dt^2}(a) = -\rho_0(a)(\nabla\Phi)(X_t(a)) - (\nabla p)(t, X_t(a)), \ a \in D,$$

where X_t is a volume-preserving map of D, $\forall t \geq 0$. As a matter of fact, the local well-posedness of the Muskat system written in Lagrangian coordinates directly follows from the techniques developed in [20].

2.1 A time discrete version of the Muskat system based on the polar factorization of maps

A natural time discretization of the Muskat system, written in material coordinates, rely on the following "polar factorization" theorem (cf. [8,11] and [25] in the Riemannian case):

Theorem 1 (Polar factorization of maps.) *Let $D \subset \mathbb{R}^d$ be a compact convex domain (for simplicity), with Lebesgue measure $\mathcal{L}_D$. Let $T \in L^2(D, \mathbb{R}^d)$ s.t. $\mathcal{L}_D \circ T^{-1} << \mathcal{L}_{\mathbb{R}^d}$. Then T admits a unique factorization $T = \nabla U \circ X$, where:*

(i) *$U : D \to \mathbb{R}$ is convex;*
(ii) *$X : D \to D$ is volume-preserving (i.e. $\mathcal{L}_D \circ X^{-1} = \mathcal{L}_D$).*

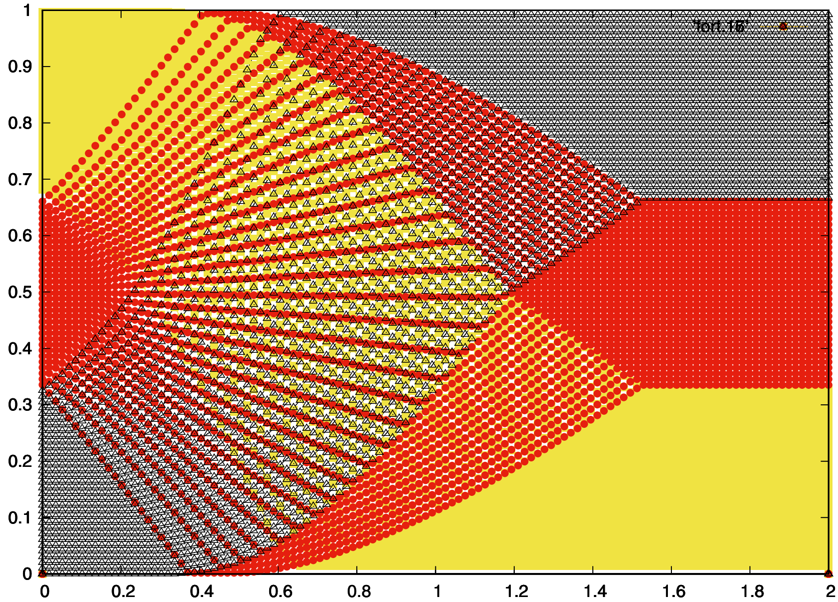

Fig. 2 The horizontal axis corresponds to time while the vertical axis corresponds to space. Particles have three different possible weights: light particles are drawn in black, heavy ones in yellow and neutral in red. At time 0, heavy particles lie at the top of the column, while light ones lie at the bottom. We can see how the neutral particles are forced to give way to the other ones but eventually return to their original positions. The computation is performed according to the polar factorization scheme

This result generalizes the polar factorization of real square matrices (which corresponds to the particular case when D is the unit ball and T a linear map from D to $\mathbb{R}^d$, both factors ∇U and X being also linear in this very special case). It can also be seen as a nonlinear version of the Helmholtz decomposition of vector fields which simply amounts to linearizing the polar factorization about the identity map. The polar factorization of maps suggests the following time discretization with time step δt of the Muskat system

$$\frac{d}{dt} X_t = -\rho_0 \, \nabla \Phi \circ X_t - \nabla p \circ X_t .$$

We polar factorize, at each time step $n \in \mathbb{N}$, the "predictor"

$$X^n - \delta t \;\rho_0 \; \nabla \Phi \circ X^n$$

as

$$(Id + \delta t \nabla p^{n+1}) \circ X^{n+1},$$

with $X^{n+1} : D \to D$ volume preserving and $x \in D \to |x|^2/2 + \delta t \;\; p^{n+1}(x)$ convex.

Surprisingly enough, as $d = 1$, this "predictor–corrector" scheme still makes sense and can be trivially coded through a standard sorting algorithm! In some sense, the polar factorization scheme already suggests a natural relaxation of the Muskat model where fluid particles can cross each other as shown on Fig. 2.

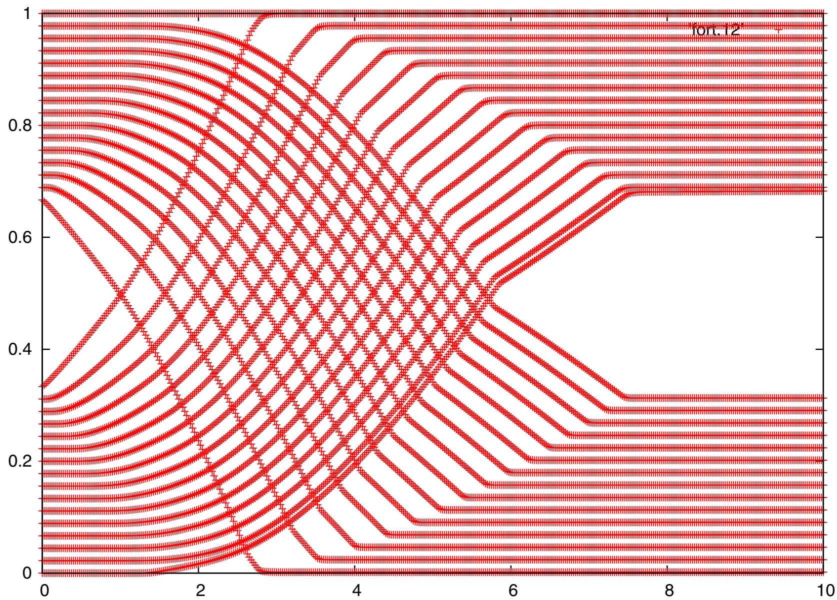

Fig. 3 The horizontal axis corresponds to time while the vertical axis corresponds to space. Particles have three different possible weights: only light and heavy particles are drawn in red. At time 0, heavy particles lie at the top of the column, while light ones lie at the bottom. The computation is performed according to a standard CFD method (first order upwind scheme). Trajectories are recovered through numerical integration of the velocity fields associated to the heavy and light phases

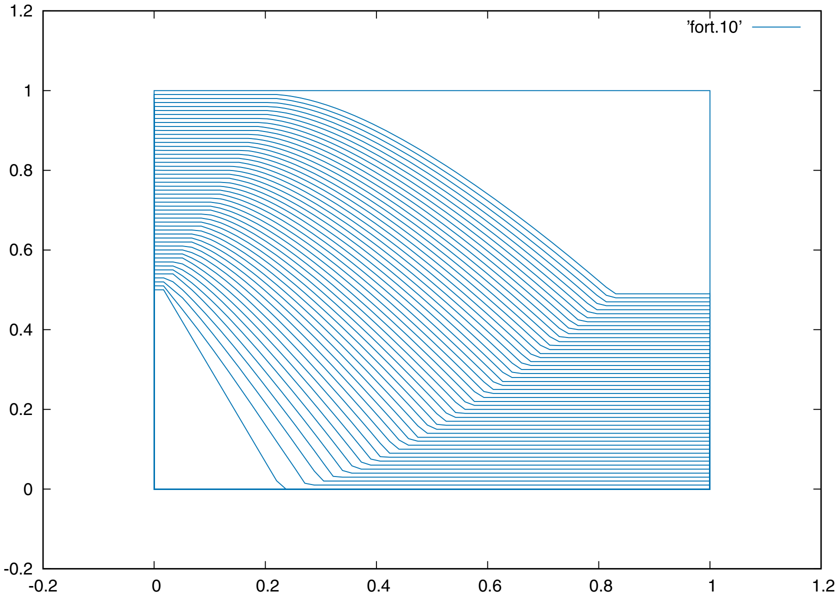

Fig. 4 Muskat model. The horizontal axis corresponds to time while the vertical axis corresponds to space. Particles have two different possible weights, but only heavy particles are drawn in blue. At time 0, heavy particles lie at the top of the column, while light ones lie at the bottom. The computation is performed according to the polar factorization scheme

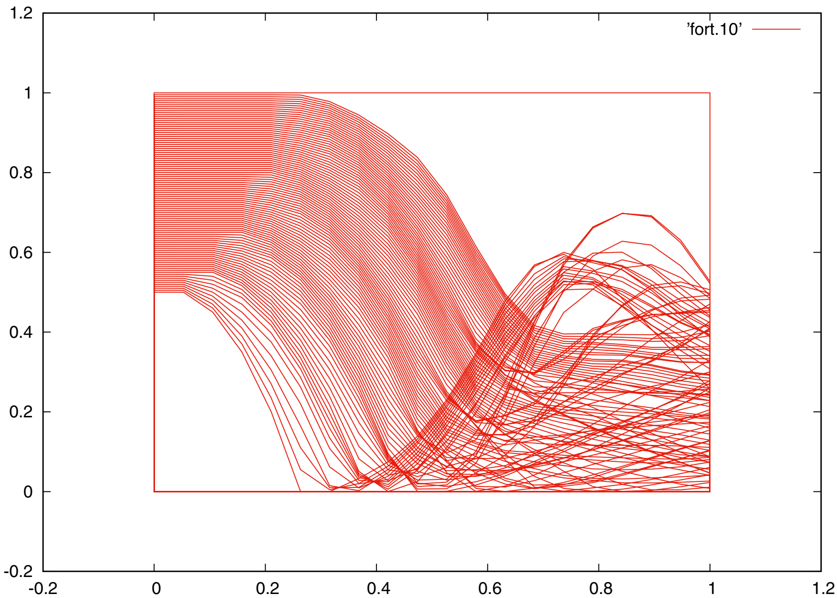

Fig. 5 Inhomogeneous Euler model. The horizontal axis corresponds to time while the vertical axis corresponds to space. Particles have two different possible weights, but only heavy particles are drawn in red. At time 0, heavy particles lie at the top of the column, while light ones lie at the bottom. The computation is performed according to a suitable polar factorization scheme

2.2 The Eulerian–Lagrangian relaxed Muskat system

Just as we did for the Euler equation, we easily find a relaxed version of the Muskat system

$$\frac{dX_t}{dt}(a) = -\rho_0(a)(\nabla\Phi)(X_t(a)) - (\nabla p)(t, X_t(a))$$

in terms of

$$c(t,x,a) = \delta(x - X_t(a)), \quad q(t,x,a) = \frac{dX_t}{dt}(a)\delta(x - X_t(a)),$$

We immediately get

$$\partial_t c(t,x,a) + \nabla_x q(t,x,a) = 0, \quad q(t,x,a) = c(t,x,a)(\rho_0(a)\nabla\Phi(x) + \nabla p(t,x)),$$

and obtain for c the self-consistent first-order pseudo-differential system of conservation laws:

$$\partial_t c(t,x,a) = \nabla_x \cdot (c(t,x,a)[\rho_0(a)\nabla\Phi(x) + \nabla p(t,x)])$$
$$-\Delta p(t,x) = \nabla_x \cdot \left(\int_a c(t,x,a)\rho_0(a)\nabla\Phi(x)\right).$$

This is somewhat surprising since the Muskat equations are often rather seen as a gradient flow (as in [22,27])! In one space dimension, as $d = 1$, this system is no longer pseudo-differential and further reduces to

$$\partial_t c(t,x,a) = \partial_x \left(c(t,x,a)[\rho_0(a) - \int_b \rho_0(b)c(t,x,b)]\partial_x\Phi(x)\right).$$

These equations can be discretized by standard CFD methods. The agreement with the polar factorization scheme already discussed is remarkable as shown by comparing Figs. 2 and 3.

2.3 Entropy conservation and local well-posedness

In the special case when the potential Φ is linear, so that $\nabla\Phi$ is just a constant, one can easily check that the relaxed "Eulerian–Lagrangian" Muskat system admits an extra conservation law for the Boltzmann entropy

$$\int_a (c \log c - c)(t, x, a),$$

which is strictly convex in c. This property essentially suffices for the local well-posedness of the relaxed Muskat system (in the spirit of [16]), in sharp contrast with the frequent ill-posedness of the relaxed Euler equations. This is well illustrated, at the numerical level and in dimension $d = 1$, by Figs. 4 and 5, where one can compare the rather wild behavior of the relaxed (inhomogeneous) Euler equations and the much smoother behavior of the Muskat system!

Conclusion

The relaxed Euler equations are well suited for the minimizing geodesic problem à la Arnold (leading to the existence and uniqueness of a pressure gradient), but not so much for the Cauchy problem which is rarely well-posed. In sharp contrast, in the friction dominated case describing incompressible flows in porous media, the corresponding relaxed Muskat system is at least locally well-posed and well suited for stable numerical computations ot the Cauchy problem.

References

1. L. Ambrosio, A. Figalli, *On the regularity of the pressure field of Brenier's weak solutions to incompressible Euler equations,* Calc. Var. Partial Differential Equations 31 (2008) 497-509.
2. L. Ambrosio, A. Figalli, *Geodesics in the space of measure-preserving maps and plans,* Archive for Rational Mechanics and Analysis 194 (2009) 421-462.
3. V. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluides parfaits,* Ann. Institut Fourier 16 (1966) 319-361.
4. V. Arnold, B. Khesin, *Topological methods in hydrodynamics, Applied Mathematical Sciences, 125,* Springer-Verlag, New York 1998.
5. A. Baradat, *Continuous dependence of the pressure field with respect to endpoints for ideal incompressible fluids,* Calc. Var. Partial Differential Equations 58 (2019), no. 1, Art. 25, 22
6. A. Baradat, *Nonlinear instability in Vlasov type equations around rough velocity profiles,* Ann. Inst. H. Poincaré Anal. Non Linéaire 37 (2020) 489-547.
7. A. Baradat, L. Monsaingeon, *Small noise limit and convexity for generalized incompressible flows, Schrödinger problems, and optimal transport,* Arch. Ration. Mech. Anal. 235 (2020) 1357-1403.
8. Y. Brenier, *Décomposition polaire et réarrangement monotone des champs de vecteurs,* C. R. Acad. Sci. Paris I Math. 305 (1987) 805-808.
9. Y. Brenier, *A combinatorial algorithm for the Euler equations of incompressible flows,* Comput. Methods Appl. Mech. Engrg. 75 (1989) 325-332.
10. Y. Brenier, *The least action principle and the related concept of generalized flows for incompressible perfect fluids,* J.of the AMS 2 (1989) 225-255.
11. Y. Brenier, *Polar factorization and monotone rearrangement of vector-valued functions,* Comm. Pure Appl. Math. 44 (1991) 375-417.

12. Y. Brenier, *Minimal geodesics on groups of volume-preserving maps,* Comm. Pure Appl. Math. 52 (1999) 411-452.
13. Y. Brenier, *Extended Monge-Kantorovich theory, Optimal transportation and applications,* pp. 91-121, Lecture Notes in Math., 1813, Springer 2003.
14. Y. Brenier, *Remarks on the Minimizing Geodesic Problem in Inviscid Incompressible Fluid Mechanics,* Calc. Var. Partial Differential Equations 47 (2013) 55-64.
15. Y. Brenier, W. Gangbo, *L^p approximation of maps by diffeomorphisms,* Calc. Var. Partial Differential Equations 16 (2003) 147-164.
16. C. Dafermos, *Hyperbolic conservation laws in continuum physics, Fourth edition,* Springer-Verlag, Berlin, 2016.
17. D. Dauvergne, *The Archimedean limit of random sorting networks,* arXiv:1802.08934
18. D. Dauvergne, B. Virág, *Circular support in random sorting networks,* Trans. Am. Math. Soc. 373 (2020) 1529-1553.
19. C. De Lellis, L. Székelyhidi Jr, *The Euler equations as a differential inclusion,* Ann. of Math. (2) 170 (2009) 1417-1436.
20. D. Ebin, J. Marsden, *Groups of diffeomorphisms and the notion of an incompressible fluid,* Ann. of Math. 92 (1970) 102-163.
21. Y.Eliashberg, T.Ratiu, *The diameter of the symplectomorphism group is infinite,* Invent. Math. 103 (1991) 327-340.
22. N. Gigli, F. Otto, *Entropic Burgers' equation via a minimizing movement scheme based on the Wasserstein metric,* Calc. Var. Partial Differential Equations 47 (2013) 181-206.
23. R. Hamilton, *The inverse function theorem of Nash and Moser,* Bull. Am. Math. Soc. 7 (1982) 65-222.
24. H. Lavenant, *Time-convexity of the entropy in the multiphasic formulation of the incompressible Euler equation,* Calc. Var. Partial Differential Equations 56 (2017) Art. 170, 29 pp.
25. R. McCann, *Polar factorization of maps on Riemannian manifolds,* Geom. Funct. Anal. 11 (2001) 589-608.
26. Y. Neretin, *Categories of bistochastic measures and representations of some infinite-dimensional groups,* Sb. 183 (1992), no. 2, 52-76.
27. F. Otto, *Evolution of microstructure in unstable porous media flow: a relaxational approach,* Comm. Pure Appl. Math. 52 (1999) 873-915.
28. F. Otto, *The geometry of dissipative evolution equations: the porous medium equation,* Comm. Partial Differential Equations 26 (2001) 101-174.
29. M. Rahman, B. Virág, M. Vizer, *Geometry of Permutation Limits,* Combinatorica 39 (2019) 933-960
30. A. Shnirelman, *On the geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid,* Math. Sbornik USSR 56 (1987) 79-105.
31. A. Shnirelman, *Generalized fluid flows, their approximation and applications,* Geom. Funct. Anal. 4 (1994) 586-620.
32. A. Shnirelman, *On the nonuniqueness of weak solution of the Euler equation,* Comm. Pure Appl. Math. 50 (1997) 1261-1286.
33. L. Székelyhidi, *Relaxation of the incompressible porous media equation,* Ann. Sci. Ec. Norm. Supér. (4) 45 (2012) 491-509.