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# A sharp scalar curvature estimate for CMC hypersurfaces satisfying an Okumura type inequality

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**Abstract** We obtain a sharp estimate to the scalar curvature of stochastically complete hypersurfaces immersed with constant mean curvature in a locally symmetric Riemannian space obeying standard curvature constraints (which includes, in particular, a Riemannian space with constant sectional curvature). For this, we suppose that these hypersurfaces satisfy a suitable Okumura-type inequality recently introduced by Meléndez (Bull Braz Math Soc 45:385–404, 2014), which is a weaker hypothesis than to assume that they have two distinct principal curvatures. Our approach is based on the equivalence between stochastic completeness and the validity of the weak version of the Omori–Yau's generalized maximum principle, which was established by Pigola et al. (Proc Am Math Soc 131:1283–1288, 2002; Mem Am Math Soc 174:822, 2005).

**Keywords** Locally symmetric spaces · Stochastically complete hypersurfaces · Constant mean curvature hypersurfaces · Scalar curvature · Isoparametric hypersurfaces

**Résumé** Nous obtenons une estimation optimale de la courbure scalaire des hypersurfaces stochastiquement complètes immergées avec courbure moyenne constante dans un espace Riemannien localement symétrique, obéissant aux contraintes de courbure standard (qui comprend, en particulier, un espace Riemannien avec courbure sectionnelle constante). Pour cela, nous supposons que ces hypersurfaces satisfont une inégalité appropriée de type Okumura récemment introduite par Meléndez (Bull Braz Math Soc 45:385–404, 2014), ce qui est une hypothèse plus faible que de supposer qu'elles ont deux courbures principales distinctes.

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Notre approche est basée sur l'équivalence entre la complétude stochastique et la validité de la version faible du principe maximal généralisé de Omori–Yau, qui a été établi par Pigola et al. (Proc Am Math Soc 131:1283–1288, 2002; Mem Am Math Soc 174:822, 2005).

**Mathematics Subject Classification** Primary 53C42; Secondary 53C40

## 1 Introduction

The problem of characterizing hypersurfaces immersed with constant mean curvature in a Riemannian space form $\overline{M}_c^{n+1}$ of constant sectional curvature $c$ constitutes an important and fruitful topic in the theory of isometric immersions, which has being widely approached by many authors. For instance, Alencar and do Carmo [1] studied compact hypersurfaces immersed with constant mean curvature in the Euclidean sphere. Specifically, they introduced a tensor $\Phi$, the so-called total umbilicity tensor of the hypersurface, and showed that if the squared norm of $\Phi$ is bounded from above for a certain constant $\beta_H$ depending only on mean curvature and dimension of hypersurface, then either the hypersurface is totally umbilical or the equality $|\Phi|^2 = \beta_H$ holds. In the last case, they characterized all hypersurfaces having this property. This result extended previous ones due to Simons [12], Lawson [6] and Chern et al. [4].

More recently, Alías and García-Martínez [2] used the weak Omori–Yau maximum principle due to Pigola et al. [10,11] to study the behavior of the scalar curvature $R$ (or, equivalently, of the norm of $\Phi$) of a hypersurface immersed with constant mean curvature in a Riemannian space form $\overline{M}_c^{n+1}$, with $n \geq 3$, by deriving a sharp estimate for the infimum of $R$ (or equivalently, for the supremum of the norm of $\Phi$). In particular, they gave a generalization of result due Alencar and do Carmo to complete parabolic hypersurfaces in space forms. Afterwards, following the approach introduced in [2] and by assuming an Okumura type inequality, Meléndez [7] proved results similar to the above cited.

We recall that a Riemannian manifold is said to be *locally symmetric* when all the covariant derivative components of its curvature tensor vanish identically. This class of Riemannian manifolds consists in an interesting generalization of constant sectional curvature spaces and, consequently, it is a natural question to revisit in this ambient spaces the known results of constant sectional curvature spaces. So, following the ideas introduced in [2,7], our main purpose in this paper is to obtain a sharp estimate for the infimum of the scalar curvature of stochastically complete hypersurfaces with constant mean curvature in a locally symmetric Riemannian manifold obeying standard curvature constraints. For this, we will suppose that these hypersurfaces satisfy an Okumura-type inequality recently introduced by Meléndez in [7] and which is a weaker hypothesis compared with the assumption that such hypersurfaces have two distinct principal curvatures (for more details, see Sect. 3).

This manuscript is organized in the following way: in Sect. 2 we introduce some basic facts and notations that will appear along the paper. In particular, we establish our curvatures constraints related to a hypersurface immersed in a locally symmetric space. In Sect. 3 we recall an Okumura type inequality due to Meléndez in [7] and we quote two key lemmas. In Sect. 4 we apply the weak Omori–Yau's generalized maximum principle to obtain a sharp estimate for the infimum of the scalar curvature of a stochastically complete hypersurface with constant mean curvature, characterizing the equality by showing that, if it holds, then the hypersurface must be either totally umbilical or isometric to an isoparametric hypersurface having two distinct principal curvatures (see Theorems 1 and 2).

## 2 Locally symmetric spaces

In this work, we will deal with $n$-dimensional, orientable and connected hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed into an $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$. We choose a local field of orthonormal frame $\{e_1, \ldots, e_{n+1}\}$ in $\overline{M}^{n+1}$ with dual coframe $\{\omega_1, \ldots, \omega_{n+1}\}$ such that, at each point of $\Sigma^n$, $e_1, \ldots, e_n$ are tangent to $\Sigma^n$ and $e_{n+1}$ is normal to $\Sigma^n$. We will use the following convention for the indices:

$$1 \le A, B, C, \ldots \le n+1 \quad \text{and} \quad 1 \le i, j, k, \ldots \le n.$$

In this setting, $\overline{R}_{ABCD}$, $\overline{R}_{AC}$ and $\overline{R}$ denote respectively the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of the Riemannian manifold $\overline{M}^{n+1}$. So, we have

$$\overline{R}_{AC} = \sum_B \overline{R}_{ABCB} \quad \text{and} \quad \overline{R} = \sum_A \overline{R}_{AA}.$$

Now, restricting all the tensor to $\Sigma^n$, we see that $\omega_{n+1} = 0$ on $\Sigma^n$. Hence, $0 = d\omega_{n+1} = -\sum_i \omega_{n+1i} \wedge \omega_i$ and as it well known we get

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}.$$

This gives the second fundamental form of $\Sigma^n$, $A = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$ and its squared norm $|A|^2 = \sum_{i,j} h_{ij}^2$. Furthermore, the mean curvature $H$ of $\Sigma^n$ is defined by $H = \frac{1}{n}\sum_i h_{ii}$.

On the other hand, it follows from Gauss equation that the Ricci curvature and the scalar curvature of $\Sigma^n$ are given, respectively, by

$$R_{ij} = \sum_k \overline{R}_{ikjk} + nHh_{ij} - \sum_k h_{ik}h_{kj} \quad \text{and} \quad R = \sum_i R_{ii}, \tag{2.1}$$

where $R_{ijkl}$ are the components of the Riemannian curvature tensor of $\Sigma^n$. So, by (2.1) we obtain

$$R = \sum_{i,j} \overline{R}_{ijij} + n^2H^2 - |A|^2. \tag{2.2}$$

We also remember that the squared norm of the covariant differential of the second fundamental form $A$ is given by

$$|\nabla A|^2 = \sum_{i,j,k} h_{ijk}^2, \tag{2.3}$$

where $h_{ijk}$ denote the first covariant derivatives of $h_{ij}$.

Taking a local orthonormal frame $\{e_1, \ldots, e_n\}$ on $\Sigma^n$ such that $h_{ij} = \lambda_i\delta_{ij}$, from equation (2.10) of [5] we have the following Simons type formula:

$$\begin{aligned}\frac{1}{2}\Delta|A|^2 &= |\nabla A|^2 + \sum_i \lambda_i (nH)_{ii} + nH\sum_i \lambda_i^3 - |A|^4 \\ &\quad - \sum_{i,j,k} h_{ij}(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}) + \sum_i \overline{R}_{(n+1)i(n+1)i}\left(nH\lambda_i - |A|^2\right) \\ &\quad + \sum_{i,j}(\lambda_i - \lambda_j)^2\overline{R}_{ijij}.\end{aligned} \tag{2.4}$$

Proceeding as in [3,5], we will assume that there exist constants $c_1$ and $c_2$ such that the sectional curvature $\overline{K}$ of the ambient space $\overline{M}^{n+1}$ satisfies the following two constraints

$$\overline{K}(\eta, v) = \frac{c_1}{n}, \tag{2.5}$$

for vectors $\eta \in T^{\perp}\Sigma$ and $v \in T\Sigma$, and

$$\overline{K}(u, v) \geq c_2, \tag{2.6}$$

for vectors $u, v \in T\Sigma$.

From now on, we will consider $\overline{M}^{n+1}$ being a locally symmetric Riemannian manifold, which means that all the covariant derivative components $\overline{R}_{ABCD;E}$ of its curvature tensor vanish identically.

*Remark 1* Obviously, when the ambient manifold $\overline{M}^{n+1}$ has constant sectional curvature $\bar{c}$, then it is locally symmetric and the curvature conditions (2.5) and (2.6) are satisfied for every hypersurface $\Sigma^n$ immersed in $\overline{M}^{n+1}$, with $c_1/n = c_2 = \bar{c}$. Therefore, in some sense our assumptions are a natural generalization of the case where the ambient space has constant sectional curvature. Moreover, when the ambient manifold is a Riemannian product of two Riemannian manifolds of constant sectional curvature, say $\overline{M} = M_1(\kappa_1) \times M_2(\kappa_2)$, then $\overline{M}$ is locally symmetric and, if $\kappa_1 = 0$ and $\kappa_2 \geq 0$, then every hypersurface of type $\Sigma = \Sigma_1 \times M_2(\kappa_2)$, where $\Sigma_1$ is an orientable and connected hypersurface immersed into $M_1(\kappa_1)$, satisfies the curvature constraints (2.5) and (2.6) with $c_1 = c_2 = 0$ (for more details, see Remark 3.1 of [3]).

Supposing that $\overline{M}^{n+1}$ satisfies condition (2.5) and denoting by $\overline{R}_{AB}$ the components of its Ricci tensor, we have that its scalar curvature $\overline{R}$ is such that

$$\overline{R} = \sum_{A=1}^{n+1} \overline{R}_{AA} = \sum_{i,j=1}^{n} \overline{R}_{ijij} + 2\sum_{i=1}^{n} \overline{R}_{(n+1)i(n+1)i} = \sum_{i,j=1}^{n} \overline{R}_{ijij} + 2c_1. \tag{2.7}$$

Since the scalar curvature of a locally symmetric Riemannian manifold is constant, from (2.7) we see that $\sum_{i,j} \overline{R}_{ijij}$ is a constant naturally attached to $\overline{M}^{n+1}$. So, for sake of simplicity, in the course of this work we will denote the constant $\frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij}$ by $\overline{\mathcal{R}}$ and, assuming that $\overline{M}^{n+1}$ also satisfies condition (2.6), the parameter $c$ will stand for the quantity $2c_2 - \dfrac{c_1}{n}$.

## 3 Key lemmas

Given $\Phi_{ij} = h_{ij} - H\delta_{ij}$, we will also consider the following symmetric tensor

$$\Phi = \sum_{i,j} \Phi_{ij} \omega_i \otimes \omega_j.$$

Let $|\Phi|^2 = \sum_{i,j} \Phi_{ij}^2$ be the squared norm of $\Phi$. It is not difficult to check that $\Phi$ is traceless and $|\Phi|^2 = |A|^2 - nH^2 \geq 0$, with equality if and only if $\Sigma^n$ is totally umbilical. For that reason, $\Phi$ is called the total umbilicity tensor of $\Sigma^n$. Moreover, from (2.2) we get

$$|\Phi|^2 = n(n-1)\left(H^2 + \overline{\mathcal{R}}\right) - R. \tag{3.1}$$

Motivated by classical Okumura's lemma which was established in [8], we will consider the following Okumura type inequality introduced by Meléndez in [7]:

$$|\mathrm{tr}(\Phi^3)| \leq \frac{(n-2p)}{\sqrt{np(n-p)}}|\Phi|^3, \tag{3.2}$$

where $1 \leq p \leq n/2$. It is worth to point out that, since $\Phi$ is traceless, the classical Okumura's lemma [8] guarantees that the inequality (3.2) is automatically true when $p = 1$. Moreover, to suppose that inequality (3.2) holds is weaker than to assume that the hypersurface has two distinct principal curvatures with multiplicities $p$ and $n - p$. Indeed, in this case there exist real numbers $\mu$ and $\nu$ such that

$$\kappa_1 = \cdots = \kappa_p = \mu$$
$$\kappa_{p+1} = \cdots = \kappa_n = \nu,$$

where $\kappa_i$ stand for the eigenvalues of $\Phi$. Then, it is not difficult see that

$$0 = \sum_i \kappa_i = p\mu + (n-p)\nu \quad \text{and} \quad |\Phi|^2 = \sum_i \kappa_i^2 = p\mu^2 + (n-p)\nu^2.$$

Hence, we infer that

$$\mathrm{tr}(\Phi^3) = \sum_i \kappa_i^3 = p\mu^3 + (n-p)\nu^3 = \pm\frac{(2p-n)}{\sqrt{np(n-p)}}|\Phi|^3.$$

In order to establish our characterization results, we will need of the following Okumura type result (for more details, see Lemma 2.2 of [7]).

**Lemma 1** *Let $\kappa_1, \ldots, \kappa_n$, $n \geq 3$, be real numbers such that $\sum_i \kappa_i = 0$ and $\sum_i \kappa_i^2 = \beta^2$, where $\beta \geq 0$. Then,*

$$\sum_i \kappa_i^3 = \frac{(n-2p)}{\sqrt{np(n-p)}}\beta^3 \quad \left(\sum_i \kappa_i^3 = -\frac{(n-2p)}{\sqrt{np(n-p)}}\beta^3\right), \quad 1 \leq p \leq n-1,$$

*holds if and only if $p$ of the numbers $\kappa_i$ are nonnegative (resp. nonpositive) and equal and the rest $n - p$ of the numbers $\kappa_i$ are nonpositive (resp. nonnegative) and equal.*

For the proof of our results, we will also make use of the well known weak Omori–Yau maximum principle. Let us recall that, following the terminology introduced by Pigola et al. in [10,11], the Omori–Yau maximum principle is said to hold on a (not necessarily complete) $n$-dimensional Riemannian manifold $\Sigma^n$ if, for any smooth function $u \in C^2(\Sigma)$ with $u^* = \sup u < +\infty$ there exists a sequence of points $(p_k) \subset \Sigma^n$ satisfying

$$u(p_k) > u^* - \frac{1}{k}, \quad |\nabla u(p_k)| < \frac{1}{k} \quad \text{and} \quad \Delta u(p_k) < \frac{1}{k}.$$

In this sense, the classical result given by Omori and Yau in [9,13] states that Omori–Yau maximum principle holds on every complete Riemannian manifold with Ricci curvature bounded from below.

On the other hand, as observed also by Pigola et al. in [11], the validity of Omori–Yau maximum principle on $\Sigma^n$ does not depend on curvatures bounds as much as one would expect. For instance, the Omori–Yau maximum principle holds on every Riemannian manifolds which is properly immersed into a Riemannian space form with controlled mean curvature (see [11, Example 1.14]). In particular, it holds for every constant mean curvature hypersurface properly immersed into a Riemannian space form.

More generally, and following again the terminology introduced in [11], the weak Omori–Yau maximum principle is said to hold on a (not necessarily complete) $n$-dimensional Riemannian manifold $\Sigma^n$ if, for any smooth function $u \in C^2(\Sigma)$ with $u^* < +\infty$ there exists a sequence of points $(p_k) \subset \Sigma^n$ with the properties

$$u(p_k) > u^* - \frac{1}{k} \quad \text{and} \quad \Delta u(p_k) < \frac{1}{k}.$$

As it was proved by Pigola et al. [10,11], the fact that the weak Omori–Yau maximum principle holds on $\Sigma^n$ is equivalent to the stochastic completeness of the manifold. More precisely,

**Lemma 2** *A Riemannian manifold $\Sigma^n$ is stochastically complete if and only if, for every $u \in C^2(\Sigma)$ satisfying $\sup_\Sigma u < +\infty$, there exists a sequence of points $(p_k) \subset \Sigma^n$ such that*

$$\lim u(p_k) = \sup_\Sigma u \quad \text{and} \quad \limsup \Delta u(p_k) \leq 0.$$

A direct consequence of Lemma 2 is that the weak Omori–Yau maximum principle holds on every parabolic Riemannian manifold.

## 4 Main results

This section is devoted to obtain a sharp estimate for the infimum of the scalar curvature of a stochastically complete hypersurface immersed with constant mean curvature in a locally symmetric Riemannian manifold. So, we state our first result.

**Theorem 1** *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$, $n \geq 4$, be a stochastically complete hypersurface immersed in a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions (2.5) and (2.6). Suppose that $\Sigma^n$ has constant mean curvature $H$ with $H^2 + c > 0$, where $c = 2c_2 - c_1/n$. If its total umbilicity tensor $\Phi$ satisfies (3.2) for some $1 \leq p \leq n/2$, then*

(i) *either* $\inf_\Sigma R = n(n-1)(H^2 + \overline{\mathcal{R}})$ *and $\Sigma^n$ is a totally umbilical hypersurface,*
(ii) *or*

  (a) $\inf_\Sigma R \leq n(n-2)H^2 + n(n-1)\overline{\mathcal{R}} - nc$, *if $p = n/2$,*
  (b) *and if $p < n/2$,*

$$\inf_\Sigma R \leq \frac{n(n-2p)}{2p(n-p)} \left( 2p(n-p)c_{n,p} + nd_{n,p}H^2 + |H|\sqrt{n^2H^2 + 4p(n-p)c} \right),$$

  *where the constants $c_{n,p}$ and $d_{n,p}$ are given by*

$$c_{n,p} = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2p)} \quad \text{and} \quad d_{n,p} = \frac{2np - 2p^2 - n}{n-2p}.$$

*Moreover, if the equality holds and this infimum is attained at some point of $\Sigma^n$ (in the case $c > 0$, assume in addition that $H \neq 0$), then $\Sigma^n$ is an isoparametric hypersurface with two distinct principal curvatures of multiplicities $p$ and $n-p$.*

It follows from (3.1) that $\inf_\Sigma R = n(n-1)(H^2 + \overline{\mathcal{R}}) - \sup_\Sigma |\Phi|^2$. Hence, Theorem 1 can be rewritten equivalently in terms of the total umbilicity tensor as follows.

**Theorem 2** *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$, $n \geq 4$, be a stochastically complete hypersurface immersed in a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions* (2.5) *and* (2.6). *Suppose $\Sigma^n$ has constant mean curvature $H$ such that $H^2 + c > 0$, where $c = 2c_2 - c_1/n$. If its total umbilicity tensor $\Phi$ satisfies* (3.2) *for some $1 \leq p \leq n/2$, then*

(i) *either* $\sup_\Sigma |\Phi| = 0$ *and $\Sigma^n$ is a totally umbilical hypersurface,*
(ii) *or*

$$\sup_\Sigma |\Phi| \geq \alpha_{|H|,c,p} = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left(\sqrt{n^2H^2 + 4p(n-p)c} - (n-2p)|H|\right) > 0.$$

*Moreover, if the equality holds and this supremum is attained at some point of $\Sigma^n$, then $\Sigma^n$ is an isoparametric hypersurface with (in the case $c > 0$, assume that $H \neq 0$) two distinct principal curvatures of multiplicities $p$ and $n - p$.*

*Proof* Firstly, taking a local orthonormal frame field $\{e_1, \ldots, e_n\}$ in $\Sigma^n$ such that

$$h_{ij} = \lambda_i \delta_{ij} \quad \text{and} \quad \Phi_{ij} = \kappa_i \delta_{ij},$$

we can check that

$$\sum_i \kappa_i = 0, \quad \sum_i \kappa_i^2 = |\Phi|^2 \quad \text{and} \quad \sum_i \lambda_i^3 = \sum_i \kappa_i^3 + 3H|\Phi|^2 + nH^3.$$

Now, since $\overline{M}^{n+1}$ is locally symmetric and $\Sigma^n$ has constant mean curvature, it follows from (2.4) that

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2 &= \frac{1}{2}\Delta|A|^2 \\ &= |\nabla A|^2 + nH\sum_i \lambda_i^3 - |A|^4 + \sum_i \overline{R}_{(n+1)i(n+1)i}\left(nH\lambda_i - |A|^2\right) \\ &\quad + \sum_{i,j}(\lambda_i - \lambda_j)^2\overline{R}_{ijij}. \end{aligned} \tag{4.1}$$

From curvature conditions (2.5) and (2.6), we get

$$\sum_i \overline{R}_{(n+1)i(n+1)i}(nH\lambda_i - |A|^2) = c_1(nH^2 - |A|^2) = -c_1|\Phi|^2 \tag{4.2}$$

and

$$\begin{aligned}\sum_{i,j}\overline{R}_{ijij}(\lambda_i - \lambda_j)^2 &\geq c_2\sum_{i,j}(\lambda_i - \lambda_j)^2 \\ &= 2nc_2(|A|^2 - nH^2) = 2nc_2|\Phi|^2. \end{aligned} \tag{4.3}$$

Moreover, it follows from our hypothesis (3.2) that

$$\begin{aligned} nH\sum_i \lambda_i^3 - |A|^4 &= n^2H^4 + 3nH^2|\Phi|^2 + nH\sum_i \kappa_i^3 - \left(|\Phi|^2 + nH^2\right)^2 \\ &\geq n^2H^4 + 3nH^2|\Phi|^2 - n|H|\left|\sum_i \kappa_i^3\right| - |\Phi|^4 - 2nH^2|\Phi|^2 - n^2H^4 \\ &\geq -|\Phi|^4 - \frac{n(n-2p)}{\sqrt{np(n-p)}}|H||\Phi|^3 + nH^2|\Phi|^2. \end{aligned} \tag{4.4}$$

Hence, since $c = 2c_2 - c_1/n$, inserting (4.2), (4.3) and (4.4) into (4.1) we obtain that

$$\frac{1}{2}\Delta|\Phi|^2 \geq |\nabla A|^2 - |\Phi|^4 - \frac{n(n-2p)}{\sqrt{np(n-p)}}|H||\Phi|^3 + n(H^2+c)|\Phi|^2$$
$$\geq -|\Phi|^2 P_{|H|,c,p}(|\Phi|), \tag{4.5}$$

where

$$P_{|H|,c,p}(x) = x^2 + \frac{n(n-2p)}{\sqrt{np(n-p)}}|H|x - n(H^2+c).$$

Observe that, since $H^2+c>0$, the polynomial $P_{|H|,c,p}(x)$ has an unique positive root given by

$$\alpha_{|H|,c,p} = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left(\sqrt{n^2H^2+4p(n-p)c} - (n-2p)|H|\right).$$

If $\sup_\Sigma |\Phi| = +\infty$, then (ii) holds trivially and there is nothing to prove. If $\sup_\Sigma |\Phi| < +\infty$, then we can apply Lemma 2 to the function $|\Phi|^2$ to assures that there exists a sequence of points $(p_k) \subset \Sigma^n$ such that

$$\lim |\Phi|(p_k) = \sup_\Sigma |\Phi| \quad \text{and} \quad \limsup \Delta|\Phi|^2(p_k) \leq 0,$$

which jointly with (4.5) implies

$$\left(\sup_\Sigma |\Phi|\right)^2 P_{|H|,c,p}\left(\sup_\Sigma |\Phi|\right) \geq 0.$$

It follows from here that either $\sup_\Sigma |\Phi| = 0$, which means that $|\Phi|$ vanishes identically and the hypersurface is totally umbilical, or $\sup_\Sigma |\Phi| > 0$ and then $P_{|H|,c,p}(\sup_\Sigma |\Phi|) \geq 0$. In the latter case, it must be $\sup_\Sigma |\Phi| \geq \alpha_{|H|,c,p}$, which gives the inequality in (ii).

Moreover, assume that equality $\sup_\Sigma |\Phi| = \alpha_{|H|,c,p}$ holds. In this case, $P_{|H|,c,p}(|\Phi|) \leq 0$ on $\Sigma^n$, which jointly with (4.5) implies that function $|\Phi|^2$ is subharmonic on $\Sigma^n$. Therefore, if this supremum is attained at some point of $\Sigma^n$, it follows from stronger maximum principle that $|\Phi| = \alpha_{|H|,c,p}$ is constant. Thus, (4.5) becomes trivially an equality,

$$\frac{1}{2}\Delta|\Phi|^2 = 0 = -|\Phi|^2 P_{|H|,c,p}(|\Phi|).$$

From here we obtain that $|\nabla A|^2 = 0$ and, consequently, from (2.3) we conclude that $\Sigma^n$ is isoparametric hypersurface. Finally, using once more the equality (4.5) we also obtain the equality in Lemma 1, which implies that the hypersurface has exactly two distinct principal curvatures of multiplicities $p$ and $n-p$. This finishes the proof from theorem. □

In the particular case where $\Sigma^n$ is complete, we obtain from Theorem 1 (or Theorem 2) the following consequence.

**Corollary 1** *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$, $n \geq 4$, *be a complete hypersurface immersed in a locally symmetric Riemannian manifold* $\overline{M}^{n+1}$ *satisfying curvature conditions* (2.5) *and* (2.6). *Suppose that* $\Sigma^n$ *has constant mean curvature* $H$ *with* $H^2+c>0$, *where* $c = 2c_2 - c_1/n$. *If its total umbilicity tensor* $\Phi$ *satisfies* (3.2) *for some* $1 \leq p \leq n/2$, *then*

(i) *either* $\inf_\Sigma R = n(n-1)(H^2+\overline{\mathcal{R}})$ *and* $\Sigma^n$ *is a totally umbilical hypersurface,*
(ii) *or*

(a) $\inf_\Sigma R \leq n(n-2)H^2 + n(n-1)\overline{\mathcal{R}} - nc$, *if* $p = n/2$,
(b) *and if* $p < n/2$,

$$\inf_\Sigma R \leq \frac{n(n-2p)}{2p(n-p)}\left(2p(n-p)c_{n,p} + nd_{n,p}H^2 + |H|\sqrt{n^2H^2 + 4p(n-p)c}\right),$$

*where the constants* $c_{n,p}$ *and* $d_{n,p}$ *are given by*

$$c_{n,p} = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2p)} \quad \text{and} \quad d_{n,p} = \frac{2np - 2p^2 - n}{n - 2p}.$$

*Moreover, if the equality holds and this infimum is attained at some point of* $\Sigma^n$ *(in the case* $c > 0$*, assume in addition that* $H \neq 0$*), then* $\Sigma^n$ *is an isoparametric hypersurface with two distinct principal curvatures of multiplicities* $p$ *and* $n - p$.

As mentioned before, we can rewritten Corollary 1 equivalently in terms of the total umbilicity tensor as follows.

**Corollary 2** *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$, $n \geq 4$, *be a complete hypersurface immersed in a locally symmetric Riemannian manifold* $\overline{M}^{n+1}$ *satisfying curvature conditions* (2.5) *and* (2.6). *Suppose that* $\Sigma^n$ *has constant mean curvature* $H$ *with* $H^2 + c > 0$*, where* $c = 2c_2 - c_1/n$. *If its total umbilicity tensor* $\Phi$ *satisfies* (3.2) *for some* $1 < p \leq n/2$*, then*

(i) *either* $\sup_\Sigma |\Phi| = 0$ *and* $\Sigma^n$ *is a totally umbilical hypersurface,*
(ii) *or*

$$\sup_\Sigma |\Phi| \geq \alpha_{|H|,c,p} = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left(\sqrt{n^2H^2 + 4p(n-p)c} - (n-2p)|H|\right) > 0.$$

*Moreover, if the equality holds and this supremum is attained at some point of* $\Sigma^n$ *(in the case* $c > 0$*, assume in addition that* $H \neq 0$*), then* $\Sigma^n$ *is an isoparametric hypersurface with two distinct principal curvatures of multiplicities* $p$ *and* $n - p$.

*Proof* We note that when $\sup_\Sigma |\Phi| = +\infty$ the result is clearly true. So, we can suppose that $\sup_\Sigma |\Phi| < +\infty$. In this case, since $\Sigma^n$ has constant mean curvature, we have that $\sup |A|^2 < +\infty$. Hence, from Eq. (2.1) and our hypothesis on sectional curvature of $\overline{M}^{n+1}$, we get

$$R_{ii} \geq (n-1)c_2 - nH \sup_\Sigma |A| - \sup_\Sigma |A|^2 > -\infty,$$

that is, the Ricci curvature of $\Sigma^n$ is bounded from below. In particular, $\Sigma^n$ is stochastically complete and the result follows from Theorem 2. □

Another consequence of Theorem 1 is the following result for complete parabolic hypersurfaces in locally symmetric spaces.

**Corollary 3** *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$, $n \geq 4$, *be a complete parabolic hypersurface immersed in a locally symmetric Riemannian manifold* $\overline{M}^{n+1}$ *satisfying curvature conditions* (2.5) *and* (2.6). *Suppose that* $\Sigma^n$ *has constant mean curvature* $H$ *with* $H^2 + c > 0$*, where* $c = 2c_2 - c_1/n$. *If its total umbilicity tensor* $\Phi$ *satisfies* (3.2) *for some* $1 \leq p \leq n/2$*, then*

(i) *either* $\inf_\Sigma R = n(n-1)(H^2 + \overline{\mathcal{R}})$ *and* $\Sigma^n$ *is a totally umbilical hypersurface,*
(ii) *or*

(a) $\inf_\Sigma R \le n(n-2)H^2 + n(n-1)\overline{\mathcal{R}} - nc$, *if* $p = n/2$,
(b) *and if* $p < n/2$,

$$\inf_{\Sigma} R \le \frac{n(n-2p)}{2p(n-p)}\left(2p(n-p)c_{n,p} + nd_{n,p}H^2 + |H|\sqrt{n^2H^2 + 4p(n-p)c}\right),$$

*where the constants* $c_{n,p}$ *and* $d_{n,p}$ *are given by*

$$c_{n,p} = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2p)} \quad \text{and} \quad d_{n,p} = \frac{2np - 2p^2 - n}{n-2p}.$$

*Moreover, if the equality holds (in the case* $c > 0$*, assume in addition that* $H \neq 0$*), then* $\Sigma^n$ *is an isoparametric hypersurface with two distinct principal curvatures of multiplicities* $p$ *and* $n - p$.

Equivalently, we can prove the following

**Corollary 4** *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$, $n \ge 4$, *be a complete parabolic hypersurface immersed in a locally symmetric Riemannian manifold* $\overline{M}^{n+1}$ *satisfying curvature conditions* (2.5) *and* (2.6)*. Suppose that* $\Sigma^n$ *has constant mean curvature* $H$ *with* $H^2 + c > 0$*, where* $c = 2c_2 - c_1/n$*. If its total umbilicity tensor* $\Phi$ *satisfies* (3.2) *for some* $1 \le p \le n/2$*, then*

(i) *either* $\sup_\Sigma |\Phi| = 0$ *and* $\Sigma^n$ *is a totally umbilical hypersurface,*
(ii) *or*

$$\sup_{\Sigma} |\Phi| \ge \alpha_{|H|,c,p} = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left(\sqrt{n^2H^2 + 4p(n-p)c} - (n-2p)|H|\right) > 0.$$

*Moreover, if the equality holds (in the case* $c > 0$*, assume in addition that* $H \neq 0$*), then* $\Sigma^n$ *is an isoparametric hypersurface with two distinct principal curvatures of multiplicities* $p$ *and* $n - p$.

*Proof* Firstly, we recall that every parabolic Riemannian manifold is stochastically complete. Then, by the first part of Theorem 2 we obtain that either $\sup_\Sigma |\Phi| = 0$ and $\Sigma^n$ is totally umbilical hypersurface, or $\sup_\Sigma |\Phi| \ge \alpha_{|H|,c,p}$. Moreover, if equality $\sup_\Sigma |\Phi| = \alpha_{|H|,c,p}$ holds, then as in the proof of Theorem 2 we have $P_{|H|,c,p}(|\Phi|) \le 0$ and $|\Phi|^2$ is a subharmonic function on $\Sigma^n$ which is bounded from above. Since $\Sigma^n$ is parabolic, it must be constant $|\Phi| = \alpha_{|H|,c,p}$. Therefore, at this point we can reason in a similar way to the proof of Theorem 2. □

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