Ann. Math. Québec (2019) 43:99–124
https://doi.org/10.1007/s40316-017-0094-x

Applications of Kronecker's limit formula for elliptic Eisenstein series

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Received: 12 June 2017 / Accepted: 16 October 2017 / Published online: 31 October 2017

Abstract We develop two applications of the Kronecker's limit formula associated to elliptic Eisenstein series: A factorization theorem for holomorphic modular forms, and a proof of Weil's reciprocity law. Several examples of the general factorization results are computed, specifically for certain moonshine groups, congruence subgroups, and, more generally, non-compact subgroups with one cusp. In particular, we explicitly compute the Kronecker limit function associated to certain elliptic fixed points for a few small level moonshine groups.

Keywords Eisenstein series · Kronecker limit formula · Modular forms

Résumé Dans cet article nous développons deux applications de la formule limite de Kronecker associée aux series d'Eisenstein elliptiques: Un théorème de factorisation pour des formes modulaires holomorphes et une preuve de la loi de réciprocité de Weil. Plusieurs exemples de notre résultat général de factorisation sont donnés, particulièrement pour quelques groupes de type moonshine, groupes de congruence et, plus généralement, pour des groupes

The first named author acknowledges grant support from the NSF and PSC-CUNY. We thank Professor Floyd Williams for making available to us the unpublished dissertation [30] which was written by his student I. N. Vassileva. The results in the manuscript were of great interest to us, and we hope the document will become available to the mathematical community.

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non-cocompactes à une seule pointe. En particulier, nous calculons la fonction limite de Kronecker associée à certains points elliptiques pour des groupes de type moonshine de petit niveau.

Mathematics Subject Classification 11M36 · 11F11

1 Introduction and statement of results

1.1 Non-holomorphic Eisenstein series.

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind which acts on the hyperbolic space $\mathbb{H}$ by fractional linear transformations, and let $M = \Gamma \backslash \mathbb{H}$ be the finite volume quotient. One can view M as a finite volume hyperbolic Riemann surface, possibly with cusps and elliptic fixed points. For convenience, we will use M to denote both the Riemann surface as well as a (Ford) fundamental domain for Γ acting on $\mathbb{H}$.

The abelian subgroups of Γ are classified in three distinct types: Parabolic, hyperbolic, and elliptic. Accordingly, there are three types of scalar-valued non-holomorphic Eisenstein series, whose definitions we now recall.

Parabolic subgroups are characterized by having a unique fixed point P on the boundary of the extended upper-half plane $\widehat{\mathbb{H}}$. The fixed point P is known as a cusp of M, and the associated parabolic subgroup is denoted by Γ_P. The parabolic Eisenstein series $\mathcal{E}_P^{\mathrm{par}}(z, s)$ associated to P is defined, for $z \in M$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, by the series

$$\mathcal{E}_P^{\mathrm{par}}(z, s) = \sum_{\eta \in \Gamma_P \backslash \Gamma} \mathrm{Im}(\sigma_P^{-1} \eta z)^s,$$

where σ_P is a *scaling matrix* for the cusp P, i.e. an element of $\mathrm{PSL}_2(\mathbb{R})$ such that $\sigma_P \infty = P$ and $\sigma_P^{-1} \Gamma_P \sigma_P = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$.

Hyperbolic subgroups have two fixed points on the extended upper-half plane $\widehat{\mathbb{H}}$. Let us denote a hyperbolic subgroup by Γ_γ for $\gamma \in \Gamma$, and let $\mathcal{L}_\gamma$ denote the geodesic path in $\mathbb{H}$ connecting the two fixed points of the hyperbolic element γ. Let σ_γ be a *scaling matrix* for $\mathcal{L}_\gamma$, i.e. an element of $\mathrm{PSL}_2(\mathbb{R})$ such that $\sigma_\gamma \mathcal{L} = \mathcal{L}_\gamma$, where $\mathcal{L}$ denotes the imaginary axis. Following Kudla and Millson from [22], one then defines the scalar-valued hyperbolic Eisenstein series, for $z \in M$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, by the series

$$\mathcal{E}_\gamma^{\mathrm{hyp}}(z, s) = \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \cosh(d_{\mathrm{hyp}}(\sigma_\gamma^{-1} \eta z, \mathcal{L}))^{-s}, \tag{1}$$

where $d_{\mathrm{hyp}}(\sigma_\gamma^{-1} \eta z, \mathcal{L})$ denotes the hyperbolic distance from the point $\sigma_\gamma^{-1} \eta z$ to the geodesic $\mathcal{L}$. Using the identity $\cosh(d_{\mathrm{hyp}}(w, \mathcal{L})) = |w| / \mathrm{Im}(w)$, one can rewrite the hyperbolic Eisenstein series in terms of the entries of γ.

The difference between (1) and the series $\Omega(s-1, z)$ defined in [22] is that our series is a scalar-valued hyperbolic Eisenstein series whereas the series $\Omega(s-1, z)$ is form-valued and scaled with the factor $\Gamma((s+1)/2) / \left(\sqrt{\pi}\, \Gamma(s/2)\right)$. The hyperbolic Eisenstein series is also similar to the form-valued series which appears at the bottom of p. 189 of [7].

Elliptic subgroups have finite order and have a unique fixed point within $\mathbb{H}$. In fact, for any point $w \in M$, there is an elliptic subgroup Γ_w which fixes w, where in all but finitely many cases Γ_w is reduced to the identity element. Elliptic Eisenstein series were defined in an unpublished manuscript from 2004 by Jorgenson and Kramer and were studied in depth in the dissertation [24] by von Pippich and in [25]. Specifically, for $z \in M$, $z \neq w$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the elliptic Eisenstein series is defined by the series

$$\mathcal{E}_w^{\mathrm{ell}}(z,s) = \sum_{\eta \in \Gamma_w \backslash \Gamma} \sinh(d_{\mathrm{hyp}}(\eta z, w))^{-s} = \sum_{\eta \in \Gamma_w \backslash \Gamma} \left(\frac{2\,\mathrm{Im}(w)\mathrm{Im}(\eta z)}{|\eta z - w|\,|\eta z - \overline{w}|} \right)^s, \tag{2}$$

where $d_{\mathrm{hyp}}(\eta z, w)$ denotes the hyperbolic distance from ηz to w. It is immediate from (2) that, for $z, w \in M$ with $z \neq w$, one has the relation $\mathrm{ord}(w)\mathcal{E}_w^{\mathrm{ell}}(z,s) = \mathrm{ord}(z)\mathcal{E}_z^{\mathrm{ell}}(w,s)$, where $\mathrm{ord}(w)$ resp. $\mathrm{ord}(z)$ denotes the order of Γ_w and Γ_z, respectively.

1.2 Known properties and relations

There are some fundamental differences between the three types of Eisenstein series defined above. Hyperbolic Eisenstein series are in $L^2(M)$, whereas parabolic and elliptic series are not. Elliptic Eisenstein series are defined as a sum over cosets of Γ by a finite subgroup of Γ, and indeed the series (2) can be extended to all Γ which would introduce a multiplicative factor equal to the order of Γ_w. However, hyperbolic and parabolic series are necessarily formed by sums over cosets of Γ by an infinite subgroup of Γ. Parabolic Eisenstein series are eigenfunctions of the hyperbolic Laplacian Δ_{hyp}; however, hyperbolic and elliptic Eisenstein series satisfy a differential-difference equation which involves the value of the series at $s+2$. Specifically, if $\mathrm{Re}(s) > 1$, then one can differentiate term-by-term and prove that

$$(\Delta_{\mathrm{hyp}} - s(1-s))\mathcal{E}_\gamma^{\mathrm{hyp}}(z,s) = s^2\mathcal{E}_\gamma^{\mathrm{hyp}}(z,s+2)$$

and

$$(\Delta_{\mathrm{hyp}} - s(1-s))\mathcal{E}_w^{\mathrm{ell}}(z,s) = -s^2\mathcal{E}_w^{\mathrm{ell}}(z,s+2). \tag{3}$$

Despite their differences, there are several intriguing ways in which these Eisenstein series interact. Since the hyperbolic Eisenstein series are in $L^2(M)$, the expression (1) admits a spectral expansion which involves the parabolic Eisenstein series; see [15] and [22]. If one considers a degenerating sequence of Riemann surfaces obtained by pinching a geodesic, then the associated hyperbolic Eisenstein series converges to a parabolic Eisenstein series on the limit surface; see [4] and [8]. If one studies a family of elliptically degenerating surfaces obtained by re-uniformizing at a point with increasing order, then the corresponding elliptic Eisenstein series converges to a parabolic Eisenstein series on the limit surface; see [9].

Finally, there are some basic similarities amongst the series. Each series admits a meromorphic continuation to all $s \in \mathbb{C}$. The poles of the meromorphic continuations have been identified and are closely related, in all cases involving data associated to the continuous and non-cuspidal discrete spectrum of the hyperbolic Laplacian and, for hyperbolic and elliptic series, involving data associated to the cuspidal spectrum as well; see [15] and [25]. Finally, and most importantly for this article, for all known instances, the parabolic Eisenstein series is holomorphic at $s = 0$, which implies that the hyperbolic and elliptic Eisenstein series are holomorphic at $s = 0$ as well. In all of these cases, the value of each Eisenstein series at $s = 0$ is independent of z. The coefficient of s in the Taylor series expansion about $s = 0$ of each of these Eisenstein series, which is a function of z, shall be called the *Kronecker limit function.*

1.3 Kronecker limit functions

Recall that Dedekind's delta function $\Delta(z)$ is defined by

$$\Delta(z) = \left[q_z^{1/24} \prod_{n=1}^{\infty} \left(1 - q_z^n\right) \right]^{24} = \eta(z)^{24}$$

with $q_z = e^{2\pi i z}$ and $\eta(z)$ denoting the classical eta function. With this, the classical Kronecker's limit formula for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ reads as (see [28], Theorem 1, p. 14, with $\zeta_Q(s) = 2\zeta(2s)\mathcal{E}_\infty^{\mathrm{par}}(z, s)$)

$$\mathcal{E}_\infty^{\mathrm{par}}(z, s) = \frac{3}{\pi(s-1)} - \frac{1}{2\pi} \log\bigl(|\Delta(z)| \, \mathrm{Im}(z)^6\bigr) + C + O(s-1), \quad \text{as } s \to 1,$$

where $C := 6(1 - 12\,\zeta'(-1) - \log(4\pi))/\pi$; see also [19], where a detailed proof is provided.

By employing the well-known functional equation for $\mathcal{E}_\infty^{\mathrm{par}}(z, s)$, the Kronecker's limit formula can be reformulated as

$$\mathcal{E}_\infty^{\mathrm{par}}(z, s) = 1 + \log\bigl(|\Delta(z)|^{1/6} \, \mathrm{Im}(z)\bigr)s + O(s^2), \quad \text{as } s \to 0.$$

For general Fuchsian groups of the first kind, Goldstein [10] studied analogues of the Kronecker's limit formula associated to parabolic Eisenstein series. We shall use the results from [10] throughout this article.

The hyperbolic Eisenstein series in [22] are form-valued, and the series are defined by an infinite sum which converges for $\mathrm{Re}(s) > 0$. The main result in [22] is that the one-form valued hyperbolic Eisenstein series is holomorphic at $s = 0$, and its value at $s = 0$ is equal to the harmonic form which is Poincaré dual to the geodesic determined by the two fixed points of γ.

The analogue of the Kronecker's limit formula for the elliptic Eisenstein series was first proved in [24] and [25]. Specifically, it is shown that for any $w \in \mathbb{H}$, at $s = 0$, the series (2) admits the Laurent expansion

$$\begin{aligned} \mathrm{ord}(w)\, \mathcal{E}_w^{\mathrm{ell}}(z, s) &- \frac{2^s \sqrt{\pi}\, \Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{k=1}^{p_\Gamma} \mathcal{E}_{P_k}^{\mathrm{par}}(w, 1-s)\, \mathcal{E}_{P_k}^{\mathrm{par}}(z, s) \\ &= -c - \log\bigl(|H_\Gamma(z, w)|^{\mathrm{ord}(w)} (\mathrm{Im}(z))^c\bigr) \cdot s + O(s^2), \quad \text{as } s \to 0, \end{aligned} \tag{4}$$

where P_k, $k = 1, \ldots, p_\Gamma$, are the cusps of M and $c = 2\pi / \mathrm{vol}_{\mathrm{hyp}}(M)$. Let us write the coefficient of s in (4) as

$$\log\bigl(|H_\Gamma(z, w)|^{\mathrm{ord}(w)} (\mathrm{Im}(z))^c\bigr) = \log\bigl(|\tilde{H}_\Gamma(z, w)| (\mathrm{Im}(z))^c (\mathrm{Im}(w))^c\bigr), \tag{5}$$

where we have set $\tilde{H}_\Gamma(z, w) = H_\Gamma(z, w)^{\mathrm{ord}(w)} / \mathrm{Im}(w)^c$. As a function of z, $H_\Gamma(z, w)$, hence $\tilde{H}_\Gamma(z, w)$, is locally holomorphic on $\mathbb{H}$. Furthermore, $H_\Gamma(z, w)$ is uniquely determined up to multiplication by a complex constant of absolute value one. In addition, $H_\Gamma(z, w)$ is an automorphic form with a non-trivial multiplier system, which depends on w, with respect to Γ acting on z. The function $H_\Gamma(z, w)$ vanishes if and only if $z = \eta w$ for some $\eta \in \Gamma$. Many properties of $\tilde{H}_\Gamma(z, w)$ can be derived from properties of $H_\Gamma(z, w)$. In particular, since

$$\mathrm{ord}(w)\, \mathcal{E}_w^{\mathrm{ell}}(z, s) = \mathrm{ord}(z)\, \mathcal{E}_z^{\mathrm{ell}}(w, s),$$

one can combine Corollary 7.4 of [25] together with the properties of the automorphic Green's function (see, e.g., [13]) to conclude that $\tilde{H}_\Gamma(z, w)$ is locally holomorphic in the variable w.

Two explicit computations are given in [24] and [25] for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ when considering the elliptic Eisenstein series $E_w^{\mathrm{ell}}(z, s)$ associated to the points $w = i$ and $w = \rho = (1 +$

$i\sqrt{3})/2$. In these cases, the elliptic Kronecker limit function $H_\Gamma(z, w)$ at points $w = i$ and $w = \rho$ is such that

$$|H_\Gamma(z,i)| = \exp(-B_i)\,|E_6(z)|\,, \quad \text{where } B_i = -3(24\zeta'(-1) - \log(2\pi) + 4\log\Gamma(1/4)), \tag{6}$$

$$|H_\Gamma(z,\rho)| = \exp(-B_\rho)\,|E_4(z)|\,, \quad \text{where } B_\rho = -2(24\zeta'(-1) - 2\log(2\pi/\sqrt{3}) + 6\log\Gamma(1/3)). \tag{7}$$

The Kronecker's limit formula for each elliptic Eisenstein series is given by the asymptotic formulas

$$\mathcal{E}_i^{\mathrm{ell}}(z,s) = -\log(|E_6(z)||\Delta(z)|^{-1/2})\cdot s + O(s^2), \quad \text{as } s \to 0, \tag{8}$$

and

$$\mathcal{E}_\rho^{\mathrm{ell}}(z,s) = -\log(|E_4(z)||\Delta(z)|^{-1/3})\cdot s + O(s^2), \quad \text{as } s \to 0, \tag{9}$$

where E_4 and E_6 are the classical holomorphic Eisenstein series for $\mathrm{PSL}_2(\mathbb{Z})$ of weight four and six, respectively. More generally, in [25], it is shown that

$$\mathrm{ord}(w)\mathcal{E}_w^{\mathrm{ell}}(z,s) = -\log\big(|j(z) - j(w)|\big)\cdot s + O(s^2), \quad \text{as } s \to 0, \tag{10}$$

where $j(z)$ is the classical j-invariant. Equations (3) and (10) are reminiscent of relations involving the well-studied automorphic Green's function; see, in particular, equations (5.2) and (5.3) of [13]. Indeed, Corollary 7.4 of [25] states that, in a sense, a first order approximation to the elliptic Eisenstein series at $s = 0$ is given by a multiple of the automorphic Green's function. We refer the interested reader to Sect. 7 of [25] for a detailed development of various identities and asymptotic formulas.

Before continuing, let us state what we believe to be an interesting side comment. As we show below, one can realize the Kronecker limit function for parabolic Eisenstein series for groups with one cusp as the coefficient of s in the Laurent expansion of the parabolic Eisenstein series at $s = 0$. One has yet to study the Laurent expansion near $s = 0$, in particular the coefficient of s, for the scalar-valued hyperbolic Eisenstein series; for that matter, we have not fully understood the analogous question for the vector of parabolic Eisenstein series for general groups. We expect that one can develop a systematic theory by focusing on coefficients of s in all cases.

1.4 Important comment and assumption

At the present moment, we do not have a complete understanding of the behavior of the parabolic Eisenstein series $\mathcal{E}_P^{\mathrm{par}}(z,s)$ near $s = 0$. If the group has one cusp, the functional equation of the Eisenstein series implies that $\mathcal{E}_P^{\mathrm{par}}(z,0) = 1$. In the notation to be set below, the evaluation that $\mathcal{E}_P^{\mathrm{par}}(z,0) = 1$ is equivalent to the statement that its scattering determinant $\det(\Phi_M(s))$ is zero at $s = 0$. However, this equivalence could fail to be true when there is more than one cusp. For example, on p. 536 of [11], the author computes the scattering matrix for $\Gamma_0(N)$ for square-free N, from which it is clear that $\det(\Phi_M(s))$ is holomorphic but not zero at $s = 0$. Specifically, it remains to determine if the parabolic Eisenstein series is holomorphic at $s = 0$, which is a question we were unable to answer in complete generality. For that reason, we shall work throughout this article under the following assumption.

Assumption A For every cusp P of M, the parabolic Eisenstein series $\mathcal{E}_P^{\mathrm{par}}(z,s)$ is holomorphic at $s = 0$.

Assumption A is true in all the instances where specific examples are developed.

1.5 Main results

The purpose of the present paper is to further study the Kronecker limit function associated to elliptic Eisenstein series. We develop two applications. To begin, we examine the relation (4) and study the contribution near $s = 0$ of the term involving the parabolic Eisenstein series. As with the parabolic Eisenstein series, the resulting expression is particularly simple in the case when the group Γ has one cusp. However, in all cases, we obtain an asymptotic formula for $\mathcal{E}^{\text{ell}}_w(z, s)$ near $s = 0$ which allows us to prove asymptotic bounds for the elliptic Kronecker limit function at any cusp of M. From this analysis, we obtain the main result of this article, namely a factorization theorem which expresses holomorphic forms on M of arbitrary weight as products of the elliptic Kronecker limit functions.

The product formulas are developed in detail in the case of the so-called moonshine groups, which are discrete groups obtained by adding the Fricke and Atkin-Lehner involutions to the congruence subgroups $\Gamma_0(N)$. As an application of the factorization theorem, we establish further examples of relations similar to (6), (7), (8), and (9). For example, the moonshine group $\Gamma = \overline{\Gamma_0(2)^+} = \Gamma_0(2)^+/\{\pm\text{Id}\}$ has $e_2 = 1/2 + i/2$ as an elliptic fixed point of order four. In Sect. 6.2, we prove that the elliptic Kronecker limit function $H_2(z, e_2)$ associated to the point e_2 is such that

$$|H_2(z, e_2)| = \exp(-B_{2,e_2})|E_4^{(2)}(z)|^{1/2},$$

where $E_4^{(2)}(z)$ is the weight four holomorphic Eisenstein series associated to $\Gamma_0(2)^+$ and

$$B_{2,e_2} = -\left(24\zeta'(-1) + \log(8\pi^2) - \frac{11}{6}\log 2 + \frac{1}{12}\log\left(|\Delta(1/2 + i/2)\cdot\Delta(1+i)|\right)\right).$$

In this case, the Kronecker's limit formula for the elliptic Eisenstein series $\mathcal{E}^{\text{ell}}_{e_2}(z, s)$ reads as

$$\mathcal{E}^{\text{ell}}_{e_2}(z, s) = -\log\left(|E_4^{(2)}(z)|^{1/2}|\Delta(z)\Delta(2z)|^{-1/12}\right)\cdot s + O(s^2), \quad \text{as } s \to 0,$$

or, equivalently, as

$$\mathcal{E}^{\text{ell}}_{e_2}(z, s) = -\log\left(\frac{1}{\sqrt{5}}|E_4(z) + 4E_4(2z)|^{1/2}|\Delta(z)\Delta(2z)|^{-1/12}\right)\cdot s + O(s^2), \quad \text{as } s \to 0. \tag{11}$$

The factorization theorem (Theorem 9 below) allows one to formulate numerous examples of this type, of which we develop a few for certain moonshine and congruence subgroups.

Second, we use the elliptic Kronecker's limit formula to give a new proof of Weil's reciprocity formula. A number of authors have obtained generalizations of Weil's reciprocity law; see, for example, the elegant presentation in [21] which discusses various reciprocity laws over $\mathbb{C}$ as well as Deligne's article [3] where the author re-interprets Tate's local symbol and obtains a number of generalizations and applications. It would be interesting to study the possible connection between the functional analytic method of the article [17] with the algebraic ideas in [3] and results surveyed in [21].

Let us finish this introduction by giving an outline of the paper. In Sect. 2, we establish notation and recall various known results. In Sect. 3, we reformulate the Kronecker's limit formula for parabolic Eisenstein series as an asymptotic statement near $s = 0$. From the results in Sect.3, we then prove, in Sect. 4, the asymptotic behavior of the elliptic Kronecker limit function at each cusp of M. Specific examples are given for moonshine groups $\overline{\Gamma_0(N)^+}$ with square-free level N and congruence subgroups $\overline{\Gamma_0(p)}$ with prime level p. In Sect. 5 we prove the factorization theorem which states, in somewhat vague terms, that any holomorphic form on M can be written as a product of elliptic Kronecker limit functions, up to a multiplicative constant. In addition, from the asymptotic formula from Sect. 4, one is able to obtain specific information associated to the multiplicative constant in the aforementioned description of the factorization theorem. In Sect. 6 we give examples of the factorization theorem for holomorphic Eisenstein series for the modular group, for moonshine groups of levels 2 and 5, for general moonshine groups, and for congruence subgroups $\overline{\Gamma_0(p)}$ of prime level. Finally, in Sect. 7, we present our proof of Weil's reciprocity using the elliptic Kronecker limit functions and state a few concluding remarks.

2 Background material

2.1 Basic notation

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ denote a Fuchsian group of the first kind acting by fractional linear transformations on the hyperbolic upper half-plane $\mathbb{H} := \{z = x+iy \in \mathbb{C} \mid x, y \in \mathbb{R};\ y > 0\}$. We let $M := \Gamma\backslash\mathbb{H}$, which is a finite volume hyperbolic Riemann surface, and denote by $p : \mathbb{H} \longrightarrow M$ the natural projection. We assume that M has e_Γ elliptic fixed points and p_Γ cusps. We identify M locally with its universal cover $\mathbb{H}$.

We let μ_{hyp} denote the hyperbolic metric on M, which is compatible with the complex structure of M, and has constant negative curvature equal to minus one. The hyperbolic line element ds^2_{hyp}, resp. the hyperbolic Laplacian Δ_{hyp} acting on functions, are given as

$$ds^2_{\mathrm{hyp}} := \frac{dx^2 + dy^2}{y^2}, \quad \text{resp.} \quad \Delta_{\mathrm{hyp}} := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

By $d_{\mathrm{hyp}}(z, w)$ we denote the hyperbolic distance between the two points $z, w \in \mathbb{H}$.

2.2 Moonshine groups

Let $N = p_1 \cdot \ldots \cdot p_r$ be a square-free, non-negative integer. The subset of $\mathrm{SL}_2(\mathbb{R})$, defined by

$$\Gamma_0(N)^+ := \left\{ e^{-1/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) :\ ad - bc = e,\ \ a, b, c, d, e \in \mathbb{Z},\ \ e \mid N,\ e \mid a,\ e \mid d,\ N \mid c \right\}$$

is an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{R})$. We use the terminology "moonshine group" of level N to describe $\Gamma_0(N)^+$ because of the important role these groups play in "monstrous moonshine". Previously, the groups $\Gamma_0(N)^+$ were studied in [12] where it was proved that if a subgroup $G \subset \mathrm{SL}_2(\mathbb{R})$ is commensurable with $\mathrm{SL}_2(\mathbb{Z})$, then there exists a square-free, non-negative integer N such that G is a subgroup of $\Gamma_0(N)^+$. We also refer to p. 27 of [27] where the groups $\Gamma_0(N)^+$ are cited as examples of groups which are commensurable with $\mathrm{SL}_2(\mathbb{Z})$ but not necessarily conjugate to a subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Let $\{\pm\mathrm{Id}\}$ denote the set of two elements, where Id is the identity matrix. In general, if Γ is a subgroup of $\mathrm{SL}_2(\mathbb{R})$, we let $\overline{\Gamma} := \Gamma/\{\pm\mathrm{Id}\}$ denote its projection into $\mathrm{PSL}_2(\mathbb{R})$.

With this notation, we introduce the quotient space $Y_N^+ := \overline{\Gamma_0^+(N)}\backslash\mathbb{H}$. The topological features of Y_N^+ in terms of the integer N are developed in detail in [2]. In particular, for any square-free N, Y_N^+ has one cusp. Generally speaking, the spaces Y_N^+ serve as interesting examples for general considerations, as developed in [18] in the study of Weyl's law, in [19] in the study of q-expansions of holomorphic modular functions, and in [20] in the study of the ring of holomorphic modular forms when Y_N^+ has genus zero. In the present paper, the groups $\Gamma_0(N)^+$ will yield explicit and interesting examples of Kronecker limit functions; see Sect. 6.

2.3 Holomorphic Eisenstein series

Following [26], we define a weakly modular form f of weight $2k$ for $k \geq 1$ associated to Γ to be a function f which is meromorphic on $\mathbb{H}$ and satisfies the transformation property

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z), \quad \text{for any} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Let Γ be a Fuchsian group of the first kind that has at least one class of parabolic elements. By conjugating, if necessary, we may always assume that the group Γ has a subgroup isomorphic to $\mathbb{Z}$ with ∞ as a fixed point and scaling matrix equal to the identity. In this situation, any weakly modular form f will satisfy the relation $f(z+1) = f(z)$, so we can write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q_z^n, \quad \text{where } q_z = e(z) = e^{2\pi i z}.$$

If $a_n = 0$ for all $n < 0$, then f is said to be holomorphic at the cusp at ∞.

A holomorphic modular form with respect to Γ is a weakly modular form which is holomorphic on $\mathbb{H}$ and at all the cusps of Γ. Examples of holomorphic modular forms are the holomorphic Eisenstein series, which are defined as follows. Let Γ_∞ denote the subgroup of Γ which stabilizes the cusp at ∞. For $k \geq 2$, let

$$E_{2k,\Gamma}(z) := \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} (cz+d)^{-2k}. \tag{12}$$

It is elementary to show that the series on the right-hand side of (12) is absolutely convergent for all integers $k \geq 2$ and defines a holomorphic modular form of weight $2k$ with respect to Γ. Furthermore, the series $E_{2k,\Gamma}$ is bounded and non-vanishing at the cusps and such that

$$E_{2k,\Gamma}(z) = 1 + O(\exp(-2\pi \operatorname{Im}(z))), \quad \text{as } \operatorname{Im}(z) \to \infty.$$

When $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, we denote $E_{2k,\mathrm{PSL}_2(\mathbb{Z})}$ by E_{2k}. The holomorphic forms $E_{2k}(z)$ have the q-expansions

$$E_{2k}(z) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q_z^n,$$

where B_{2k} denotes the $2k$-th Bernoulli number and σ_l is the generalized divisor function, which is defined by $\sigma_l(m) = \sum_{d|m} d^l$. By convention, we set $\sigma(m) = \sigma_1(m)$.

On the full modular surface, there is no weight two holomorphic modular form. Consider, however, the function $E_2(z)$ defined by its q-expansion

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q_z^n,$$

which transforms according to the formula

$$E_2(\gamma z) = (cz+d)^2 E_2(z) + \frac{6}{\pi i} c(cz+d),$$

for $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$. It is elementary to show that for a prime p, the function

$$E_{2,p}(z) := E_2(z) - pE_2(pz) \tag{13}$$

is a weight two holomorphic form associated to the congruence subgroup $\overline{\Gamma_0(p)}$ of $\mathrm{PSL}_2(\mathbb{Z})$. The q-expansion of $E_{2,p}$ is given by

$$E_{2,p}(z) = (1-p) - 24\sum_{n=1}^{\infty} \sigma(n)(q_z^n - pq_z^{pn}). \tag{14}$$

When $\Gamma = \overline{\Gamma_0^+(N)}$, we denote the forms $E_{2k,\overline{\Gamma_0^+(N)}}$ by $E_{2k}^{(N)}$. In [19] it is proved that $E_{2k}^{(N)}(z)$ may be expressed as a linear combination of forms $E_{2k}(z)$, with dilated arguments, namely

$$E_{2k}^{(N)}(z) = \frac{1}{\sigma_k(N)} \sum_{v|N} v^k E_{2k}(vz), \tag{15}$$

where the sum is taken over all positive divisors of N.

2.4 Scattering matrices

Assume that the surface M has p_Γ cusps, we let P_j with $j = 1, \ldots, p_\Gamma$ denote the individual cusps. Denote by $\phi_{jk}(s)$, with $j, k = 1, \ldots, p_\Gamma$, the entries of the hyperbolic scattering matrix $\Phi_M(s)$ which are computed from the constant terms in the Fourier expansion of the parabolic Eisenstein series $\mathcal{E}_{P_j}^{\mathrm{par}}(z, s)$ around the cusp P_k. For all $j, k = 1, \ldots, p_\Gamma$, each function ϕ_{jk} has a simple pole at $s = 1$ with residue equal to $1/\operatorname{vol}_{\mathrm{hyp}}(M)$. Furthermore, ϕ_{jk} has a Laurent series expansion at $s = 1$ which we write as

$$\phi_{jk}(s) = \frac{1}{\operatorname{vol}_{\mathrm{hyp}}(M)(s-1)} + \beta_{jk} + \gamma_{jk}(s-1) + O((s-1)^2), \quad \text{as } s \to 1. \tag{16}$$

After a slight renormalization and a trivial generalization, Theorem 3-1 from [10] asserts that the parabolic Eisenstein series $\mathcal{E}_{P_j}^{\mathrm{par}}(z, s)$ admits the Laurent expansion

$$\begin{aligned}\mathcal{E}_{P_j}^{\mathrm{par}}(z, s) = {} & \frac{1}{\operatorname{vol}_{\mathrm{hyp}}(M)(s-1)} + \beta_{jj} - \frac{1}{\operatorname{vol}_{\mathrm{hyp}}(M)} \log|\eta_{P_j}^4(z)\operatorname{Im}(z)| + f_j(z)(s-1) \\ & + O((s-1)^2),\end{aligned} \tag{17}$$

as $s \to 1$, for $j = 1, \ldots, p_\Gamma$.

As the notation suggests, the function $\eta_{P_j}(z)$ is a holomorphic form for Γ and is a generalization of the eta function $\eta(z)$ for the full modular group. To be precise, $\eta_{P_j}(z)$ is an automorphic form corresponding to the multiplier system $v(\sigma) = \exp(i\pi S_{\Gamma,j}(\sigma))$, where $S_{\Gamma,j}(\sigma)$ is a generalization of a Dedekind sum attached to the cusp P_j for each $j = 1, \ldots, p_\Gamma$ of M, i.e. $S_{\Gamma,j}(\sigma)$ is a real number depending on $\sigma \in \Gamma$ which satisfies the relation

$$\log \eta_{P_j}(\sigma(z)) = \log \eta_{P_j}(z) + \frac{1}{2}\log(cz+d) + \pi i S_{\Gamma,j}(\sigma).$$

The coefficient $f_j(z)$ multiplying $(s-1)$ in formula (17) is a certain function, whose behavior is not relevant for this paper. This term would probably yield a definition of generalized Dedekind sums; see, for example, [29].

The main focus of this paper is on the Kronecker limit functions corresponding to the Taylor series expansion at $s = 0$; hence, we set the following notation

$$\phi_{jk}(s) = a_{jk} + b_{jk}s + c_{jk}s^2 + O(s^3), \quad \text{as } s \to 0 \tag{18}$$

for the coefficients in the Laurent expansion of ϕ_{jk} at $s = 0$. Note that the form of this expansion is justified by Assumption A made in Sect. 1.4.

3 Kronecker's limit formula for parabolic Eisenstein series

In this section we will re-write the Kronecker's limit formula for the parabolic Eisenstein series as an expression involving the Laurent expansion at $s = 0$. We begin with the following lemma which states certain relations amongst coefficients appearing in (16) and (18).

Let us recall that throughout the paper we assume that Assumption A holds true.

Lemma 1 *With notation as in* (16) *and* (18)*, we have, for each* $k, l = 1, \ldots, p_\Gamma$*, the following relations:*

$$\sum_{j=1}^{p_\Gamma} a_{jk} = 0, \tag{19}$$

$$\sum_{j=1}^{p_\Gamma} \left(-\frac{b_{jk}}{\mathrm{vol}_{\mathrm{hyp}}(M)} + a_{jk}\beta_{jl} \right) = \delta_{kl}, \tag{20}$$

$$\sum_{j=1}^{p_\Gamma} \left(-\frac{c_{jk}}{\mathrm{vol}_{\mathrm{hyp}}(M)} + b_{jk}\beta_{jl} \right) = \sum_{j=1}^{p_\Gamma} a_{jk}\gamma_{jl}, \tag{21}$$

where δ_{kl} *is the Kronecker symbol.*

Proof The relations (19) through (21) are immediate consequences of the functional equation for the scattering matrix, namely the formula $\Phi_M(s)\Phi_M(1-s) = \mathrm{Id}$; see, e.g., [13], Theorem 6.6. In particular, the formulae are obtained by computing the coefficients of s^{-1}, 1, and s in the Laurent expansion at $s = 0$. □

Proposition 2 *With the notation introduced in* (16) *and* (18)*, the parabolic Eisenstein series* $\mathcal{E}^{\mathrm{par}}_{P_j}(z, s)$ *has a Taylor series expansion at* $s = 0$ *which can be written as*

$$\begin{aligned}\mathcal{E}^{\mathrm{par}}_{P_j}(z, s) = &\sum_{k=1}^{p_\Gamma} \left[-\frac{b_{jk}}{\mathrm{vol}_{\mathrm{hyp}}(M)} + a_{jk}\left(\beta_{kk} - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M)} \log \left| \eta^4_{P_k}(z)\,\mathrm{Im}(z) \right| \right) \right] \\ &+ s \cdot \sum_{k=1}^{p_\Gamma} \left[-\frac{c_{jk}}{\mathrm{vol}_{\mathrm{hyp}}(M)} + b_{jk}\left(\beta_{kk} - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M)} \log \left| \eta^4_{P_k}(z)\,\mathrm{Im}(z) \right| \right) \right. \\ &\left. + a_{jk} f_k(z) \right] + O(s^2).\end{aligned} \tag{22}$$

Proof The result is a straightforward computation based on the functional equation

$$(\mathcal{E}^{\text{par}}_{P_1}(z,s)\ \ldots\ \mathcal{E}^{\text{par}}_{P_{p_\Gamma}}(z,s))^T = \Phi_M(s)(\mathcal{E}^{\text{par}}_{P_1}(z,1-s)\ \ldots\ \mathcal{E}^{\text{par}}_{P_{p_\Gamma}}(z,1-s))^T,$$

see, e.g., [13], Theorem 6.5., together with the expansions (17) and (18). □

In the case when $p_\Gamma = 1$, the relations (19) through (21) and Proposition 2 become particularly simple and yield an elegant statement. As is standard, the cusp is normalized to be at ∞, and the associated Eisenstein series, eta function, scattering coefficients, etc. are written with the subscript ∞.

Corollary 3 *The Kronecker's limit formula for parabolic Eisenstein series $\mathcal{E}^{\text{par}}_\infty(z,s)$ on a finite volume Riemann surface with one cusp at ∞ can be written as*

$$\mathcal{E}^{\text{par}}_\infty(z,s) = 1 + \log(|\eta^4_\infty(z)|\operatorname{Im}(z)) \cdot s + O(s^2), \quad \text{as } s \to 0. \tag{23}$$

Example 4 In the case when $\Gamma = \overline{\Gamma^+_0(N)}$, where N is a square-free, positive integer with r prime factors , the quotient space Y^+_N has one cusp. The automorphic form η_∞ is explicitly computed in [19], where it is proved that

$$\eta_\infty(z) = \sqrt[2^r]{\prod_{v|N}\eta(vz)},$$

where the product is taken over positive divisors of N.

Example 5 In the case when Γ is the congruence group $\overline{\Gamma_0(N)}$, for a positive integer N, the corresponding quotient space $M_N := \overline{\Gamma_0(N)}\backslash\mathbb{H}$ has many cusps. Using a standard fundamental domain, M_N has cusps at ∞, at 0 and, in the case when N is not prime, at the rational points $1/v$, where $v \mid N$ is such that $(v, \frac{N}{v}) = 1$, where $(\cdot, \cdot)$ stands for the greatest common divisor. As in the above example, let us use the subscript ∞ to denote data associated to the cusp at ∞. In particular, the automorphic form η_∞ in the example under consideration was explicitly computed in [30], where it is proved that

$$\eta_\infty(z) = \sqrt[\varphi(N)]{\prod_{v|N}\eta(vz)^{v\mu(N/v)}},$$

where $\varphi(N)$ is the Euler $\varphi-$function and μ denotes the Möbius function. In the case of other cusps P_k, the automorphic form η_{P_k} was also computed in [30], but the expressions are more involved so we omit giving the formulas here.

Also, for the cusp at ∞ and the principal congruence subgroup $\Gamma(N)$, the eta-function is computed in Theorem 1, p. 405, of [29].

4 Kronecker's limit formula for elliptic Eisenstein series

The function $H_\Gamma(z,w)$, defined in (4) is called the *elliptic Kronecker limit function at* w. It satisfies the transformation rule

$$H_\Gamma(\gamma z, w) = \varepsilon_w(\gamma)(cz+d)^{2C_w} H_\Gamma(z,w), \quad \text{for any } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma, \tag{24}$$

where $\varepsilon_w(\gamma) \in \mathbb{C}$ is a constant of absolute value 1, independent of z and

$$C_w = 2\pi/(\operatorname{ord}(w)\operatorname{vol}_{\text{hyp}}(M)), \tag{25}$$

see [24], Proposition 6.1.2., or [25]. Since $H_\Gamma(z, w)$, as a function of z, is finite and non-zero at the cusp $P_1 = \infty$, we decide to re-scale so that $H_\Gamma(z, w)$ is real at the cusp ∞.

We begin by studying the asymptotic behavior of $H_\Gamma(\sigma_{P_\ell} z, w)$ as $y = \mathrm{Im}(z) \to \infty$, for $\ell = 1, \ldots, p_\Gamma$.

Proposition 6 *For any cusp P_ℓ, with $\ell = 1, \ldots, p_\Gamma$, let*

$$B_{w,P_\ell} = -C_w \left(2 - \log 2 + \log \left|\eta^4_{P_\ell}(w)\,\mathrm{Im}(w)\right| - \beta_{\ell\ell}\,\mathrm{vol}_{\mathrm{hyp}}(M)\right). \tag{26}$$

Then there exists a constant $a_{w,P_\ell} \in \mathbb{C}$ of modulus one such that

$$H(\sigma_{P_\ell} z, w) = a_{w,P_\ell} \exp(-B_{w,P_\ell})|c_\ell z + d_\ell|^{2C_w} + O(\exp(-2\pi\,\mathrm{Im}(z))), \quad \textit{as } \mathrm{Im}(z) \to \infty,$$

where $\sigma_{P_\ell} = \left(\begin{smallmatrix} * & * \\ c_\ell & d_\ell \end{smallmatrix}\right)$ is a scaling matrix for the cusp P_ℓ and C_w is defined by (25).

Proof The proof closely follows the proof of [24], Proposition 6.2.2. when combined with the Taylor series expansion (22) of the parabolic Eisenstein series at $s = 0$. For the convenience of the reader, we present the complete argument.

Combining the Eq. (4) with the proof of Proposition 6.1.1 from [24], taking $e_j = w$, we can write

$$-\log(|H_\Gamma(z, w)|\,\mathrm{Im}(z)^{C_w}) = \mathcal{K}_w(z),$$

where the function $\mathcal{K}_w(z)$ can be expressed as the sum of two terms: A term $\mathcal{F}_w(z)$ arising from the spectral expansion and a term $\mathcal{G}_w(z)$ which can be expressed as the sum over the group. Furthermore, for $z \in \mathbb{H}$ such that $\mathrm{Im}\, z > \mathrm{Im}(\gamma w)$ for all $\gamma \in \Gamma$ the parabolic Fourier expansion of $\mathcal{K}_w(\sigma_{P_\ell} z)$ is given by

$$\mathcal{K}_w(\sigma_{P_\ell} z) = \sum_{m \in \mathbb{Z}} b_{m,w,P_\ell}(y) e(mx)$$

with coefficients $b_{m,w,P_\ell}(y)$ given by

$$b_{m,w,P_\ell}(y) = \int_0^1 \mathcal{K}_w(\sigma_{P_\ell} z) e(-mx)\,dx.$$

Since the hyperbolic Laplacian is SL_2-invariant, we easily generalize computations from p. 128 of [24] to deduce that

$$\mathcal{K}_w(\sigma_{P_\ell} z) = -C_w \log y + A_{w,P_\ell} y + B_{w,P_\ell} + \sum_{m=1}^{\infty} (A_{m;w,P_\ell} e(mz) + \overline{A}_{m;w,P_\ell} e(-m\bar{z})),$$

for some constants $A_{w,P_\ell}, B_{w,P_\ell} \in \mathbb{R}$ and complex constants $A_{m;w,P_\ell}$.

Let us introduce the notation

$$f_{w,P_\ell}(z) := \exp\left(-2 \sum_{m=1}^{\infty} A_{m;w,P_\ell} e(mz)\right), \tag{27}$$

from which one immediately can write

$$\mathcal{K}_w(\sigma_{P_\ell} z) = A_{w,P_\ell} y + B_{w,P_\ell} - \log(\left|f_{w,P_\ell}(z)\right| \mathrm{Im}(z)^{C_w}). \tag{28}$$

When employing (28), we can re-write (4) as

$$\mathcal{E}^{\mathrm{ell}}_{w}(\sigma_{P_\ell}z,s)-h_w(s)\sum_{j=1}^{p_\Gamma}\mathcal{E}^{\mathrm{par}}_{P_j}(w,1-s)\mathcal{E}^{\mathrm{par}}_{P_j}(\sigma_{P_\ell}z,s) \tag{29}$$
$$=-\,C_w+(A_{w,P_\ell}y+B_{w,P_\ell}-\log(|f_{w,P_\ell}(z)|\operatorname{Im}(z)^{C_w}))\cdot s+O(s^2),$$

as $s\to 0$, where

$$h_w(s):=\frac{2^s\sqrt{\pi}\,\Gamma(s-1/2)}{\operatorname{ord}(w)\Gamma(s)}. \tag{30}$$

As in [24], pp. 129–130, we use the functional equation of the parabolic Eisenstein series and consider the constant term in the Fourier series expansion, as a function of z, of the function

$$\mathcal{E}^{\mathrm{ell}}_{w}(\sigma_{P_\ell}z,s)-h_w(s)\sum_{j=1}^{p_\Gamma}\mathcal{E}^{\mathrm{par}}_{P_j}(w,1-s)\mathcal{E}^{\mathrm{par}}_{P_j}(\sigma_{P_\ell}z,s)$$
$$=\mathcal{E}^{\mathrm{ell}}_{w}(\sigma_{P_\ell}z,s)-h_w(s)\sum_{j=1}^{p_\Gamma}\mathcal{E}^{\mathrm{par}}_{P_j}(w,s)\mathcal{E}^{\mathrm{par}}_{P_j}(\sigma_{P_\ell}z,1-s). \tag{31}$$

The constant term is given by

$$-h_w(s)\sum_{j=1}^{p_\Gamma}\phi_{j\ell}(1-s)y^s\mathcal{E}^{\mathrm{par}}_{P_j}(w,s)=-\frac{\sqrt{\pi}}{\operatorname{ord}(w)}\frac{\Gamma(s-1/2)}{\Gamma(s)}(2y)^s\sum_{j=1}^{p_\Gamma}\phi_{j\ell}(1-s)\mathcal{E}^{\mathrm{par}}_{P_j}(w,s).$$

Recall the expansions

$$\Gamma(s-1/2)=-2\sqrt{\pi}\left(1+(2-\gamma-2\log 2)s+O(s^2)\right), \tag{32}$$
$$\frac{1}{\Gamma(s)}=s\left(1+\gamma s+O(s^2)\right),\quad\text{and}\quad(2y)^s=1+s\log(2y)+O(s^2), \tag{33}$$

which hold as $s\to 0$, where, as usual, γ denotes the Euler constant. When combining these expressions with (16), we can write the asymptotic expansions near $s=0$ of the constant term in the Fourier series expansion of (31) as

$$\frac{2\pi}{\operatorname{ord}(w)}\left(1+(2+\log y-\log 2)s+O(s^2)\right)$$
$$\cdot\sum_{j=1}^{p_\Gamma}\left(-\frac{1}{\operatorname{vol}_{\mathrm{hyp}}(M)}+\beta_{j\ell}s+O(s^2)\right)\mathcal{E}^{\mathrm{par}}_{P_j}(w,s). \tag{34}$$

Let us now compute the first two terms in the Taylor series expansion at $s=0$ of the expression

$$\sum_{j=1}^{p_\Gamma}\left(-\frac{1}{\operatorname{vol}_{\mathrm{hyp}}(M)}+\beta_{j\ell}s+O(s^2)\right)\mathcal{E}^{\mathrm{par}}_{P_j}(w,s). \tag{35}$$

By applying (22), we conclude that the constant term in the Taylor series expansion of (35) is

$$\sum_{j=1}^{p_\Gamma}\sum_{k=1}^{p_\Gamma}\frac{-1}{\operatorname{vol}_{\mathrm{hyp}}(M)}\left(-\frac{b_{jk}}{\operatorname{vol}_{\mathrm{hyp}}(M)}+a_{jk}\beta_{kk}-\frac{a_{jk}}{\operatorname{vol}_{\mathrm{hyp}}(M)}\log\left|\eta^4_{P_k}(w)\operatorname{Im}(w)\right|\right).$$

Applying relations (19) and (20) we then obtain, by computing the sums, that the constant term in (35) is equal to $-1/\operatorname{vol}_{\mathrm{hyp}}(M)$. The factor multiplying s is equal to

$$\sum_{j=1}^{p_\Gamma}\sum_{k=1}^{p_\Gamma}\frac{-1}{\operatorname{vol}_{\mathrm{hyp}}(M)}\left(-\frac{c_{jk}}{\operatorname{vol}_{\mathrm{hyp}}(M)}+b_{jk}\beta_{kk}-\frac{b_{jk}}{\operatorname{vol}_{\mathrm{hyp}}(M)}\log\left|\eta^4_{P_k}(w)\operatorname{Im}(w)\right|+a_{jk}f_k(w)\right)$$
$$+\sum_{j=1}^{p_\Gamma}\sum_{k=1}^{p_\Gamma}\beta_{j\ell}\left(-\frac{b_{jk}}{\operatorname{vol}_{\mathrm{hyp}}(M)}+a_{jk}\beta_{kk}-\frac{a_{jk}}{\operatorname{vol}_{\mathrm{hyp}}(M)}\log\left|\eta^4_{P_k}(w)\operatorname{Im}(w)\right|\right).$$

Applying relations (19) to (21) we get that

$$\sum_{j=1}^{p_\Gamma}\sum_{k=1}^{p_\Gamma}a_{jk}f_k(w)=0$$

and

$$\sum_{k=1}^{p_\Gamma}\left(-\frac{1}{\operatorname{vol}_{\mathrm{hyp}}(M)}\log\left|\eta^4_{P_k}(w)\operatorname{Im}(w)\right|+\beta_{kk}\right)\sum_{j=1}^{p_\Gamma}\left(-\frac{b_{jk}}{\operatorname{vol}_{\mathrm{hyp}}(M)}+a_{jk}\beta_{j\ell}\right)$$
$$=-\frac{1}{\operatorname{vol}_{\mathrm{hyp}}(M)}\log\left|\eta^4_{P_\ell}(w)\operatorname{Im}(w)\right|+\beta_{\ell\ell},$$

as well as

$$\sum_{j=1}^{p_\Gamma}\sum_{k=1}^{p_\Gamma}\left(-\frac{c_{jk}}{\operatorname{vol}_{\mathrm{hyp}}(M)}+b_{jk}\beta_{j\ell}\right)=\sum_{j=1}^{p_\Gamma}\sum_{k=1}^{p_\Gamma}a_{jk}\gamma_{j\ell}=0.$$

Therefore, the factor multiplying s in the Taylor series expansion of (35) is equal to

$$-\frac{1}{\operatorname{vol}_{\mathrm{hyp}}(M)}\log\left|\eta^4_{P_\ell}(w)\operatorname{Im}(w)\right|+\beta_{\ell\ell}.$$

Inserting this into (34) we see that the constant term in the Fourier series expansion of (31) is given by

$$-C_w-C_w\left(2-\log 2+\log y+\log\left|\eta^4_{P_\ell}(w)\operatorname{Im}(w)\right|-\beta_{\ell\ell}\operatorname{vol}_{\mathrm{hyp}}(M)\right)s+O(s^2),$$

as $s\to 0$. Comparing this result with the right-hand side of formula (29) and having in mind the definition of the number C_w, we immediately deduce the identities $A_{w,P_\ell}=0$,

$$B_{w,P_\ell}=-C_w\left(2-\log 2+\log\left|\eta^4_{P_\ell}(w)\operatorname{Im}(w)\right|-\beta_{\ell\ell}\operatorname{vol}_{\mathrm{hyp}}(M)\right),$$

and

$$\begin{aligned}\mathcal{K}_w(\sigma_{P_\ell}z)&=-\log(|H_\Gamma(\sigma_{P_\ell}z,w)||c_\ell z+d_\ell|^{-2C_w}\operatorname{Im}(z)^{C_w})\\&=B_{w,P_\ell}-\log(|f_{w,P_\ell}(z)|\operatorname{Im}(z)^{C_w}),\end{aligned}$$

where the function f_{w,P_ℓ} is defined by (27). From (27) we deduce that

$$\left|f_{w,P_\ell}(z)\right|=\exp\left(-2\operatorname{Re}\left(\sum_{m=1}^{\infty}A_{m;w,P_\ell}e(mz)\right)\right)=1+O(\exp(-2\pi\operatorname{Im}(z))),$$

as $\operatorname{Im}(z)\to\infty$. Therefore,

$$\left|H_\Gamma(\sigma_{P_\ell}z,w)\right|=\exp(-B_{w,P_\ell})|c_\ell z+d_\ell|^{2C_w}+O(\exp(-2\pi\operatorname{Im}(z))),\quad \text{as } \operatorname{Im}(z)\to\infty.$$

This completes the proof. □

Example 7 (*Moonshine groups*) Let $N = p_1 \cdot \ldots \cdot p_r$ be a square-free number and recall that the surface $Y_N^+ = \overline{\Gamma_0(N)^+}\backslash\mathbb{H}$ possesses one cusp at ∞ with the identity as a scaling matrix. The scattering determinant φ_N associated to the only cusp of Y_N^+ at ∞ is computed in [18], where it was shown that

$$\varphi_N(s) = \sqrt{\pi}\,\frac{\Gamma(s-1/2)}{\Gamma(s)}\,\frac{\zeta(2s-1)}{\zeta(2s)}\cdot D_N(s),$$

where $\zeta(s)$ is the Riemann zeta function and

$$D_N(s) = \prod_{j=1}^{r}\frac{p_j^{1-s}+1}{p_j^s+1} = \frac{1}{N^{s-1}}\prod_{j=1}^{r}\frac{p_j^{s-1}+1}{p_j^s+1}.$$

Let b_N denote the constant term in the Laurent series expansion of $\varphi_N(s)$ at $s = 1$. One can compute b_N by expanding the functions $D_N(s)$, $\Gamma(s)$, and $\zeta(s)$ at $s = 1$, which yields, at $s = 1$, the Laurent expansions

$$D_N(s) = \frac{2^r}{\sigma(N)}\left(1 + (s-1)\left(\sum_{j=1}^{r}\frac{(1-p_j)\log p_j}{2(p_j+1)} - \log N\right) + O((s-1)^2)\right),$$

$$\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)} = \pi\left(1 - 2\log 2(s-1) + O((s-1)^2)\right), \tag{36}$$

as well as

$$\frac{\zeta(2s-1)}{\zeta(2s)} = \frac{6}{\pi^2}\left(\frac{1}{2(s-1)} - \log(2\pi) + 1 - 12\zeta'(-1) + O(s-1)\right). \tag{37}$$

Multiplying the expansions (36) and (37) and using that (see, for example, [19])

$$\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(Y_N^+)} = \frac{3\cdot 2^r}{\pi\,\sigma(N)},$$

we arrive at the expression

$$b_N = -\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(Y_N^+)}\left(\sum_{j=1}^{r}\frac{(p_j-1)\log p_j}{2(p_j+1)} - \log N + 2\log(4\pi) + 24\zeta'(-1) - 2\right).$$

With this formula, Proposition 6, and Example 4 we conclude that the elliptic Kronecker limit function $H_N(z,w) := H_{\overline{\Gamma_0^+(N)}}(z,w)$ associated to the point $w \in Y_N^+$ may we written as

$$H_N(z,w) = a_{N,w}\exp(-B_{N,w}) + O(\exp(-2\pi\,\mathrm{Im}(z))), \quad \text{as } \mathrm{Im}(z)\to\infty,$$

where $a_{N,w}$ is a complex constant of modulus one and

$$B_{N,w} = -\frac{2\pi}{\mathrm{ord}(w)\,\mathrm{vol}_{\mathrm{hyp}}(Y_N^+)} \times\left(\sum_{j=1}^{r}\frac{(p_j-1)\log p_j}{2(p_j+1)} - \log N + C + \log\left(\sqrt[2^r]{\prod_{v|N}|\eta(vw)|^4}\cdot\mathrm{Im}(w)\right)\right)$$

with $C := \log(8\pi^2) + 24\zeta'(-1)$.

Example 8 (*Congruence subgroups of prime level*) Let $M_p = \overline{\Gamma_0(p)}\backslash\mathbb{H}$, where p is a prime. The surface M_p has two cusps, at ∞ and 0. The scaling matrix for the cusp at ∞ is the identity matrix. The scattering matrix in this setting is computed in [11], p. 536, and is given by

$$\Phi_{M_p}(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} \cdot \frac{1}{p^{2s}-1}\begin{pmatrix} p-1 & p^s - p^{1-s} \\ p^s - p^{1-s} & p-1 \end{pmatrix}.$$

Using the expansions (36) and (37), together with $\mathrm{vol}_{\mathrm{hyp}}(M_p) = \pi(p+1)/3$ and the expansion

$$\frac{p-1}{p^{2s}-1} = \frac{1}{p+1} - \frac{2p^2\log p}{(p-1)(p+1)^2}(s-1) + O((s-1)^2), \quad \text{as } s \to 1\,,$$

we conclude that the coefficients β_{11} and β_{22} in the Laurent series expansion (16) are given by

$$\beta_{11} = \beta_{22} = -\frac{2}{\mathrm{vol}_{\mathrm{hyp}}(M_p)}\left(\log(4\pi p) + 12\zeta'(-1) - 1 + \frac{\log p}{p^2-1}\right).$$

Therefore, from Proposition 6, when applied to the cusp ∞, and Example 5, we conclude that the elliptic Kronecker limit function $\widetilde{H}_p(z, w) := H_{\overline{\Gamma_0(p)}}(z, w)$ associated to the point $w \in M_p$ can be written as

$$\widetilde{H}_p(z, w) = \widetilde{a}_{p,w}\exp(-\widetilde{B}_{p,w}) + O(\exp(-2\pi\,\mathrm{Im}(z))), \quad \text{as } \mathrm{Im}(z) \to \infty,$$

where $\widetilde{a}_{p,w}$ is a complex constant of modulus one and

$$\widetilde{B}_{p,w} = -\frac{2\pi}{\mathrm{ord}(w)\,\mathrm{vol}_{\mathrm{hyp}}(M_p)}\left(\frac{2p^2\log p}{p^2-1} + C + \log\left(\left|\sqrt[p-1]{\frac{\eta(pw)^p}{\eta(w)}} \cdot \mathrm{Im}(w)\right|\right)\right)$$

with $C := \log(8\pi^2) + 24\zeta'(-1)$.

5 A factorization theorem

In (6) and (7) one has an evaluation of the elliptic Kronecker limit function in the special case when $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and $w = i$ or $w = \rho = \exp(2\pi i/3)$ are the elliptic fixed points of $\mathrm{PSL}_2(\mathbb{Z})$. The following theorem generalizes these results. A further extension is discussed in Remark 10 below.

Theorem 9 *Let $M = \Gamma\backslash\mathbb{H}$ be a finite volume Riemann surface with at least one cusp. Without loss of generality, we assume that one cusp is at ∞ with the identity as a scaling matrix. Let k be a fixed positive integer such that there exists a weight $2k$ holomorphic form f_{2k} on M which is non-vanishing at all cusps and with the $q-$expansion at ∞ given by*

$$f_{2k}(z) = b_{f_{2k}} + \sum_{n=1}^{\infty} b_{f_{2k}}(n)q_z^n. \tag{38}$$

Let $Z(f_{2k})$ denote the set of all zeros f_{2k} counted according to their multiplicities and let us define the function

$$H_{f_{2k}}(z) := \prod_{w\in Z(f_{2k})} H_\Gamma(z, w),$$

where, as above, $H_\Gamma(z,w)$ is the elliptic Kronecker limit function. Then there exists a complex constant $c_{f_{2k}}$ such that

$$f_{2k}(z) = c_{f_{2k}} H_{f_{2k}}(z) \tag{39}$$

and

$$\left|c_{f_{2k}}\right| = \left|b_{f_{2k}}\right| \exp\left(\sum_{w \in Z(f_{2k})} B_{w,\infty}\right),$$

where $B_{w,\infty}$ is defined in (26).

Proof Assume that f_{2k} possesses $m+l \geq 1$ zeros on M, where m zeros are at the elliptic points e_j of M, $j = 1, \dots, m$, and l zeros are at the non-elliptic points $w_i \in M$, where, of course, all zeros are counted with multiplicities. Then $H_{f_{2k}}(z)$ is a holomorphic function on M which is vanishing if and only if $z \in Z(f_{2k})$ and which according to (24) satisfies the transformation rule

$$H_{f_{2k}}(\gamma z) = \varepsilon_{f_{2k}}(\gamma)(cz+d)^{C_{f_{2k}}} H_{f_{2k}}(z), \quad \text{for any } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma,$$

where $\varepsilon_{f_{2k}}(\gamma)$ is a constant of modulus one,

$$C_{f_{2k}} = \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(M)} \left(\sum_{j=1}^{m} \frac{1}{n_{e_j}} + l\right),$$

and n_e is the order of the elliptic point e.

The classical Riemann-Roch theorem relates the number of zeros of a holomorphic form to its weight and the genus of M in the case M is smooth and compact. A generalization of the relation follows from Proposition 7, p. II-7, of [1] which, in the case under consideration, yields the formula

$$k \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(M)}{2\pi} = \sum_{e \in \mathcal{E}_N} \frac{1}{n_e} v_e(f) + \sum_{z \in M \setminus \mathcal{E}_N} v_z(f), \tag{40}$$

where $\mathcal{E}_N$ denotes the set of elliptic points of M and $v_z(f)$ denotes the order of the zero z of f.

Since $Z(f_{2k})$ is the set of all vanishing points of f_{2k}, formula (40) implies that

$$2k \cdot \frac{\mathrm{vol}_{\mathrm{hyp}}(M)}{4\pi} = \sum_{j=1}^{m} \frac{1}{n_{e_j}} + l,$$

hence $C_{f_{2k}} = 2k$. In other words, $H_{f_{2k}}(z)$ is a holomorphic function on M, vanishing if and only if $z \in Z(f_{2k})$ and satisfying the transformation rule

$$H_{f_{2k}}(\gamma z) = \varepsilon_{f_{2k}}(\gamma)(cz+d)^{2k} H_{f_{2k}}(z), \quad \text{for any } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma.$$

By Proposition 6, we have that for any $w \in Z(f_{2k})$ and any cusp P_l of M, with $l = 1, \dots, p_\Gamma$, the function

$$F_{f_{2k}}(z) := \frac{H_{f_{2k}}(z)}{f_{2k}(z)}$$

is a non-vanishing holomorphic function on M, bounded and non-zero at the cusp at ∞ and has at most polynomial growth in the variable $\mathrm{Im}(z)$ at any other cusp of M. Therefore, the function $\log|F_{f_{2k}}(z)|$ is harmonic on M whose growth at any cusp is such that $\log|F_{f_{2k}}(z)|$ is L^2 on M. As a result, $\log|F_{f_{2k}}(z)|$ admits a spectral expansion; see [11] or [13]. Since $\log|F_{f_{2k}}(z)|$ is harmonic, one can use integration by parts to show that $\log|F_{f_{2k}}(z)|$ is orthogonal to any eigenfunction corresponding to a non-zero eigenvalue of the hyperbolic Laplacian. Therefore, from the spectral expansion, one concludes that $\log|F_{f_{2k}}(z)|$ is constant, hence so is $F_{f_{2k}}(z)$. The evaluation of the constant is obtained by considering the limiting behavior as z approaches ∞. This completes the proof of (39). □

Remark 10 It is evident that one can generalize Theorem 9 to the case when the holomorphic form f_{2k} vanishes at a cusp, or at several cusps. In such an instance, one should include factors of the parabolic Kronecker limit function in the construction of $H_{f_{2k}}$. The parabolic Kronecker limit function is bounded and non-vanishing at each of the cusps other than the one to which it is associated, and the (fractional) order to which it vanishes follows from Theorem 1 of [29]. As with Theorem 9, one can express any holomorphic modular form as a product of parabolic and elliptic Kronecker limit functions, up to a multiplicative constant. Furthermore, the multiplicative constant can be computed, up to a factor of modulus one, from the value of the various functions at a cusp.

6 Examples of factorization

6.1 An arbitrary surface with one cusp

In the case when a surface M has one cusp, we get the following special case of Theorem 9.

Corollary 11 *Let $M = \Gamma\backslash\mathbb{H}$ be a finite volume Riemann surface with one cusp, which we assume to be at ∞ with the identity as a scaling matrix. Then the weight $2k$ holomorphic Eisenstein series $E_{2k,\Gamma}$ defined in (12) can be represented as*

$$E_{2k,\Gamma}(z) = a_{E_{2k,\Gamma}} B_{E_{2k,\Gamma}} \prod_{w\in Z(E_{2k,\Gamma})} H_\Gamma(z,w),$$

where $a_{E_{2k,\Gamma}}$ is a complex constant of modulus one and

$$B_{E_{2k,\Gamma}} = \prod_{w\in Z(E_{2k,\Gamma})} \exp\left(C_w\left(\log 2 - 2 + \beta_M \,\mathrm{vol}_{\mathrm{hyp}}(M)\right)\right)\cdot\left|\eta_\infty^4(w)\,\mathrm{Im}(w)\right|^{-C_w}.$$

As before, η_∞ is the parabolic Kronecker limit function defined in Sect. 3, formula (17), and β_M is the constant term in the Laurent series expansion of the scattering determinant on M.

In this case, due to a very simple form of the Kronecker's limit formula for parabolic Eisenstein series as $s \to 0$, the factorization theorem yields an interesting form of the Kronecker's limit formula for elliptic Eisenstein series, which we state as the following proposition.

Proposition 12 *Let $M = \Gamma\backslash\mathbb{H}$ be a finite volume Riemann surface with one cusp, which we assume to be at ∞ with the identity as a scaling matrix. Let k be a fixed positive integer such that there exists a weight $2k$ holomorphic form f_{2k} on M with the q-expansion at ∞ given by (38). Then*

$$\sum_{w\in Z(f_{2k})} \mathcal{E}^{\mathrm{ell}}_w(z,s) = -s\log\left(|f_{2k}(z)||\eta^4_\infty(z)|^{-k}\right) + s\log|b_{f_{2k}}| + O(s^2), \tag{41}$$

as $s \to 0$*, where* $b_{f_{2k}}$ *is the non-zero constant term in the expansion* (38) *and* $Z(f_{2k})$ *denotes the set of all zeros of* f_{2k} *counted with multiplicities.*

Proof We start with formula (4), which we divide by $\mathrm{ord}(w)$, and take the sum over all $w \in Z(f_{2k})$ to get

$$\begin{aligned}&\sum_{w\in Z(f_{2k})} \mathcal{E}^{\mathrm{ell}}_w(z,s) - \mathcal{E}^{\mathrm{par}}_\infty(z,s) \sum_{w\in Z(f_{2k})} h_w(s)\mathcal{E}^{\mathrm{par}}_\infty(w,1-s)\\ &= -\sum_{w\in Z(f_{2k})} C_w\,(1+s\log(\mathrm{Im}\,z)) - \log\left(\prod_{w\in Z(f_{2k})} |H_\Gamma(z,w)|\right)\cdot s + O(s^2),\end{aligned} \tag{42}$$

as $s \to 0$, where C_w and $h_w(s)$ are defined by (25) and (30), respectively. Recall that for any $w \in Z(f_{2k})$, we let $\mathrm{ord}(w)$ denote the order of the elliptic subgroup Γ_w of Γ which fixes w. One now expands the second term on the left hand side of (42) into a Taylor series at $s = 0$ by applying formulas (32), (33), (23), and (17). After multiplication, we get, as $s \to 0$, the expression

$$\begin{aligned}&\mathcal{E}^{\mathrm{par}}_\infty(z,s) \sum_{w\in Z(f_{2k})} h_w(s)\mathcal{E}^{\mathrm{par}}_\infty(w,1-s)\\ &= \sum_{w\in Z(f_{2k})} C_w\big(1 + s\big(2 - \log 2 - \beta_M\,\mathrm{vol}_{\mathrm{hyp}}(M) + \log|\eta^4_\infty(w)\,\mathrm{Im}(w)|\\ &\quad + |\eta^4_\infty(z)\,\mathrm{Im}(z)|\big)\big) + O(s^2),\end{aligned} \tag{43}$$

as $s \to 0$. Theorem 9 yields that

$$\log\left(\prod_{w\in Z(f_{2k})} |H_\Gamma(z,w)|\right) = \log|f_{2k}(z)| - \sum_{w\in Z(f_{2k})} B_{w,\infty} - \log|b_{f_{2k}}|, \tag{44}$$

where $B_{w,\infty}$ is defined by (26) for the cusp $P_l = \infty$. Finally, from formula (40), we get that

$$\sum_{w\in Z(f_{2k})} C_w = k.$$

Therefore, by inserting (26), (44), and (43) into (42), we immediately deduce (41). The proof is complete. □

Remark 13 In the case $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, the parabolic Kronecker limit function is given by $\eta_\infty(z) = \eta(z) = \Delta(z)^{1/24}$. Then, for $k = 3$ and $f_{2k} = E_6$, we have $b_{E_6} = 1$ and $Z(E_6) = \{i\}$, hence Proposition 12 yields (8). Analoguously, for $k = 2$ and $f_{2k} = E_4$, we have $b_{E_4} = 1$ and $Z(E_4) = \{\rho\}$, and Proposition 12 gives (9). Furthermore, we have $B_{E_6,\Gamma} = \exp(B_i)$ and $B_{E_4,\Gamma} = \exp(B_\rho)$, where B_i and B_ρ are given by (6) and (7), respectively; see [24] and [25].

Let us now develop further examples of surfaces with one cusp and explicitly compute the constant $B_{E_{2k},\Gamma}$ in these special cases.

6.2 Moonshine groups of square-free level

Example 14 Consider the surface Y_2^+. There exists one elliptic fixed point of order two, $e_1 = i/\sqrt{2}$, and one elliptic fixed point of order four, $e_2 = 1/2 + i/2$. The surface Y_2^+ has genus zero and one cusp, hence $\mathrm{vol}_{\mathrm{hyp}}(Y_2^+) = \pi/2$. The transformation rule for $E_6^{(2)}$ implies that the form must vanish at the points e_1 and e_2. Furthermore, formula (40) when applied to Y_2^+ becomes

$$\frac{2k}{8} = \nu_\infty(f) + \frac{1}{4}\nu_{e_2}(f) + \frac{1}{2}\nu_{e_1}(f) + \sum_{z \in Y_2^+ \setminus \{e_1, e_2\}} \nu_z(f). \tag{45}$$

Taking $k = 3$, we conclude that e_1 and e_2 are the only vanishing points of $E_6^{(2)}$ and the order of vanishing is one at each point. Therefore, in the notation of Theorem 9 and Example 7, we have that the form $H_6^{(2)}(z) = H_{E_6^{(2)}}(z)$ is given by $H_6^{(2)}(z) := H_2(z, e_1) H_2(z, e_2)$. Assuming that the phase of $H_6^{(2)}(z)$ is such that it attains real values at the cusp ∞, we have that

$$E_6^{(2)}(z) = C_{2,6} H_6^{(2)}(z), \tag{46}$$

where the absolute value of the constant $C_{2,6}$ is given by $|C_{2,6}| = e^{B_{2,e_1} + B_{2,e_2}}$ with

$$B_{2,e_1} = -2\left(24\zeta'(-1) + \log(8\pi^2) - \frac{4}{3}\log 2 + \frac{1}{12}\log(|\Delta(i\sqrt{2}) \cdot \Delta(i/\sqrt{2})|)\right)$$

and

$$B_{2,e_2} = -\left(24\zeta'(-1) + \log(8\pi^2) - \frac{11}{6}\log 2 + \frac{1}{12}\log(|\Delta(1/2 + i/2) \cdot \Delta(1 + i)|)\right).$$

Let us now consider the case when $k = 2$. From (45), we have that only e_1 and e_2 can be vanishing points of $E_4^{(2)}$. However, there are two possibilities: Either e_2 is an order two vanishing point, and $E_4^{(2)}(z) \neq 0$ for all $z \neq e_2$ in the fundamental domain $\mathcal{F}_2$ of Y_2^+, or e_1 is an order one vanishing point and $E_4^{(2)}(z) \neq 0$ for all points $z \neq e_1$ in $\mathcal{F}_2$. If the latter possibility is true, then $E_6^{(2)}(z)/E_4^{(2)}(z)$ would be a weight two holomorphic modular form which vanishes only at e_2, which is not possible since there is no weight two modular form on Y_N^+ for any square-free N such that the surface Y_N^+ has genus zero; see, for example, [20]. Therefore, $E_4^{(2)}$ vanishes at e_2 of order two, and there are no other vanishing points of $E_4^{(2)}$ on Y_2^+.

Hence, in the notation of Theorem 9, we have $H_4^{(2)}(z) := H_{E_4^{(2)}}(z) = H_2(z, e_2)^2$, implying that

$$E_4^{(2)}(z) = C_{2,4} H_2(z, e_2)^2, \tag{47}$$

where $|C_{2,4}| = e^{2B_{2,e_2}}$. This proves that $H_2(z, e_2)^2$ is a weight four holomorphic modular function on $\overline{\Gamma_0(2)^+}$. If we combine (46) with (47) we get

$$H_2(z, e_1)^2 = \frac{C_{2,4}}{C_{2,6}^2} \cdot \frac{(E_6^{(2)}(z))^2}{E_4^{(2)}(z)};$$

in other words, $H_2(z, e_1)^2$ is a weight eight holomorphic modular function on $\overline{\Gamma_0(2)^+}$.

Furthermore, an application of Proposition 12 with $f_{2k} = E_4^{(2)}$ and $Z_{f_{2k}} = \{e_2\}$ (with multiplicity two) together with Example 4 and with the representation formula (15) yields (11).

By applying Proposition 12 with $f_{2k} = E_6^{(2)}$ and $Z_{f_{2k}} = \{e_1, e_2\}$ together with formula (11) we get the following elliptic Kronecker's limit formula

$$\mathcal{E}_{e_1}^{\mathrm{ell}}(z,s) = -s\log\left(|E_6^{(2)}(z)||E_4^{(2)}(z)|^{-1/2}|\Delta(z)\Delta(2z)|^{-1/6}\right) + O(s^2), \quad \text{as } s \to 0.$$

Example 15 Consider the surface Y_5^+. There exist three order two elliptic fixed points, namely $e_1 = i/\sqrt{5}$, $e_2 = 2/5 + i/5$, and $e_3 = 1/2 + i/(2\sqrt{5})$. The surface Y_5^+ has genus zero and one cusp, hence $\mathrm{vol}_{\mathrm{hyp}}(Y_5^+) = \pi$. Using the transformation rule for $E_6^{(5)}$, one concludes that the holomorphic form $E_6^{(5)}$ must vanish at e_1, e_2, and e_3. By the dimension formula (40), one sees that e_1, e_2, and e_3 are the only zeros of $E_6^{(5)}$. Theorem 9 then implies that

$$E_6^{(5)}(z) = C_{5,6}H_6(z,e_1)H_6(z,e_2)H_6(z,e_3) := C_{5,6}H_6^{(5)}(z), \tag{48}$$

where the absolute value of the constant $C_{5,6}$ is given by $|C_5| = e^{B_{5,e_1}+B_{5,e_2}+B_{5,e_3}}$ and

$$B_{5,e_1} + B_{5,e_2} + B_{5,e_3} = -3\left(24\zeta'(-1) + \log(8\pi^2)\right) - \log 50$$
$$+\frac{1}{12}\log\left(\left|\Delta(i/\sqrt{5})\Delta(i\sqrt{5})\Delta(2/5+i/5)\Delta(2+i)\Delta(1/2+i/(2\sqrt{5}))\Delta(5/2+i\sqrt{5}/2)\right|\right).$$

One can view (48) as analogue of the Jacobi triple product formula.

Remark 16 Let $N = p_1 \cdot \ldots \cdot p_r$ be a square-free number. Then, according to the properties of the surface Y_N^+ listed in Sect. 2.2, one can develop a number of results similar to the above examples when $N = 2$ or $N = 5$. In particular, Theorem 9 holds, so one can factor any holomorphic Eisenstein series $E_{2k}^{(N)}$ of weight $2k$ into a product of elliptic Kronecker limit functions, up to a factor of modulus one.

6.3 Congruence subgroups of prime level

Consider the surface $M_p = \overline{\Gamma_0(p)}\backslash\mathbb{H}$ for a prime p. The smallest positive integer k such that there exists a weight $2k$ holomorphic form is $k = 1$. As a result, we have the following corollary of Theorem 9.

Corollary 17 *Let $f_{2k,p}$ denote a weight $2k \geq 2$ holomorphic form on the surface M_p bounded at cusps and such that the constant term in its q-expansion is equal to $b_{f_{2k},p}$. Assume that $b_{f_{2k},p} \neq 0$. Then,*

$$f_{2k,p}(z) = a_{f_{2k},p}\widetilde{B}_{f_{2k},p}\prod_{w\in Z(f_{2k},p)}\widetilde{H}_p(z,w),$$

where $a_{f_{2k},p}$ is a complex constant of modulus one and

$$\widetilde{B}_{f_{2k},p} = |b_{f_{2k},p}|\prod_{w\in Z(f_{2k},p)}\left(\exp\left[-C_w\left(\frac{2p^2\log p}{p^2-1} + C\right)\right]\left|\sqrt[p-1]{\frac{\eta(pw)^p}{\eta(w)}}\,\mathrm{Im}(w)\right|^{-C_w}\right)$$

with $C := \log(8\pi^2) + 24\zeta'(-1)$.

Let us now compute the constants $\widetilde{B}_{f_{2k},p}$ for two cases.

Example 18 If $p = 2$, then the surface M_2 has only one elliptic fixed point, $e = 1/2 + i/2$, which has order two. Furthermore, $\mathrm{vol}_{\mathrm{hyp}}(M_p) = \pi$, hence formula (40) with $k = 1$ implies that the holomorphic form $E_{2,2}$ defined by (13) with $p = 2$ vanishes only at e, and the vanishing order is one. From the $q-$expansion (14) we have that $\left|b_{E_{2,2},2}\right| = 2 - 1 = 1$. Since $C_e = 1$, we get

$$E_{2,2}(z) = a_2 \cdot \frac{1}{16\sqrt[3]{4}\pi^2} \exp(-24\zeta'(-1)) \left|\frac{\eta(1/2 + i/2)}{\eta(1+i)^2}\right| \widetilde{H}_2(z, e),$$

for some complex constant a_2 of modulus one. In other words, the elliptic Kronecker limit function $\widetilde{H}_2(z, e)$ is a weight two modular form on $\overline{\Gamma_0(2)}$.

Example 19 If $p = 3$, then the surface M_3 has only one elliptic fixed point $e = 1/2 + \sqrt{3}i/6$, which has order three. The hyperbolic volume of the surface M_3 is $4\pi/3$, hence formula (40) with $k = 1$ implies that the holomorphic form $E_{2,3}$ vanishes only at e, of order two. Furthermore, $\left|b_{E_{2,3},2}\right| = 2$ and $C_e = 1/2$, so then

$$E_{2,3}(z) = a_3 \cdot \frac{1}{12\sqrt[4]{27}\pi^2} \exp(-24\zeta'(-1)) \left|\left(\frac{\eta\left(1/2 + i\sqrt{3}/6\right)}{\eta\left(3/2 + i\sqrt{3}/2\right)^3}\right)^{1/2}\right| \widetilde{H}_3(z, e)^2,$$

for some complex constant a_3 of modulus one.

7 Additional considerations

In this section, we use the elliptic Kronecker limit function to prove the Weil's reciprocity law. In addition, we state a few concluding remarks.

7.1 The factorization theorem for compact surfaces

Assume that the Riemann surface M is compact. In the notation of the proof of Theorem 9, the quotient

$$F_{f_{2k}}(z) := \frac{H_{f_{2k}}(z)}{f_{2k}(z)}$$

is a non-vanishing, bounded, and holomorphic function on M, hence the quotient is constant. In other words, there is a constant $c_{f_{2k}}$ such that

$$f_{2k}(z) = c_{f_{2k}} H_{f_{2k}}(z) := c_{f_{2k}} \prod_{w \in Z(f_{2k})} H_\Gamma(z, w).$$

The point now is to develop a strategy by which one can evaluate $c_{f_{2k}}$. Perhaps the most natural approach would be to study the limiting value of

$$\widetilde{H}_\Gamma(z) := \lim_{w \to z} \frac{H_\Gamma(z, w)}{z - w},$$

which needs to be considered in the correct sense as a holomorphic form on M. One can then express $c_{f_{2k}}$ in terms of the first non-zero coefficient of f_{2k} about a point $z \in Z(f_{2k})$, a product of the forms $H_{f_{2k}}(z, w)$ for two different points in $Z(f_{2k})$ and $\widetilde{H}_\Gamma(z)$. Such formulae could be quite interesting in various cases of arithmetic interest. We will leave the development of such identities for future investigation.

7.2 Symmetric form of the elliptic Kronecker limit function

For simplicity, assume that the Riemann surface M is smooth and compact. Let us re-write (4) as

$$\mathcal{E}_w^{\mathrm{ell}}(z,s) = -c - \log\big(|\tilde{H}_\Gamma(z,w)|(\operatorname{Im}(z)\operatorname{Im}(w))^c\big)\cdot s + O(s^2), \quad \text{as } s \to 0,$$

where, as before, $c = 2\pi/\operatorname{vol}_{\mathrm{hyp}}(M)$ and $\tilde{H}_\Gamma(z,w) = H_\Gamma(z,w)/\operatorname{Im}(w)^c$. Recall that the function $\tilde{H}_\Gamma(z,w)$ is locally holomorphic in both z and w; see Sect. 1.3. Using the transformation rule (24) for the function $H_\Gamma(z,w)$, the fact that, in our setting $C_w = 2\pi/(\operatorname{ord}(w)\operatorname{vol}_{\mathrm{hyp}}(M)) = c$, for all $w \in M$, together with the symmetry $\mathcal{E}_w^{\mathrm{ell}}(z,s) = \mathcal{E}_z^{\mathrm{ell}}(w,s)$, we may write the transformation rules of $\tilde{H}_\Gamma$ as

$$\tilde{H}_\Gamma(\gamma z, w) = \varepsilon_{1,w}(\gamma)(cz+d)^{2c}\tilde{H}_\Gamma(z,w), \quad \text{for any } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma,$$

and

$$\tilde{H}_\Gamma(z, \gamma w) = \varepsilon_{2,z}(\gamma)(cw+d)^{2c}\tilde{H}_\Gamma(z,w), \quad \text{for any } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma,$$

where $\varepsilon_{1,w}$ and $\varepsilon_{2,z}$ denote complex numbers of absolute value equal to one. For fixed z and γ, $\varepsilon_{1,w}(\gamma)$ is locally holomorphic in w. However, since M is compact, and $\varepsilon_{1,w}(\gamma)$ has modulus one, it follows that $\varepsilon_{1,w}(\gamma)$ is independent of w. Similarly, $\varepsilon_{2,z}(\gamma)$ is independent of z, so then we may write

$$\varepsilon_1(\gamma) := \varepsilon_{1,w}(\gamma) \quad \text{and} \quad \varepsilon_2(\gamma) := \varepsilon_{2,z}(\gamma).$$

7.3 Weil reciprocity

We now will use the discussion in Sects. 7.1 and 7.2 to prove Weil's reciprocity law. For simplicity, we assume that the Riemann surface M is smooth and compact, though it is evident that with careful consideration the approach can be extended.

Theorem 20 (Weil reciprocity) *Let f and g be meromorphic functions on the smooth, compact Riemann surface M. Let D_f and D_g denote the divisors of f and g, respectively, which we write as*

$$D_f = \sum m_f(P)P \quad \textit{and} \quad D_g = \sum m_g(P)P.$$

Assume that D_f and D_g are disjoint. Then, we have

$$\prod_{w_j \in D_g} f(w_j)^{m_g(w_j)} = \prod_{z_i \in D_f} g(z_i)^{m_f(z_i)}.$$

Proof The factorization theorem developed in Sect. 7.1 together with the discussion from Sect. 7.2 implies that we can write

$$f(z) = c_f \prod_{w_j \in D_f} \tilde{H}_\Gamma(z, w_j)^{m_f(w_j)}$$

for some constant c_f. However, since $\tilde{H}_\Gamma(z,w)$ is locally holomorphic in w, and vanishes to first order when w approaches z, one can also write any holomorphic form as a product using the second variable in $\tilde{H}_\Gamma$. Thus we can write g as

$$g(w) = c_g \prod_{z_i \in D_g} \tilde{H}_\Gamma(z_i, w)^{m_g(z_i)}.$$

Both f and g have degree zero, so then

$$\sum_{z_i \in D_g} m_g(z_i) = \sum_{w_j \in D_f} m_f(w_j) = 0,$$

hence

$$\prod_{z_i \in D_g} c_f^{m_g(z_i)} = \prod_{w_j \in D_f} c_g^{m_g(w_j)} = 1.$$

Therefore, we obtain

$$\prod_{w_j \in D_g} f(w_j)^{m_g(w_j)} = \prod_{w_j \in D_g} \prod_{z_i \in D_f} \tilde{H}_\Gamma(z_i, w_j)^{m_f(w_j) m_g(z_i)} = \prod_{z_i \in D_f} g(z_i)^{m_f(z_i)},$$

which completes the proof of Theorem 20. □

7.4 Unitary characters and Artin formalism

As with parabolic Eisenstein series, one can extend the study of elliptic Eisenstein series to include the presence of a unitary character. More precisely, let $\pi : \Gamma \to U(n)$ denote an n-dimensional unitary representation of the group Γ with associated character χ_π. Let us define

$$\mathcal{E}_w^{\text{ell}}(z, s; \pi) = \sum_{\eta \in \Gamma} \chi_\pi(\eta) \sinh(d_{\text{hyp}}(\eta z, w))^{-s} \tag{49}$$

to be the elliptic Eisenstein series twisted by χ_π. Note that if $n = 1$ and π is trivial, then the above definition is equal to $\text{ord}(w)$ times the series in (2). In general terms, the meromorphic continuation of (49) could be proven by using the methodology of [17], which depended on the spectral expansion and small time asymptotics of the associated heat kernel.

Having established the meromorphic continuation of (49), one then can study the elliptic Kronecker limit functions. It would be interesting to place the study in the context of the Artin formalism relations; see [16] and references therein. The system of elliptic Eisenstein series associated to the representations π will satisfy additive Artin formalism relations, and, through exponentiation, the corresponding elliptic Kronecker limit functions will satisfy multiplicative Artin formalism relations. It would be interesting to carry out these computations in the setting of the congruence groups $\Gamma_0(N)$ as subgroups of the moonshine groups $\Gamma_0(N)^+$, for instance, in order to relate the above-mentioned computations for parabolic Kronecker limit functions. It is possible that a similar approach could yield further relations amongst the elliptic Kronecker limit functions.

7.5 Fay's prime form

In a separate consideration, it may be interesting to express $\tilde{H}_\Gamma(z, w)$ in terms of Fay's prime form. This investigation will not be undertaken in the present article, though we can, at this time, comment on the matter.

Assume for now that M is smooth and compact. In chapter 2 of [6] the author constructs the prime form $E(z, w)$ for points $z, w \in M$ with certain characterizing properties, including: It is a locally holomorphic function on $M \times M$, it vanishes if and only if $z = w$, it is anti-symmetric, meaning $E(z, w) = -E(w, z)$, and its periodicity properties when continuing along a path on M (meaning, when lifting to $\mathbb{H} \times \mathbb{H}$ and analytically continuing) can be determined. We refer to p. 19 of [6] for precise statements.

In other terms, Fay describes $E(z, w)$ as a holomorphic section of a particular line bundle on $M \times M$; see statement (i) on p. 16 of [6]. In the language of algebraic geometry, one can equip the line bundle with a canonical norm $\|\cdot\|^2$ so that $\|E(z, w)\|^2$ is a well-defined smooth function on $M \times M$; see p. 401 of [5] which is elaborated upon in [14]. In general terms, Fay's definition of the prime form is a quotient of the Riemann theta function with characteristics and certain holomorphic one forms. The norm of the prime form is obtained using the norm of the Riemann theta function (see p. 228 of [14]) and the norm of holomorphic one-forms using the Weil–Petterson metric on the canonical bundle (see p. 232 of [14]).

With this discussion, one can relate the Kronecker limit function and $\log \|E(z, w)\|$. Indeed, both (5) and $\log \|E(z, w)\|$ have logarithmic singularities as z approaches w. Also, both functions are such that away from the singularity, the hyperbolic Laplacian of each function is constant. Therefore, we have, in somewhat vague terms, that the prime form can be expressed in terms of $\tilde{H}_{\Gamma}(z, w)$, trivial theta functions (see chapter VI of [23]), and constants which may depend M.

Going further, it would be interesting to study the means by which the Kronecker limit function could extend the definition of the prime form to Riemann surfaces with singularities or with cusps. Additionally, one could examine specific situations, such as compact surfaces of small genus, to determine if the above discussion would lead to precise formulas involving Riemann's theta function and related quantities. We will leave this point of study for future investigations.

Acknowledgements We thank the anonymous referee for carefully reading a preliminary manuscript and providing many comments. His/her remarks helped greatly in improving this article.

References

1. Borel, A., Chowla, S., Herz, C.S., Iwasawa, K., Serre, J.P.: Seminar on complex multiplication. Lecture Notes in Math, vol. 21. Springer, Berlin, New-York (1966)
2. Cummins, C.: Congruence subgroups of groups commensurable with PSL(2, $\mathbb{Z}$) of genus 0 and 1. Exp. Math. **13**, 361–382 (2004)
3. Deligne, P.: Le symbole modéré. Inst. Hautes Études Sci. Publ. Math. **73**, 147–181 (1991)
4. Falliero, T.: Dégénérescence de séries d'Eisenstein hyperboliques. Math. Ann. **339**, 341–375 (2007)
5. Faltings, G.: Calculus on arithmetic surfaces. Ann. Math. **119**, 387–424 (1984)
6. Fay, J.D.: Theta functions on Riemann surfaces. Lecture Notes in Math, vol. 352. Springer, Berlin (1973)
7. Fay, J.D.: Fourier coefficients of the resolvent for a Fuchsian group. J. Reine Angew. Math. **293/294**(1977), 143–203 (1977)
8. Garbin, D., Jorgenson, J., Munn, M.: On the appearance of Eisenstein series through degeneration. Comment. Math. Helv. **83**, 701–721 (2008)
9. Garbin, D., von Pippich, A.-M.: On the behavior of Eisenstein series through elliptic degeneration. Comm. Math. Phys. **292**, 511–528 (2009)
10. Goldstein, L.J.: Dedekind sums for a Fuchsian group. I. Nagoya Math. J. **80**, 21–47 (1973)
11. Hejhal, D.: The Selberg trace formula for PSL(2, $\mathbb{R}$). II, Lecture Notes in Math, vol. 1001, Springer, Berlin (1983)
12. Helling, H.: Bestimmung der Kommensurabilitätsklasse der Hilbertschen Modulgruppe. Math. Z. **92**, 269–280 (1966)
13. Iwaniec, H.: Spectral methods of automorphic forms. Graduate Studies in Mathematics, vol. 53, American Mathematical Society, Providence, RI (2002)
14. Jorgenson, J.: Asymptotic behavior of Faltings's delta function. Duke Math. J. **61**, 221–254 (1990)
15. Jorgenson, J., Kramer, J., von Pippich, A.-M.: On the spectral expansion of hyperbolic Eisenstein series. Math. Ann. **346**, 931–947 (2010)
16. Jorgenson, J., Lang, S.: Artin formalism and heat kernels. J. Reine Angew. Math. **447**, 165–200 (1994)

17. Jorgenson, J., von Pippich, A.-M., Smajlović, L.: On the wave representation of hyperbolic, elliptic, and parabolic Eisenstein series. Adv. Math. **288**, 887–921 (2016)
18. Jorgenson, J., Smajlović, L., Then, H.: On the distribution of eigenvalues of Maass forms on certain moonshine groups. Math. Comp. **83**, 3039–3070 (2014)
19. Jorgenson, J., Smajlović, L., Then, H.: Kronecker's limit formula, holomorphic modular functions and q-expansions on certain arithmetic groups. Exp. Math. **54**, 295–320 (2016)
20. Jorgenson, J., Smajlović, L., Then, H.: Certain aspects of holomorphic function theory on some genus zero arithmetic groups. LMS J. Comput. Math. **19**, 360–381 (2016)
21. Khovanskii, A.: Logarithmic functional and reciprocity laws. In: Toric Topology, M. Harada, Y. Karshon, M. Masuda, T. Panov (eds.) Contemp. Math, vol. 460, pp. 221–229. Amer. Math. Soc., Providence, RI (2008)
22. Kudla, S.S., Millson, J.J.: Harmonic differentials and closed geodesics on a Riemann surface. Invent. Math. **54**, 193–211 (1979)
23. Lang, S.: Introduction to algebraic and abelian functions. Graduate Texts in Mathematics, vol. 89, Springer, New York (1982)
24. von Pippich, A.-M.: The arithmetic of elliptic Eisenstein series. PhD thesis, Humboldt-Universität zu Berlin (2010)
25. von Pippich, A.-M.: A Kronecker limit type formula for elliptic Eisenstein series. arXiv:1604.00811 (2016)
26. Serre, J.-P.: A course in arithmetic. Graduate Texts in Mathematics, vol. 7, Springer, New York (1973)
27. Shimura, G.: Introduction to the Arithmetic Theory of Automorphic Forms. Publications of the Mathematical Society of Japan. Princeton University Press, Princeton (1971)
28. Siegel, C.L.: Advanced analytic number theory. Tata Institute of Fundamental Research Studies in Mathematics, vol. 9, Tata Institute of Fundamental Research, Bombay (1980)
29. Takada, I.: Dedekind sums of $\Gamma(N)$. Japan J. Math. **12**, 401–411 (1986)
30. Vassileva, I.N.: Dedekind eta function, Kronecker limit formula and Dedekind sum for the Hecke group. PhD thesis, University of Massachusetts Amherst (1996)