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# On Sandon-type metrics for contactomorphism groups

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**Abstract** For certain contact manifolds admitting a 1-periodic Reeb flow we construct a conjugation-invariant norm on the universal cover of the contactomorphism group. With respect to this norm the group admits a quasi-isometric monomorphism of the real line. The construction involves the partial order on contactomorphisms and symplectic intersections. This norm descends to a conjugation-invariant norm on the contactomorphism group. As a counterpoint, we discuss conditions under which conjugation-invariant norms for contactomorphisms are necessarily bounded.

**Keywords** Contact manifold · Contactomorphism · Conjugation Invariant norm

**Resume** On construit une norme invariante par conjugaison sur le revêtement universel du groupe des contactomorphismes associé à certaines variétés de contact admettant un flot de Reeb 1-périodique. Par rapport à cette norme, le groupe admet un monomorphisme quasi-isométrique des réels. La construction utilise l'ordre partiel sur les contactomorphismes et des propriétés des intersections symplectiques. Cette norme induit une norme invariante par conjugaison sur le groupe des contactomorphismes. Par contraste avec cette construction, nous discutons de conditions sous lesquelles des normes invariantes par conjugaison sur des groupes des contactomorphismes sont nécessairement bornées.

**Mathematics Subject Classification** 53Dxx

## 1 Introduction

A *conjugation-invariant norm* on a group $G$ is a function $\nu\colon G \to [0, \infty)$ satisfying the following properties:

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(1) $\nu(\mathbb{1}) = 0$ and $\nu(g) > 0$ for all $g \neq \mathbb{1}$.
(2) $\nu(gh) \leq \nu(g) + \nu(h)$ for all $g, h \in G$.
(3) $\nu(g^{-1}) = \nu(g)$ for all $g \in G$.
(4) $\nu(h^{-1}gh) = \nu(g)$ for all $g, h \in G$.

A function $\nu$ satisfying only 1–3 is a *norm* on $G$ , while if the non-degeneracy condition $\nu(g) > 0$ for $g \neq \mathbb{1}$ is dropped $\nu$ is said to be a *pseudo-norm* (or *semi-norm*).

Given any bi-invariant metric $d$ on $G$, distance to the identity defines a conjugation-invariant norm, $\nu(f) := d(f, \mathbb{1})$, and vice versa, any conjugation-invariant norm $\nu$ defines a bi-invariant metric $d(f, g) := \nu(fg^{-1})$.

Following the terminology of [8] we say a norm on $G$ is *bounded* when there exists $C < \infty$ such that $\nu(g) \leq C$ for all $g \in G$. A norm is called *stably unbounded* if for some $g \in G$, $\nu(g^n) \geq c|n|$ for all $n \in \mathbb{Z}$ with some $c > 0$. A norm $\nu$ is *discrete* if there exists a constant $c > 0$ such that $c \leq \nu(g)$ for any $g \neq \mathbb{1}$, and a norm is *trivial* if it is both discrete and bounded (i.e., equivalent to the trivial norm). In many cases we will consider, a general argument of [8] implies that all conjugation-invariant norms are discrete, and hence boundedness is equivalent to triviality.

In this paper, we focus on conjugation-invariant norms on contactomorphism groups and in particular on their (un)boundedness. Such norms were discovered by Sandon in [31] and further studied in recent papers [39] by Zapolsky and [9] by Colin and Sandon. Their geometric properties turn out to be sensitive to the contact topology of $(V, \xi)$. The above norms are:

- unbounded for $T^*\mathbb{R}^n \times S^1$, but not stably unbounded [9,31];
- stably unbounded for $T^*X \times S^1$ with compact $X$ [39] and for $\mathbb{R}P^{2n+1}$ [9];
- bounded for $S^{2n+1}$ and $\mathbb{R}^{2n+1}$ [9],

where the manifolds in the list are equipped with the standard contact structures. All these norms are studied by using Legendrian spectral invariants. Sandon's norm and the related norm by Zapolsky are actually defined through these invariants, while the Colin–Sandon norms have geometric and/or dynamical definitions. It is also possible to define non-conjugation-invariant (pseudo-) norms purely in terms of Hamiltonians; such norms, in the spirit of Hofer's norm, have been studied by Shelukhin [35] and Müller and Spaeth [25] (cf. also [29,30]).

Conjugation-invariant norms are closely related to quasi-morphisms. Indeed if $\mu$ is a homogeneous quasi-morphism on a group $G$—a real-valued function on $G$ such that $\phi(f^n) = n\phi(f)$ for any $f \in G$, $n \in \mathbb{Z}$ and for which there is $D > 0$ such that $|\phi(fg) - \phi(f) - \phi(g)| \leq D$ for all $f, g \in G$—then it is easily checked that $\phi$ is conjugation-invariant and $\mu(f) := |\phi(f)| + D$ for $f \neq \mathbb{1}$, $\mu(\mathbb{1}) := 0$ defines a stably unbounded conjugation-invariant norm on $G$. Quasi-morphisms on contactomorphism groups are constructed for real projective spaces by Givental [18] and generalizing his construction, for Lens spaces by Granja et al. [20], as well as for certain prequantizations by Borman and Zapolsky [7]. Thus, in all these cases, the contactomorphism groups carry stably unbounded conjugation invariant norms.

Finally, we note that, as with diffeomorphisms, a fragmentation property holds for contactomorphisms (see Banyaga [4]) and any open cover therefore induces a corresponding fragmentation norm (c.f. [8] for diffeomorphisms, [9] for contactomorphisms). With some care in the choice of cover this norm will be conjugation-invariant and, as we show in Sect. 3.4, will dominate all known conjugation-invariant norms, a situation analogous to that for diffeomorphism groups [8].

The aim of the present paper is twofold. In Sect. 2 for certain contact manifolds admitting a 1-periodic Reeb flow we give yet another construction of a stably unbounded conjugation-invariant norm on the (universal cover of) contactomorphism groups. The construction involves the partial order on contactomorphism groups introduced in [14]. Stable unboundedness is automatic for compact manifolds, and for non-compact manifolds it is deduced from basic results on symplectic intersections. The examples include various prequantization spaces such as $T^*X \times S^1$ with closed $X$, prequantizations of symplectically aspherical manifolds containing a closed Bohr–Sommerfeld Lagrangian submanifold and the standard projective spaces $\mathbb{R}P^{2n+1}$.

These results are contrasted with the following statement proved in Sect. 3: if the contact fragmentation norm is bounded, which in particular holds for $V = S^{2n+1}$, then every known conjugation invariant norm on the identity component of the contactomorphism group is trivial. The proof follows closely [8].

## 2 Constructions

### 2.1 Preliminaries

Let $(V, \xi)$ be a contact manifold, not necessarily closed, with co-oriented contact structure $\xi$. Assume moreover that $\lambda$ is a contact form which obeys the co-orientation and whose Reeb vector field generates a circle action $e_t$, $t \in S^1$.

Let us fix some notation for the groups we will be dealing with. We write $\mathcal{G}(V, \xi)$ for the identity component of the group of compactly supported contactomorphisms of $(V, \xi)$. This is shortened to $\mathcal{G}(V)$ or $\mathcal{G}$ when clear from the context. We denote by $\widetilde{\mathcal{G}}(V)$ or $\widetilde{\mathcal{G}}$ the universal cover of $\mathcal{G}(V) = \mathcal{G}$.

Contact isotopies supported in a given open set $X \subset V$ give rise to subgroups of $\mathcal{G}(V)$ and $\widetilde{\mathcal{G}}(V)$ generated by these isotopies. We denote them respectively by $\mathcal{G}(X) \subset \mathcal{G}(V)$ and $\widetilde{\mathcal{G}}(X, V) \subset \widetilde{\mathcal{G}}(V)$. Let us mention that in general $\widetilde{\mathcal{G}}(X, V)$ does not coincide with the universal cover $\widetilde{\mathcal{G}}(X)$ of $\mathcal{G}(X)$. However there exists a natural epimorphism

$$\widetilde{\mathcal{G}}(X) \to \widetilde{\mathcal{G}}(X, V). \tag{1}$$

When $V$ is closed, the construction and properties of our norm on $\widetilde{\mathcal{G}}(V)$, resp. $\mathcal{G}(V)$, are quite direct, whereas for open $V$ they are more involved. Indeed, to handle the open case, we first define and study a norm on the auxiliary group $\mathcal{G}_e(V, \lambda)$ consisting of contactomorphisms of the form $e_t \cdot \phi$ where $\phi \in \mathcal{G}(V)$. When $V$ is closed these groups coincide, $\mathcal{G}(V) = \mathcal{G}_e(V)$, and the general construction we give in Sect. 2.2 for both open or closed $V$ reduces to a direct construction in the case of closed $V$. The reader who wishes to see the simpler case of closed $V$ first may proceed directly to Sect. 2.2, where Remark 2.13 gives the needed background.

We emphasize that by *contactomorphism*, we always mean a diffeomorphism preserving the co-oriented contact structure. We do not, however, require contactomorphisms to preserve any specific contact form. We denote by $\widetilde{\mathcal{G}}_e$ the universal cover of $\mathcal{G}_e$.

When $V$ is an open manifold, every $f \in \mathcal{G}_e(V)$ coincides with some $e_t$ outside a sufficiently large compact subset, so we have a fibration $\mathcal{G}(V) \to \mathcal{G}_e(V) \to S^1$. The exact homotopy sequence yields in this case that

$$0 = \pi_2(S_1) \to \pi_1(\mathcal{G}) \to \pi_1(\mathcal{G}_e) \ ,$$

and hence $\pi_1(\mathcal{G}) \to \pi_1(\mathcal{G}_e)$ is a monomorphism. This implies that $\widetilde{\mathcal{G}}$ can be considered as a subgroup of $\widetilde{\mathcal{G}}_e$.

Let $SV = (V \times \mathbb{R}_+, d(s\lambda))$ be the symplectization of $V$. For a contactomorphism $f$ of $V$ we write $\bar{f}$ for the corresponding $\mathbb{R}_+$-equivariant symplectomorphism of $SV$. A time-dependent function $F_t : SV \to \mathbb{R}$ which is $\mathbb{R}_+$-equivariant, i.e. such that $F_t(sx) = sF_t(x)$ for all $s \in \mathbb{R}_+, x \in SV$, is called a *contact Hamiltonian*. The Hamiltonian flow it defines is also $\mathbb{R}_+$-equivariant and so produces a contact isotopy of $(V, \lambda)$. Moreover, every contact isotopy $f_t$ is given uniquely by such an $F_t$. In particular, the Reeb flow $e_t$ is given by the (time-independent) contact Hamiltonian $s$.

Any $f \in \widetilde{\mathcal{G}}_e$ is a homotopy class of path connecting the identity to a fixed contactomorphism. We write $\{f_t\}_{t\in[a,b]}$ for a specified path in the class $f$, $\{f_t\}$ when $[a, b] = [0, 1]$, and $\{f_t\}_{t\in S^1}$ for a loop. We remind that the contact Hamiltonian depends on this choice of path within the class $f$ and is not uniquely defined by $f$. We write $H(f_t)$ to denote the Hamiltonian for the path $\{f_t\}$. We recall that the product $fg$ of $f, g \in \widetilde{\mathcal{G}}_e$ can be represented by the path $\{f_t g_t\}$, and by the cocycle formula

$$H(f_t g_t) = H(f_t) + H(g_t) \circ (\bar{f}_t)^{-1}. \tag{2}$$

*Stabilization and suspension* We consider $\mathrm{Stab}(SV) := SV \times T^*S^1$, the stabilization of $SV$, with symplectic form $d[s\lambda + rdt]$. For any subset $X \subset SV$ we then define $\mathrm{Stab}_0(X) = X \times \mathcal{O}_{S^1} \subset SV \times T^*S^1$, where in general in this paper we write $\mathcal{O}_M$ for the zero section of $T^*M$. The stabilization $\mathrm{Stab}(SV)$ is symplectomorphic to the symplectization of the contact manifold $(V \times T^*S^1, \lambda + rdt)$, since under the diffeomorphism $\sigma : S(V \times T^*S^1) \to SV \times T^*S^1$, $(s, u, r, t) \mapsto (s, u, sr, t)$ for $s \in \mathbb{R}_+, u \in V, r \in \mathbb{R}, t \in S^1$, the Liouville form $s\lambda + rdt$ on the exact symplectic manifold $SV \times T^*S^1$ pulls back to the form $s(\lambda + rdt)$ on $S(V \times T^*S^1)$. This identification induces an $\mathbb{R}_+$-action on $SV \times T^*S^1$ given by $c \cdot (z, r, t) = (c \cdot z, cr, t)$ where $z \in SV, r \in \mathbb{R}, t \in S^1$.

**Definition 2.1** We say that a compact set $B \subset SV$ has *stable intersection property* if $\mathrm{Stab}_0(B)$ cannot be displaced from $SV \times \mathcal{O}_{S^1}$ by an $\mathbb{R}_+$-equivariant Hamiltonian diffeomorphism of $SV \times T^*S^1$.

*Remark 2.2* Our terminology related to stabilizations is slightly different from that defined in [14, Sect. 2.2]. In that paper the notion is used in the contact category and the stabilization of the contact manifold $V$ is the contact manifold $\mathrm{Stab}(V)$ obtained by quotienting $SV \times T^*S^1$ via the $\mathbb{R}_+$-action; by contrast, we remain in the symplectic category and refer to the symplectic manifold $SV \times T^*S^1$ as the stabilization $\mathrm{Stab}(SV)$ of $SV$. In [14] the stabilization $\mathrm{Stab}(K)$ of subsets $K \subset V$ is only defined for certain $K$ (Legendrians and pre-Lagrangians) which lift (non-canonically) to Lagrangians $L \subset SK$ and $\mathrm{Stab}(K)$ is taken to be the (canonical) image of $L \times \mathcal{O}_{S^1}$ after quotienting. Their definition of $\mathrm{Stab}(K)$ could also apply to $K = V$ but this is not done and would produce a conflict with the definition of $\mathrm{Stab}(V)$ mentioned earlier. In the symplectic category we do not need to worry about lifts and we define the stabilization $\mathrm{Stab}_0(X) = X \times \mathcal{O}_{S^1}$ for any set $X \subset SV$. The subscript prevents conflict with $\mathrm{Stab}(SV) = SV \times T^*S^1$ in the case $X = SV$.

*Remark 2.3* The definition of stable intersection property in [14] is for a pair of sets $(K, A)$ in $V$ and corresponds to non-displaceability of $\mathrm{Stab}(K)$ from $\mathrm{Stab}(A)$ via contactomorphisms; in our case we consider de facto only $K = V$ and say that $B \subset SV$ has stable intersection property when $\mathrm{Stab}_0(B)$ cannot be displaced from $\mathrm{Stab}_0(SV)$ by an $\mathbb{R}_+$-equivariant Hamiltonian diffeomorphism. It is important to note that stable intersection property of $(K, A)$ in the sense of [14] for some $K$ implies it for $K = V$ and, thus, results establishing stable intersection property of $(K, A)$ in the sense of [14] always imply a lift $B \subset SV$ of $A$ has stable intersection property in our sense.

In the present paper every use of the stable intersection property in fact uses only non-existence of a displacing $\mathbb{R}_+$-equivariant Hamiltonian diffeomorphism of $\mathrm{Stab}(SV)$ which is a *suspension* of a contractible loop of contactomorphisms of $V$ (see [14]). We recall the notion of suspension next. This construction associates to a loop of contactomorphisms of $V$ an $\mathbb{R}_+$-equivariant symplectomorphism of $SV \times T^*S^1$ as follows. Let $\varphi = \{\phi_t\}_{t\in S^1}$, $\phi_0 = \phi_1 = \mathbb{1}$ be a loop of contactomorphisms in $\mathcal{G}_e(V)$ generated by a contact Hamiltonian $\Phi_t$ on $SV$. Define *the suspension map*

$$\Sigma_\varphi : SV \times T^*S^1 \to SV \times T^*S^1$$

of $\varphi$ by

$$(z, r, t) \mapsto \left(\bar{\phi}_t z, r - \Phi_t\left(\bar{\phi}_t z\right), t\right).$$

The map $\Sigma_\varphi$ is an $\mathbb{R}_+$-equivariant symplectomorphism of $SV \times T^*S^1$. Given two loops $\varphi$ and $\theta$, the co-cycle formula implies $\Sigma_{\varphi\circ\theta} = \Sigma_\varphi \circ \Sigma_\theta$. Furthermore, if $\varphi$ is contractible and $\varphi^{(s)}$ is a homotopy of $\varphi = \varphi^{(0)}$ to the constant loop $\varphi^{(1)} = \mathbb{1}$, the family of suspension maps $\Sigma_{\varphi^{(s)}}$ is a Hamiltonian isotopy of $SV \times T^*S^1$.

For certain purposes, the following modification of the stable intersection property will be useful. We write $\Pi$ for the image of the fundamental group $\pi_1(\mathcal{G}, \mathbb{1})$ in $\pi_1(\mathcal{G}_e, \mathbb{1})$ under the natural inclusion morphism. Each loop in $\mathcal{G}_e$ representing an element of $\Pi$ can be written as a product of a contractible loop in $\mathcal{G}_e$ and a loop in $\mathcal{G}$ (note that the order of factors is not important since $\mathcal{G}$ is a normal subgroup of $\mathcal{G}_e$). We consider a compact subset $B \subset SV$ which satisfies the following condition:

$$\Sigma_\varphi(SV \times \mathcal{O}_{S^1}) \cap (B \times \mathcal{O}_{S^1}) \neq \emptyset, \quad \forall \,\text{loop}\, \varphi \text{ s.t. } [\varphi] \in \Pi. \tag{3}$$

The importance of these stable intersection properties for us rests on the following Lemma.

**Lemma 2.4** (c.f. Proposition [14, 2.3.B]) *Let $B \subset SV$ be a compact set with stable intersection property. Then for every contractible loop $\varphi = \{\phi_t\}_{t\in S^1}$ its contact Hamiltonian $\Phi$ vanishes for some $t_0 \in S^1$ and $y \in B$: $\Phi_{t_0}(y) = 0$. If $B$ satisfies condition (3) then the same holds true for every loop $\varphi$ representing an element of $\Pi$.*

*Proof* The stable intersection property implies that the sets $\Sigma_\varphi(SV \times \mathcal{O}_{S^1})$ and $B \times \mathcal{O}_{S^1}$ intersect. Thus there exist $z \in SV$ and $t_0 \in S^1$ such that $\bar{\phi}_{t_0} z \in B$ and $\Phi_{t_0}(\bar{\phi}_{t_0} z) = 0$. Setting $y = \bar{\phi}_{t_0} z$, we get the first statement of the lemma. The second statement is analogous. □

*Partial order on contactomorphisms* The following binary relation introduced and studied in [14] plays a crucial role in our story: we write $f \succeq \mathbb{1}$, $f \in \widetilde{\mathcal{G}}_e$ if $f$ "can be given by a non-negative contact Hamiltonian", i.e. there is some path $\{f_t\}$ in the class of $f$ having a non-negative contact Hamiltonian. Observe that in this case $\frac{d}{dt} f_t(x) \in T_{f_t x} V$ belongs to the non-negative half-space bounded by the contact hyperplane $\ker(\lambda_{f_t x})$. We remark that having a non-negative Hamiltonian is a coordinate-free condition and so $f \succeq \mathbb{1}$ is invariant under conjugation of $f$ in $\widetilde{\mathcal{G}}_e$. We write $f \succeq g$ if $fg^{-1} \succeq \mathbb{1}$. By conjugating with $g^{-1}$, we have the equivalent definition: $f \succeq g$ if $g^{-1} f \succeq \mathbb{1}$. Observe that the relation $\succeq$ is reflexive.

We denote by $e$ the element of $\widetilde{\mathcal{G}}_e$ represented by the path $\{e_t\}_{t\in[0,1]}$ and denote by $e^c$, $c \in \mathbb{R}$ the element represented by $\{e_t\}_{t\in[0,c]}$ (so $e$ is shorthand for $e^1$).

**Lemma 2.5** (1) *For $f, g \in \widetilde{\mathcal{G}}_e$, $f \succeq g$ if and only if $f$ and $g$ can be given by Hamiltonians $F$ and $G$ such that $F \geq G$. Moreover one can prescribe either $F$ or $G$ in advance, hence $\succeq$ is transitive;*
(2) *$f \succeq g$ and $a \succeq b$ implies that $fa \succeq gb$, i.e. $\succeq$ is bi-invariant;*
(3) *Any element $\phi \in \widetilde{\mathcal{G}}_e$ generated by a positive contact Hamiltonian bounded away from zero on the hypersurface $\{s = 1\}$ is* dominant*, i.e. $\forall f \in \widetilde{\mathcal{G}}_e, \exists p \in \mathbb{N}$ s.t. $\phi^p \succeq f$. In particular any $e^c$, $c > 0$ is dominant.*

*Proof* To prove the *if* direction of the first property, fix paths $\{f_t\}$ and $\{g_t\}$ for $f$ and $g$ such that $H(f_t) \geq H(g_t)$. By (2)

$$\begin{aligned} H(g_t^{-1} f_t)(z,t) &= H(g_t^{-1})(z,t) + H(f_t)(\bar{g}_t z, t) \\ &= -H(g_t)(\bar{g}_t z, t) + H(f_t)(\bar{g}_t z, t). \end{aligned}$$

So $\{g_t^{-1} f_t\}$ is a path in the class $g^{-1} f$ having non-negative Hamiltonian. Now, to prove the *only if* direction, let $\{h_t\}$ be a path in the class $g^{-1} f$ having non-negative Hamiltonian $H(h_t)$. Let $\{g_t\}$ be an arbitrary path for $g$ and set $f_t = g_t h_t$ (or if we wish to prescribe $f_t$ then set $g_t$ accordingly). Then $h_t = g_t^{-1} f_t$ and so the earlier computation shows that $H(f_t) \geq H(g_t)$.

The second property is also proved using the cocycle formula (2) (comparing Hamiltonians for $f(ab^{-1})$ and $g$). The third property is straightforward (note that $cs$ is a contact Hamiltonian for $e^c$). □

*Remark 2.6* The Hamiltonians $F$ and $G$ in Lemma 2.5 item 1 can moreover be taken to be 1-periodic. This is because a non-negative (resp. positive) isotopy can be homotoped within the class of non-negative (resp. positive) isotopies to one with 1-periodic Hamiltonian. For positive isotopies this is given by Lemma [14, 3.1.A], see also the argument within the proof of Theorem [15, 1.19]. The same arguments go through for non-negative isotopies.

*Remark 2.7* The definition of $\succeq$ summarized above (due to [14]) applies very generally in the universal cover of any connected contactomorphism group when the underlying contact structure is co-oriented. Reflexivity, transitivity and bi-invariance of $\succeq$ are given by the previous Lemma. Anti-symmetry however is not automatic.As the reader may verify, it fails on the universal cover of a group of contactomorphisms if and only if that group contains a non-constant contractible loop with non-negative Hamiltonian (c.f. [14, Proposition 2.1.A]).

**Definition 2.8** For closed $V$, $\widetilde{\mathcal{G}}(V)$—or for short-hand simply $V$—is said to be *orderable*[1] [15] when $\succeq$ is anti-symmetric on $\widetilde{\mathcal{G}}(V)$. For $V$ with 1-periodic Reeb flow we define $\widetilde{\mathcal{G}}_e(V)$ to be *orderable* when $\succeq$ is anti-symmetric on $\widetilde{\mathcal{G}}_e(V)$.

We note that when $V$ is closed and admits 1-periodic Reeb flow $\widetilde{\mathcal{G}}_e(V) = \widetilde{\mathcal{G}}(V)$ so the two definitions of orderability coincide.

*Remark 2.9* We note that the notion of orderability in Definition 2.8 depends, in general, on the choice of contact form with 1-periodic Reeb flow. In practice, however, we will deduce orderability from Theorem 2.10 whose assumptions are not related to a specific contact form. In particular, when the theorem is applicable it implies orderability of $\widetilde{\mathcal{G}}_e(V)$ for *any* 1-periodic Reeb flow $e$.

[1] In fact, although the study of $\succeq$ on $\widetilde{\mathcal{G}}(V)$ was initiated in [14], the terminology orderable was introduced later in [15] and was applied to $V$ itself; to avoid confusion we specify instead the relevant group.

The next result is an analogue to our setting of Theorem [14, 2.3A] which was stated and proved in [14] only for closed $V$.

**Theorem 2.10** *Suppose that $V$ admits 1-periodic Reeb flow. If $SV$ contains a compact set $B$ with stable intersection property then $\succeq$ is anti-symmetric on $\widetilde{\mathcal{G}}_e(V)$, i.e. $\widetilde{\mathcal{G}}_e(V)$ is orderable.*

*Remark 2.11* (*Proof overview and comparison with* [14]) As mentioned in Remark 2.7, anti-symmetry of $\succeq$ is equivalent to the non-existence of a non-constant, non-negative contractible loop of contactomorphisms. The proof of Proposition [14, 2.3.A] derives this non-existence from the stable intersection property using two results: Propositions [14, 2.1.B] and [14, 2.3.B].

The first, Proposition [14, 2.1.B], applies to $\mathcal{G}(V)$ for closed $V$ and says that existence of a non-negative, non-constant contractible loop implies existence of a positive one. The proof in [14] uses compactness of $V$ in an essential way; we modify it significantly to deal with our setting in Proposition 2.12 below.

The second, Proposition [14, 2.3.B], shows that stable intersection property prevents existence of a positive contractible loop. It generalizes immediately to our setting and was stated and proved already as Lemma 2.4.

**Proposition 2.12** (c.f. Proposition [14, 2.1.B]) *Assume there exists a non-negative non-constant loop $\{f_t\}_{t\in S^1}$ of contactomorphisms. Then for any compact set $C \subset V$ there exists a loop $\{g_t\}_{t\in S^1}$ of contactomorphisms whose contact Hamiltonian $G_t$ is positive on $SC$ for all $t \in S^1$. Moreover, when $\{f_t\}_{t\in S^1}$ is contractible so is $\{g_t\}_{t\in S^1}$. Compact support is also retained.*

*Proof of Theorem 2.10* As described in Remark 2.11, Proposition 2.12 and Lemma 2.4 together imply Theorem 2.10. More precisely, given compact $B \subset SV$ with stable intersection property we let $C$ be its projection to $V$ and apply Proposition 2.12. □

*Proof of Proposition 2.12* At first we proceed as in the first two steps of the proof of Proposition [14, 2.1.B], and then we resort to a modification.

As in step 1 of [14, 2.1.B], without loss of generality we may assume $F(z,0) \neq 0$ for some $z \in SV$. Let $U \subset V$ be such that $F(z,0) > 0$ for all $z \in SU$.

As in step 2 of [14, 2.1.B], we take a sequence $\varphi_1, \ldots, \varphi_d$ of elements of $\mathcal{G}_e$ such that $C \subset \bigcup_{k=0}^{d} \psi_k(U)$, where $\psi_0 = \mathbb{1}$, $\psi_k = \varphi_1 \cdots \varphi_k$ for $k = 1, \ldots, d$. This can be done since $C$ is compact. Define

$$g_t = f_t \varphi_1 f_t \ldots \varphi_d f_t (\varphi_1 \cdots \varphi_d)^{-1}. \tag{4}$$

This forms a loop $\{g_t\}_{t\in S^1}$ generated by the Hamiltonian

$$\begin{aligned} G(z,t) = F(z,t) &+ F\left(\bar{\varphi}_1^{-1} \bar{f}_t^{-1} z, t\right) \\ &+ \cdots + F\left(\bar{\varphi}_d^{-1} \bar{f}_t^{-1} \cdots \bar{\varphi}_1^{-1} \bar{f}_t^{-1} z, t\right). \end{aligned}$$

For $z \in SU$ the first summand is positive when $t = 0$. On the other hand, for $z \in SC \backslash SU$ there exists $k$ such that $\bar{\psi}_k^{-1} z \in SU$, in which case the $k$'th summand is positive when $t = 0$. Since all summands are non-negative we conclude $G(z,0) > 0$ for all $z \in SC$. Note that $\{g_t\}_{t\in S^1}$ is contractible if $\{f_t\}_{t\in S^1}$ is, and compact support is also retained, so without loss of generality we now assume $F(z,t) > 0$ for all $z \in SC$, $t = 0$ and hence for all $z \in SC$ and $t \in \Delta \subset S^1$ a closed interval containing 0.

Let $H$ be an autonomous contact Hamiltonian $H : SV \to \mathbb{R}$ which is positive on the set $\cup_{t\in S^1} \bar{f}_t(SC)$. Moreover assume $H$ vanishes outside $SK$ for some compact $K \subset V$ so the associated contact isotopy $h_t$ is compactly supported. By construction, $H(\bar{f}_t^{-1}z) > 0$ for all $t \in S^1$, $z \in SC$. We claim there exists a smooth function $u : [0, 1] \to \mathbb{R}$ with the following properties:

- $u(0) = u(1) = 0$.
- For all $t \notin \Delta$, $u'(t) > 0$.
- For all $t \in \Delta$,

$$u'(t) > -\min_{z\in SC} \frac{F(z,t)}{H\left(\bar{f}_t^{-1}z\right)}. \tag{5}$$

Indeed, the minimum in (5) exists and is positive, since for each $t \in \Delta$, $F(z,t)/H(\bar{f}_t^{-1}z)$ is a well defined, positive, and $\mathbb{R}_+$-invariant function on $SC$; therefore, we can allow $u'$ to be negative on part of $\Delta$ and so make it positive outside of $\Delta$.

Consider now the loop $g_t = f_t h_{u(t)}$. Clearly, it is homotopic to $\{f_t\}$ via the endpoint-preserving homotopy $\{f_t h_{su(t)}\}$. It is also compactly supported when $f_t$ is (because $\{h_t\}$ has compact support).

Moreover, its Hamiltonian is $G(z,t) = F(z,t) + u'(t)H(\bar{f}_t^{-1}z)$ which, by condition (5), is positive for all $z \in SC$ and $t \in \Delta$. We claim that $G(z,t) > 0$ for all $z \in SC$ and $t \in S^1$. Indeed, if $t \notin \Delta$, then $u'(t) > 0$ and, as observed above, $H(\bar{f}_t^{-1}z) > 0$ for $z \in SC$, thus implying $G(z,t) > 0$ (since $F \geq 0$). □

## 2.2 The norm $\nu$

Throughout this section we assume that $\widetilde{\mathcal{G}}_e(V)$ is orderable; that is, the relation $\succeq$ is a partial order on $\widetilde{\mathcal{G}}_e(V)$ (see Remark 2.7).

*Remark 2.13* (*Compact manifolds*) When restricting attention to closed $V$ only, Sect. 2.2 can be read independently of the previous section. Let us note that in this case the group $\mathcal{G}_e$ is simply $\mathcal{G}$, the identity component of the contactomorphism group of $V$. For the sake of readers who skipped the previous section, we recall the relation defined on $\widetilde{\mathcal{G}}$ in [14] by $f \succeq g$ if and only if $f$ and $g$ can be generated by contact Hamiltonians $F$ and $G$, respectively, such that $F \geq G$. This is a bi-invariant, transitive and reflexive relation on $\widetilde{\mathcal{G}}$ (see Lemma 2.5 or [14] for more details). $V$ is called *orderable* when $\succeq$ is anti-symmetric (i.e., a partial order), and we assume this is the case throughout this section. We recall additionally that the Reeb loop $e = [\{e_t\}_{t\in S^1}]$ is *dominant*, that is, for every $f \in \widetilde{\mathcal{G}}$ there is $k \in \mathbb{N}$ such that $e^k \succeq f$.

Observe that $e$ lies in the center of $\widetilde{\mathcal{G}}_e$. For an element $f \in \widetilde{\mathcal{G}}_e$ consider the following invariants:

$$\nu_+(f) := \min\{n \in \mathbb{Z} \ : \ e^n \succeq f\}$$

and

$$\nu_-(f) := \max\{n \in \mathbb{Z} \ : \ e^n \preceq f\}.$$

Note that $\nu_- \leq \nu_+$ by transitivity of $\succeq$.
It is readily checked that $\nu_+$ and $\nu_-$ are conjugation-invariant (since $e$ is in the center) and $\nu_-(f) = -\nu_+(f^{-1})$, using the bi-invariance of $\succeq$. We also observe that $\nu_+$ and $\nu_-$ are respectively sub- and super-additive:

$$\nu_+(fg) \le \nu_+(f) + \nu_+(g)$$
$$\nu_-(fg) \ge \nu_-(f) + \nu_-(g).$$

We write

$$\nu(f) := \max(|\nu_+(f)|, |\nu_-(f)|).$$

*Remark 2.14* Let $k \in \mathbb{Z}_{\ge 0}$. We remark that $\nu(f) \le k$ if and only if $e^{-k} \preceq f \preceq e^k$. This in turn is equivalent to the property: $f$ can be generated by a contact Hamilonian $F_+$ such that $F_+ \le ks$ and also by a contact Hamiltonian $F_-$ such that $F_- \ge -ks$ (see Lemma 2.5). By Remark 2.6, the Hamitonians $F_\pm$ may moreover be assumed to be 1-periodic.

We have the following.

**Theorem 2.15** *$\nu$ is a conjugation-invariant norm on $\widetilde{\mathcal{G}}_e$.*

*Proof* Clearly $\nu \ge 0$. Observe moreover that $\nu(f) = 0$ if and only if $\mathbb{1} \succeq f \succeq \mathbb{1}$ and hence $f = \mathbb{1}$. Since both $\nu_+$ and $\nu_-$ are conjugation-invariant, $\nu$ is as well. It remains to prove the triangle inequality. Observe that, by Remark 2.14,

$$\nu(f) = \min\{k \in \mathbb{Z}_{\ge 0} \,:\, e^{-k} \preceq f \preceq e^k\}.$$

Now, let $f, g \in \widetilde{\mathcal{G}}_e$, and put $m = \nu(f)$, $n = \nu(g)$. Then

$$e^{-m} \preceq f \preceq e^m; \quad e^{-n} \preceq g \preceq e^n.$$

By bi-invariance of $\succeq$, this implies $e^{-(m+n)} \preceq fg \preceq e^{m+n}$, which gives $\nu(fg) \le m + n = \nu(g) + \nu(g)$. □

*Remark 2.16* Note that $a \succeq b \succeq \mathbb{1}$ implies $\nu(a) \ge \nu(b)$. In other words, $(\widetilde{\mathcal{G}}(V), \nu, \succeq)$ is a partially ordered metric space in the sense of [14, Section 1.7].

*Remark 2.17* We note that the norm $\nu$ depends not only on the contact structure $\xi$, but rather on the specific choice of contact form with 1-periodic Reeb orbit. When $M$ is closed, however, the *equivalence class* of the norm $\nu$ depends only on the contact structure. Indeed, given two contact forms with 1-periodic Reeb flows represented respectively by elements $e, f \in \widetilde{\mathcal{G}}$, denote by $\nu_e$ and $\nu_f$ the corresponding norms on $\widetilde{\mathcal{G}}$. One easily verifies the inequality $\nu_e \le \nu_e(f) \cdot \nu_f$, which shows that the two norms are equivalent.

When $V$ is open, however, we cannot compare the two norms as above. It would be interesting to study whether in this case one may find two contact forms with 1-periodic Reeb flows giving rise to non-equivalent norms. We thank an anonymous referee for posing this question to us.

*Remark 2.18* We note $\nu(e^n) = |n|$ for all $n \in \mathbb{Z}$; therefore, $\nu$ is always stably unbounded on $\widetilde{\mathcal{G}}_e(V)$. In particular when $V$ is closed, $\nu$ is stably unbounded on $\widetilde{\mathcal{G}}(V) = \widetilde{\mathcal{G}}_e(V)$. In general the interesting question is when stable unboundedness passes to $\widetilde{\mathcal{G}}$ and $\mathcal{G}$. The following results give sufficient conditions for this to occur. They appeal to the stable intersection property and its modification (3).

**Theorem 2.19** *Suppose that a compact set $B \subset SV$ has stable intersection property and in addition $B$ is invariant under the flow $\bar{e}_t$. Assume $f \in \widetilde{\mathcal{G}}_e$ can be generated by a contact Hamiltonian $F_t$ satisfying $F_t > cs$ on $B$ for some $c > 0$ and all $t \in S^1$. Then $\nu_+(f) \ge [c]$, the integer part of $c$, and so $\nu(f) \ge [c]$.*

**Corollary 2.20** *Suppose that a compact set $B \subset SV$ has stable intersection property and in addition $B$ is invariant under the flow $\bar{e}_t$. Then $\nu$ is stably unbounded on $\widetilde{\mathcal{G}}(V) \subset \widetilde{\mathcal{G}}_e(V)$. Moreover, $\widetilde{\mathcal{G}}(V)$ admits a quasi-isometric monomorphism of the real line. In fact these statements already hold for $\widetilde{\mathcal{G}}(X, V) \subset \widetilde{\mathcal{G}}(V)$ for any open set $X \subset V$ which contains the projection $C = \pi(B)$ of $B$ to $V$.*

*Proof* Suppose $c \in \mathbb{N}$ and $\epsilon > 0$. Let $C = \pi(B) \subset V$ be the (compact) projection of $B$. Define $F$ to be an autonomous contact Hamiltonian equal to $(c+\epsilon)s$ on $SC$ and supported in a larger $SK$, with $K \subset V$ compact. Then $F$ generates an element $f \in \mathcal{G}(V)$ which satisfies the hypotheses of Theorem 2.19. In particular if $n \in \mathbb{Z}$ then $nF$ is a Hamiltonian for $f^n$. For $n > 0$ it strictly exceeds $cn$ on $B$ so the Theorem gives $\nu_+(f^n) \geq cn$, while for $n < 0$ we obtain $\nu_-(f^n) \geq c|n|$. We conclude $\nu(f^n) \geq c|n|$ for all $n \in \mathbb{Z}$. Moreover, taking $\epsilon < 1$, we can arrange that $|F| \leq (c+1)s$ on $SV$, which by Remark 2.14 gives $\nu(f^n) \leq (c+1)|n|$. Since $F$ is autonomous its Hamiltonian flow $t \mapsto f_t$ thus defines a quasi-isometric monomorphism $\mathbb{R} \to \widetilde{\mathcal{G}}(V)$. □

*Proof of Theorem 2.19* Without loss of generality we may assume $c$ to be an integer. Suppose it is not true that $\nu_+(f) \geq c$. Then $f \preceq e^c$ (recall $e^c$ denotes the class of the path $\{e_{ct}\}$). This means that

$$H\left(\bar{f}_t\right) \leq H\left(\bar{e}_{ct}\bar{\phi}_t\right) \tag{6}$$

for some contractible loop $\varphi = \{\phi_t\}$ on $\mathcal{G}_e$. By the cocycle formula, this yields $F_t \leq cs + \Phi_t \circ \bar{e}_{-ct}$. Applying Lemma 2.4, we see that $\Phi_{t_0}(y) = 0$ for some $t_0 \in S^1$ and $y \in B$. Since $x := \bar{e}_{ct_0} y \in B$, we get that $F_{t_0}(x) \leq cs(x)$, contradicting the assumption $F_t|_B > cs$ for all $t$. □

If we consider only compactly supported contactomorphisms, i.e. restrict the above norm to $\widetilde{\mathcal{G}}$, it descends to $\mathcal{G}$ as follows. Given $f' \in \mathcal{G}$, define

$$\nu_*(f') := \inf \nu(f)\ , \tag{7}$$

where the infimum is taken over all lifts $f$ of $f'$ to $\widetilde{\mathcal{G}} \subset \widetilde{\mathcal{G}}_e$. Observe that $\nu_*$ is non-degenerate: indeed, since $\nu(f)$ is integer, the infimum is necessarily attained on some lift $f$, but $\nu(f) = 0$ yields $f = \mathbb{1}$ and hence $f' = \mathbb{1}$.

**Theorem 2.21** *Suppose that a compact set $B$ satisfies condition (3) and in addition $B$ is invariant under the flow $\bar{e}_t$. Assume $f \in \widetilde{\mathcal{G}}$ is generated by a contact Hamiltonian $F_t$ satisfying $F_t > cs$ on $B$, for some $c > 0$ and all $t \in S^1$. Let $f' \in \mathcal{G}$ be the time-one endpoint of $f$. Then $\nu_*(f') \geq [c]$.*

**Corollary 2.22** *Suppose that a compact set $B \subset SV$ satisfies condition (3) and in addition $B$ is invariant under the flow $\bar{e}_t$. Then $\nu_*$ is stably unbounded on $\mathcal{G}(V)$. Moreover, $\mathcal{G}(V)$ admits a quasi-isometric monomorphism of the real line. In fact these statements already hold for $\mathcal{G}(X) \subset \mathcal{G}(V)$ for any open set $X \subset V$ which contains the projection $C = \pi(B)$ of $B$ to $V$.*

The proof of Theorem 2.21 repeats verbatim that of Theorem 2.19 with the following observation: given $f$ and $f'$ as in our hypotheses, inequality (6) holds for a loop $\varphi = \{\phi_t\}$ representing an element of $\Pi$. The derivation of Corollary 2.22 from Theorem 2.21 is analogous to that of Corollary 2.20 from Theorem 2.19.

### 2.3 Examples

In this section we discuss some settings where our norm is well-defined.

*Remark 2.23* In Examples 2.24, 2.26, 2.28 below we consider $V$ of the form $T^*Z \times S^1$. In all these settings, given a set $Y \subset T^*Z$ we write $\widehat{Y} := Y \times S^1$ to denote its lift to $V$.

*Example 2.24* Assume that $V = T^*X \times S^1$, where $X$ is a closed manifold, equipped with the contact form $\lambda = d\tau - pdq$ whose Reeb flow is given by $e_t(p, q, \tau) = (p, q, \tau + t)$. In this setting we have the following result.

**Proposition 2.25** *The norm $\nu$ is well-defined and stably unbounded on $\widetilde{\mathcal{G}}$. Moreover, the norm $\nu_*$ on $\mathcal{G}$, defined by (7), is stably unbounded.*

Let us mention before the proof that the norm $\nu_*$ is greater than or equal to the norm defined by Zapolsky in [39] (this readily follows from [39]).

*Proof* Consider the Lagrangian submanifold

$$B := \mathcal{O}_X \times \{s = 1\} \subset SV,$$

where $\{s = 1\} \subset S^1 \times \mathbb{R}_+$. We claim that $B$ has stable intersection property. Indeed, the symplectic embedding $SV \to T^*X \times T^*S^1$, $(q, p, \tau, s) \mapsto (q, -sp, \tau, s - 1)$ maps $B$ to the zero-section $\mathcal{O}_{X\times S^1}$ so the claim follows from the standard fact that the zero section of $T^*(X \times S^1 \times S^1)$ cannot be displaced from itself by any Hamiltonian diffeomorphism [16,19,21]. Thus by Theorems 2.10 and 2.15 and Corollary 2.20, $\nu$ is well-defined on $\widetilde{\mathcal{G}}_e$ and stably unbounded on $\widetilde{\mathcal{G}} \subset \widetilde{\mathcal{G}}_e$. Next we turn to $\nu_*$. We claim that $B$ in fact satisfies condition (3). Indeed, for any loop $\varphi = \{\phi_t\}$ in $\mathcal{G}_e$ representing an element of $\Pi$, we claim that $\Sigma_\varphi(\mathrm{Stab}_0(B))$ has the same Liouville class as $\mathrm{Stab}_0(B)$ and hence these Lagrangian submanifolds intersect by a theorem of Gromov [19, 2.3.$B_4''$]. We prove this claim below. Therefore by Corollary 2.22 we deduce that the norm $\nu_*$ is stably unbounded on the group $\mathcal{G}$.

Finally we address the claim regarding Liouville classes. We need to show $[\Lambda|_{\mathrm{Stab}_0(B)}] = [\Lambda|_{\Sigma_\varphi(\mathrm{Stab}_0(B))}]$ where $\Lambda := s\lambda + rdt$ is the Liouville form on $SV \times T^*S^1$ and $\varphi$ represents an element of $\Pi$. By definition, $\Pi$ is the image of $\pi_1(\mathcal{G}, \mathbb{1}) \to \pi_1(\mathcal{G}_e, \mathbb{1})$. Thus there exists a loop $\psi$ on $\mathcal{G}$ such that $\psi$ and $\varphi$ are homotopic as loops in $\mathcal{G}_e$. It follows that $\Sigma_\varphi$ and $\Sigma_\psi$ differ by a Hamiltonian diffeomorphism (because a homotopy between $\varphi$ and $\psi$ yields a path between their suspensions and thus a Hamiltonian path, since we are in an exact setting). But the latter preserve Liouville classes, thus without loss of generality assume $\varphi = \{\phi_t\}$ is a loop on $\mathcal{G}$. By the Künneth formula $H_1(\mathrm{Stab}_0(B))$ is generated by loops of the form $\{(\gamma(\rho), 0, 0)\}_{\rho\in S^1}$ and $\{(z_0, 0, t)\}_{t\in S^1}$ where $\{\gamma(\rho)\}_{\rho\in S^1}$ is a loop in $B$ and $z_0 \in B$. Thus $H_1(\Sigma_\varphi(\mathrm{Stab}_0(B)))$ is generated by loops $\{(\gamma(\rho), -\Phi_0(\gamma(\rho)), 0)\}_{\rho\in S^1}$ and $\{(\bar{\phi}_t z_0, -\Phi_t(\bar{\phi}_t z_0), t)\}_{t\in S^1}$. $\Lambda$ coincides on loops $\{(\gamma(\rho), 0, 0)\}_{\rho\in S^1}$ and $\{(\gamma(\rho), -\Phi_0(\gamma(\rho)), 0)\}_{\rho\in S^1}$ (since $rdt$ vanishes). Before comparing the other loops note that we may deform $\mathcal{O}_X \times S^1$ by a contact isotopy so that at least one point $w$ of the image lies outside the compact support of the isotopy $\phi_t$. This deformation lifts and extends trivially to a Hamiltonian isotopy $h_t$ of $SV \times T^*S^1$ so without loss of generality, appealing also to the Hamiltonian isotopy $\Sigma_\varphi h_t \Sigma_\varphi^{-1}$ for $\Sigma_\varphi(\mathrm{Stab}_0(B))$, we may replace $z_0$ by $(s, w)$ for some $s \in \mathbb{R}_+$ when evaluating $\Lambda$ on loops $\{(z_0, 0, t)\}_{t\in S^1}$ vs. $\{(\bar{\phi}_t z_0, -\Phi_t(\bar{\phi}_t z_0), t)\}_{t\in S^1}$. These loops now lie outside the support of $\Sigma_\varphi$ and so coincide. □

*Example 2.26* As above, let $V = T^*X \times S^1$. Under extra hypotheses, one can make an even stronger statement than just unboundedness of our norm $\nu$. Namely, we have the following result:

**Proposition 2.27** *In the setting of Example 2.26, suppose that $T^*X$ admits a closed Lagrangian submanifold $L \subset T^*X$ such that*

(a) $HF(L, L) \neq 0$ *(Floer homology with coefficients in a field, say $\mathbb{Z}_2$)*
(b) $(a \cdot L) \cap L = \emptyset,\ \forall a > 0, a \neq 1$, *where* $a \cdot (p, q) = (ap, q)$.

*Then for any bounded domain $\mathcal{U} \subset T^*X$ containing the zero section and any $N \in \mathbb{N}$, $\widetilde{\mathcal{G}}(\widehat{\mathcal{U}}, V)$ admits a quasi-isometric monomorphism of $\mathbb{R}^N$.*

Some examples of $X$ and $L$ satisfying the hypotheses of Proposition 2.27 are:

(1) $X$ is a closed manifold admitting a closed 1-form $\alpha$ without zeroes, and $L$ is the graph of $\alpha$.
(2) $X = S^2$ and $L$ is the Lagrangian torus studied by Albers and Frauenfelder in [1] with $HF(L, L; \mathbb{Z}_2) \neq 0$.

*Proof* Fix $N \in \mathbb{N}$ and fix a bounded tube $\mathcal{U} \subset T^*X$ around the zero section. Choose distinct real numbers $a_1, \dots, a_N, a_j \neq 1$ such that $L_j := a_j L \subset \mathcal{U}$. Thus $L_j,\ j = 1, \dots, N$ are pairwise disjoint. We now identify $SV$ with a domain $\mathcal{W} := \Theta(SV) \subset T^*X \times T^*S^1$ via the $\mathbb{R}_+$-equivariant symplectic embedding

$$\Theta : SV \to T^*X \times T^*S^1, (p, q, s, \tau) \mapsto (-s \cdot p, q, s, \tau)$$

where $(p, q) \in T^*X, s \in \mathbb{R}_+, \tau \in S^1$. In $\mathcal{W}$, put $\widehat{L}_j := L_j \times S^1 \subset V$. Let $W_j \subset T^*X$ be tubular neighborhoods of $L_j$ respectively such that $\overline{W_j} \cap \overline{W_i} = \emptyset$ when $i \neq j$, where $\overline{W_j}$ denotes the closure of $W_j$. Then the closures of their lifts $\widehat{W}_j$ to $V$ are pairwise disjoint.

Take contact Hamiltonians $H_j$, with *supp* $H_j \subset S(\widehat{W}_j)$, $H_j = 1$ on $\widehat{L}_j$ and $0 \leq H_j \leq s$. Let $h_j^t$ be the corresponding Hamiltonian flow. Consider the map $\Psi : \mathbb{R}^N \to \widetilde{\mathcal{G}}(\widehat{\mathcal{U}}, V)$ given by $(t_1, \dots, t_N) \mapsto h_1^{t_1} \dots h_N^{t_N}$. This is by construction a homomorphism, which is injective since $W_j$'s are pairwise disjoint. On the one hand, $h_1^{t_1} \dots h_N^{t_N}$ is generated by $H = \sum_{j=1}^N t_j H_j$ so

$$\nu\left(h_1^{t_1} \dots h_N^{t_N}\right) \leq \max_j(|t_j| + 1).$$

This follows by Lemma 2.5 (1) since the supports of $H_j$ are pairwise disjoint and thus $H \leq \max_j |t_j| s$. On the other hand, $H|_{\widehat{L}_j} = t_j$ so by Theorem 2.19

$$\nu\left(h_1^{t_1} \dots h_N^{t_N}\right) \geq \max_j(|t_j| - 1).$$

Indeed, if $\max_j |t_j|$ occurs for $t_i > 0$ apply the Theorem verbatim, while if $t_i < 0$ apply it to the inverse of $h_1^{t_1} \dots h_N^{t_N}$ and use symmetry of $\nu$. Thus

$$||t||_\infty - 1 \leq \nu\left(h_1^{t_1} \dots h_N^{t_N}\right) \leq ||t||_\infty + 1\ ,$$

where $||t||_\infty := \max_j |t_j|$. We conclude that

$$\Psi : \left(\mathbb{R}^N, ||\cdot||_\infty\right) \to \left(\widetilde{\mathcal{G}}(\mathcal{U}, V), \nu\right)$$

is a quasi-isometric monomorphism. □

*Example 2.28* Let $X$ be a closed manifold equipped with a Riemannian metric $\rho$ *without contractible geodesics* and consider $V = T^*X \times T^*\mathbb{T}^k \times S^1$, where $\mathbb{T}^k = (S^1)^k$ is the $k$-torus. For $c > 0$ put

$$\Xi_c := \{(p, q) \in T^*X \ : \ |p|_\rho = c \}.$$

In this setting we prove the following results.

**Proposition 2.29** *Let $\mathcal{U} \subset T^*X \times T^*\mathbb{T}^k$ be a bounded domain containing the zero section and any $N \in \mathbb{N}$, $\widetilde{\mathcal{G}}(\widehat{\mathcal{U}}, V)$ admits a quasi-isometric monomorphism of $\mathbb{R}^N$.*

We thank Michael Usher [38] for his suggestion to consider hypersurfaces $\Xi_c$ in a similar Hofer-geometric context.

*Proof* We claim that for every $c > 0$ and $k \geq 1$, the subset

$$\Xi_c \times \mathbb{T}^k \subset T^*X \times T^*\mathbb{T}^k$$

(identifying $\mathbb{T}^k$ with the zero section $\mathcal{O}_{\mathbb{T}^k}$) is non-displaceable. Indeed, observe that $\Xi' = \Xi_c \times \mathbb{T}^k$ is a coistropic submanifold of $T^*X \times T^*\mathbb{T}^k$. Moreover, it is stable in the sense of [17, Theorem 1.5]. Assume on the contrary that $\Xi'$ is displaceable. By Ginzburg's Theorem 1.5, there exists a disc of positive symplectic area with boundary lying on one of the fibers of $\Xi'$. Every fiber of $\Xi'$ is of the form $L := \gamma \times \mathbb{T}^k$, where $\gamma$ is a trajectory of the geodesic flow on $\Xi$. Since all closed geodesics of $\rho$ are non-contractible, the inclusion $L \to T^*X \times T^*\mathbb{T}^k$ induces a monomorphism of fundamental groups. Thus every disc with boundary on $L$ has vanishing symplectic area, a contradiction.

We now argue as in Example 2.26. Namely, given a bounded domain $\mathcal{U} \subset T^*X \times T^*\mathbb{T}^k$ containing the zero section and an integer $N$, fix distinct positive numbers $c_1, \ldots, c_N$ such that $L_j := \Xi_{c_j} \times \mathbb{T}^k \subset \mathcal{U}$ for $1 \leq j \leq N$, and let $H_j$ be contact Hamiltonians supported in pair-wise disjoint neighbourhoods of the symplectizations of $\widehat{L}_j$ such that $H_j = 1$ on $\widehat{L}_j$ and $0 \leq H_j \leq s$. Denote by $h_j^t$ the Hamiltonian flow generated by $H_j$. Exactly as in the proof of Proposition 2.27, we find that the map

$$\left(\mathbb{R}^N, ||\cdot||_\infty\right) \to \left(\widetilde{\mathcal{G}}(\widehat{\mathcal{U}}), V\right), \quad (t_1, \ldots, t_N) \mapsto h_1^{t_1} \cdots h_N^{t_N},$$

is a quasi-isometric monomorphismn. □

*Example 2.30* [Compact manifolds] Let $V$ be a closed orderable contact manifold, that is the group $\widetilde{\mathcal{G}}(V)$ is orderable, and suppose that $V$ admits a contact form with 1-periodic Reeb flow. By Remark 2.18, the norm $\nu$ is automatically stably unbounded on $\widetilde{\mathcal{G}}(V)$. Examples of such manifolds include:

- Real projective space $V = \mathbb{R}P^{2n+1}$, with its standard contact structure. Orderability in this case was proven in [14, Theorem 1.3.E], using the work of Givental [18].

- More generally, consider lens spaces $V = S^{2n+1}/\mathbb{Z}_k$, with their standard contact structure. Orderability in this case was proven by Milin [23] (see also [33]). The situation changes drastically when we pass to the $k$-fold cover $S^{2n+1}$ of $V$: as we shall see in the next section, all known conjugation-invariant norms on $\widetilde{\mathcal{G}}(S^{2n+1})$ are bounded, provided $n \geq 1$. Let us note that while $\nu$ descends to a conjugation-invariant norm $\nu_*$ on $\mathcal{G}(V)$, it is not clear if $\nu_*$ is unbounded. In the case $2n + 1 = 3$, is it possible that every conjugation-invariant norm on $\mathcal{G}(M^3)$, $M^3$ a contact three-manifold, is bounded? The analog of this statement holds for diffeomorphisms by a result of Burago-Ivanov-Polterovich (Theorem [8, 1.11(iii)]) and work in progress by Patrick Massot seeks to use open book decompositions to develop a contact version of that argument.
- Let $V \to M$ be a prequantization of a closed symplectic manifold with $[\omega] \in H^2(M, \mathbb{Z})$. Suppose that $M$ contains a closed Lagrangian submanifold $L$ such that the connection on $V$ defined by $\lambda$ has trivial holonomy when restricted to $L$ (the Bohr-Sommerfeld condition), and the relative homotopy group $\pi_2(M, L)$ vanishes. Under these assumptions, orderability of $\widetilde{\mathcal{G}}(V)$ is proven in [14, Theorem 1.3.D].

### 2.4 Norm $\nu$ and $k$-translated fixed points

In this section we study the interplay between three aspects of the group $\widetilde{\mathcal{G}}$: the geometry of the norm $\nu$, a "Hofer type" topology on $\widetilde{\mathcal{G}}$, and dynamical features of contact isotopies. We prove, roughly speaking, that elements of $\widetilde{\mathcal{G}}$ which lie on the "Hofer type" boundary of a $\nu$-ball must possess translated points. Let us begin by a more precise formulation of these concepts.

**Definition 2.31** We say that $f \in \widetilde{\mathcal{G}}$ has a *k-translated fixed point* $x \in V$, $k \in \mathbb{Z}$, if there exists a contact isotopy $\{f_t\}$, $t \in [0, 1]$, in the class $f$ such that $\bar{f}_t x = \bar{e}_{kt} x$. In particular, $f_1 x = x$, since the Reeb flow is 1-periodic.

This definition is adapted to our setting of $f \in \widetilde{\mathcal{G}}$. It is closely related to the notion of a translated point of a contactomorphism. Recall that $x \in V$ is a translated point of a contactomorphism $\varphi$ if $\varphi(x)$ and $x$ belong to the same Reeb orbit and $(\varphi^*\alpha)_x = \alpha_x$. Translated points were introduced by Sandon [32,34] and further studied by Colin and Sandon [9]. They form a special case of *leaf-wise intersection points*, which were studied by Moser [24] and more recently by Albers and Fraunfelder [2] and Albers and Merry [3]. Finally, a translated point which is also a fixed point is called a *discriminant point*, a notion which goes back to Givental [18].

Thus if $x$ is a $k$-translated fixed point of $f \in \widetilde{\mathcal{G}}$ there exists a contact isotopy $\{f_t\}$ in the class $f$ for which $x$ is a translated point of $f_t$, for all $t$. Moreover, as $k$ is integer, $x$ is a discriminant point of $f_1$.

Next we turn to the "Hofer type" topology. Define, for $\varepsilon > 0$, $\mathring{\mathcal{B}}(\varepsilon)$ (resp. $\mathcal{B}(\varepsilon)$) as the set of $f \in \widetilde{\mathcal{G}}$ which can be generated by Hamiltonians $F_\pm$ such that $F_+(x, s, t) < \varepsilon s$ and $F_-(x, s, t) > -\varepsilon s$ (resp. $F_+(x, s, t) \leq \varepsilon s$ and $F_-(x, s, t) \geq -\varepsilon s$). As in Remark 2.6 we may assume such Hamiltonians are 1-periodic. Recall that for integer $\varepsilon = k$, $\mathcal{B}(k)$ is the closed ball of radius $k$ in the norm $\nu$, i.e.

$$\mathcal{B}(k) = \{f \in \widetilde{\mathcal{G}} \colon \nu(f) \leq k\}.$$

We will in fact not deal explicitly with the topology these balls generate, but rather restrict out attention to the following notion.

**Definition 2.32** We say that $f \in \widetilde{\mathcal{G}}$ is *k-robust*, for $k \in \mathbb{N}$, if $f\mathring{\mathcal{B}}(\varepsilon) \subset \mathcal{B}(k)$ for some $\varepsilon > 0$.

One readily checks that $f$ is $k$-robust if and only if $f \in \mathring{\mathcal{B}}(k)$. Indeed, for the 'only if' direction, let $\epsilon > 0$ such that $f\mathring{\mathcal{B}}(\epsilon) \subset \mathcal{B}(k)$ and take positive $c < \epsilon$ and elements $g_\pm \in \mathring{\mathcal{B}}(\epsilon)$ such that $g_\pm$ coincides with $e^{\pm c}$ on the support of $f$. Then $fe^{\pm c} \in \mathcal{B}(k)$ yields the existence of Hamiltonians $F_\pm$ as needed to conclude $f \in \mathring{\mathcal{B}}(k)$. In particular, we see that $f$ is $k$-robust if and only if $gfg^{-1}$ is $k$-robust (since, if $f$ can be generated by a Hamiltonian $F$, then $gfg^{-1}$ can be generated by $F \circ g_1^{-1}$).

**Theorem 2.33** *Let $k \in \mathbb{N}$ and suppose $f \in \mathcal{B}(k)$ has neither $k$-translated fixed points nor $-k$-translated fixed points. Then $f$ is $k$-robust.*

*Proof* Assume $f \in \mathcal{B}(k) \subset \widetilde{\mathcal{G}}$ has no $k$-translated fixed points. Let $W := SV \times T^*S^1$, with coordinates $(s, x, r, t)$. Recall that $W$ is equipped with the symplectic form $\Omega := d(s\lambda) + dr \wedge dt$ and with the $\mathbb{R}_+$-action $c \cdot (s, x, r, t) = (cs, x, cr, t)$, $c \in \mathbb{R}_+$.

Since $f \in \mathcal{B}(k)$, in particular $f \preceq e^k$ and so $f$ is generated by a 1-periodic Hamiltonian $F(s, x, t)$ with $F(s, x, t) \leq ks$. Moreover $F$ vanishes when $x$ lies outside some compact subset of $V$. We will show, using the assumption that $f$ has no $k$-translated fixed points, that $f$ can in fact be generated by a Hamiltonian $F_+$ satisfying the strict inequality $F_+(s, x, t) < ks$. We proceed in the following five steps.

**Step 1** Put $H(s, x, r, t) = r + F(s, x, t)$ and $K(s, x, r, t) = r + ks$. Since $H \leq K$ the hypersurface $\Xi = \{H = 0\} \subset W$ lies in the closed domain $U = \{K \geq 0\} \subset W$. Moreover, $dH = dK$ at each point of the set $Y = \Xi \cap \partial U$, since on $Y$ the function $H - K$ attains its maximal value. Thus the Hamiltonian vector fields sgrad $H$ and sgrad $K$ coincide on $Y$.

Observe that all orbits of the Hamiltonian flow of $K$ on $\partial U$ are (up to time shifts $\tau \mapsto \tau + \tau_0$, where $\tau$ stands for the time variable of the flow) circles of the form $\gamma_{x,s}(\tau) = (e_{k\tau}x, s, -ks, \tau)$.

**Step 2** We claim that *the set $Y \subset \Xi$ does not contain a compact invariant set of the Hamiltonian flow $h_\tau$ of $H$ on $\Xi$*. Indeed, otherwise this invariant set necessarily contains some $h_\tau$-orbit, which, since sgrad $H$ = sgrad $K$ on $Y$, must be a circle of the form $\gamma_{x,s}(\tau)$. Denote by $p : W \to SV$ the natural projection, and note that

$$\bar{f}_\tau(x, s) = p\big(h_\tau(x, s, -ks, 0)\big) = p(\gamma_{x,s}(\tau)) = \bar{e}_{k\tau}(x, s).$$

Therefore $(x, s)$ is a $k$-translated fixed point of $f$, a contradiction with the assumption of the theorem. The claim follows.

**Step 3** Observe that the Hamiltonians $H, K$ are equivariant with respect to the $\mathbb{R}_+$-action on $W$. Since $H(x, s, r, t) = r$ for $x$ outside a compact subset of $V$ and $K(x, s, r, t) = r + ks$ with $k > 0$, the set $Y/\mathbb{R}_+$ is compact. By an $\mathbb{R}_+$-equivariant application of a theorem of Sullivan [22,36] there exists an $\mathbb{R}_+$-equivariant function $\Phi(x, s, r, t)$ on $W$ with $d\Phi(\text{sgrad } H) < 0$ at every point of $Y$ (namely, apply Sullivan's theorem to the induced flow on $\{s = 1\}$ identified with $W/\mathbb{R}_+$ and extend the resulting function equivariantly). Here we use the fact that $Y$ does not contain a compact invariant set of the Hamiltonian flow $h_\tau$ on $\Xi$, see Step 2. Since on $Y$

$$d\Phi(\text{sgrad } H) = d\Phi(\text{sgrad } K) = \Omega(\text{sgrad } K, \text{sgrad } \Phi) = -dK(\text{sgrad } \Phi),$$

it follows that sgrad $\Phi$ is transversal to $\partial U$ at the points of $Y$ and moreover sgrad $\Phi$ looks inside $U$ at points of $Y$ (cf. [26, the proof of Theorem 1.5]). Denoting by $\phi_t$ the Hamiltonian flow of $\Phi$, we get that for a sufficiently small $\epsilon > 0$

$$\phi_\epsilon(\Xi) \subset \text{Interior}(U). \tag{8}$$

**Step 4** The hypersurface $\Xi' := \phi_\epsilon(\Xi)$ is transversal to the lines parallel to the $r$-axis and hence has the form $\Xi' = \{r + F'(x, s, t) = 0\}$ for some $\mathbb{R}_+$ equivariant Hamiltonian $F'$ on $SV$. Put $H'(x, s, r, t) = r + F'(x, s, t)$. We claim that the time one map $f'$ of $F'$ is conjugate to $f$ in $\widetilde{\mathcal{G}}(V)$. Indeed, $S = \Xi \cap \{t = 0\}$ is a Poincaré section of the Hamiltonian flow $h_\tau$ on $\Xi$. Similarly, $S' = \Xi' \cap \{t = 0\}$ is a Poincare section of the Hamiltonian flow $h'_\tau$ of $H'$ on $\Xi'$. Denote by $\psi$ and $\psi'$ the corresponding return maps.

Further, $\phi_\epsilon(S)$ is a Poincaré section of $f'_t$ with return map

$$\psi'' = \phi_\epsilon \psi \phi_\epsilon^{-1}. \tag{9}$$

The orbits of $\bar{f}'_t$ establish an $\mathbb{R}_+$-equivariant symplectomorphism, say $\eta$, between $S'$ and $\phi_\epsilon(S)$. Thus

$$\psi' = \eta^{-1}\psi''\eta. \tag{10}$$

Finally, let $\pi : S \to SV$ and $\pi' : S' \to SV$ be the restrictions of the natural projection. Then $\bar{f} = \pi\psi\pi^{-1}$ and $\bar{f}' = \pi'\psi'(\pi')^{-1}$. Combining this with (9) and (10) we get that $\bar{f}$ and $\bar{f}'$ are conjugate by $\mathbb{R}_+$-equivariant symplectomorphisms, and hence $f$ and $f'$ are conjugate as well.

**Step 5** By (8) we have the strict inequality $F' < ks$. Since $f$ and $f'$ are conjugate, we deduce that $f$ can be generated by a Hamiltonian $F_+$ satisfying $F_+ < ks$, as asserted.

Finally, as $f \in \mathcal{B}(K)$ we also have $f \succeq e^{-k}$, and so $f$ is generated by a periodic Hamiltonian such that $F \geq -ks$. Repeating Steps 1–5 above, using this time that $f$ has no $-k$-translated fixed points, we get that $f$ is also generated by a Hamiltonian $F_-$ satisfying $F_- > -ks$. This shows that $f \in \mathring{\mathcal{B}}(k)$, which as noted above is equivalent to $f$ being $k$-robust. □

# 3 Obstructions

## 3.1 Overview

In the next sections we discuss some restrictions on conjugation-invariant norms on $\mathcal{G}(V)$ or $\widetilde{\mathcal{G}}(V)$ for certain contact manifolds $V$. Our first result concerns discreteness, our second result boundedness. Recall that a conjugation invariant norm $\mu$ is called discrete if $\mu(g) \geq c$ for some $c > 0$ and all $g \neq \mathbb{1}$.

**Theorem 3.1** *Let $V$ be any contact manifold.*

(1) *Any conjugation-invariant norm on $\mathcal{G}(V)$ is discrete.*
(2) *Any conjugation-invariant norm on $\widetilde{\mathcal{G}}(V)$ is discrete on $\widetilde{\mathcal{G}}(V)\backslash\pi_1(\mathcal{G}(V))$.*

*Example 3.2* This example shows the second part of Theorem 3.1 cannot be improved. It follows from [10] that $\pi_1(\mathcal{G}(S^3)) = \mathbb{Z}$. Let $\phi \in \pi_1(\mathcal{G}(S^3))$ be a generator, and let $r \in (0, 1)$ be an irrational number. Define a norm $\mu$ on $\widetilde{\mathcal{G}}(S^3)$ by setting

$$\mu(\phi^n) = |e^{2\pi i n r} - 1|,$$

and $\mu(g) = 1$ for $g \notin \pi_1(\mathcal{G}(S^3))$. One readily checks that $\mu$ defines a norm on $\widetilde{\mathcal{G}}(S^3)$, which is conjugation-invariant since $\pi_1(\mathcal{G}(S^3))$ is a normal subgroup. Moreover, $\mu$ is clearly not discrete.

Next we address boundedness. To this end, we consider the fragmentation norm. Recall that any compactly supported contact isotopy can be represented as a finite product of contact isotopies each supported in a Darboux ball (see [4]). Here by Darboux ball we mean a contact embedded image of an open ball centred at the origin in the standard Euclidean space. The contact fragmentation norm $\nu_F(f)$ of $f \in \widetilde{\mathcal{G}}(V)$ is the minimal number of factors in such a representation of $f$. One can analogously define the contact fragmentation norm on $\mathcal{G}(V)$. These norms are useful for us as they are maximal in the following sense:

**Theorem 3.3** *Let $V$ be a contact manifold and let $\mu$ be a conjugation-invariant norm on $\mathcal{G}(V)$ or $\widetilde{\mathcal{G}}(V)$ which is bounded on a $C^1$-neighborhood of the identity. Then there is a constant $C = C(V, \mu)$ such that $\mu \leq C \cdot \nu_F$.*

This automatically implies:

**Corollary 3.4** *Let $V$ be a contact manifold and suppose the fragmentation norm on $\mathcal{G}(V)$ (resp. $\widetilde{\mathcal{G}}(V)$) is bounded. Then any conjugation-invariant norm on $\mathcal{G}(V)$ (resp. $\widetilde{\mathcal{G}}(V)$) which is bounded on a $C^1$-neighborhood of the identity is bounded.*

As an example, consider the sphere $S^{2n+1}$ with its standard contact structure, for $n \geq 1$.

**Proposition 3.5** *The fragmentation norm on $\widetilde{\mathcal{G}}(S^{2n+1})$ is bounded by 2 when $n \geq 1$.*

*Proof* For $z \in S^{2n+1}$ put $V_z = S^{2n+1} \setminus \{z\}$. Observe that $V_z \subset S^{2n+1}$ is a Darboux ball. Now, let $\{f_t\}$ be a contact isotopy representing $f \in \widetilde{\mathcal{G}}(S^{2n+1})$. Take a sufficiently small ball $B \subset V$ such that $X := \cup_t f_t(B) \neq S^{2n+1}$. Fix any point $z \notin X$. Let $\{g_t\}$ be a contact isotopy supported in $V_z$ with $g_t|_B = f_t|_B$ for all $t \in [0, 1]$. Set $h_t = g_t^{-1} f_t$. Observe that $h_t \in \mathcal{G}(V_w)$ for any point $w \in B$. Then $f = gh$ and hence $\nu_F(f) \leq 2$. □

As an immediate consequence of Proposition 3.5 and Corollary 3.4 we get:

**Corollary 3.6** *Let $n \geq 1$. Any conjugation-invariant norm on $\mathcal{G}(S^{2n+1})$ or $\widetilde{\mathcal{G}}(S^{2n+1})$ which is bounded on a $C^1$-neighborhood of the identity is bounded.*

*Remark 3.7* All boundedness results in this paper—both Theorem 3.3 and Corollary 3.4 above and the analogous results in Sect. 3.5—involve a $C^1$-boundedness hypothesis and their proofs use perfectness of groups of contactomorphisms of finite smoothness [37]. If one instead appeals to perfectness of the group of smooth contactomorphisms [28] then this additional hypothesis is not needed and one obtains analogous boundedness statements which hold for any conjugation-invariant norm. We note, nevertheless, that all the norms mentioned in this paper do satisfy the $C^1$-boundedness assumption appearing above:

- The norm $\nu$ (and consequently also $\nu_*$) defined in Sect. 2 satisfies this assumption, since any element of $\widetilde{\mathcal{G}}(V)$ sufficiently $C^1$-close to $\mathbb{1}$ can be represented by a flow generated by a Hamiltonian satisfying $|H| \leq \epsilon s$.
- Zapolsky's norms $\rho_{\text{osc}}$ and $\rho_{\text{sup}}$ [39] satisfy this assumption since, as mentioned in Example 2.24, they are bounded above by the norm $\nu_*$ (as follows from [39, Proposition 2.9(iii)]).
- The discriminant, zig-zag, and oscillation norms [9] satisfy this assumption. Indeed, both the oscillation and discriminant norms are bounded above by the zig-zag norm, so it suffices to consider the latter. An easy modification of the proof of [9, Lemma 2.1] shows that any element $\phi \in \widetilde{\mathcal{G}}$ sufficiently $C^1$-close to the identity can be represented as a product $\phi = fg$, where $f$ is positive, $g$ is negative, and both are $C^1$-close to the identity. By the proof of [9, Lemma 2.1] again, both $f$ and $g$ are embedded, and so $\phi$ has zig-zag norm $\leq 2$.

- Sandon's spectral norm [31] satisfies this assumption, as follows from the previous item and the fact, proven in [9, Sect. 6], that for $f = [\{f_t\}]$, the spectral number $\lceil c^+(f_1)\rceil$ is bounded from above by the discriminant norm of $f$.
- The norms of Borman and Zapolsky and Granja, Karshon, Pabiniak and Sandon, coming from homogeneous quasi-morphisms (see Sect. 1) satisfy this assumption. Indeed, the quasi-morphisms $\phi$ they construct are *monotone*, meaning that $\phi(g) \le \phi(h)$ if $g \preceq h$ (see [7, Lemma 1.33] and [20, Theorem 1.2]). This implies that the corresponding conjugation-invariant norms $\mu$ are dominated by our norm $\nu$ in the sense that for each such $\mu$ here exists $K > 0$ such that $\mu(g) \le K \cdot \nu(g)$ for all $g \in \widetilde{\mathcal{G}}(V)$ .

*Remark 3.8* In the previous remark we mentioned comparison of various known norms on contactomorphism groups. Let us mention one more such comparison, between our norm $\nu$ and the oscillation norm of Colin–Sandon [9]. We claim that we have the following inequality: $\nu_{osc} \le 3\nu+2$. Indeed, this follows easily from the compatibility of $\nu_{osc}$ with the partial order $\preceq$ [9, Proposition 3.4]: $\mathbb{1} \preceq f \preceq g$ implies $\nu_{osc}(f) \le \nu_{osc}(g)$. Let $f \in \widetilde{\mathcal{G}}$ and denote $k = \nu(f)$. By definition, we have $e^{-k} \preceq f \preceq e^k$, which by bi-invariance of $\preceq$ gives $\mathbb{1} \preceq e^k f \preceq e^{2k}$. Therefore, the compatibility of $\nu_{osc}$ with $\preceq$ gives

$$\nu_{osc}(f) \le \nu_{osc}(e^{-k}) + \nu_{osc}(e^k f) \le \nu_{osc}(e^{-k}) + \nu_{osc}(e^{2k}).$$

Finally, noting that by the definition of the oscillation norm, $\nu(e^m) \le m + 1$, this gives the asserted inequality. We note that an inequality in the converse direction is not known to us at the moment.

The rest of Sect. 3 is organized as follows. Theorem 3.1 is proved in Sect. 3.3, and Theorem 3.3 in Sect. 3.4. Beforehand, in Sect. 3.2 we recall some algebraic results used in those proofs. Finally, in Sect. 3.5 we discuss a class of sub-domains of contact manifolds for which analogous boundedness results can be obtained.

### 3.2 Algebraic results

The following definition is taken from [8].

**Definition 3.9** Let $G$ be a group, and let $H \subset G$ be a subgroup. We say that an element $g \in G$ *m-displaces* $H$ if the subgroups

$$H, gHg^{-1}, g^2Hg^{-2}, \dots, g^mHg^{-m}$$

pairwise commute.

The geometric meaning of $m$-displacement in our context is as follows. Let $U \subset V$ be an open subset. We say that a contactomorphism $\phi \in \mathcal{G}(V)$ $m$-displaces $U$ if the subsets

$$U, \phi(U), \dots, \phi^m(U)$$

are pairwise disjoint. If this holds then $\phi$ $m$-displaces the subgroup $\mathcal{G}(U)$ of $\mathcal{G}(V)$. Similarly, if $\widetilde{\phi} = \{\phi_t\} \in \widetilde{\mathcal{G}}$ is a path such that $\phi_1$ $m$-displaces $U$ then $\widetilde{\phi}$ $m$-displaces the subgroup $\widetilde{\mathcal{G}}(U, V)$ of $\widetilde{\mathcal{G}}(V)$.

Returning to the general algebraic setting, given a subgroup $H \subset G$ and an element $h$ in the commutator subgroup $[H, H]$, we denote by $cl_H(h)$ the commutator length of $h$, which is the minimal number of commutators needed to represent $h$ as a product of commutators.

We will need the following result (see [8], Theorem 2.2). Suppose that $\mu$ is a conjugation-invariant norm on a group $G$. Let $H \subset G$ be a subgroup such that there exists $g \in G$ which $m$-displaces $H$. Then for any $h \in [H, H]$ with $cl_H(h) = m$ one has

$$\mu(h) \leq 14\mu(g). \tag{11}$$

Finally, we will use the following result of Tsuboi [37] dealing with contactomorphisms of finite smoothness. Let $W$ be a connected contact manifold of dimension $2n+1$. For $1 \leq r < \infty$, denote by $\mathcal{G}^r(W)$ the identity component of the group of compactly supported $C^r$-contactomorphisms of $W$, and by $\widetilde{\mathcal{G}}^r(W)$ its universal cover. Moreover, for an open subset $X \subset W$, denote by $\widetilde{\mathcal{G}}^r(X, W)$ the subgroup of $\widetilde{\mathcal{G}}^r(W)$ containing those contact isotopies which are supported in $X$.

Tsuboi's theorem states that for $r \leq n+3/2$, the groups $\mathcal{G}^r(W)$ and $\widetilde{\mathcal{G}}^r(W)$ are perfect, i.e. equal to their commutator subgroups. In particular if $X \subset W$ is a connected open subset, the groups $\mathcal{G}^r(X)$ and $\widetilde{\mathcal{G}}^r(X, W)$ are perfect. The latter group is perfect since it is an epimorphic image of the perfect group $\widetilde{\mathcal{G}}^r(X)$ [similarly to (1) above].

### 3.3 Discreteness

In this section we prove Theorem 3.1 on discreteness. In what follows by an embedded open ball we mean the interior of an embedded closed ball. We use the following fact which we prove in Sect. 3.4 (see Example 3.12).

**Lemma 3.10** *Let $D_1, D_2 \subset V$ be Darboux balls and let $U$ be an open subset of $D_1$ such that Closure$(U) \subset D_1$. Then there exists $\phi \in \mathcal{G}(V)$ such that $\phi(U) \subset D_2$.*

*Proof of Theorem 3.1* First we note that (2) follows from (1). Indeed, denote by $\pi : \widetilde{\mathcal{G}} \to \mathcal{G}$ the natural projection. Assume that (1) holds, and let $\mu$ be a conjugation-invariant norm on $\widetilde{\mathcal{G}}$. Define a conjugation-invariant norm $\mu_*$ on $\mathcal{G}$ by $\mu_*(f) = \inf\{\mu(f') : \pi(f') = f\}$. Then by assumption $\mu_*$ is discrete. Observe that $\mu(f) \geq \mu_*(\pi f)$ for all $f \in \widetilde{\mathcal{G}}$. Since for all $f \notin \pi_1(\mathcal{G})$, $\pi(f) \neq \mathbb{1}$, we get (2).

Next we prove (1). Assume the result does not hold. Fix an open ball $U$ with closure contained in a Darboux ball $D$, and a pair of elements $\phi, \psi \in \mathcal{G}(U)$ with $[\phi, \psi] \neq \mathbb{1}$. We claim $\mu([\phi, \psi]) = 0$, a contradiction.

By assumption, for any $\varepsilon > 0$ we can find $\theta \in \mathcal{G}$ with $\mu(\theta) < \varepsilon$. Since $\theta \neq \mathbb{1}$, $\theta$ moves some point, so there must exist an open ball $B \subset V$ such that $\theta(B) \cap B = \emptyset$. Let $\eta \in \mathcal{G}$ such that $\eta(U) \subset B$ (which exists by Lemma 3.10). Then $\eta^{-1}\theta\eta$ displaces $\mathcal{G}(U)$ and hence by (11)

$$\mu([\phi, \psi]) \leq 14\mu(\eta^{-1}\theta\eta) = 14\mu(\theta) < 14\varepsilon.$$

Our claim follows. □

### 3.4 Boundedness

In this section we prove Theorem 3.3.

**Definition 3.11** An *open* connected contact manifold $(V, \xi)$ is called *contact portable* if there exists a connected compact set $V_0 \subset V$ and a contact isotopy $\{P_t\}$ of $V$, $t \geq 0$, $P_0 = \mathbb{1}$ such that the following hold:

- The set $V_0$ is an *attractor* of $\{P_t\}$, i.e. for every compact set $K \subset V$ and every neighborhood $U_0 \supset V_0$ there exists some $t > 0$ such that $P_t(K) \subset U_0$.
- There exists a contactomorphism $\theta$ of $V$ displacing $V_0$.

Note that $\theta$ is not assumed to be compactly supported. Definition 3.11 is a contact version of the notion of portable manifold defined in [8].

*Example 3.12* An example of a contact portable manifold is $\mathbb{R}^{2n+1}$, equipped with the standard contact structure given by the kernel of the 1-form $\alpha = dz - ydx$. Here we use the coordinates $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. The contact isotopy is given by

$$P_t : (x, y, z) \mapsto (e^{-t}x, e^{-t}y, e^{-2t}z).$$

The attractor $V_0$ can be taken to be the closed ball $\{|x|^2 + |y|^2 + z^2 \leq 1\}$ and the contactomorphism $\theta$ can be given, for example, by

$$\theta(x, y, z) = (x, y, z + 3).$$

Similarly, any Euclidean ball $\{|x|^2 + |y|^2 + z^2 < R\}$ is contact portable.

In particular, this example shows that every compact subset of $\mathbb{R}^{2n+1}$ can be contact isotoped into an arbitrary small neighborhood of the origin. Applying an appropriate cut-off function to the generating contact Hamiltonian, this can be done inside a Darboux chart in an arbitrary contact manifold. This, together with the transitivity of $\mathcal{G}$, proves Lemma 3.10.

The proof of the following proposition is analogous to that of Theorem 1.17 in [8].

**Proposition 3.13** *Let $(V, \xi)$ be a contact portable manifold. Then any conjugation-invariant norm on $\mathcal{G}(V)$ or $\widetilde{\mathcal{G}}(V)$, which is bounded on a $C^1$-neighborhood of the identity, is bounded.*

We prove the case of $\widetilde{\mathcal{G}}(V)$; the proof for $\mathcal{G}(V)$ is similar.

*Proof* Let $\mu$ be such a norm, and fix a constant $\delta > 0$ and a $C^1$-neighborhood $\mathcal{V} \subset \widetilde{\mathcal{G}}$ of the identity such that $\mu(g) < \delta$ for $g \in \mathcal{V}$. Let $V_0$, $P_t$ and $\theta$ be as in Definition 3.11. We can find a small connected neighborhood $U$ of $V_0$ with compact closure such that $\theta(U) \cap U = \emptyset$. First we show that $\mu$ is bounded on the subgroup $H := \widetilde{\mathcal{G}}(U, V)$ of $\widetilde{\mathcal{G}}$.

Indeed, let $U' := \theta(U)$. Then $U'$ is a neighborhood of $\theta(V_0)$, which is an attractor for the isotopy $\{g_t = \theta \circ P_t \circ \theta^{-1}\}$. Therefore, for some $T > 0$, $g_T(U \cup U') \subset U'$. Truncating the contact Hamiltonian generating $\{g_t\}$ and re-parametrizing gives a contact isotopy $\psi = \{\psi_t\} \in \widetilde{\mathcal{G}}(V)$ such that $\psi_1(U \cup U') \subset U'$. We claim $\psi_1$ $m$-displaces $U$ for all $m \geq 0$. Indeed $U$ lies in the complement of $U'$, so $\psi_1(U) \subset U' \backslash \psi_1(U')$ and $\forall k \in \mathbb{N}$, $\psi_1^k(U) \subset \psi_1^{k-1}(U') \backslash \psi_1^k(U')$, which implies the $\psi_1^k(U)$ are pairwise disjoint.

Now, by Tsuboi's theorem any $h \in H$ can be written as a product $h = h_1 \cdots h_m$, $h_i = [\sigma_i, \tau_i]$ where $\sigma_i, \tau_i \in \widetilde{\mathcal{G}}^1(U, V)$. Since the $2m$-fold product of $\widetilde{\mathcal{G}}(U, V)$ is dense in the $2m$-fold product of $\widetilde{\mathcal{G}}^1(U, V)$ and the product of $m$ commutators defines a continuous map to $\widetilde{\mathcal{G}}^1(U, V)$, there is some $g \in \widetilde{\mathcal{G}}(U, V)$ satisfying $g^{-1}h \in \mathcal{V}$ such that $g = g_1 \cdots g_m$ where each $g_i$ is a product of commutators of elements of $\widetilde{\mathcal{G}}(U, V)$. In particular, $cl_H(g) \leq m$, and hence

$$\mu(g) \leq 14\mu(\psi).$$

But then

$$\mu(h) \leq \mu(g) + \mu(g^{-1}h) \leq 14\mu(\psi) + \delta =: C.$$

This proves that $\mu$ is bounded on $H$.

Now, given $f = \{f_t\} \in \widetilde{\mathcal{G}}(V)$, let $K$ be a compact set such that $\cup_t supp\, f_t \subset K$. There exists $T$ such that $P_T(K) \subset U$. As before, truncating and re-parametrizing the contact Hamiltonian which generates $\{P_t\}_{t \in [0,T]}$, we can produce $\eta \in \widetilde{\mathcal{G}}(V)$ with $\eta(K) \subset U$. Then $\eta \circ f \circ \eta^{-1} \in H = \widetilde{\mathcal{G}}(U, V)$ and so,

$$\mu(f) = \mu(\eta \circ f \circ \eta^{-1}) \leq C.$$

□

*Remark 3.14* Observe that the proof of Proposition 3.13 works equally well for a conjugation-invariant pseudo-norm; the non-degeneracy of $\mu$ was never used.

We can now prove the maximality of the fragmentation norm. As above, we prove it for $\widetilde{\mathcal{G}}(V)$; the proof for $\mathcal{G}(V)$ is similar.

*Proof of Theorem 3.3* Let $\mu$ be a conjugation-invariant norm on $\widetilde{\mathcal{G}}(V)$ which is bounded on a $C^1$-neighborhood of the identity. Fix a Darboux ball $B \subset V$. By Proposition 3.13, $\mu$ is bounded on $\widetilde{\mathcal{G}}(B, V)$—indeed, pulling back $\mu$ by the epimorphism $\widetilde{\mathcal{G}}(B) \to \widetilde{\mathcal{G}}(B, V)$ (recall (1)) yields a conjugation-invariant pseudo-norm on $\widetilde{\mathcal{G}}(B)$, which is bounded by Remark 3.14 and Example 3.12, and so $\mu$ is bounded on $\widetilde{\mathcal{G}}(B, V)$, say by $C > 0$. Now, let $f \in \widetilde{\mathcal{G}}(V)$. Write $f = h_1 \cdots h_N$, where $N = \mu_F(f)$, and each $h_i$ is represented by an isotopy supported in a Darboux ball $B_i \subset V$. By Lemma 3.10, one can find contact isotopies mapping each *supp* $h_i$ into $B$, and so each $h_i$ is conjugate to a contact isotopy supported in $B$. Then for any $1 \leq i \leq N$, $\mu(h_i) \leq C$. We get $\mu(f) \leq CN = C\nu_F(f)$. □

### 3.5 Boundedness on sub-domains

**Definition 3.15** Suppose $(V, \xi)$ is a contact manifold and $V' \subset V$ an open subset. Then we say that $V'$ is a *portable sub-domain* of $V$ if there exists a compact set $V_0 \subset V'$, and a contact isotopy $\{P_t\}_{t\in\mathbb{R}}$ of $V$ such that the following hold:

- For every compact set $K \subset V'$ and every neighborhood $U_0 \supset V_0$ there exists some $t > 0$ such that $P_t(K) \subset U_0$.
- There exists a contactomorphism $\theta$ supported in $V'$ displacing $V_0$.

Observe that a portable sub-domain $V' \subset V$ need not be a contact portable manifold, since we allow more "squeezing room" (the isotopy $P_t$ may have support outside $V'$). By the same argument as in the proof of Proposition 3.13, we have the following result:

**Proposition 3.16** *Let $(V, \xi)$ be a contact manifold and $V' \subset V$ be a portable sub-domain of $V$. Then any conjugation-invariant norm on $\mathcal{G}(V)$ (resp. $\widetilde{\mathcal{G}}(V)$) which is bounded on a $C^1$-neighborhood of the identity is necessarily bounded on $\mathcal{G}(V')$ (resp. on $\widetilde{\mathcal{G}}(V', V)$).*

*Example 3.17* Consider the contact manifold $V = \mathbb{R}^{2n} \times S^1$ equipped with the contact structure $\xi = \mathrm{Ker}(dt - \alpha)$, where $\alpha = \frac{1}{2}(pdq - qdp)$. In what follows we assume that $n \geq 2$. Put $\mathcal{U}(r) := B^{2n}(r) \times S^1$, where $B^{2n}(r)$ stands for the ball $\{\pi(|p|^2 + |q|^2) < r\}$. In [31] Sandon defined a conjugation invariant norm on $\mathcal{G}(V)$ which is bounded on all subgroups $\mathcal{G}(\mathcal{U}(r))$. We claim that every conjugation-invariant norm on $\mathcal{G}(V)$ is necessarily bounded on $\mathcal{G}(\mathcal{U}(r))$ if $r < 1$.

To prove the claim, take any Hamiltonian symplectomorphism $\theta$ supported in $B^{2n}(r) \subset \mathbb{R}^{2n}$ which displaces the origin. Let $r' < r$ be sufficiently small so that $B^{2n}(r')$ is also displaced and let $\widehat{\theta} \in \mathcal{G}(\mathcal{U}(r))$ be the lift of $\theta$ to a contactomorphism of $V$ supported in $\mathcal{U}(r)$. We have $\widehat{\theta}(\mathcal{U}(r')) \cap \mathcal{U}(r') = \emptyset$. Further, by the Squeezing Theorem [15, Theorem 1.3] there exists $P \in \mathcal{G}(V)$ such that $P(\mathcal{U}(r)) \subset \mathcal{U}(r')$. Therefore $\mathcal{U}(r)$ is a portable sub-domain of $V$, and the claim follows from Proposition 3.16.

In fact, the above argument can be applied more generally. Recall that a symplectic manifold $(M^{2n}, \omega = d\alpha)$ is called *Liouville* if it admits a vector field $v$ and a compact $2n$-dimensional submanifold $\overline{\mathcal{U}}$ with connected boundary $Q = \partial\overline{\mathcal{U}}$ with the following properties:

- $i_\eta \omega = \alpha$. This yields that the flow $\eta_t$ of $v$ is conformally symplectic;
- $v$ is transversal to $Q$.

One can show that $(Q, \mathrm{Ker}(\alpha))$ is a contact manifold and for all specific choices of $\overline{\mathcal{U}}$ these are naturally contactomorphic. We refer to $(Q, \mathrm{Ker}(\alpha))$ as *the ideal contact boundary* of $M$. The set $C := \bigcap_{t>0} \eta_{-t}(\overline{\mathcal{U}})$ is called *the core* of $M$.

Consider now the contact manifold $V = M \times S^1$ equipped with the contact form $\lambda = dt - \alpha$. Put $\mathcal{U}(r) := \eta_{\log r}(\mathcal{U}) \times S^1$, where $\mathcal{U}$ is the interior of $\overline{\mathcal{U}}$.

**Proposition 3.18** *Suppose that the ideal contact boundary of $(M, d\alpha)$ is non-orderable and the core $C$ is displaceable by a Hamiltonian diffeomorphism in its arbitrary small neighborhood. Then there exists $r_0 > 0$ so that any conjugation-invariant norm on $\mathcal{G}(V)$ (resp. $\widetilde{\mathcal{G}}(V)$) which is bounded on a $C^1$-neighborhood of the identity is necessarily bounded on $\mathcal{G}(\mathcal{U}(r))$ (resp. $\widetilde{\mathcal{G}}(\mathcal{U}(r), V)$) for all positive $r < r_0$.*

*Proof* By [15, Theorem 1.19] there is some $r_0 > 0$ such that (by iterating the Theorem enough times), one can obtain an isotopy which squeezes $\mathcal{U}(r_0)$ arbitrarily close to $C \times S^1$, in fact within[2] some larger $\mathcal{U}(r') \subset V$. Since the core $C$ is Hamiltonian displaceable in its arbitrarily small neighborhood, the set $C \times S^1$ is displaceable in its arbitrarily small neighborhood by a contact isotopy of $V$. It follows that for $0 < r < r_0$ the set $\mathcal{U}(r)$ is a portable sub-domain of $V$ and hence by Proposition 3.16 any conjugation-invariant norm on $\mathcal{G}(V)$ (resp. $\widetilde{\mathcal{G}}(V)$) is bounded on $\mathcal{G}(\mathcal{U}(r))$ (resp. on $\widetilde{\mathcal{G}}(\mathcal{U}(r), V)$). □

An important class of Liouville manifolds is formed by complete Stein manifolds, that is by Kähler manifolds $(M, J, d\alpha)$ admitting a proper bounded from below Morse function $F$ with $\alpha = JdF$ and $\eta = -\mathrm{grad} F$, where the gradient is taken with respect to the metric $d\alpha(\cdot, J\cdot)$. (By a result of Eliashberg [11] these manifolds admit an alternative description as Weinstein manifolds provided $\dim M \geq 3$). For a generic $F$, the core of $M$ is an isotropic $CW$-complex of dimension $n - k$ with $k \in [0, n]$ (see [5,12]). We say that $M$ is *$k$-subcritical* if $k \geq 1$ and *critical* if $k = 0$.

The assumptions of Proposition 3.18 implying boundedness of suitable conjugation-invariant norms on some $\mathcal{G}(\mathcal{U}(r))$ hold true, for instance, when the Liouville manifold $M$ is $k$-critical with $k \geq 2$ : indeed, the ideal contact boundary is non-orderable by Theorem 1.16 of [15], while the core $C$ in this case is Hamiltonian displaceable in its arbitrary small neighborhood (cf. [5, Sect. 3]).

On the other hand, for some critical Liouville manifolds $M$ it is known that the core is stably non-displaceable. For instance, this is true for cotangent bundles of closed manifolds, where the core can be taken as the zero section. By Theorems 2.15, 2.19 and Corollary 2.20 this implies the conjugation invariant norm $\mu$ on $\widetilde{\mathcal{G}}(V)$ is well defined and stably unbounded when restricted to $\widetilde{\mathcal{G}}(\mathcal{U}(r))$ for all $r > 0$. Here the 1-periodic Reeb flow on $V$ is associated to the form $\lambda$ and is given by rotation along the $S^1$-factor so leaves $\mathcal{U}(r)$ invariant.

We thus see a dichotomy between boundedness of conjugation-invariant norms in small $\mathcal{U}(r) \subset M \times S^1$ in the case of sub-critical $M$, and stable unboundedness of the norm $\nu$ on any $\mathcal{U}(r)$ in the case of certain critical $M$, those with stably non-displaceable core. It is natural to ask:

[2] See the argument in Remark 1.23 of that paper and the proof of Theorem 1.3 on the same page.

**Question 3.19** Is the norm $\nu$ well-defined and stably unbounded on any $\mathcal{U}(r) \subset M \times S^1$ for all critical $M$?

It could be that $V = M \times S^1$ is orderable for every Liouville manifold $(M, d\alpha)$ which would confirm at least well-definedness of the norm $\nu$ for all $V$ of this kind. This is true, for instance, for some critical $M$ such as cotangent bundles of closed manifolds (as explained in Example 2.24). Albers pointed out that the methods of [3] should prove the result for general $M$.

Regarding boundedness vs. stable unboundedness, however, little is known so far. On the one hand, in case one wished to prove boundedness by using squeezing as in Proposition 3.18 above, it is unknown whether the ideal contact boundary of critical $M$ is orderable or not. On the other hand, it is unlikely that the technique of Sect. 2 based on stable intersection property might be applicable to proving stable unboundedness for subcritical manifolds with $\dim M \geq 2$. Indeed, by [6, Theorem 6.1.1], there are no "hard" symplectic obstructions to Hamiltonian displacement of a compact subset of $SV = M \times \mathbb{R}_+ \times S^1$ from a given closed subset, so existence of sets with stable intersection property is quite problematic. It would be interesting to explore this point further.

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