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Gaussian measures on the of space of Riemannian metrics

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Abstract We introduce Gaussian-type measures on the manifold of all metrics with a fixed volume form on a compact Riemannian manifold of dimension ≥3. For this random model we compute the characteristic function for the L^2 (Ebin) distance to the reference metric. In the Appendix, we study Lipschitz-type distance between Riemannian metrics, and give applications to the diameter, eigenvalue and volume entropy functionals.

Keywords Manifolds of metrics · L^2 distance · Gaussian measures · Diameter · Eigenvalue · Parallelizable · Frame bundle · Cartan decomposition · Symmetric space

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Résumé Nous introduisons des mesures de type gaussien sur la variété des métriques riemanniennes a forme volume fixée definies sur une variété compacte de dimension ≥3. Pour ce modèle aléatoire, nous calculons la fonction caractéristique de la distance L^2 (Ebin) à une métrique de référence donnée. Dans un appendice, nous étudions une distance de type Lipschitz entre métriques riemanniennes, et donnons des applications aux fonctionnelles associées ai diamètre, aux valeurs propres du laplacien et à l'entropie de volume.

1 Introduction

In this paper we construct Gaussian-type measures on the space of Riemannian metrics on a fixed manifold and make some elementary observations about them, leaving deeper results for further work. We will being with several motivations for our construction and with directions for further work.

Let (M, g) be a Riemannian manifold. *Quantum Chaos* is a general term for the study of connections between the dynamics of the associated geodesic flow on TM (corresponding to the physics of a classical particle moving freely on M) and the spectrum of the Laplace–Beltrami operator on $L^2(M)$ (corresponding to the physics of a quantum particle moving freely on M). We note the conjectures of Bohigas, Giannoni and Schmidt [7] about asymptotic behaviour of level spacings between Laplace eigenvalues for classically chaotic systems, and M. Berry's random wave conjectures [5] about asymptotic properties of eigenfunctions. These conjectures appear to be very difficult to prove using standard semiclassical methods, and a natural idea is instead to consider them *on average* in some sense in the space of metrics on M, or perhaps to use random methods to construct examples or counter examples.

Another motivation for our construction is of developing geometric analysis on (often infinite-dimensional) manifolds of metrics. Most important progress to date involved *differentiation* on manifolds of metrics, in particular the study of L^2 distance between metrics and related questions [10–12]. The next natural step is to define *integration* on manifolds of metrics, hence the need to define and study measures on those manifolds. Related questions have been considered in [17] (for manifolds of maps) and in [8].

We now turn to our construction. In the predecessor work [8], the authors took a fixed "reference" (or "background") metric g^0 on M, and then considered a random metric $g = e^{2\varphi} g^0$ in the conformal class of g^0, where the (logarithm of) the conformal factor φ varied as a Gaussian random field on M, constructed using the eigenfunctions of the Laplacian for the reference metric. In the present paper, we work in a transverse direction: given the reference metric g^0 we choose a random deformation g among those metrics having the same volume form as g^0. Again we parametrize those metrics by exponentiating a Gaussian random field on M. Beyond describing the construction we limit out study to the statistics of various distance functions on the space of metrics, leaving deeper investigation for later papers.

Remark 1.1 It is possible to combine the two constructions, adding a conformal factor to our deformation. We mainly avoid doing this since the (completion of the) space of all Riemannian metrics is singluar, unlike the case of a fixed volume form.

Remark 1.2 Our construction depends on a choice of a global orthonormal frame in the tangent bundle (a global section of the frame bundle). The existence of such a frame is known as *parallelizability*, and is a topological property of M. We do not believe this assumption is essential; rather it simplifies the presentation here. For example, if one patches together deformations on parts of the manifold using a partition of unity, the distance statistics would not be as nice.

With this choice in hand, our construction is equivariant under the diffeomorphism group of the manifold: the pushforward of our probability measure by a diffeomorphism is equal to the measure obtained by pushing forward the reference metric and the frame.

We give the construction in Sect. 3. It is based on viewing the space of metrics with a given volume form as the space of sections of a bundle over M with fibers diffeomophic to the symmetric space $S = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ ($n = \dim M$). This symmetric space supports an invariant Riemannian metric which can then be used to define an L^2 distance on the space of metrics, which coincides with the distance arising from a Riemannian structure on this (infinite-dimensional) space. This distance is introduced in Sect. 2.3 and is studied as a random variable in Sect. 4, where tail estimates are obtained in terms of geometric constants.

In the Appendix, similar computations are carried out for a Lispchitz-type distance, also considered in [3]. Those estimates are then applied to establish integrability and existence of exponential moments for the *diameter*, *Laplace eigenvalue* and *volume entropy* functionals of our random Riemannian metrics.

Initial directions for further work involve studying the nature of the deformation we obtain (computation of the probability of the metric to lie in a small ball around the reference metric, and the behaviour of the isoperimetic constant under the deformation). In a foundational direction, we will address in a sequel questions about *convergence* and *tightness* (i.e. relative compactness in the weak-* topology) of our families of measures.

We expect that the Gaussian measure we have introduced in this paper will have applications that extend significantly beyond the basic questions considered here, in particular to the motivating problems discussed above.

2 The space of metrics

We fix once and for all a compact smooth manifold M without boundary and write n for its dimension. We also fix a smooth volume form dv on M.

We rely crucially on the symmetric space structure of the space P of positive-definite matrices of determinant 1 and on the related structure theory of $\mathrm{SL}_n(\mathbb{R})$. In the discussion below we state the facts we use; proofs and further details may be found in the text [18], which concentrates on this case, and in [13] which develops the general theory of symmetric spaces associated to semisimple Lie groups.

2.1 The space of metrics

We start by giving a coordinate-free description of the set of Riemannian metrics with the volume form dv on M. We then restrict to a class of manifolds for which there is a coordinate system simplifying the description.

Let V be a finite-dimensional real vector space with dual space V^*, and let $\mathrm{Sym}(V) = \{g \in \mathrm{Hom}(V, V^*) | g^* = g\}$ be the space of symmetric bilinear forms on V. Among those we distinguish $\mathrm{Pos}(V) = \{g \in \mathrm{Sym}(V) | \forall v \in V : g(v, v) > 0\}$, the space of positive-definite bilinear forms on V. Let $\mathrm{SL}(V) \subset \mathrm{GL}(V)$ denote the special and general linear groups on V, and $\mathfrak{sl}(V) \subset \mathfrak{gl}(V)$ their Lie algebras. Then $\mathrm{GL}(V)$ acts on $\mathrm{Pos}(V)$ by

$$h^{-1} \cdot g = h^* \circ g \circ h. \tag{1}$$

It is well-known that this action is transitive; the stabilizer of any $h \in \mathrm{Pos}(V)$ is a maximal compact subgroup isomorphic to $\mathrm{O}(n)$. Moreover, the orbits of $\mathrm{SL}(V)$ are precisely the level sets of the determinant function $g \mapsto \det(g_0^{-1}g)$ where g_0 is a fixed isomorphism $V \to V^*$. Each level set is then of the form $\mathrm{SL}(V)/K_{g_0}$ where $K_{g_0} = \mathrm{Stab}_{\mathrm{SL}(V)}(g_0) \simeq \mathrm{SO}(n)$ and we give it the $\mathrm{SL}(V)$-invariant Riemannian structure coming from the Killing form of $\mathrm{SL}(V)$, making it into a simply connected Riemannian manifold of non-positive curvature.

Remark 2.1 Since $\mathrm{Pos}(V)$ is an open subset of the vector space $\mathrm{Sym}(V)$, we may trivialize its tangent bundle by identifying each tangent space with $\mathrm{Sym}(V)$. The reader may then verify that with this identification, the tangent space at g to the $\mathrm{SL}(V)$-orbit of g is exactly $\{X \in \mathrm{Sym}(V) | \operatorname{Tr}(g^{-1}X) = 0\}$. Here we compose the linear maps $X \in \mathrm{Hom}(V, V^*)$ and $g^{-1} \in \mathrm{Hom}(V^*, V)$ to obtain a map in $\mathrm{End}(V)$ which has a trace. The reader may also verify that, since the congruence action above is linear as an action on $\mathrm{Sym}(V)$, the derivative of the action of h^{-1} at g is the map $X \mapsto h^{-1}Xh$ (composition of linear maps).

Now the Riemannian structure on the orbit claimed above is

$$\rho_g(X, X) = \operatorname{Tr}(g^{-1}Xg^{-1}X), \tag{2}$$

and it is an immediate calculation that this metric is $\mathrm{SL}(V)$-equivariant: that $\rho_{h.g}(h.X, h.X) = \rho_g(X, X)$.

With the usual translation of notions from vector spaces to vector bundles, we associate to the tangent bundle TM the vector bundles $\mathrm{Hom}(TM, T^*M)$ and $\mathrm{Sym}(TM)$, the symmetric space-valued bundle $\mathrm{Pos}(TM)$, and the group bundles $\mathrm{GL}(TM)$ and $\mathrm{SL}(TM)$.

By definition, a *Riemannian metric* on M is a smooth section of $\mathrm{Pos}(M)$; we denote the space of sections by $\mathrm{Met}(M)$. To such a metric there is an associated Riemannian volume form, and we let $\mathrm{Met}_{\mathrm{dv}}(M)$ denote the space of metrics whose volume form is dv. Fixing a metric $g_0 \in \mathrm{Met}_{\mathrm{dv}}(M)$, the above discussion identifies $\mathrm{Met}_{\mathrm{dv}}(M)$ with the space of sections of the bundle over M whose fibers are isomorphic to $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. Moreover, the fibre at x of this bundle is equipped with a transitive isometric action of $\mathrm{SL}(T_xM)$, where the metric is the one pulled back from the identification with $S = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ (the pullback is well-defined since the metric on S is $\mathrm{SL}_n(\mathbb{R})$-invariant).

Remark 2.2 It is a classical result of Ebin [11] that the diffeomorphism group acts transitively on the space of smooth volume forms of total volume 1, and therefore that the foliation of $\mathrm{Met}(M)$ by the orbits of the diffeomorphism group $\mathrm{Diff}(M)$ descends to a foliation of $\mathrm{Met}_{\mathrm{dv}}(M)$ by the group $\mathrm{Diff}_{\mathrm{dv}}(M)$ of volume-preserving diffeomorphisms. It follows that $\mathrm{Met}(M)/\mathrm{Diff}(M) \simeq \mathrm{Met}_{\mathrm{dv}}(M)/\mathrm{Diff}_{\mathrm{dv}}(M)$; we regard this space as the *space of geometries* on M.

In local co-ordinates $(x^1, \ldots, x^n)$, the above construction reads as follows. One takes the basis $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ for T_xM and its dual basis $\{dx^i\}_{i=1}^n$ for T_x^*M. Then fibers of $\mathrm{Sym}(M)$ are represented by symmetric matrices, fibers of $\mathrm{Pos}(M)$ by positive-definite symmetric matrices. The volume form associated to $g \in \mathrm{Met}(M)$ is then given by $|\det(g_x)|^{1/2}dx^1 \wedge \cdots \wedge dx^n$. $\mathrm{Met}_{\mathrm{dv}}(M)$ is then the metrics g such that $\det(g_x) = \det(g_x^0)$ for all $x \in M$, where g^0 is any metric with Riemannian volume form dv. The group $\mathrm{GL}_n(\mathbb{R})$ then acts on the fibres via congruence transformations $h^{-1} \cdot g = h^t g h$, with the stabilizer of g_x being the orthogonal group $\mathrm{O}_{g_x}(\mathbb{R}) \simeq \mathrm{O}(n)$. Similarly, the group $\mathrm{SL}_n(\mathbb{R})$ acts transitively on the subset of the fibre with a given determinant, with point stabilizer $\mathrm{SO}_{g_x}(\mathbb{R}) \simeq \mathrm{SO}(n)$.

2.2 Deforming a metric

Fix $g^0 \in \mathrm{Met}_{\mathrm{dv}}(M)$, and for $x \in M$ let $K_x \subset G_x = \mathrm{SL}(T_x M)$ be the orthogonal group of the positive-definite quadartic form g_x^0, which is also the stabilizer of g_x^0 under the congruence action (1). Fix a frame f_x in $T_x M$, orthonormal with respect to the inner product defined by g_x^0, and let $A_x \subset G_x$ be the subgroup of matrices which are diagonal with positive entries in the basis f_x. As noted above we can identify the set of positive-definite quadratic forms on $T_x M$ with the same determinant as g^0 with the symmetric space G_x/K_x.

Remark 2.3 We warn the reader that we use the usual letter G to denote a semisimple Lie group and the letter g to denote a Riemannian metric. As such we don't have $g_x \in G_x$, and rather use h_x to denote an arbitrary element of G_x.

Recall now the *Cartan* decomposition

$$G_x = K_x A_x K_x \tag{3}$$

(see for example [18, §5.1]). This states that every $h_x \in G_x$ can be written in the form $h_x = k_{1,x} a_x k_{2,x}$ with $k_{i,x} \in K_x$ and $a_x \in A_x$, with a_x being unique up to the action of the *Weyl group* $N_{G_x}(A_x)/Z_{G_x}(A_x)$, a group isomorphic to S_n acting by permutation of the coordinates with respect to the basis f_x. Given a_x, the two elements $k_{i,x} \in K_x$ are unique up to the fact that $Z_{K_x}(a_x)$ may not be trivial (generically this centralizer is equal to $Z_{K_x}(A_x)$, which is either trivial or $\{\pm 1\}$ depending on whether n is odd or even).

Recalling that $k_{2,x} \in K_x$ stabilizes g_x^0, it follows that for $h_x \in G_x$ decomposed as above we have

$$h_x \cdot g_x^0 = (k_{1,x} a_x) \cdot g_x^0.$$

Since G_x acts transitively on the level set, it follows that every g_x^1 with the same determinant g_x^0 is of this form, and moreover that in that form the a_x is unique up to the action of S_n on A_x.

Our goal is to randomly deform g^0 by choosing elements k_x and a_x for every $x \in M$. We shall discuss the "random" aspect of the construction in the next section, and concentrate at the moment on the topological issues involved in making such constructions well-defined.

Given the orthonormal frame f_x, we can identify A_x with the space of positive diagonal matrices of determinant 1. Further, using the exponential map we may identify this group with its Lie algebra $\mathfrak{a} \simeq \mathbb{R}^{n-1}$ of diagonal matrices of trace zero. We will therefore specify a_x by choosing such a matrix at each x, that is by choosing a function $H\colon M \to \mathfrak{a}$.

While this clearly works locally, making a global identification requires a choice of frame f_x at every $x \in M$, that is an everywhere non-zero section of the frame bundle of M or equivalently a trivialization of the tangent bundle of M, something which is not possible in general. For simplicity we have decided to only discuss here the case of manifolds where such sections exist, and defer more general constructions to future papers.

Remark 2.4 We required the existence of a smooth g^0-orthonormal frame. However, this is equivalent to the topological condition ("parallelizability") of the existence of a smooth but not necessarily orthonormal frame. To see this note that starting with any non-zero smooth section of the frame bundle, applying pointwise the Gram–Schmidt procedure with respect to the metric g^0 is a smooth operation and will produce a smooth orthonormal frame.

We survey here some facts about parallelizable manifolds, mainly to note that this class is rich enough to make our construction interesting. First, a parallelizable manifold is clearly orientable. Second, a necessary condition for parallelizability is the vanishing of the second Stiefel–Whitney class of the tangent bundle, which for orientable manifolds is equivalent to M being a *spin manifold*. Examples of parallelizable manifolds include all 3-manifolds, all Lie groups, the frame bundle of any manifold and the spheres S^n with $n \in \{1, 3, 7\}$.

2.3 The L^2 metric

Once the volume form is fixed, the action of $\mathrm{SL}(T_x M)$ on the stalk of $\mathrm{Met}_{\mathrm{dv}}(M)$ at x identifies it with the symmetric space $S = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. As noted above this space supports an $\mathrm{SL}_n(\mathbb{R})$-invariant Riemannian metric of non-positive curvature. Denote its distance function d_S; we then write d_x for the well-defined metric on the stalk at x of $\mathrm{Met}_{\mathrm{dv}}(M)$. Integrating this over M then gives a metric (to be denoted Ω_2) on $\mathrm{Met}_{\mathrm{dv}}(M)$: given two Riemannian metrics $g^0, g^1 \in \mathrm{Met}_{\mathrm{dv}}(M)$ on M with the same Riemannian volume form dv, we set

$$\Omega_2^2(g^0, g^1) = \int_M d_x^2(g_x^0, g_x^1)\mathrm{dv}(x).$$

For a different point of view on this metric, recall that d_S is the distance function associated to the Riemannian metric (2). Fixing $V = \mathbb{R}^n$ with its standard metric and frame, we write $G = \mathrm{SL}_n(\mathbb{R})$, $K = \mathrm{SO}(n)$ so that $S = G/K$. In this setting one can find directly the geodesics connecting the standard metric to any metric which is diagonal in the standard basis. Using G-equivariance and the Cartan decomposition (3), the upshot is the following (for details see [18, §5.1]): let $hK, h'K \in S = G/K$ correspond to two metrics of equal determinant. Then $Kh^{-1}h'K$ is a well-defined element of $K\backslash G/K \simeq A/S_n$, where A is the group of diagonal matrices of determinant 1 and positive entries. Let $a \in A$ be a representative for $Kh^{-1}h'K$. We then say that g and h are in *relative position* a. Writing $\log a$ for the vector of n logarithms of the diagonal entries of a (note that the entries of $\log a$ sum to 1, since $\det a = 1$), it turns out that and $d_S(hK, hK) = \|\log a\|$, where and $\|\cdot\|$ is the usual ℓ^2 norm.

3 Gaussian measures on the space of metrics

We next turn to the question of actually constructing our Gaussian measures. For a general reference on Gaussian random variables see [6]. In view of the decomposition considered in Sect. 2.1, it is natural to split the construction into diagonal and orthogonal parts.

Let g^0 be our reference metric. Every other metric of $\mathrm{Met}_{\mathrm{dv}}$ is of the form $g_x^1 = k_x a_x \cdot g_x^0$ where k, a are smooth functions on M such that $k_x \in K_x$ and $a_x \in A_x$. In Sects. 3.1 and 3.2 we describe random constructions of a_x and k_x respectively.

It is not hard to verify that $\bigcup_x K_x$, $\bigcup_x A_x$, $\bigcup_x G_x$ are subbundles of the Lie-group bundle $\mathrm{GL}(TM)$, and that their Lie algebras therefore furnish subbundles of the Lie algebra bundle $\mathfrak{gl}(TM) \simeq \mathrm{End}(TM)$. Specifically, $\mathrm{Lie}(G_x)$ consists of the endomorphisms of $T_x M$ of trace zero, $\mathrm{Lie}(K_x)$ consists of the endomorphisms which are skew-symmetric in the frame f_x, and $\mathrm{Lie}(A_x)$ consists of those which are diagonal of trace zero in the frame.

For the constructions below we fix a complete orthonormal basis $\{\psi_j\}_{j=0}^{\infty} \subset L^2(M)$ such that $\Delta_{g^0}\psi_j + \lambda_j \psi_j = 0$, with λ_j being a non-decreasing ordering of the spectrum of the Laplace operator Δ_{g^0}. Our constructions are in fact independent of the choice of basis of each eigenspace, but it is more convenient to make an explicit choice.

3.1 The radial part

We begin by defining a measure on the space of smooth functions $x \mapsto H_x$ such that $H_x \in \mathrm{Lie}(A_x)$ (sections of the bundle $\bigcup_x \mathrm{Lie}(A_x)$). We follow the recipe of [17]: choose *decay coefficients* $\beta_j = F(\lambda_j)$ where $F(t)$ is an eventually monotonically decreasing function of t and $F(t) \to 0$ as $t \to \infty$. Then set

$$H_x = \sum_{j=1}^{\infty} \pi_n(\underline{\xi}_j)\beta_j \psi_j(x), \tag{4}$$

where $\underline{\xi}_j$ are i.i.d. standard Gaussians in $\mathbb{R}^n$, and $\pi_n : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection on the hyperplane $\sum_{i=1}^n x_i = 0$.

Finally, set

$$a_x = \exp(H_x)$$

where exp is the exponential map to A_x from its Lie algebra.

The smoothness of H defined by (4) is given by [17, Theorem6.3]. The following two propositions apply whenever $\underline{\xi}_j$ in (4) denotes a d-dimensional standard Gaussian, while M has dimension n.

Proposition 3.1 *If $\beta_j = O(j^{-r})$ where $r > (q+\alpha)/n + 1/2$, then H defined by* (4) *converges a.s. in $C^{q,\alpha}(M, \mathbb{R}^d)$.*

We remark that the exponents in Proposition 3.1 are independent of d (the dimension of the “target” space). Now Weyl’s law for M [2,16] states that λ_j grows roughly as $j^{2/n}$. It follows that

Proposition 3.2 *If $\beta_j = O(\lambda_j^{-s})$ where $s > q/2 + n/4$, then H defined by* (4) *converges a.s. in $C^q(M, \mathbb{R}^d)$.*

3.2 The angular part

In this paper we study invariants of g^1 that can be bound only using a, so that our later calculations will only depend on the marginal distribution of a. Thus, as long as the choices of k and a are independent, the choice of k has no effect. In future work we plan to ask more detailed questions where this choice will become relevant. For example, determining the curvature of g^1 following the ideas of [8] requires differentiating g^1_x with respect to x and this immediately involves the choice of k_x. We thus propose the following specific choice, again using the recipe of Eq. (4). We set

$$k_x = \exp_x(u_x)$$

where u_x is the Gaussian vector

$$u_x = \sum_{j=1}^{\infty} \underline{\eta}_j \delta_j \psi_j(x). \tag{5}$$

Here $\underline{\eta}_j \in \mathfrak{so}_n$ are i.i.d. standard Gaussian anti-symmetric matrices (i.e. each $\underline{\eta}_j$ is given by $d_n = n(n-1)/2$ i.i.d. standard Gaussian variables corresponding to the upper-triangular part of $\underline{\eta}_j$), and $\delta_j = F_2(\lambda_j)$ are decay factors, given as functions of the corresponding eigenvalues.

Proposition 3.1 above applies again to give the smoothness properties of our random sections. In particular, since the exponents in Proposition 3.1 are independent of d_n, substituting into Weyl's law we get a straightforward analogue of Proposition 3.2 for the expression (5).

3.3 Remarks on the construction

For readers who may not wish to refer to a textbook such as [6], we briefly recall that a random vector such as H_x is *Gaussian* if its finite-dimensional marginals are Gaussian, which in our case roughly means (though we want more) that for every k points $x_1, \dots, x_k \in M$, the joint distribution of the finite-dimensional vector $(H_{x_1}, \dots, H_{x_k})$ is Gaussian.

Our Gaussian vectors are balanced (their expectation is zero) and they are therefore determined by their covariance function (roughly, the function on $M \times M$ given by the expectation of $H_{x_1} \otimes H_{x_2}$).

Remark 3.3 For the convenience of the reader who prefers Gaussian variables to be defined by their covariance function, we note here the covariance functions relevant to our case.

Let $\mathfrak{g}_x = \mathfrak{sl}(T_x M)$ denote the Lie algebra of $\mathrm{SL}(T_x M)$. As noted above our Gaussian measure is defined on appropriate spaces of sections of subbundles of the bundle $\bigcup_x \mathfrak{g}_x$. With sufficient continuity it is enough to consider the covariance operator evaluated on linear functionals of the form $X \mapsto \alpha_x(X(x))$, where X is a section of the bundle and $\alpha_x \in \mathfrak{g}_x^*$.

Our Gaussian measure for the diagonal part then has the covariance functions

$$R((x,k),(x',k')) = \delta_{kk'} \sum_j \beta_j^2 \psi_j(x)\psi_j(x'), \tag{6}$$

where k is an index for the diagonal entries of a matrix in $\mathfrak{g}_x$, diagonal with respect to our fixed frame and (x, k) therefore denote the functional mapping the section H_x to the kth entry of the diagonal matrix at x. The angular part has a similar covariance function.

For standard choices of β_k, we note that the covariance function for analogously-defined scalar fields would be well-known spectral invariants: we'd have

$$r(x,y) = \begin{cases} Z(x,y,2s) := \sum_{k=1}^{\infty} \frac{\psi_k(x)\psi_k(y)}{\lambda_k^{2s}}, & \beta_k = \lambda_k^{-s}; \\ e^*(x,y,2t) := \sum_{k=1}^{\infty} \frac{\psi_k(x)\psi_k(y)}{e^{2t\lambda_k}}, & \beta_k = e^{-t\lambda_k}. \end{cases} \tag{7}$$

Here $Z(x, y, 2s)$ is known as the *spectral zeta function* of Δ_0 (see e.g. [16]), while $e^*(x, y, 2t)$ is the corresponding *heat kernel* (see e.g. [4] or [9, Ch.6]), both taken *without the constant term* that would correspond to the constant eigenfunction ψ_0 with eigenvalue zero.

Remark 3.4 When taking $\beta_j = \lambda_j^{-s}$, the parameter s determines the a.s. Sobolev regularity of the random metric g via Propositions 3.1 and 3.2. If the metric g^0 is real-analytic, then letting $\beta_k = e^{-t\lambda_k}$ makes the random metric g real-analytic as well, with the parameter t related to the a.s. radius of analyticity (the exponent in rate of decay of Fourier coefficients).

Remark 3.5 A similar construction applies to the space of *all* Riemannian metrics on M (without necessarily fixing the volume form). We now work in the symmetric space $\mathrm{GL}(T_x M)/\mathrm{O}(g_x^0)$. The only change is that in Eq. (4) one lets A_j be standard vector-valued Gaussians without the projection.

There is a Riemannian structure and an L^2 metric (due to Ebin) defined on the space of all metrics. A detailed study of the metric properties of this space was undertaken in [10].

4 Ω_2 as a random variable

In this section we study the statistics of Ω_2^2.

4.1 The distribution function

We recall one definition of the (fiber-wise) distance d_x introduced in Sect. 2.3. For this choose a basis for $T_x M$ orthonormal with respect to $g^0(x)$ (in this basis the reference metric g_x^0 is represented by the identity matrix). If the translation from g_x^0 to g_x^1 is given by the element $k_x a_x \in G_x$ with a_x diagonal in the chosen basis, k_x orthogonal, then the metric g_x^1 is represented by the symmetric positive-definite matrix $k_x a_x^2 k_x^{-1}$. Writing $e^{b_i(x)}$ for the diagonal entries of a_x, we have

$$d_x^2(g_x^0, g_x^1) = \sum_{i=1}^n b_i(x)^2.$$

Accordingly,

$$\Omega_2^2(g^0, g^1) = \int_M \left(\sum_{i=1}^n b_i(x)^2 \right) \mathrm{dv}(x). \tag{8}$$

In our random model, the vector-valued function $b(x)$ is a Gaussian random field, chosen according to Eq. (4), where here we choose π_n to be the orthogonal projection. In other words $b(x)$ is defined by projecting an isotropic Gaussian in $\mathbb{R}^n$ orthogonally to the hyperplane $\sum_i b_i(x) = 0$. Integrating over x, we find that the distribution of Ω_2^2 is given by:

$$\Omega_2^2 \overset{D}{=} \sum_j \beta_j^2 \sum_{i=1}^{n-1} W_{i,j}$$

where the $W_{i,j}$ are independent random variables with χ^2 distribution. We can rewrite this as

$$\Omega_2^2 \overset{D}{=} \sum_j \beta_j^2 V_j$$

with i.i.d. $V_j \sim \chi_{n-1}^2$ (χ^2 distribution with $n-1$ degrees of freedom).

Recall that the *moment generating function* of the random variable X is the function $M_X(t) = \mathcal{E}(\exp(tX))$. These can be used, for example, to estimate the probability of large deviations of the variable X. Having represented Ω_2^2 as the sum of independent variables with known distribution, we can now explicitly compute its moment generating function as the product

$$\begin{aligned} M_{\Omega_2^2}(t) &= \prod_j \prod_{i=1}^{n-1} M_{\chi_1^2}(t\beta_j^2) = \prod_j \prod_{i=1}^{n-1} (1 - 2t\beta_j^2)^{-1/2} \\ &= \prod_j (1 - 2t\beta_j^2)^{-(n-1)/2} \end{aligned}$$

The following result is proved similarly.

Proposition 4.1 *The characteristic function $E(\exp(it\Omega_2^2))$ can be computed explicitly as*

$$\prod_j \prod_{k=1}^{n-1} (1-2it\beta_j^2)^{-1/2} = \prod_j (1-2it\beta_j^2)^{-(n-1)/2}.$$

4.2 Tail estimates for Ω_2^2

Here we apply [14, Lemma1, (4.1)] to estimate the probability of the following events:

$$\text{Prob}\{\Omega_2^2 > R^2\}, \quad R \to \infty. \tag{9}$$

We let $W = \sum_i a_i Z_i^2$ with Z_i i.i.d. standard Gaussians, and for $(n-1)(j-1)+1 \le i \le (n-1)j$, we have $a_i = \beta_j^2$ (i.e. each β_j^2 is repeated $(n-1)$ times). We let $\|a\|_\infty = \sup_j a_j$. Assume from now on that $\beta_j = F(\lambda_j)$ is a *monotone decreasing* function; then $\|a\|_\infty = a_1 = \beta_1^2$.

It is shown in [14, Lemma1, (4.1)] that for $W_k = \sum_{i=1}^{k(n-1)} a_i Z_i^2$, we have

$$\text{Prob}\left\{ W_k \ge \sum_{i=1}^{k(n-1)} a_i + 2\left(\sum_{i=1}^{k(n-1)} a_i^2\right)^{1/2} \sqrt{y} + 2\,\|a\|_\infty\, y \right\} \le e^{-y}.$$

Letting $k \to \infty$, we get the following quantities:

$W := \lim_{k\to\infty} W_k = \Omega_2^2$;
$A^2 = \sum_{i=1}^\infty a_i = (n-1)\sum_{j=1}^\infty \beta_j^2$;
$B^4 = \sum_{i=1}^\infty a_i^2 = (n-1)\sum_{j=1}^\infty \beta_j^4$;
$\|a\|_\infty = a_1 = \beta_1^2$.

With x^2 instead of y, we get:

$$\text{Prob}\{W \ge A^2 + 2B^2 x + 2\,\|a\|_\infty^2\, x^2\} \le e^{-x^2}.$$

Solving

$$R^2 = 2||a||_\infty^2 x^2 + 2B^2 x + A^2.$$

for x gives (for $R \ge A$) the following root:

$$x(R) = \frac{-B^2 + \sqrt{B^4 + 2(R^2 - A^2)||a||_\infty^2}}{2||a||_\infty^2}. \tag{10}$$

We conclude that

$$\text{Prob}\{\Omega_2 \ge R\} \le e^{-(x(R))^2},$$

where $x(R)$ is given by (10).

It is easy to show that there exists a constant $C = C(A, B, ||a||_\infty)$ such that for $R \ge A$, we have

$$x(R)^2 \ge \frac{R^2}{2||a||_\infty^2} - CR = \frac{R^2}{2\beta_1^2} - CR.$$

We also notice that

$$\text{Prob}\{\Omega_2 \ge R\} \ge \text{Prob}\{\beta_1^2 Z_1^2 \ge R^2\} = \Psi\left(\frac{R}{\beta_1}\right) \ge C\frac{\beta_1 e^{-R^2/(2\beta_1^2)}}{R},$$

provided $R \geq \beta_1$.

To summarize:

Theorem 4.2 *For $R \geq A$, we have*

$$\frac{C\beta_1}{R}\exp\left(\frac{-R^2}{2\beta_1^2}\right) \leq \text{Prob}\{\Omega_2 \geq R\} \leq \exp\left(\frac{-R^2}{2\beta_1^2} + CR\right).$$

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Appendix by Y. Canzani, D. Jakobson and L. Silberman Lipschitz distance. Applications to the study of diameter and Laplace eigenvalues

In this section we shall prove tail estimates for a Lipschitz-type distance ρ defined below, and use those estimates to prove that the *diameter* and *Laplace eigenvalue* functionals are measurable with respect to the Gaussian measures defined in Sect. 3, and to give tail estimates for them. We maintain the hypothesis that all metrics under consideration have the same associated volume form dv, though the results could be easily modified to remove this assumption.

Lipschitz distance

Here we study a (Lipschitz-type) distance ρ related to the distance used in [3] by Bando and Urakawa. It is defined by

$$\rho(g^0, g^1) = \sup_{x\in M}\ \sup_{0\neq\xi\in T_xM}\left|\ln\frac{g^1(\xi,\xi)}{g^0(\xi,\xi)}\right| \tag{11}$$

In other words, it is determined by taking the identity map on M and considering its Lipschitz constants between the two metrics. Note that the fiber-wise constant is also the larger of the Lipschitz constants of the map and its inverse: on the one hand, clearly for any curve on M, its g^1-length is at most $\exp(\rho(g^0, g^1))$ times its g^0 length, and conversely for the (x, ξ) achieving the supremum, taking y near x in the direction ξ we see that the g^0 and g^1- distances between x, y are roughly distorted by the same factor (though we do not know which is larger).

As in the case of Ω_2, $\rho(g^1, g^0)$ depends only on a_x where $g_x^1 = k_x a_x \cdot g_x^0$ (action of G_x on $\text{Hom}(T_xM, T_x^*M)$ by composition; in the representation of metrics are positive-definite matrices this is the congruence action $g_x^2 = k_x a_x g_x^0 a_x k_x^{-1}$). In the our adapted frame, the diagonal part of g^1 has entries $e^{2b_i(x)}$, where $\sum_i b_i(x) = 0$ for every $x \in M$, and where the vector $b(x) = (b_1(x), \ldots, b_n(x))$ is defined by the formula (4). Specifically, for any x ∈M the second supremum in (11) is equal to

$$2\sup_i |b_i(x)| \tag{12}$$

The supremum is attained for $\xi = e_i$ (the ith unit vector in $T_x M$). Accordingly,

$$\rho(g^0, g^1) = 2 \sup_{1 \le i \le n} \sup_{x \in M} |b_i(x)| \tag{13}$$

Tail estimate for ρ

Now, $\rho > R$ iff $\sup_j \sup_{x \in M} |2b_i(x)| > R$. Accordingly,

$$\mathrm{Prob}\{\rho(g^0, g^1) > R\} \le \mathrm{Prob}\left\{\sup_{x \in M} \sup_i |b_i(x)| > R/2\right\}. \tag{14}$$

Recall that $\mathrm{diag}(b_1, \ldots, b_n)$ is given by projecting a random vector on a particular hyperplane, which does not increase the maximum norm. It follows that

$$\mathrm{Prob}\{\rho(g^1, g^0) > R\} \le \mathrm{Prob}\left\{\sup_{x \in M} \sup_j |a_j(x)| > R/2\right\},$$

where a_j are the components of an $\mathbb{R}^n$-valued Gaussian vector. By symmetry we have for fixed i that

$$\mathrm{Prob}\{\sup_{x \in M} |a_i(x)| > u\} \le 2 \cdot \mathrm{Prob}\left\{\sup_{x \in M} a_i(x) > u\right\}.$$

Taking the union bound we find that

$$\mathrm{Prob}\{\rho(g^0, g^1) > R\} \le 2n \cdot \mathrm{Prob}\left\{\sup_{x \in M} a_1(x) > R/2\right\}. \tag{15}$$

We would like to estimate this probability as $R \to \infty$. We will need the covariance function for the scalar random field $a_1(x)$, given by (see (6))

$$r_{a_1}(x, y) = \sum_{k=1}^{\infty} \beta_k^2 \psi_k(x) \psi_k(y),$$

where ψ_k denote the L^2-normalized eigenfunctions of $\Delta(g_0)$.

The following result now follows in a standard way from the Borell-TIS theorem; it can be easily deduced from the calculations in [8, §3] and [1, §2, (2.1.3)]. We denote by σ^2 the supremum of the variance $r_{a_1}(x, x)$:

$$\sigma^2 := \sigma(a_1)^2 := \sup_{x \in M} r_{a_1}(x, x). \tag{16}$$

Proposition 5.1 *Let $\sigma(a_j)$ be as in (16). Then*

$$\lim_{R \to \infty} \frac{\ln \mathrm{Prob}\{\sup_{x \in M} a_1(x) > R/2\}}{R^2} = \frac{-1}{8\sigma^2}.$$

Proposition 5.1 and (15) imply the following

Corollary 5.2 *Let $\sigma^2 := \sup_{x \in M} r_{a_1}(x, x)$. Then*

$$\lim_{R \to \infty} \frac{\ln \mathrm{Prob}\{\rho(g_0, g_1) > R\}}{R^2} \le \frac{-1}{8\sigma^2}. \tag{17}$$

In the sequel, we shall need a slightly more precise estimate; it follows from the previous discussion and the estimates in [1, §2,p.50].

Proposition 5.3 *There exists* $\alpha > 0$ *such that for a fixed* $\epsilon > 0$ *and for large enough* R*, we have*

$$\text{Prob}\{\rho(g_1, g_0) > R\} \leq 2n \exp\left(\frac{\alpha R}{2} - \frac{R^2}{8\sigma^2}\right).$$

Diameter and eigenvalue functionals

In this section we use Corollary 5.2 to give estimates for the diameter and Laplace eigenvalues of the random metric g_1.

Lemma 5.4 *Assume that* $d\text{vol}(g_0) = d\text{vol}(g_1)$*, and that in addition* $\rho(g_0, g_1) < R$*. Then*

$$e^{-R} \leq \frac{\text{diam}(M, g_1)}{\text{diam}(M, g_0)} \leq e^R. \tag{18}$$

and also

$$e^{-2R} \leq \frac{\lambda_k(\Delta(g_1))}{\lambda_k(\Delta(g_0))} \leq e^{2R}. \tag{19}$$

Proof The definition (11) implies that for any fixed path $\gamma : [0, 1] \to M$, the ratio of its lengths with respect to the metrics g_0 and g_1 is satisfies

$$e^{-R} \leq \frac{L_{g_1}(\gamma)}{L_{g_0}(\gamma)} \leq e^R.$$

Since

$$\text{diam}(M, g) = \sup_{x,y\in M} \inf_{\gamma:\gamma(0)=x,\gamma(1)=y} L_g(\gamma),$$

the inequality (18) follows.

To prove (19), we let $h \in H^1(M)$, $h \not\equiv 0$ be a test function. Then $||h||_g^2 := \int_M h^2 \text{dv}$ is independent of the metric, since the volume form dv is fixed. The Rayleigh quotient of h is equal to

$$\frac{\langle dh, dh\rangle_{g^{-1}}}{||h||_g^2},$$

where g^{-1} denotes the co-metric corresponding to g. Since the Lipschitz distance is symmetric in its two arguments, we conclude that if $\rho(g_0, g_1) < R$, then $\rho(g_0^{-1}, g_1^{-1}) < R$ as well. It follows that

$$e^{-2R} \leq \frac{\langle dh, dh\rangle_{g_0^{-1}}}{\langle dh, dh\rangle_{g_1^{-1}}} \leq e^{2R}. \tag{20}$$

By the min-max characterization of the eigenvalues (see e.g. [3, §2]),

$$\lambda_k(\Delta(g)) = \inf_{U\subset H^1(M):\, \dim U=k+1} \sup_{h\in U,\, h\not\equiv 0} \frac{||dh||_{g^{-1}}^2}{||h||_g^2}.$$

The estimate (19) now follows from (20). □

We next establish some integrability results for the *diameter* functional $\text{diam}(M, g_1)$. They follow from Lemma 5.4 and the stronger form of Corollary 5.2.

Theorem 5.5 *Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonically increasing function such that for some $\delta > 0$*

$$h(e^y) = O\left(\exp\left[y^2(1/(8\sigma^2) - \delta)\right]\right).$$

Then $h(\mathrm{diam}(g_1))$ is integrable with respect to the probability measure $d\omega(g_1)$ constructed in Sect. 3.

In the proof we shall use Proposition 5.3.

Proof Without loss of generality, assume that we have normalized g_0 so that $\mathrm{diam}(g_0) = 1$. We have shown in (18) that if $\rho(g_0, g_1) < R$, then

$$\mathrm{diam}(g_1) \leq \mathrm{diam}(g_0) \cdot e^R = e^R,$$

so (h being monotone) we have under the same assumption that

$$h(\mathrm{diam}(g_1)) < h(e^R).$$

Since $h \geq 0$, the function $h(\mathrm{diam}(g_1))$ is integrable provided the sum

$$\sum_{k=N}^{\infty} h(e^k) \cdot \mathrm{Prob}\{g_1 : k - 1 \leq \rho(g_1, g_0) \leq k\}$$

converges. By the hypotheses on h and Corollary 5.2, that sum is dominated by

$$2n \sum_{k=N}^{\infty} h(e^k) \exp\left(\frac{\alpha(k-1)}{2} - \frac{(k-1)^2}{8\sigma^2}\right) \leq$$
$$2n \sum_{k=N}^{\infty} \exp\left[\frac{\alpha(k-1)}{2} + \left(\frac{k^2}{8\sigma^2} - \delta k^2\right) - \frac{(k-1)^2}{8\sigma^2}\right]$$

Choosing N large enough, we find that the last sum is dominated by

$$2n \sum_{k=N}^{\infty} \exp\left[\frac{-\delta k^2}{2}\right],$$

and the last expression clearly converges. □

Remark 5.6 The proof of Theorem 5.5 can be easily modified to establish analogous results for *averages* of the distance function. For example, given $t > 0$, consider the functional

$$E_t(g) := \int_M \int_M (\mathrm{dist}_g(x, y))^t \, \mathrm{dv}(x) \, \mathrm{dv}(y).$$

We leave the details to the reader.

Another corollary is the following

Theorem 5.7 *Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonically increasing function such that for some $\delta > 0$*

$$h(e^{2y}) = O\left(\exp\left[y^2(1/(8\sigma^2) - \delta)\right]\right).$$

Then $h(\lambda_k(\Delta(g_1)))$ is integrable with respect to the probability measure $d\omega(g_1)$ constructed in Sect. 3.

Proof The proof is similar to the proof of Theorem 5.5. We let $\lambda_k(g_0) =: e^{2\beta_k} =: e^{2\beta}$.

It follows from (19) that if $\rho(g_0, g_1) < R$, then $\lambda_k(g_1) \leq \lambda_k(g_0) \cdot e^{2R} = e^{2(R+\beta)}$. By monotonicity of the function h, we have

$$h(\lambda_k(g_1)) < h(e^{2(R+\beta)}).$$

Since $h \geq 0$, the function $h(\lambda_k(g_1))$ is integrable provided the sum

$$\sum_{m=N}^{\infty} h(e^{2(m+\beta)}) \cdot \text{Prob}\{g_1 : m-1 \leq \rho(g_1, g_0) \leq m\}$$

converges.

By the assumptions on h and Corollary 5.2, that sum is dominated by

$$2n \sum_{m=N}^{\infty} h(e^{2(m+\beta)}) \exp\left(\frac{\alpha(m-1)}{2} - \frac{(m-1)^2}{8\sigma^2}\right) \leq$$
$$2n \sum_{m=N}^{\infty} \exp\left[\frac{\alpha(m-1)}{2} + \left(\frac{(m+\beta)^2}{8\sigma^2} - \delta(m+\beta)^2\right) - \frac{(m-1)^2}{8\sigma^2}\right]$$

Choosing N large enough, we find that the last sum is dominated by

$$2n \sum_{m=N}^{\infty} \exp\left[\frac{-\delta m^2}{2}\right],$$

and the last expression clearly converges. □

Remark 5.8 Theorems 5.5 and 5.7 prove integrability results about the diameter and eigenvalue functionals. We plan to further study those and other functionals in future papers.

Volume entropy functional

The *volume entropy* functional $h_{vol}(g)$ of a metric g was defined by Manning in [15] as the exponential growth rate of volume in the universal cover (by showing that this growth rate is independent of the point of reference). In other words, it was shown that for any point x in the universal cover N of a compact Riemannian manifold M, the limit

$$h_{vol} = \lim_{s\to\infty} \frac{1}{s} \ln \text{vol}(B(x, s)), \tag{21}$$

exists and is independent of the choice of x. Here, volumes and distances (and thus balls) in N are with respect to the metric lifted from M.

We first prove the following counterpart of Lemma 5.4.

Lemma 5.9 *Assume that* $d\text{vol}(g_0) = d\text{vol}(g_1)$*, and that in addition* $\rho(g_0, g_1) < R$*. Then*

$$e^{-R} \leq \frac{h_{vol}(M, g_1)}{h_{vol}(M, g_0)} \leq e^R. \tag{22}$$

Proof By symmetry it is enough to prove the right-side inequality. Since ρ bounds the Lipschitz constant of the identity map (the argument above lifts to the universal cover), we have (balls in N with respect to the lifts of the respective metrics)

$$B_{g_1}(x, s) \subset B_{g_0}(x, e^R \cdot s). \tag{23}$$

By definition of h_{vol}, for any $\epsilon > 0$ there exists $s_0 > 0$ such that for every $s > s_0$, we have

$$\frac{1}{s} \ln \mathrm{vol} B_{g_0}(x, s) \leq h_{vol}(g_0) + \epsilon.$$

Combining the two claims, it follows that for $s > s_0$,

$$\frac{1}{s} \ln \mathrm{vol} B_{g_1}(x, s) \leq e^R \frac{1}{se^R} \ln \mathrm{vol} B_{g_0}(x, e^R s) \leq e^R (h_{vol}(g_0) + \epsilon)$$

(we used here the assumption that g_0, g_1 have the same volume form, so that the set-theoretic inclusion of balls implied an inequality on the volumes; without the assumption the volume would be an additional factor from the distortion of the volume form, but note that this factor would not affect the inequality in the limit $s \to \infty$).

Letting $s \to \infty$ we obtain $h_{vol}(g_1) \leq e^R(h_{vol}(g_0) + \epsilon)$, and letting $\epsilon \to 0$ we finally get

$$h_{vol}(g_1) \leq e^R h_{vol}(g_0).$$

□

Lemma 5.9 now easily implies

Theorem 5.10 *The conclusion of the Theorem 5.5 remains true if the diameter functional is replaced by the volume entropy functional h_{vol}.*

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