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"*Enlacements asymptotiques*" revisited

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Abstract We give an alternative proof of a theorem of Gambaudo and Ghys (Topology 36(6):1355–1379, 1997) and Fathi (Transformations et homéomorphismes préservant la mesure. Systèmes dynamiques minimaux. Thèse Orsay, 1980) on the interpretation of the Calabi homomorphism for the standard symplectic disc as an average rotation number. This proof uses only basic complex analysis.

Keywords Calabi homomorphism · Asymptotic linking · Hamiltonian diffeomorphism

Résumé Nous donnons une preuve alternative d'un théorème de Gambaudo and Ghys (Topology 36(6):1355–1379, 1997) et Fathi (Transformations et homéomorphismes préservant la mesure. Systèmes dynamiques minimaux. Thèse Orsay, 1980) sur l'interpretation de l'homomorphisme de Calabi pour le disque symplectique standard comme un nombre de rotation moyen. Cette preuve utilise seulement l'analyse compléxe de base.

Mathematics Subject Classification Primary 53D; Secondary 57

Contents

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1 The theorem of Gambaudo–Ghys and Fathi

Let $\mathcal{G} = Ham_c(\mathbb{D}, \omega)$ be the group of compactly supported Hamiltonian diffeomorphisms of the standard disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ endowed with the standard symplectic form $\omega = \frac{i}{2} dz \wedge d\bar{z}$. The Calabi homomorphism [2] from $\mathcal{G}$ to $\mathbb{R}$ is defined as

$$\mathrm{Cal}(\phi) = \int_0^1 \left(\int_{\mathbb{D}} H_t \, \omega \right) dt,$$

where H_t is the compactly supported Hamiltonian of a Hamiltonian isotopy $\{\phi_t\}_{t \in [0,1]}$ with $\phi_1 = \phi$. In other words this isotopy is generated by a time-dependent vector field X_t, that satisfies the relation

$$\iota_{X_t} \omega = -dH_t.$$

The mean rotation number is defined in terms convenient for the proof as follows. Consider the differential form

$$\alpha = \frac{1}{2\pi} \frac{d(z_1 - z_2)}{z_1 - z_2}$$

(used by Arnol'd [1] in his study of the cohomology of the pure braid groups) on the configuration space $X_2 = X_2(\mathbb{D}) = \{(z_1, z_2) | z_j \in \mathbb{D},\ z_1 \neq z_2\} = \mathbb{D} \times \mathbb{D} \backslash \Delta$, where $\Delta \subset \mathbb{D} \times \mathbb{D}$ is the diagonal. Denote by

$$\theta = Im(\alpha)$$

its imaginary part. Note that the two forms α and θ are closed. For each pair of points $(z_1, z_2) \in \mathbb{D} \times \mathbb{D}$ such that $z_1 \neq z_2$, that is for each point $x = (z_1, z_2) \in X_2$, consider the curve $\{\phi_t \cdot x\}$ in X_2 defined by

$$\phi_t \cdot x = (\phi_t(z_1), \phi_t(z_2))$$

for each $t \in [0, 1]$. The average rotation number is

$$\Phi(\phi) = \int_{X_2} dm^2(x) \int_{\{\phi_t \cdot x\}} \theta,$$

where $dm^2(x) = dm(z_1) dm(z_2)$ is the Lebesgue measure on $\mathbb{D} \times \mathbb{D}$ restricted to X_2 (here $dm(z)$ denotes the Lebesgue measure on $\mathbb{D}$). By preservation of volume, it is clear that Φ is a homomorphism $\mathcal{G} \to \mathbb{R}$.

The theorem of Gambaudo and Ghys [5] and Fathi [4] is the following equality.

Theorem 1

$$\Phi = -2\,\mathrm{Cal},$$

as homomorphisms $\mathcal{G} \to \mathbb{R}$.

Gambaudo and Ghys have presented several proofs of this result, and in [4] a different proof of Fathi is found. More proofs of this result are known today (cf. [3]). Here we present an alternative short proof, which is in fact a complex-variable version of the proof of Fathi.

2 The alternative proof

Put $\xi_t = dz(X_t)$, for the natural complex coordinate z on $\mathbb{D}$. Hence ξ_t is a smooth compactly supported complex-valued function on $\mathbb{D}$. The computations

$$\iota_{X_t}\left(\frac{i}{2}dz \wedge d\overline{z}\right) = \frac{i}{2}\xi_t d\overline{z} - \frac{i}{2}\overline{\xi}_t dz$$

and

$$-dH_t = -\frac{\partial H_t}{\partial \overline{z}}d\overline{z} - \frac{\partial H_t}{\partial z}dz$$

give us

$$\xi_t = 2i\frac{\partial H_t}{\partial \overline{z}}. \tag{1}$$

Now

$$\Phi(\phi) = Im\left(\int_{X_2} dm^2(x)\int_{\{\phi_t \cdot x\}} \alpha\right),$$

and hence it is sufficient to compute

$$\int_{X_2} dm^2(x)\int_{\{\phi_t\cdot x\}} \alpha = \frac{1}{2\pi}\int_{X_2} dm^2(x)\int_{\{\phi_t\cdot x\}} \frac{d(z_1 - z_2)}{z_1 - z_2} =$$

$$= \frac{1}{2\pi}\int_{X_2} dm(z_1)dm(z_2)\int_0^1 dt\ \frac{\xi_t(\phi_t(z_1)) - \xi_t(\phi_t(z_2))}{\phi_t(z_1) - \phi_t(z_2)} =$$

as the function is absolutely integrable (see Lemma 1), by Fubini,

$$= \frac{1}{2\pi}\int_0^1 dt\ \int_{X_2} dm(z_1)dm(z_2)\ \frac{\xi_t(\phi_t(z_1)) - \xi_t(\phi_t(z_2))}{\phi_t(z_1) - \phi_t(z_2)}$$

$$= \frac{1}{2\pi}\int_0^1 dt\ \int_{X_2} dm(z_1)dm(z_2)\ \frac{\xi_t(z_1) - \xi_t(z_2)}{z_1 - z_2} =$$

as both terms of the sum are absolutely integrable (a consequence of the proof of Lemma 1 as well),

$$= 2\cdot\frac{1}{2\pi}\int_0^1 dt\ \int_{\mathbb{D}} dm(w)\int_{\mathbb{D}\setminus\{w\}} \frac{\xi_t(z)}{z - w}dm(z) =$$

by Eq. 1,

$$= -2i\int_0^1 dt\ \int_{\mathbb{D}} dm(w)\int_{\mathbb{D}\setminus\{w\}} \frac{1}{2\pi i}\frac{\partial H_t}{\partial \overline{z}}\frac{dz \wedge d\overline{z}}{z - w}$$

$$= -2i\int_0^1 dt\ \int_{\mathbb{D}} dm(w)H_t(w) = -2i\,\mathrm{Cal}(\phi).$$

The penultimate equality is a consequence of the Cauchy formula for smooth functions [6, Theorem 1.2.1]. Indeed for any C^1 function $f : \mathbb{D} \to \mathbb{C}$, we have

$$f(w) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f(z)}{z - w}dz + \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{\partial f}{\partial \overline{z}}\frac{dz \wedge d\overline{z}}{z - w}.$$

It remains to note that as H_t is zero near the boundary, the first term of the sum vanishes.

Now we show the absolute integrability that we use to change the order of integration.

Lemma 1

$$\int_{X_2}\int_0^1 dm(z_1)dm(z_2)dt\ \frac{|\xi_t(\phi_t(z_1))-\xi_t(\phi_t(z_2))|}{|\phi_t(z_1)-\phi_t(z_2)|}<\infty$$

By the Tonnelli theorem, the following chain of inequalities suffices:

$$\begin{aligned}
&\int_0^1 dt\int_{X_2} dm(z_1)dm(z_2)\ \frac{|\xi_t(\phi_t(z_1))-\xi_t(\phi_t(z_2))|}{|\phi_t(z_1)-\phi_t(z_2)|}\\
&=\int_0^1 dt\int_{X_2} dm(z_1)dm(z_2)\ \frac{|\xi_t(z_1)-\xi_t(z_2)|}{|z_1-z_2|}\\
&\le 2\int_0^1 dt\int_{\mathbb{D}} dm(z)|\xi_t(z)|\int_{\mathbb{D}\setminus\{z\}}\ \frac{1}{|z-w|}dm(w)\\
&\le 8\pi\int_0^1 dt\int_{\mathbb{D}}|\xi_t|dm<\infty,
\end{aligned}$$

because

$$\int_{\mathbb{D}\setminus\{z\}}\ \frac{1}{|z-w|}dm(w)\le 4\pi,$$

as one verifies by direct calculation.

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