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Bergman Projections Induced by Doubling Weights on the Unit Ball of $\mathbb{C}^n$

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Abstract
Let $p > 1$ and $\omega, \upsilon \in \mathcal{D}$. The boundedness of $P_\omega : L^\infty(\mathbb{B}) \to \mathcal{B}(\mathbb{B})$ and $P_\omega(P_\omega^+) : L^p(\mathbb{B}, \upsilon dV) \to L^p(\mathbb{B}, \upsilon dV)$ are investigated in this paper.

Keywords Weighted Bergman space · Bergman projection · Doubling weight

Mathematics Subject Classification 32A36 · 47B33

1 Introduction

Let $\mathbb{B}$ be the open unit ball of $\mathbb{C}^n$ and $\mathbb{S}$ the boundary of $\mathbb{B}$. When $n = 1$, $\mathbb{B}$ is the open unit disk in the complex plane $\mathbb{C}$ and always denoted by $\mathbb{D}$. Let $H(\mathbb{B})$ denote the space of all holomorphic functions on $\mathbb{B}$. For any two points

$$z = (z_1, z_2, \ldots, z_n) \quad \text{and} \quad w = (w_1, w_2, \ldots, w_n)$$

in $\mathbb{C}^n$, define $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ and $|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$.

Suppose ω is a radial weight (i.e., ω is a positive, measurable and integrable function on $[0, 1)$ and $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{B}$). Let $\hat{\omega}(r) = \int_r^1 \omega(t)dt$. We say that

- ω is a doubling weight, denoted by $\omega \in \hat{\mathcal{D}}$, if there is a constant $C > 0$ such that

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$$\hat{\omega}(r) < C\hat{\omega}\left(\frac{1+r}{2}\right), \quad \text{when } 0 \le r < 1;$$

- ω is a regular weight, denoted by $\omega \in \mathcal{R}$, if ω is continuous and there exist $C > 0$ and $\delta \in (0, 1)$ such that

$$\frac{1}{C} < \frac{\hat{\omega}(t)}{(1-t)\omega(t)} < C, \quad \text{when } t \in (\delta, 1);$$

- ω is a rapidly increasing weight, denoted by $\omega \in \mathcal{I}$, if (see [10])

$$\lim_{r\to 1} \frac{\hat{\omega}(r)}{(1-r)\omega(r)} = \infty;$$

- ω is a reverse doubling weight, denoted by $\omega \in \check{\mathcal{D}}$, if there exist $K > 1$ and $C > 1$, such that

$$\hat{\omega}(t) \ge C\hat{\omega}\left(1 - \frac{1-t}{K}\right), \quad t \in (0, 1). \tag{1}$$

See [9, 10] and the references therein for more details about $\mathcal{I}, \mathcal{R}, \hat{\mathcal{D}}$. Let $\mathcal{D} = \hat{\mathcal{D}} \cap \check{\mathcal{D}}$. More information about $\check{\mathcal{D}}$ and $\mathcal{D}$ can be found in [7, 14].

Let $d\sigma$ and dV be the normalized surface and volume measures on $\mathbb{S}$ and $\mathbb{B}$, respectively. For $0 < p < \infty$, the Hardy space $H^p(\mathbb{B})$(or H^p) is the space consisting of all functions $f \in H(\mathbb{B})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\int_{\mathbb{S}} |f(r\xi)|^p d\sigma(\xi)\right)^{1/p}, \quad 0 < p < \infty.$$

H^∞ is the space consisting of all $f \in H(\mathbb{B})$ such that $\|f\|_{H^\infty} = \sup_{z\in\mathbb{B}} |f(z)| < \infty$.

For any $f \in H(\mathbb{B})$, let $\Re f$ be the radial derivative of f, that is,

$$\Re f(z) = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z), \quad z = (z_1, z_2, \ldots, z_n) \in \mathbb{B}.$$

Then the Bloch space $\mathcal{B}(\mathbb{B})$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{\mathcal{B}(\mathbb{B})} = |f(0)| + \sup_{z\in\mathbb{B}} (1 - |z|^2)|\Re f(z)| < \infty.$$

When $n = 1$, $\|\cdot\|_{\mathcal{B}(\mathbb{D})}$ differs from the norm defined in the classical way, but the two norms are equivalent. See [19] for example. We denote $\mathcal{B}(\mathbb{B})$ by $\mathcal{B}$ for simplicity.

Suppose μ is a positive Borel measure on $\mathbb{B}$ and $0 < p < \infty$. The Lebesgue space $L^p(\mathbb{B}, d\mu)$ consists of all measurable complex functions f on $\mathbb{B}$ such that $|f|^p$ is integrable with respect to μ, that is, $f \in L^p(\mathbb{B}, d\mu)$ if and only if

$$\|f\|_{L^p(\mathbb{B},d\mu)} = \left(\int_{\mathbb{B}} |f(z)|^p d\mu(z)\right)^{1/p} < \infty.$$

$L^\infty(\mathbb{B}, d\mu)$ consists of all measurable complex functions f on $\mathbb{B}$ such that f is essentially bounded, that is, $f \in L^\infty(\mathbb{B}, d\mu)$ if and only if

$$\|f\|_{L^\infty(\mathbb{B},d\mu)} = \inf_{E\subset\mathbb{B},\mu(E)=0} \sup_{z\in\mathbb{B}\setminus E} |f(z)| < \infty.$$

More details about $L^p(\mathbb{B}, d\mu)$ can be found in [18, 20]. For a positive and measurable function ω on $\mathbb{B}$, letting $d\mu(z) = \omega(z)dV(z)$, μ is a Borel measure on $\mathbb{B}$ if $\omega \in L^1(\mathbb{B}, dV)$. Then, we will write $L^p(\mathbb{B}, d\mu)$ as $L^p(\mathbb{B}, \omega dV)$. When $n = 1$ and $z \in \mathbb{D}$, let $dV(z) = \frac{1}{\pi}dA(z)$ be the normalized area measure on $\mathbb{D}$. Then we can define the Lebesgue space on the unit disk in the same way.

In [10], Peláez and Rättyä introduced a new class of weighted Bergman spaces $A^p_\omega(\mathbb{D})$, which is induced by rapidly increasing weights ω in $\mathbb{D}$. That is

$$A^p_\omega(\mathbb{D}) = L^p(\mathbb{D}, \omega dA) \cap H(\mathbb{D}), \quad \text{where } 0 < p < \infty.$$

See [9–13, 15, 16] for more results on $A^p_\omega(\mathbb{D})$ with $\omega \in \hat{\mathcal{D}}$. In [4], we extended some results of the Bergman space $A^p_\omega(\mathbb{D})$ to the unit ball $\mathbb{B}$ of $\mathbb{C}^n$. That is,

$$A^p_\omega(\mathbb{B}) = L^p(\mathbb{B}, \omega dA) \cap H(\mathbb{B}), \quad \text{where } 0 < p < \infty.$$

In brief, let $A^p_\omega = A^p_\omega(\mathbb{B})$. As a subspace of $L^p(\mathbb{B}, \omega dV)$, the norm on A^p_ω will be written as $\|\cdot\|_{A^p_\omega}$. It is easy to check that A^p_ω is a Banach space when $p \geq 1$ and a complete metric space with distance $\rho(f, g) = \|f - g\|^p_{A^p_\omega}$ when $0 < p < 1$. When $\alpha > -1$ and $c_\alpha = \Gamma(n+\alpha+1)/[\Gamma(n+1)\Gamma(\alpha+1)]$, if $\omega(z) = c_\alpha(1-|z|^2)^\alpha$, the space A^p_ω becomes the classical weighted Bergman space A^p_α, and we write $dV_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dV(z)$. When $\alpha = 0$, $A^p_0 = A^p$ is the standard Bergman space. See [18, 20] for the theory of H^p and A^p_α.

When $p = 2$, the space A^2_ω is a Hilbert space with the inner product

$$\langle f, g\rangle_{A^2_\omega} = \int_{\mathbb{B}} f(z)\overline{g(z)}\omega(z)dV(z) \quad \text{for all } f, g \in A^2_\omega.$$

In a standard way, for every $z \in \mathbb{B}$, the point evaluation $L_z f = f(z)$ is a bounded linear functional on A^2_ω. By Riesz's Representation Theorem, we see that there exists a unique function B^ω_z such that

$$f(z) = \langle f, B^\omega_z\rangle_{A^2_\omega} = \int_{\mathbb{B}} f(w)\overline{B^\omega_z(w)}\omega(w)dV(w) \quad \text{for all } f \in A^2_\omega.$$

For any $f \in L^1(\mathbb{B}, \omega dV)$, the Bergman projection $P_\omega f$ is defined by

$$P_\omega f(z) = \int_{\mathbb{B}} f(\xi)\overline{B_z^\omega(\xi)}\omega(\xi)dV(\xi),$$

while the maximal Bergman projection P_ω^+ is defined by

$$P_\omega^+(f)(z) = \int_{\mathbb{B}} f(\xi)\left|B_z^\omega(\xi)\right|\omega(\xi)dV(\xi).$$

When $\omega(z) = c_\alpha(1-|z|^2)^\alpha(\alpha > -1)$, P_ω and P_ω^+ will be denoted by P_α and P_α^+, respectively.

The study of the Bergman projection has a long history. If $s \in \mathbb{C}$ such that $\Re s > -1$ for all $f \in L^1(\mathbb{B}, dV)$, let

$$T_s f(z) = \frac{\Gamma(n+s+1)}{\Gamma(n+1)\Gamma(s+1)} \int_{\mathbb{B}} \frac{(1-|w|^2)^s}{(1-\langle z, w\rangle)^{n+1+s}} f(w)dV(w).$$

Obviously, when $s = 0$, we have $T_0 = P_0$. In [5], Forelli and Rudin proved that T_s is bounded on $L^p(\mathbb{B}, dV)$ if and only if $(1+\Re s)p > 1$ under the assumption that $1 \le p < \infty$. In [3], Choe proved that T_s is bounded on $L^p(\mathbb{B}, dV_\alpha)$ if and only if $(1+\Re s)p > 1+\alpha$ when $p \ge 1$ and $\alpha > -1$. From [20, Thm. 2.11], we see that P_α is bounded on $L^p(\mathbb{B}, dV_\beta)$ if and only if $p(\alpha+1) > \beta+1$ when $p \ge 1$, $\alpha, \beta \in (-1, \infty)$. In [8], Liu gave a sharp estimate for the norm of P_0 on $L^p(\mathbb{B}, dV)$.

In the setting of the unit disk, Bekollé and Bonami showed that, if $1 < p < \infty$, υ is positive on $\mathbb{D}$ and $\int_{\mathbb{D}} \upsilon(z)dA_\alpha(z) < \infty$, $P_\alpha : L^p(\mathbb{D}, \upsilon dA_\alpha) \to L^p(\mathbb{D}, \upsilon dA_\alpha)$ is bounded if and only if υ satisfies the Bekollé–Bonami condition, see [1, 2]. The result was extended in [17] for some $\omega \in \mathcal{R}$. In [13], the Bergman projections P_ω and the maximal Bergman projection P_ω^+ on some function spaces on $\mathbb{D}$ were studied when $\omega \in \mathcal{R}$. In [14], Peláez and Rättyä studied Bergman projections and the maximal Bergman projection P_ω^+ induced by radial weights ω on some function spaces on $\mathbb{D}$.

Motivated by [13, 14], in this paper we investigate the boundedness of $P_\omega : L^\infty(\mathbb{B}, dV) \to \mathcal{B}(\mathbb{B})$ and $P_\omega(P_\omega^+) : L^p(\mathbb{B}, \upsilon dV) \to L^p(\mathbb{B}, \upsilon dV)$ on the unit ball of $\mathbb{C}^n$ with $p > 1$ and $\omega, \upsilon \in \mathcal{D}$.

This paper is organized as follows. In Sect. 2, we recall some results and notation. In Sect. 3, we give some estimates for B_z^ω with $\omega \in \hat{\mathcal{D}}$. In Sect. 4, we investigate the boundedness of P_ω and P_ω^+ with $\omega \in \mathcal{D}$.

Throughout this paper, the letter C will denote a constant which may differ from one occurrence to the other. The notation $A \lesssim B$ means that there is a positive constant C such that $A \le CB$. The notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

2 Preliminary Results

For any $\xi, \tau \in \overline{\mathbb{B}}$, let $d(\xi, \tau) = |1 - \langle \xi, \tau \rangle|^{1/2}$. Then $d(\cdot, \cdot)$ is the non-isotropic metric. For $r > 0$ and $\xi \in \mathbb{S}$, let

$$Q(\xi, r) = \left\{ \eta \in \mathbb{S} : |1 - \langle \xi, \eta \rangle| \le r^2 \right\}.$$

$Q(\xi, r)$ is a ball in $\mathbb{S}$ for all $\xi \in \mathbb{S}$ and $r \in (0, 1)$. More information about $d(\cdot, \cdot)$ and $Q(\xi, r)$ can be found in [18, 20].

For any $a \in \mathbb{B}\backslash\{0\}$, let $Q_a = Q(a/|a|, \sqrt{1 - |a|})$ and

$$S_a = S(Q_a) = \left\{ z \in \mathbb{B} : \frac{z}{|z|} \in Q_a, |a| < |z| < 1 \right\}.$$

When $a = 0$, let $Q_a = \mathbb{S}$ and $S_a = \mathbb{B}$. We call S_a the Carleson block. See [4] for more information about the Carleson block. As usual, for a measurable set $E \subset \mathbb{B}$, $\omega(E) = \int_E \omega(z) dV(z)$.

Lemma 1 *Let ω be a radial weight.*

(i) *The following statements are equivalent.*

 (a) $\omega \in \hat{\mathcal{D}}$;
 (b) *there is a constant $b > 0$ such that $\hat{\omega}(t)/(1 - t)^b$ is essentially increasing;*
 (c) *for all $x \ge 1$, $\int_0^1 s^x \omega(s) ds \approx \hat{\omega}(1 - 1/x)$.*

(ii) *$\omega \in \check{\mathcal{D}}$ if and only if there is a constant $a > 0$ such that $\hat{\omega}(t)/(1-t)^a$ is essentially decreasing.*

(iii) *If ω is continuous, then $\omega \in \mathcal{R}$ if and only if there are $-1 < a < b < +\infty$ and $\delta \in [0, 1)$, such that*

$$\frac{\omega(t)}{(1 - t)^b} \nearrow \infty, \quad and \quad \frac{\omega(t)}{(1 - t)^a} \searrow 0, \quad when\ \delta \le t < 1. \tag{2}$$

Lemma 1 plays an important role in this research and can be found in many papers. Here, we refer to [7, Lem. B, Lem. C] and observation (v) in [10, Lem. 1.1].

For any radial weight ω, its associated weight ω^* is defined by

$$\omega^*(z) = \int_{|z|}^1 \omega(s) \log \frac{s}{|z|} s ds, \quad z \in \mathbb{D}\backslash\{0\}.$$

The following lemma gives some properties and applications of ω^*.

Lemma 2 *Let $\omega \in \hat{\mathcal{D}}$. The following statements hold.*

(i) *$\omega^*(r) \approx (1 - r) \int_r^1 \omega(t) dt$ when $r \in (1/2, 1)$.*
(ii) *For any $\alpha > -2$, $(1 - t)^\alpha \omega^*(t) \in \mathcal{R}$.*

(iii) $\omega(S_a) \approx (1-|a|)^n \int_{|a|}^1 \omega(r)dr$.
(iv) $\hat{\omega}(z) \approx \hat{\omega}(a)$, *if* $1/C < (1-|z|)/(1-|a|) < C$ *for some fixed* $C > 1$.

Proof (i) and (ii) are [10, Lem. 1.6, Lem. 1.7], respectively. (iii) was proved in [4]. (iv) can be proved directly by (i), (ii) and Lemma 1. For the benefit of readers, we give a proof here.

Suppose $\omega \in \hat{\mathcal{D}}$. Then there exist $a, b > -1$ and $\delta \in (0, 1)$ such that (2) holds for ω^*. Then, for all $\delta < x \leq y < 1$ such that $1/C \leq (1-x)/(1-y) \leq C$, we have

$$1 \approx \left(\frac{1-x}{1-y}\right)^a \leq \frac{\omega^*(x)}{\omega^*(y)} \leq \left(\frac{1-x}{1-y}\right)^b \approx 1.$$

If $x \leq \delta$ and $1/C \leq (1-x)/(1-y) \leq C$, $\hat{\omega}(x) \approx \hat{\omega}(y)$ is obvious. So,

$$\hat{\omega}(z) \approx \hat{\omega}(a), \quad \text{if } \frac{1}{C} < \frac{1-|z|}{1-|a|} < C.$$

The proof is complete. □

For a Banach space or a complete metric space X and a positive Borel measure μ on $\mathbb{B}$, we say that μ is a q-Carleson measure for X if the identity operator $I_d : X \to L^q_\mu$ is bounded. When $0 < p \leq q < \infty$ and $\omega \in \hat{\mathcal{D}}$, a characterization of the q-Carleson measure for A^p_ω was given in [4].

Theorem A *Let* $0 < p \leq q < \infty$, $\omega \in \hat{\mathcal{D}}$, *and* μ *be a positive Borel measure on* $\mathbb{B}$. *Then* μ *is a* q*-Carleson measure for* A^p_ω *if and only if*

$$\sup_{a\in\mathbb{B}} \frac{\mu(S_a)}{(\omega(S_a))^{\frac{q}{p}}} < \infty. \tag{3}$$

Moreover, if μ *is a* q*-Carleson measure for* A^p_ω, *then the identity operator* $I_d : A^p_\omega \to L^q_\mu$ *satisfies*

$$\|I_d\|^q_{A^p_\omega \to L^q_\mu} \approx \sup_{a\in\mathbb{B}} \frac{\mu(S_a)}{(\omega(S_a))^{q/p}}.$$

3 Some Estimates About B^{ω}_z with $\omega \in \hat{\mathcal{D}}$

In this section, we consider the reproducing kernel of A^2_ω and give some estimates for it. Let's recall some notations. For all $f \in H(\mathbb{B})$, the Taylor series of f at origin, which converges absolutely and uniformly on each compact subset of $\mathbb{B}$, is

$$f(z) = \sum_m a_m z^m, \quad z \in \mathbb{B}.$$

Here the summation is over all multi-indexs $m = (m_1, m_2, \dots, m_n)$, m_k is a non-negative integer and $z^m = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$. Let $|m| = m_1 + m_2 + \cdots + m_n$, $m! = m_1! m_2! \cdots m_n!$ and $f_k(z) = \sum_{|m|=k} a_m z^m$. Then the Taylor series of f can be written as $f(z) = \sum_{k=0}^{\infty} f_k(z)$, which is called the homogeneous expansion of f.

Lemma 3 *Let $\omega \in \hat{\mathcal{D}}$. Then*

$$B_z^{\omega}(w) = \frac{1}{2n!} \sum_{k=0}^{\infty} \frac{(n-1+k)!}{k! \omega_{2n+2k-1}} \langle w, z \rangle^k$$

and

$$\|B_z^{\omega}\|_{\mathcal{B}} \approx \frac{1}{\omega(S_z)} \approx \|B_z^{\omega}\|_{H^{\infty}}, \quad z \in \mathbb{B}.$$

Here and henceforth, $\omega_s = \int_0^1 r^s \omega(r) dr$.

Proof Suppose $f \in A_{\omega}^2$ and $f(z) = \sum_m a_m z^m$, $z \in \mathbb{B}$. For any fixed $z \in \mathbb{B}$, let

$$B_z^{\omega}(w) = \sum_m b_m(z) w^m.$$

By [20, Lem. 1.8, Lem. 1.11], we have

$$\begin{aligned} f(z) &= \int_{\mathbb{B}} f(w) \overline{B_z^{\omega}(w)} \omega(w) dV(w) \\ &= 2n \sum_m \frac{(n-1)! m!}{(n-1+|m|)!} a_m \overline{b_m(z)} \int_0^1 r^{2n+2|m|-1} \omega(r) dr \\ &= 2n! \sum_m \frac{m!}{(n-1+|m|)!} a_m \overline{b_m(z)} \omega_{2n+2|m|-1}. \end{aligned}$$

Set

$$z^m = \frac{2n! m!}{(n-1+|m|)!} \omega_{2n+2|m|-1} \overline{b_m(z)}.$$

Then,

$$\begin{aligned} B_z^{\omega}(w) &= \frac{1}{2n!} \sum_{k=0}^{\infty} \frac{(n-1+k)!}{k! \omega_{2n+2k-1}} \sum_{|m|=k} \frac{|m|!}{m!} w^m \overline{z}^m \\ &= \frac{1}{2n!} \sum_{k=0}^{\infty} \frac{(n-1+k)!}{k! \omega_{2n+2k-1}} \langle w, z \rangle^k. \end{aligned}$$

Therefore,

$$\Re B_z^\omega(w) = \frac{1}{2n!}\sum_{k=1}^{\infty}\frac{(n-1+k)!}{(k-1)!\omega_{2n+2k-1}}\langle w,z\rangle^k.$$

By Stirling's estimate and Lemma 1, when $1/2 \le |z| < 1$, we obtain

$$|B_z^\omega(w)| \lesssim \sum_{k=1}^{\infty}\frac{k^{n-1}|z|^k}{\omega_{2n+2k-1}} \approx \sum_{k=n}^{\infty}\frac{(k+1)^{n-1}|z|^k}{\omega_{2k+1}}$$

and

$$|\Re B_z^\omega(z)| \approx \sum_{k=1}^{\infty}\frac{k^n|z|^{2k}}{\omega_{2n+2k-1}} \approx \sum_{k=n+1}^{\infty}\frac{(k+1)^n|z|^{2k}}{\omega_{2k+1}}.$$

Let $\hat{\omega}_\alpha(t) = (1-t)^\alpha\hat{\omega}(t)$ for any fixed $\alpha \in \mathbb{R}$. Using [13, eq. (20)] and Lemma 2, we get

$$\sum_{k=n}^{\infty}\frac{(k+1)^{n-1}|z|^k}{\omega_{2k+1}} \approx \int_0^{|z|}\frac{1}{\hat{\omega}_{n+1}(t)}dt \lesssim \frac{1}{(1-|z|)^n\hat{\omega}(z)} \approx \frac{1}{\omega(S_z)}$$

and

$$\sum_{k=n+1}^{\infty}\frac{(k+1)^n|z|^{2k}}{\omega_{2k+1}} \approx \int_0^{|z|^2}\frac{1}{(1-t)^{n+2}\hat{\omega}(t)}dt.$$

By Lemma 1, there exists a constant $b > 0$ such that $\hat{\omega}(t)/(1-t)^b$ is essentially increasing. So, by Lemma 2,

$$\int_0^{|z|^2}\frac{1}{(1-t)^{n+2}\hat{\omega}(t)}dt \gtrsim \frac{(1-|z|^2)^b}{\hat{\omega}(|z|^2)}\int_0^{|z|^2}\frac{1}{(1-t)^{n+2+b}}dt \gtrsim \frac{1}{(1-|z|)\omega(S_z)}.$$

Therefore, when $1/2 \le |z| < 1$, we have

$$\|B_z^\omega\|_{H^\infty} \lesssim \frac{1}{\omega(S_z)}, \quad \frac{1}{\omega(S_z)} \lesssim \|B_z^\omega\|_{\mathcal{B}}. \tag{4}$$

When $|z| < 1/2$, since $\omega(S_z) \approx 1$, $\|B_z^\omega\|_{\mathcal{B}} \ge |B_z^\omega(0)| \gtrsim 1$, and

$$|B_z^\omega(w)| \le \frac{1}{2n!}\sum_{k=0}^{\infty}\frac{(n-1+k)!}{k!\omega_{2n+2k-1}}\frac{1}{2^k} < \infty,$$

(4) also holds. By the fact that $\|f\|_{\mathcal{B}} \lesssim \|f\|_{H^\infty}$, we obtain the desired result. The proof is complete. ◻

Lemma 4 *Let $0 < p < \infty$, $\omega, \upsilon \in \hat{\mathcal{D}}$. Then the following assertions hold.*

(i) *When $|rz| > 1/4$,*

$$M_p^p(r, B_z^\omega) \approx \int_0^{r|z|} \frac{1}{\hat{\omega}(t)^p (1-t)^{np-n+1}} dt$$

and

$$M_p^p(r, \Re B_z^\omega) \approx \int_0^{r|z|} \frac{1}{\hat{\omega}(t)^p (1-t)^{(n+1)p-n+1}} dt.$$

(ii) *When $|z| > 6/7$,*

$$\|B_z^\omega\|_{A_\upsilon^p}^p \approx \int_0^{|z|} \frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p (1-t)^{np-n+1}} dt$$

and

$$\|\Re B_z^\omega\|_{A_\upsilon^p}^p \approx \int_0^{|z|} \frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p (1-t)^{(n+1)p-n+1}} dt.$$

Proof When $n = 1$, the lemma was proved in [13], so we always assume $n \geq 2$. Since we will use some results on $A_\omega^p(\mathbb{D})$, for brief, the symbol A_ω^p only means $A_\omega^p(\mathbb{B})$ with $n \geq 2$. Meanwhile, let $B_z^{\omega,1}$ denote the reproducing kernel of $A_\omega^2(\mathbb{D})$. Recall that, on the unit disk, $dA_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dA(z)$, where $dA(z)$ is the normalized area measure on $\mathbb{D}$.

By Lemma 3,

$$\Re B_z^\omega(w) = \frac{1}{2n!} \sum_{k=1}^\infty \frac{(n-1+k)!}{(k-1)!\omega_{2n+2k-1}} \langle w, z\rangle^k.$$

Let $e_1 = (1, 0, \ldots, 0)$. When $|rz| > 0$, by a rotation transformation and [20, Lem. 1.9], we have

$$\begin{aligned}
M_p^p(r, \Re B_z^\omega) &= M_p^p(r, \Re B_{|z|e_1}^\omega) = \frac{1}{2n!} \int_{\mathbb{S}} \left| \sum_{k=1}^\infty \frac{(n-1+k)!}{(k-1)!\omega_{2n+2k-1}} \langle r\eta, |z|e_1\rangle^k \right|^p d\sigma(\eta) \\
&\approx \int_{\mathbb{D}} \left| \sum_{k=1}^\infty \frac{(n-1+k)!}{(k-1)!\omega_{2n+2k-1}} (r|z|\xi)^k \right|^p (1-|\xi|^2)^{n-2} dA(\xi) \\
&= \frac{1}{|rz|^{(n-1)p}} \int_{\mathbb{D}} \left| \sum_{k=1}^\infty \frac{|rz|^{n+k-1}(\xi^{n+k-1})^{(n)}}{\omega_{2(n+k-1)+1}} \right|^p |\xi|^p dA_{n-2}(\xi) \\
&\approx \frac{1}{|rz|^{(n-1)p}} \int_{\mathbb{D}} \left| \sum_{k=0}^\infty \frac{|rz|^k(\xi^k)^{(n)}}{\omega_{2k+1}} \right|^p dA_{n-2}(\xi) \\
&= \frac{1}{|rz|^{(n-1)p}} \|(B_{r|z|}^{\omega,1})^{(n)}\|_{A_{n-2}^p(\mathbb{D})}^p.
\end{aligned}$$

When $r|z| > 1/4$, by [13, Thm. 1],

$$M_p^p(r, \Re B_z^\omega) \approx \int_0^{r|z|} \frac{(1-t)^{n-1}}{\hat{\omega}(t)^p(1-t)^{p(n+1)}}dt = \int_0^{r|z|} \frac{1}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt.$$

Therefore, when $|z| > 6/7$, by Fubini's theorem we obtain

$$\|\Re B_z^\omega\|_{A_\upsilon^p}^p \approx \int_{1/2}^1 r^{2n-1}\upsilon(r)M_p^p(r, \Re B_z^\omega)dr = \int_0^{|z|} \frac{\int_{\max\{t/|z|,1/2\}}^1 r^{2n-1}\upsilon(r)dr}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt.$$

When $0 \le t \le |z|/2$,

$$\int_{\max\{t/|z|,1/2\}}^1 r^{2n-1}\upsilon(r)dr = \int_{\frac{1}{2}}^1 r^{2n-1}\upsilon(r)dr \approx 1 \approx \hat{\upsilon}(t). \tag{5}$$

When $|z|/2 \le t \le |z|$, we get

$$\int_{\max\{t/|z|,1/2\}}^1 r^{2n-1}\upsilon(r)dr = \int_{\frac{t}{|z|}}^1 r^{2n-1}\upsilon(r)dr \le \hat{\upsilon}(t). \tag{6}$$

By Lemma 1 and the fact that $\upsilon \in \hat{\mathcal{D}}$, there exists a constant $b > 0$ such that $\frac{\hat{\upsilon}(t)}{(1-t)^b}$ is essentially increasing. So,

$$\begin{aligned}\int_{|z|/2}^{|z|} \frac{\int_{\max\{t/|z|,1/2\}}^1 r^{2n-1}\upsilon(r)dr}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt &\gtrsim \int_{|z|/2}^{2|z|-1} \frac{\hat{\upsilon}(\frac{t}{|z|})}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt \\ &\gtrsim \int_{|z|/2}^{2|z|-1} \frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}\left(\frac{1-\frac{t}{|z|}}{1-t}\right)^b dt \\ &\gtrsim \int_{|z|/2}^{2|z|-1} \frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt, \end{aligned} \tag{7}$$

where the last estimate follows from

$$\frac{1}{|z|}\frac{|z|-t}{1-t} \ge \frac{1}{|z|}\frac{|z|-(2|z|-1)}{1-(2|z|-1)} \gtrsim 1 \quad \text{for all } t \in \left(\frac{|z|}{2}, 2|z|-1\right) \text{ and } |z| > \frac{6}{7}.$$

Meanwhile, $\omega, \upsilon \in \hat{\mathcal{D}}$ and Lemma 2 imply

$$\begin{aligned}\int_{|z|/2}^{2|z|-1} \frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt &\ge \int_{4|z|-3}^{2|z|-1} \frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt \\ &\approx \frac{\hat{\upsilon}(2|z|-1)}{\hat{\omega}(2|z|-1)^p(1-|z|)^{p(n+1)-n}} \\ &\approx \int_{2|z|-1}^{|z|} \frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt. \end{aligned} \tag{8}$$

Then (7) and (8) imply that

$$\begin{aligned}\int_{|z|/2}^{|z|}\frac{\int_{\max\{t/|z|,1/2\}}^{1}r^{2n-1}\upsilon(r)dr}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt &\gtrsim 2\int_{|z|/2}^{2|z|-1}\frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt\\ &\gtrsim\left(\int_{|z|/2}^{2|z|-1}+\int_{2|z|-1}^{|z|}\right)\frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt\\ &=\int_{|z|/2}^{|z|}\frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt. \qquad (9)\end{aligned}$$

So, if $|z|>6/7$, by (5) and (6),

$$\int_0^{|z|}\frac{\int_{\max\{t/|z|,1/2\}}^{1}r^{2n-1}\upsilon(r)dr}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt\lesssim\int_0^{|z|}\frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt.$$

By (5) and (9), we get

$$\int_0^{|z|}\frac{\int_{\max\{t/|z|,1/2\}}^{1}r^{2n-1}\upsilon(r)dr}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt\gtrsim\int_0^{|z|}\frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt.$$

Therefore,

$$\|\Re B_z^{\omega}\|_{A_\upsilon^p}^p\approx\int_0^{|z|}\frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p(1-t)^{p(n+1)-n+1}}dt.$$

The rest of the lemma can be proved in the same way. The proof is complete. □

4 Main Results and Proofs

In this section, we give the main results and proofs of this paper. We note that

$$\|f\|_{L^\infty(\mathbb{B},\omega dV)}=\|f\|_{L^\infty(\mathbb{B},dV)},$$

when $\omega\in\hat{\mathcal{D}}$. So, let $L^\infty=L^\infty(\mathbb{B},\omega dV)=L^\infty(\mathbb{B},dV)$ in this section.

Theorem 1 *When $\omega\in\mathcal{D}$, $P_\omega:L^\infty\to\mathcal{B}$ is bounded and onto.*

Proof For all $f\in L^\infty$, by Lemma 4,

$$|\Re(P_\omega f)(z)|\le\int_{\mathbb{B}}|f(w)||\Re B_z^{\omega}(w)|\omega(w)dV(w)\le\|f\|_{L^\infty}\|\Re B_z^{\omega}\|_{A_\omega^1}\lesssim\frac{\|f\|_{L^\infty}}{1-|z|}.$$

So, $P_\omega:L^\infty\to\mathcal{B}$ is bounded.

By [4, eq. (14)], we see that

$$\|f\|_{A_\omega^2}^2=\omega(\mathbb{B})|f(0)|^2+4\int_{\mathbb{B}}\frac{|\Re f(z)|^2}{|z|^{2n}}\omega^{n*}(z)dV(z),$$

where

$$\omega^{n*}(z) := \int_{|z|}^{1} r^{2n-1} \log \frac{r}{|z|} \omega(r) dr.$$

So, for $f, g \in A^2_\omega$,

$$\langle f, g \rangle_{A^2_\omega} = \omega(\mathbb{B}) f(0) \overline{g(0)} + 4 \int_{\mathbb{B}} \frac{\Re f(z) \overline{\Re g(z)}}{|z|^{2n}} \omega^{n*}(z) dV(z). \tag{10}$$

Let

$$W_1(t) := \frac{\hat{\omega}(t)}{1-t}, \quad \text{and } W_1(z) := W_1(|z|).$$

Since $\omega \in \mathcal{D}$, by Lemma 1, there are constants $a, b > 0$ such that $\hat{\omega}(t)/(1-t)^a$ is essentially decreasing and $\hat{\omega}(t)/(1-t)^b$ is essentially increasing. Thus,

$$\int_r^1 \frac{\hat{\omega}(t)}{1-t} dt \lesssim \frac{\hat{\omega}(r)}{(1-r)^a} \int_r^1 (1-t)^{a-1} dt \approx \hat{\omega}(r)$$

and

$$\int_r^1 \frac{\hat{\omega}(t)}{1-t} dt \gtrsim \frac{\hat{\omega}(r)}{(1-r)^b} \int_r^1 (1-t)^{b-1} dt \approx \hat{\omega}(r).$$

Then,

$$\hat{W}_1(r) = \int_r^1 \frac{\hat{\omega}(t)}{1-t} dt \approx \hat{\omega}(r) = (1-r) W_1(r).$$

Therefore, $W_1 \in \mathcal{R}$. By Lemma 2 and Theorem A, $\| \cdot \|_{A^p_\omega} \approx \| \cdot \|_{A^p_{W_1}}$. Then for all $p > 0$, by [6, Thm. 1], we get

$$\|f\|^p_{A^p_\omega} \approx \|f\|^p_{A^p_{W_1}} \approx |f(0)|^p + \int_{\mathbb{B}} |\Re f(z)|^p (1-|z|)^p W_1(z) dV(z). \tag{11}$$

For any $f \in H(\mathbb{B})$ and $|z| \le 1/2$, let $f_r(z) = f(rz)$ for $r \in (0, 1)$. By Cauchy's fomula, see [20, Thm. 4.1] for example, we have

$$f(z) = f_{3/4}\left(\frac{4z}{3}\right) = \int_{\mathbb{S}} \frac{f_{3/4}(\eta)}{\left(1 - \langle \frac{4z}{3}, \eta \rangle\right)^n} d\sigma(\eta).$$

After a calculation, when $|z| \le 1/2$,

$$|f(z)| \lesssim \|f\|_{A^1_\omega}, \quad |\Re f(z)| \lesssim |z| \|f_{3/4}\|_{H^\infty}, \quad \text{and} \quad |\Re f(z)| \lesssim |z| \|f\|_{A^1_\omega}.$$

We note that, when $|z| \geq 1/2$,

$$\omega^{n*}(z) = \int_{|z|}^{1} t^{2n-1} \log \frac{t}{|z|} \omega(t)dt \approx \int_{|z|}^{1} t \log \frac{t}{|z|} \omega(t)dt = \omega^*(z).$$

So, when $g \in \mathcal{B}$ and $f \in A^1_\omega$, by (10), (11) and Lemma 2, there exists a $C = C(n, \omega, g)$, such that

$$\begin{aligned}
|\langle f_r, g\rangle_{A^2_\omega}| &\leq C\left(\|f_r\|_{A^1_\omega} + \|f_r\|_{A^1_\omega}\int_{\frac{1}{2}\mathbb{B}} \frac{\omega^{n*}(z)}{|z|^{2n-2}}dV(z) + \int_{\mathbb{B}\setminus\frac{1}{2}\mathbb{B}} \frac{|\Re f_r(z)\Re g(z)|}{|z|^{2n}}\omega^{n*}(z)dV(z)\right)\\
&\approx \|f_r\|_{A^1_\omega} + \int_{\mathbb{B}\setminus\frac{1}{2}\mathbb{B}} |\Re f_r(z)\Re g(z)|(1-|z|)\hat{\omega}(z)dV(z)\\
&\leq \|f_r\|_{A^1_\omega} + \|g\|_{\mathcal{B}}\int_{\mathbb{B}} |\Re f_r(z)|\hat{\omega}(z)dV(z)\\
&\approx \|f_r\|_{A^1_\omega} + \|g\|_{\mathcal{B}}\int_{\mathbb{B}} |\Re f_r(z)|(1-|z|)W_1(z)dV(z)\\
&\lesssim \|f\|_{A^1_\omega} + \|g\|_{\mathcal{B}}\|f\|_{A^1_\omega}.
\end{aligned}$$

Therefore, $g \in \mathcal{B}$ induces a bounded linear functional on A^1_ω defined by $F_g(f) = \lim_{r\to 1}\langle f_r, g\rangle_{A^2_\omega}$ for all $f \in A^1_\omega$.

On the other hand, the Hahn–Banach theorem and the well known fact (see [19, Thm. 1.1] for example) that

$$(L^1(\mathbb{B}, \omega dV))^* \simeq L^\infty(\mathbb{B}, \omega dV)$$

guarantee the existence of $\varphi \in L^\infty$ such that

$$\lim_{r\to 1}\langle f_r, g\rangle_{A^2_\omega} = F_g(f) = \int_{\mathbb{B}} f(z)\overline{\varphi(z)}\omega(z)dV(z) = \lim_{r\to 1}\int_{\mathbb{B}} f_r(z)\overline{\varphi(z)}\omega(z)dV(z)$$

for all $f \in A^1_\omega$. Since P_ω is self-adjoint and $P_\omega(f_r) = f_r$, we have

$$\int_{\mathbb{B}} f_r(z)\overline{\varphi(z)}\omega(z)dV(z) = \int_{\mathbb{B}} P_\omega(f_r)(z)\overline{\varphi(z)}\omega(z)dV(z) = \int_{\mathbb{B}} f_r(z)\overline{P_\omega(\varphi)(z)}\omega(z)dV(z).$$

By the first part of the proof, $P_\omega\varphi \in \mathcal{B}$. Thus, $g - P_\omega\varphi \in \mathcal{B}$ and represents the zero functional. So, $g = P_\omega\varphi$. The proof is complete. □

Remark 1 By the above proof, we see that $P_\omega : L^\infty \to \mathcal{B}$ is bounded when $\omega \in \hat{\mathcal{D}}$.

Theorem 2 *Suppose* $1 < p < \infty$ *and* $\omega, \upsilon \in \mathcal{D}$. *Let* $q = p/(p-1)$. *Then the following statements are equivalent:*

(i) $P^+_\omega : L^p_\upsilon \to L^p_\upsilon$ *is bounded;*
(ii) $P_\omega : L^p_\upsilon \to L^p_\upsilon$ *is bounded;*
(iii) $M := \sup_{0\leq r<1} \frac{\hat{\upsilon}(r)^{1/p}}{\hat{\omega}(r)}\left(\int_r^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}}s^{2n-1}ds\right)^{1/q} < \infty;$

(iv) $N := \sup_{0\le r<1}\left(\int_0^r \frac{\upsilon(s)}{\hat{\omega}(s)^p}s^{2n-1}ds+1\right)^{1/p}\left(\int_r^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}}s^{2n-1}ds\right)^{1/q} < \infty.$

Proof When $n = 1$, the theorem was first proved in [13] and improved in [7, 14]. So, we always assume that $n \ge 2$.

$(i) \Rightarrow (ii)$. It is obvious.

$(ii) \Rightarrow (iii)$. Suppose that (ii) holds. Let P_ω^* be the adjoint of P_ω with respect to $\langle\cdot,\cdot\rangle_{L^2_\upsilon}$. For all $f, g \in L^\infty$, by Fubini's Theorem,

$$\begin{aligned}\langle f, P_\omega^* g\rangle_{L^2_\upsilon} = \langle P_\omega f, g\rangle_{L^2_\upsilon} &= \int_{\mathbb{B}} P_\omega f(z)\overline{g(z)}\upsilon(z)dV(z)\\ &= \int_{\mathbb{B}}\left(\int_{\mathbb{B}} f(\xi)\overline{B_z^\omega(\xi)}\omega(\xi)dV(\xi)\right)\overline{g(z)}\upsilon(z)dV(z)\\ &= \int_{\mathbb{B}}\left(\int_{\mathbb{B}} \overline{g(z)B_z^\omega(\xi)}\upsilon(z)dV(z)\right) f(\xi)\omega(\xi)dV(\xi)\\ &= \int_{\mathbb{B}}\left(\overline{\frac{\omega(\xi)}{\upsilon(\xi)}\int_{\mathbb{B}} g(z)B_z^\omega(\xi)\upsilon(z)dV(z)}\right) f(\xi)\upsilon(\xi)dV(\xi).\end{aligned}$$

Since L^∞ is dense in L^p_υ and L^q_υ, by the last equality we get

$$P_\omega^*(g)(\xi) = \frac{\omega(\xi)}{\upsilon(\xi)}\int_{\mathbb{B}} g(z)B_z^\omega(\xi)\upsilon(z)dV(z), \quad g \in L^q_\upsilon. \tag{12}$$

By the assumption, P_ω^* is bounded on L^q_υ. Let $g_j(z) = z_1^j$, where $z = (z_1, z_2, \ldots, z_n)$ and $j \in \mathbb{N}\cup\{0\}$. By [20, Lem. 1.11] and Lemma 3,

$$\begin{aligned}P_\omega^*(g_j)(\xi) &= \frac{\omega(\xi)}{\upsilon(\xi)}\int_{\mathbb{B}} g_j(z)B_z^\omega(\xi)\upsilon(z)dV(z)\\ &= \frac{1}{2n!}\frac{\omega(\xi)}{\upsilon(\xi)}\sum_{k=0}^\infty \frac{(n-1+k)!}{k!\omega_{2n+2k-1}}\int_{\mathbb{B}} g_j(z)\langle\xi, z\rangle^k\upsilon(z)dV(z)\\ &= \frac{2n}{2n!}\frac{\omega(\xi)}{\upsilon(\xi)}\sum_{k=0}^\infty \frac{(n-1+k)!}{k!\omega_{2n+2k-1}}\int_0^1 r^{2n+k+j-1}\upsilon(r)dr\int_{\mathbb{S}}\eta_1^j\langle\xi,\eta\rangle^k d\sigma(\eta)\\ &= \xi_1^j\frac{\omega(\xi)}{\upsilon(\xi)}\frac{\upsilon_{2n+2j-1}}{\omega_{2n+2j-1}}\frac{(n-1+j)!}{j!(n-1)!}\frac{(n-1)!j!}{(n-1+j)!}\\ &= \xi_1^j\frac{\omega(\xi)}{\upsilon(\xi)}\frac{\upsilon_{2n+2j-1}}{\omega_{2n+2j-1}}.\end{aligned}$$

By Lemmas 1 and 2, we obtain

$$\|g_j\|^q_{L^q_\upsilon} = \int_{\mathbb{B}}|z_1|^{qj}\upsilon(z)dV(z) = 2n\int_0^1 r^{2n+qj-1}\upsilon(r)dr\int_{\mathbb{S}}|\eta_1|^{qj}d\sigma(\eta)$$

$$
\begin{aligned}
&= 2n\upsilon_{2n+qj-1}\int_{\mathbb{S}}|\eta_1|^{qj}d\sigma(\eta)\\
&\approx \upsilon_{2n+2j-1}\int_{\mathbb{S}}|\eta_1|^{qj}d\sigma(\eta),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|P_\omega^*(g_j)\|_{L_\upsilon^q}^q &= \left(\frac{\upsilon_{2n+2j-1}}{\omega_{2n+2j-1}}\right)^q\int_{\mathbb{B}}|\xi_1|^{jq}\frac{\omega^q(\xi)}{\upsilon^{q-1}(\xi)}dV(\xi)\\
&\approx \left(\frac{\upsilon_{2n+2j-1}}{\omega_{2n+2j-1}}\right)^q\int_0^1 r^{2n+qj-1}\frac{\omega^q(r)}{\upsilon^{q-1}(r)}dr\int_{\mathbb{S}}|\eta_1|^{jq}d\sigma(\eta)\\
&\gtrsim \|g_j\|_{L_\upsilon^q}^q\frac{\upsilon_{2n+2j-1}^{q-1}}{\omega_{2n+2j-1}^q}\int_{1-\frac{1}{2j+1}}^1\frac{\omega^q(r)}{\upsilon^{q-1}(r)}r^{2n-1}dr\\
&\approx \|g_j\|_{L_\upsilon^q}^q\frac{\upsilon_{2j+1}^{q-1}}{\omega_{2j+1}^q}\int_{1-\frac{1}{2j+1}}^1\frac{\omega^q(r)}{\upsilon^{q-1}(r)}r^{2n-1}dr.
\end{aligned}
$$

Let $r_j = 1-1/(2j+1)$. We get

$$
\|P_\omega^*(g_j)\|_{L_\upsilon^q}^q \gtrsim \|g_j\|_{L_\upsilon^q}^q\frac{\hat{\upsilon}(r_j)^{q-1}}{\hat{\omega}(r_j)^q}\int_{r_j}^1\frac{\omega^q(r)}{\upsilon^{q-1}(r)}r^{2n-1}dr.
$$

Let

$$
H(t) = \frac{\hat{\upsilon}(t)^{q-1}}{\hat{\omega}(t)^q}\int_t^1\frac{\omega^q(r)}{\upsilon^{q-1}(r)}r^{2n-1}dr.
$$

When $r_j \le t < r_{j+1}$, $H(t) \lesssim H(r_j)$. Thus, by the assumption, we get $\sup_{t\ge 0} H(t) < \infty$, as desired.

$(iii) \Rightarrow (i)$. Suppose that (iii) holds. For $z \in \mathbb{B}$, let

$$
h(z) = \upsilon(z)^{1/p}\left(\int_{|z|}^1\frac{\omega(s)^q}{\upsilon(s)^{q-1}}s^{2n-1}ds\right)^{1/(pq)}.
$$

By the assumption we have

$$
\int_t^1\left(\frac{\omega(s)}{h(s)}\right)^q s^{2n-1}ds = q\left(\int_t^1\frac{\omega(s)^q}{\upsilon(s)^{q-1}}s^{2n-1}ds\right)^{1/q} \lesssim M\frac{\hat{\omega}(t)}{\hat{\upsilon}(t)^{1/p}}. \tag{13}
$$

If $r|z| \le 1/4$, by Lemma 3,

$$
M_1(r, B_z^\omega) \le \|B_{rz}^\omega\|_{H^\infty} \approx \frac{1}{\hat{\omega}(S_{rz})} \approx 1.
$$

If $r|z| > 1/4$, by Lemma 1, there exists a constant $a > 0$ such that $\hat{\omega}(t)/(1-t)^a$ is essentially decreasing. Then by Lemma 4,

$$M_1(r, B_z^{\omega}) \lesssim \int_0^{r|z|} \frac{dt}{\hat{\omega}(t)(1-t)} \lesssim \frac{(1-r|z|)^a}{\hat{\omega}(r|z|)} \int_0^{r|z|} \frac{dt}{(1-t)^{a+1}} \approx \frac{1}{\hat{\omega}(r|z|)}.$$

So, for all $r \in (0,1)$ and $z \in \mathbb{B}$,

$$M_1(r, B_z^{\omega}) \lesssim 1 + \int_0^{r|z|} \frac{1}{\hat{\omega}(t)(1-t)} dt \lesssim \frac{1}{\hat{\omega}(r|z|)}. \tag{14}$$

Hence, by (13), (14), Fubini's theorem and Lemma 1, we obtain

$$\begin{aligned} \int_{\mathbb{B}} |B_z^{\omega}(\xi)| \left(\frac{\omega(\xi)}{h(\xi)}\right)^q dV(\xi) &= 2n \int_0^1 \left(\frac{\omega(r)}{h(r)}\right)^q r^{2n-1} M_1(r, B_z^{\omega}) dr \\ &\lesssim \int_0^1 \left(\frac{\omega(r)}{h(r)}\right)^q r^{2n-1} \left(1 + \int_0^{r|z|} \frac{1}{\hat{\omega}(t)(1-t)} dt\right) dr \\ &\lesssim M \frac{\hat{\omega}(0)}{\hat{\upsilon}(0)^{1/p}} + \int_0^{|z|} \frac{1}{\hat{\omega}(t)(1-t)} \int_{\frac{t}{|z|}}^1 \left(\frac{\omega(r)}{h(r)}\right)^q r^{2n-1} dr dt \\ &\le M \frac{\hat{\omega}(0)}{\hat{\upsilon}(0)^{1/p}} + \int_0^{|z|} \frac{1}{\hat{\omega}(t)(1-t)} \int_t^1 \left(\frac{\omega(r)}{h(r)}\right)^q r^{2n-1} dr dt \\ &\lesssim M + M \int_0^{|z|} \frac{1}{\hat{\upsilon}(t)^{1/p}(1-t)} dt \lesssim \frac{M}{\hat{\upsilon}(|z|)^{1/p}}. \end{aligned}$$

Therefore, Hölder's inequality and Fubini's theorem imply that

$$\begin{aligned} \|P_{\omega}^{+}(f)\|_{L_{\upsilon}^p}^p &= \int_{\mathbb{B}} \upsilon(z) \left| \int_{\mathbb{B}} f(\xi) |B_z^{\omega}(\xi)| \omega(\xi) dV(\xi) \right|^p dV(z) \\ &\le \int_{\mathbb{B}} \left(\int_{\mathbb{B}} |f(\xi)|^p h(\xi)^p |B_z^{\omega}(\xi)| dV(\xi) \right) \\ &\quad \times \left(\int_{\mathbb{B}} |B_z^{\omega}(\xi)| \left(\frac{\omega(\xi)}{h(\xi)}\right)^q dV(\xi) \right)^{p/q} \upsilon(z) dV(z) \\ &\lesssim M^{\frac{p}{q}} \int_{\mathbb{B}} \left(\int_{\mathbb{B}} |f(\xi)|^p h(\xi)^p |B_z^{\omega}(\xi)| dV(\xi) \right) \frac{\upsilon(z)}{\hat{\upsilon}(z)^{1/q}} dV(z) \\ &= M^{\frac{p}{q}} \int_{\mathbb{B}} |f(\xi)|^p h(\xi)^p \left(\int_{\mathbb{B}} |B_z^{\omega}(\xi)| \frac{\upsilon(z)}{\hat{\upsilon}(z)^{1/q}} dV(z) \right) dV(\xi). \end{aligned} \tag{15}$$

Since $|B_z^{\omega}(\xi)| = |B_{\xi}^{\omega}(z)|$, by (14) we get

$$\int_{\mathbb{B} \setminus |\xi| \mathbb{B}} |B_z^{\omega}(\xi)| \frac{\upsilon(z)}{\hat{\upsilon}(z)^{1/q}} dV(z) \lesssim \int_{|\xi|}^1 \frac{\upsilon(r)}{\hat{\upsilon}(r)^{1/q}} M_1(r, B_{\xi}^{\omega}) dr \lesssim \frac{\hat{\upsilon}(\xi)^{1/p}}{\hat{\omega}(\xi)}, \tag{16}$$

and

$$\begin{aligned}\int_{|\xi|\mathbb{B}} |B_z^{\omega}(\xi)| \frac{\upsilon(z)}{\hat{\upsilon}(z)^{1/q}} dV(z) &\lesssim \int_0^{|\xi|} \frac{\upsilon(r)}{\hat{\upsilon}(r)^{1/q}} \frac{1}{\hat{\omega}(r|\xi|)} r^{2n-1} dr \\ &\lesssim \int_0^{|\xi|} \frac{\upsilon(r)}{\hat{\upsilon}(r)^{1/q}\hat{\omega}(r)} r^{2n-1} dr. \end{aligned} \tag{17}$$

By the assumption, we have

$$\int_0^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}} s^{2n-1} ds < \infty, \quad \int_0^{1/2} \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt > 0, \quad \int_{1/2}^1 \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt > 0. \tag{18}$$

When $r \leq 1/2$,

$$\hat{\omega}(r) \approx 1 \approx \hat{\upsilon}(r)^{1/p} \left(\int_r^1 \frac{\omega(t)^q}{\upsilon(t)^{q-1}} t^{2n-1} dt \right)^{1/q}.$$

When $r > 1/2$, by Hölder's inequality,

$$\begin{aligned}\hat{\omega}(r) = \int_r^1 \omega(t) dt &\leq \hat{\upsilon}(r)^{1/p} \left(\int_r^1 \frac{\omega(t)^q}{\upsilon(t)^{q-1}} dt \right)^{1/q} \\ &\approx \hat{\upsilon}(r)^{1/p} \left(\int_r^1 \frac{\omega(t)^q}{\upsilon(t)^{q-1}} t^{2n-1} dt \right)^{1/q}.\end{aligned}$$

Then, for all $r \in (0, 1)$,

$$\frac{\hat{\omega}(r)^p}{\hat{\upsilon}(r)} \int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \lesssim \left(\int_r^1 \frac{\omega(t)^q}{\upsilon(t)^{q-1}} t^{2n-1} dt \right)^{p/q} \int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt. \tag{19}$$

Now, we claim that

$$K_* := \sup_{0 \leq r < 1} \left(\int_r^1 \frac{\omega(t)^q}{\upsilon(t)^{q-1}} t^{2n-1} dt \right)^{1/q} \left(\int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \right)^{1/p} < \infty. \tag{20}$$

Take this for granted for a moment. Using (19) and (20), we have

$$\begin{aligned}\int_0^{|\xi|} \frac{\upsilon(r)}{\hat{\upsilon}(r)^{1/q}\hat{\omega}(r)} r^{2n-1} dr &\leq \int_0^{|\xi|} \frac{\upsilon(r)}{\hat{\omega}(r)} \left(\frac{K_*^p}{\hat{\omega}(r)^p \int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt} \right)^{1/q} r^{2n-1} dr \\ &= K_*^{p-1} \int_0^{|\xi|} \frac{\upsilon(r)}{\hat{\omega}(r)^p} \left(\int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \right)^{-1/q} r^{2n-1} dr \\ &\approx K_*^{p-1} \left(\int_0^{|\xi|} \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \right)^{1/p}.\end{aligned} \tag{21}$$

By (16), (17), (20) and (21),

$$
\begin{aligned}
& h(\xi)^p \int_{\mathbb{B}\setminus|\xi|\mathbb{B}} |B_z^\omega(\xi)| \frac{\upsilon(z)}{\hat{\upsilon}(z)^{1/q}} dV(z) \\
& \lesssim \upsilon(\xi) \left(\int_{|\xi|}^1 \frac{\omega(t)^q}{\upsilon(t)^{q-1}} t^{2n-1} dt \right)^{1/q} \frac{\hat{\upsilon}(\xi)^{1/p}}{\hat{\omega}(\xi)} \le M\upsilon(\xi), \qquad (22)
\end{aligned}
$$

and

$$
\begin{aligned}
& h(\xi)^p \int_{|\xi|\mathbb{B}} |B_z^\omega(\xi)| \frac{\upsilon(z)}{\hat{\upsilon}(z)^{1/q}} dV(z) \\
& \lesssim K_*^{p-1} \upsilon(\xi) \left(\int_{|\xi|}^1 \frac{\omega(t)^q}{\upsilon(t)^{q-1}} t^{2n-1} dt \right)^{1/q} \left(\int_0^{|\xi|} \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \right)^{1/p} \\
& \le K_*^p \upsilon(\xi). \qquad (23)
\end{aligned}
$$

So, by (15), (22) and (23),

$$
\|P_\omega^+(f)\|_{L_\upsilon^p}^p \lesssim \int_{\mathbb{B}} |f(\xi)|^p \upsilon(\xi) dV(\xi) = \|f\|_{L_\upsilon^p}^p.
$$

Now, we prove that (20) holds. Assume $r > 1/2$. An integration by parts and Hölder's inequality give

$$
\begin{aligned}
\int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt & \le \int_0^{1/2} \frac{\upsilon(t)}{\hat{\omega}(t)^p} dt + \int_{\frac{1}{2}}^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} dt \\
& \lesssim 1 + \int_{1/2}^r \frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p} \frac{\omega(t)}{\upsilon(t)^{1/p}} \frac{\upsilon(t)^{1/p}}{\hat{\omega}(t)} t^{2n-1} dt \\
& \le 1 + \left(\int_0^r \left(\frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p} \frac{\omega(t)}{\upsilon(t)^{1/p}} \right)^q t^{2n-1} dt \right)^{1/q} \left(\int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \right)^{1/p} \\
& = 1 + J_1^{1/q} \left(\int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \right)^{1/p},
\end{aligned}
$$

where

$$
J_1 = \int_0^r \left(\frac{\hat{\upsilon}(t)}{\hat{\omega}(t)^p} \frac{\omega(t)}{\upsilon(t)^{1/p}} \right)^q t^{2n-1} dt.
$$

Since

$$
\begin{aligned}
J_1 & = \int_0^r \left(\frac{\hat{\upsilon}(t)^{1/p}}{\hat{\omega}(t)} \left(\int_t^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}} s^{2n-1} ds \right)^{1/q} \right)^{pq} \frac{\frac{\omega(t)^q}{\upsilon(t)^{q-1}}}{\left(\int_t^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}} s^{2n-1} ds \right)^p} t^{2n-1} dt \\
& \lesssim \frac{M^{pq}}{\left(\int_r^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}} s^{2n-1} ds \right)^{p-1}},
\end{aligned}
$$

we obtain

$$\int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \lesssim 1 + M^p \left(\int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \right)^{1/p} \left(\int_r^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}} s^{2n-1} ds \right)^{-p/q^2}.$$

Hence

$$\left(\int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \right)^{1/p} \lesssim 1 + M \left(\int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \right)^{1/p^2} \left(\int_r^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}} s^{2n-1} ds \right)^{-1/q^2}.$$

Multiplying the expression by $\left(\int_r^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}} s^{2n-1} ds \right)^{1/q}$, we have

$$J_2(r) \lesssim \left(\int_r^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}} s^{2n-1} ds \right)^{1/q} + M J_2(r)^{1/p},$$

where

$$J_2(r) = \left(\int_r^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}} s^{2n-1} ds \right)^{1/q} \left(\int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \right)^{1/p}.$$

Using (18), we get

$$J_2(r)^{1/q} \lesssim \left(\int_r^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}} s^{2n-1} ds \right)^{1/q^2} \left(\int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \right)^{-1/p^2} + M < \infty.$$

Therefore,

$$\sup_{r>1/2} \left(\int_r^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}} s^{2n-1} ds \right)^{1/q} \left(\int_0^r \frac{\upsilon(t)}{\hat{\omega}(t)^p} t^{2n-1} dt \right)^{1/p} < \infty.$$

When $r \leq 1/2$, (20) holds obviously.

$(iii) \Rightarrow (iv)$. Using (18) and (20), we get the desired result.

$(iv) \Rightarrow (iii)$. Assume that (iv) holds, that is,

$$N := \sup_{0\leq r<1} \left(\int_0^r \frac{\upsilon(s)}{\hat{\omega}(s)^p} s^{2n-1} ds + 1 \right)^{1/p} \left(\int_r^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}} s^{2n-1} ds \right)^{1/q} < \infty.$$

Since $\omega \in \mathcal{D}$, by Lemma 1, there exists $b > 0$ such that $\hat{\omega}(r)^p/(1-r)^b$ is essentially increasing. Then

$$\int_0^r \frac{\upsilon(s)}{\hat{\omega}(s)^p} s^{2n-1} ds \gtrsim \frac{(1-r)^b}{\hat{\omega}(r)^p} \int_0^r \frac{\upsilon(s)}{(1-s)^b} s^{2n-1} ds.$$

Since $\upsilon \in \mathcal{D}$, there exist $C > 1$ and $K > 1$ such that

$$\hat{\upsilon}(r) \geq C\hat{\upsilon}\left(1 - \frac{1-r}{K}\right).$$

Let $r_k = 1 - K^{-k}$, $k = 0, 1, 2, \ldots$. For any $r_2 \leq r < 1$, there is an integer $x = x(r)$ such that $r_x \leq r < r_{x+1}$. Then

$$\begin{aligned}
(1-r)^b \int_0^r \frac{\upsilon(s)}{(1-s)^b} s^{2n-1} ds &\geq \sum_{k=0}^{x-1} \int_{r_k}^{r_{k+1}} \left(\frac{1-r}{1-s}\right)^b \upsilon(s) s^{2n-1} ds \\
&\geq \sum_{k=0}^{x-1} r_k^{2n-1} \left(\frac{1-r_{x+1}}{1-r_k}\right)^b \left(\hat{\upsilon}(r_k) - \hat{\upsilon}(r_{k+1})\right) \\
&\geq \sum_{k=0}^{x-1} r_k^{2n-1} \frac{C-1}{CK^{(x+1-k)b}} \hat{\upsilon}(r_k) \\
&\geq \sum_{k=0}^{x-1} r_k^{2n-1} \frac{(C-1)C^{x-1-k}}{K^{(x+1-k)b}} \hat{\upsilon}(r_x) \\
&\geq \hat{\upsilon}(r) \frac{C-1}{C^2} \sum_{s=2}^{x+1} r_{x+1-s}^{2n-1} \left(\frac{C}{K^b}\right)^s \\
&\geq r_{x-1}^{2n-1} \hat{\upsilon}(r) \frac{C-1}{K^{2b}} \geq r_1^{2n-1} \hat{\upsilon}(r) \frac{C-1}{K^{2b}}.
\end{aligned}$$

So, when $r \geq r_2$,

$$\int_0^r \frac{\upsilon(s)}{\hat{\omega}(s)^p} s^{2n-1} ds \gtrsim \frac{\hat{\upsilon}(r)}{\hat{\omega}(r)^p}.$$

Therefore,

$$\sup_{r_2 \leq r < 1} \frac{\hat{\upsilon}(r)^{1/p}}{\hat{\omega}(r)} \left(\int_r^1 \frac{\omega(s)^q}{\upsilon(s)^{q-1}} s^{2n-1} ds\right)^{1/q} < \infty.$$

When $r < r_2$, (iii) holds obviously. The proof is complete. □

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