Computational Methods and Function Theory (2022) 22:629–643
https://doi.org/10.1007/s40315-021-00427-z

Approximation by Faber–Laurent Rational Functions in Variable Exponent Morrey Spaces

Sadulla Z. Jafarov[1,2]

Received: 8 January 2021 / Revised: 11 May 2021 / Accepted: 1 June 2021 /
Published online: 30 October 2021

Abstract

Let G be a finite Jordan domain bounded by a Dini-smooth curve Γ in the complex plane $\mathbb{C}$. In this work, approximation properties of the Faber–Laurent rational series expansions in variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(\Gamma)$ are studied. Also, direct theorems of approximation theory in variable exponent Morrey–Smirnov classes, defined in domains with a Dini-smooth boundary, are proved.

Keywords Faber–Laurent rational functions · Conformal mapping · Dini-smooth curve · Variable exponent Morrey spaces · Modulus of smoothness

Mathematics Subject Classification 30E05 · 30E10 · 41A10 · 41A20 · 41A30

1 Introduction, Some Auxiliary Results and Main Results

Let J denote the interval $[0, 2\pi]$ or a Jordan rectifiable curve $\Gamma \subset \mathbb{C}$. Let us denote by $\wp$ the class of Lebesgue measurable functions $p(\cdot)\colon \Gamma \to [0, \infty)$ such that

$$1 < p_* := \operatorname{essinf}_{z\in J}\, p(z) \le p^* := \operatorname{esssup}_{z\in J}\, p(z) < \infty. \tag{1.1}$$

Let $|J|$ be the Lebesgue measure of J. We suppose that the function $p(\cdot)$ satisfies the condition

Communicated by Vladimir V. Andrievskii.

✉ Sadulla Z. Jafarov
s.jafarov@alparslan.edu.tr

1 Department of Mathematics and Science Education, Faculty of Education, Muş Alparslan University, 49250 Muş, Turkey

2 Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, 9 B. Vahabzadeh Str., 1141 Baku, Azerbaijan

$$|p(z_1) - p(z_2)| \ln\left(\frac{|J|}{|z_1 - z_2|}\right) \leq c, \quad \text{for all } z_1, z_2 \in J, \tag{1.2}$$

where the constant c is independent of z_1 and z_2. A function $p(\cdot) \in \wp$ is said to belong to the class $\wp^{\log}(J)$, if the condition (1.2) is satisfied.

For $p(\cdot) \in \wp^{\log}(\Gamma)$, we define a class $L^{p(\cdot)}(\Gamma)$ of Lebesgue measurable functions $f(\cdot)\colon \Gamma \to \mathbb{R}$ satisfying the condition

$$\int_\Gamma |f(z)|^{p(z)} \, |dz| < \infty.$$

This class $L^{p(\cdot)}(\Gamma)$ is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\Gamma)} := \inf\left\{\lambda > 0\colon \int_\Gamma \left|\frac{f(x)}{\lambda}\right|^{p(z)} |dz| \leq 1\right\}.$$

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by the rectifiable Jordan curve Γ. Without loss of generality we assume $0 \in \operatorname{Int}\Gamma$. Let $G^- := \operatorname{Ext}\Gamma$. Let also $\mathbb{T} := \{w \in \mathbb{C}\colon |w| = 1\}$, $\mathbb{D} = \operatorname{Int}\mathbb{T}$ and $\mathbb{D}^- = \operatorname{Ext}\mathbb{T}$. We recall that if for a given analytic function $f(\cdot)$ on G, there exists a sequence of rectifiable Jordan curves (Γ_n) in G tending to the boundary Γ in the sense that Γ_n eventually surrounds each compact subdomain of G such that

$$\int_{\Gamma_n} |f(z)|^p \, |dz| \leq M < \infty,$$

then we say that $f(\cdot)$ belongs to the *Smirnov class* $E^p(G^-)$, $1 \leq p < \infty$. Each function $f(\cdot) \in E^p(G)$ has non-tangential limits almost everywhere (*a.e.*) on Γ and the boundary function belongs to $L^p(\Gamma)$.

We denote by $\varphi(\cdot)$ the conformal mapping of G^- onto $\mathbb{D}^-$ normalized by

$$\varphi(\infty) = \infty, \quad \lim_{z \to \infty} \frac{\varphi(z)}{z} > 0.$$

Let $\psi(\cdot)$ be the inverse of $\varphi(\cdot)$. The functions $\varphi(\cdot)$ and $\psi(\cdot)$ have continuous extensions to Γ and $\mathbb{T}$, their derivatives $\varphi'(\cdot)$ and $\psi'(\cdot)$ have definite non-tangential limit values on Γ and $\mathbb{T}$ *a.e.*, and they are integrable with respect to the Lebesgue measure on Γ and $\mathbb{T}$, respectively. It is known that $\varphi'(\cdot) \in E^1(G^-)$ and $\psi'(\cdot) \in E^1(\mathbb{D}^-)$. Note that the general information about Smirnov classes can be found in [14, pp. 168–185], [22, pp. 438–453].

Let Γ be a rectifiable Jordan curve in the complex plane. We denote $\Gamma(t, r) = \Gamma \cap B(t, r)$, $t \subset \Gamma$, $r > 0$, where $B(t, r) = \{z \in \mathbb{C}\colon |z - t| < r\}$. The *Morrey spaces* $L^{p,\lambda}(\Gamma)$ for a given $0 \leq \lambda \leq 1$ and $p \geq 1$, are defined as the set of functions $f(\cdot) \in L^p_{loc}(\Gamma)$ such that

$$\|f\|_{L^{p,\lambda}(\Gamma)} := \sup_{z \in \Gamma,\, 0 < r < L} r^{-\lambda/p} \|f\|_{L^p(\Gamma(t,r))} < \infty,$$

where L is the length of the curve Γ.

Note that $L^{p,0}(\Gamma) = L^p(\Gamma)$, and if $\lambda < 0$ or $\lambda > 1$, then $L^{p,\lambda}(\Gamma) = \Theta$, where Θ is the set of all functions equivalent to 0 on Γ.

Let $G := \operatorname{Int}\Gamma$ and $L^{p,\lambda}(\Gamma)$, $0 < \lambda \le 1$ and $1 < p < \infty$, be a Morrey space defined on Γ. We also define the *Morrey-Smirnov classes* $E^{p,\lambda}(G)$ as

$$E^{p,\lambda}(G) := \{f(\cdot) \in E_1(G) \colon f(\cdot) \in L^{p,\lambda}(\Gamma)\}.$$

Hence for $f(\cdot) \in E^{p,\lambda}(G)$ we can define the $E^{p,\lambda}(G)$ norm as

$$\|f\|_{E^{p,\lambda}(G)} := \|f\|_{L^{p,\lambda}(\Gamma)}.$$

Let $p(\cdot)\colon \Gamma \to [1, +\infty]$ be a Lebesgue measurable function satisfying condition (1.1) and $\lambda(\cdot)\colon \Gamma \to [0, 1]$ be a measurable function. We define the variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(\Gamma)$ as the set of Lebesgue measurable functions $f(\cdot)$ defined on Γ, such that

$$S_{p(\cdot),\lambda(\cdot)}(f) = \sup_{t\in\Gamma,\, 0<r<L} r^{-\lambda(x)} \int_{\Gamma(t,r)} |f(s)|^{p(s)}\, ds < \infty.$$

The norm in $L^{p(\cdot),\lambda(\cdot)}(\Gamma)$ is defined as follows

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(\Gamma)} := \inf\left\{\nu > 0\colon S_{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\nu}\right) < 1\right\}.$$

It is known that $L^{p(\cdot),\lambda(\cdot)}(\Gamma)$ is a Banach space. Note that the properties of classical Morrey spaces and variable exponent Morrey spaces have been investigated by several authors (see, for example, [3,16–19,30,40,42,46–48,50,51,54]).

We define also the *variable exponent Morrey-Smirnov* class $E^{p(\cdot),\lambda(\cdot)}(G)$ as

$$E^{p(\cdot),\lambda(\cdot)}(G) := \left\{f(\cdot) \in E^1(G) : f(\cdot) \in L^{p(\cdot),\lambda(\cdot)}(\Gamma)\right\}.$$

Note that $E^{p(\cdot),\lambda(\cdot)}(G)$ is a Banach space with respect to the norm

$$\|f\|_{E^{p(\cdot),\lambda(\cdot)}(G)} := \|f\|_{L^{p(\cdot),\lambda(\cdot)}(\Gamma)}.$$

Let $p(\cdot)\colon \mathbb{T} \to [1, +\infty]$ and $\lambda(\cdot)\colon \mathbb{T} \to [0, 1]$ be measurable functions such that $0 \le \lambda_* \le \lambda^* < 1$. Also assume that $p(\cdot) \in \wp^{\log}$. For $f(\cdot) \in L^{p(\cdot),\lambda(\cdot)}(\mathbb{T})$ we define the operator

$$(\nu_{h_i} f)(\omega) := \frac{1}{h}\int_0^h f(\omega e^{it})dt, \omega \in \mathbb{T}, \quad 0 < h < \pi.$$

It is clear that the operator ν_h is a bounded linear operator on $L^{p(\cdot)\lambda(\cdot)}(\mathbb{T})$ [21]:

$$\|\nu_h(f)\|_{L^{p(\cdot)}(\mathbb{T})} \le c_1 \|f\|_{L^{p(\cdot)}(\mathbb{T})}.$$

The function

$$\Omega(f,\delta)_{p(\cdot),\lambda(\cdot)} := \sup_{0<h\le\delta} \|f(\cdot)-\nu_h f(\cdot)\|_{L^{p(\cdot),\lambda(\cdot)}(\mathbb{T})}, \quad \delta>0,$$

is called the *modulus of smoothness* of $f(\cdot)\in L^{p(\cdot)\lambda(\cdot)}(\mathbb{T})$.

It can easily be shown that $\Omega(f,\cdot)_{p(\cdot),\lambda(\cdot)}$ is a continuous, non-negative and non-decreasing function satisfying the conditions

$$\lim_{\delta\to 0}\Omega(f,\delta)_{p(\cdot),\lambda(\cdot)}=0,$$
$$\Omega(f+g,\delta)_{p(\cdot),\lambda(\cdot)} \le \Omega(f,\delta)_{p(\cdot),\lambda(\cdot)}+\Omega(g,\delta)_{p(\cdot),\lambda(\cdot)}, \quad \delta>0,$$

for $f(\cdot), g(\cdot)\in L^{p(\cdot),\lambda(\cdot)}(\mathbb{T})$.

We denote by $w=\phi(z)$ the conformal mapping of G^- onto the domain $\mathbb{D}\colon=\{w\in\mathbb{C}\colon |w|>1\}$ normalized by the conditions

$$\phi(\infty)=\infty, \qquad \lim_{z\to\infty}\frac{\phi(z)}{z}>0$$

and let $\psi(\cdot)$ be the inverse mapping of $\phi(\cdot)$.

We denote by $w=\phi_1(z)$ the conformal mapping of G onto the domain $\mathbb{D}=\{w\in\mathbb{C}\colon |w|>1\}$, normalized by the conditions

$$\phi_1(0)=\infty, \qquad \lim_{z\to 0}(z\phi_1(z))>0,$$

and let $\psi_1(\cdot)$ be the inverse mapping of $\phi_1(\cdot)$.

The functions $\psi(\cdot)$ and $\psi_1(\cdot)$ have in some deleted neighborhood of the point $w=\infty$ the representations

$$\psi(w)=\gamma w+\gamma_0+\frac{\gamma_1}{w}+\frac{\gamma_2}{w^2}+\cdots, \qquad \gamma>0,$$

and

$$\psi_1(w)=\frac{\alpha_1}{w}+\frac{\alpha_2}{w^2}+\cdots+\frac{\alpha_k}{w^k}+\cdots, \qquad \alpha_1>0.$$

The following expansions hold [10,14,41,49]:

$$\frac{\psi'(w)}{\psi(w)-z}=\sum_{k=0}^{\infty}\frac{\Phi_k(z)}{w^{k+1}}, \qquad z\in G \text{ and } w\in\mathbb{D}^-, \tag{1.3}$$

and

$$\frac{\psi_1'(w)}{\psi_1(w)-z}=\sum_{k=0}^{\infty}-\frac{F\left(\frac{1}{z}\right)}{w^{k+1}}, \qquad z\in G^- \text{ and } w\in\mathbb{D}^-, \tag{1.4}$$

where $\Phi_k(z)$ and $F_k(1/z)$ are the Faber polynomials of degree k with respect to z and $1/z$ for the continuums $\overline{G}$ and $\overline{\mathbb{C}}\backslash G$, respectively. Also, for the Faber polynomials $\Phi_k(z)$ and rational functions $F_k(1/z)$ the integral representations

$$\Phi_k(z) = [\phi(z)]^k + \frac{1}{2\pi i}\int_\Gamma \frac{[\phi(\zeta)]^n}{\zeta - z}d\zeta, \qquad k = 0, 1, 2, \dots, \; z \in G, \quad (1.5)$$

$$F_k\left(\frac{1}{z}\right) = [\phi_1(z)]^k - \frac{1}{2\pi i}\int_\Gamma \frac{[\phi_1(\zeta)]^n}{\zeta - z}d\zeta, \qquad k = 0, 1, 2, \dots, \; z \in G \quad (1.6)$$

hold [10,49].

Let also $\chi(\cdot)$ be a continuous function on 2π. Its modulus of continuity is defined by

$$\omega(t, \chi) := \sup_{t_1, t_2 \in [0, 2\pi], |t_1 - t_2| < t} |\chi(t_1) - \chi(t_2)|, \qquad t \geq 0.$$

The curve Γ is called Dini-smooth if it has the parametrization

$$\Gamma: \chi(t), \qquad 0 \leq t \leq 2\pi,$$

such that $\chi'(t)$ is Dini-continuous, i.e.

$$\int_0^\pi \frac{\omega(t, \chi')}{t}dt < \infty$$

and

$$\chi'(t) \neq 0$$

[45, p. 48]

Let $f(\cdot) \in L_1(\Gamma)$. Then the functions $f^+(\cdot)$ and $f^-(\cdot)$ defined by

$$f^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\psi(w))\psi'(w)}{\psi(w) - z}dw, \qquad z \in G \quad (1.7)$$

and

$$f^-(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\psi_1(w))\psi_1'(w)}{\psi_1(w) - z}dw, \qquad z \in G^- \quad (1.8)$$

are analytic in G and G^-, respectively, and $f^-(\infty) = 0$. Thus the limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon \to \infty} \frac{1}{2\pi i}\int_{\Gamma \cap \{\zeta : |\zeta - z| > \varepsilon\}} \frac{f(\zeta)}{\zeta - z}d\zeta$$

exists and is finite for almost all $z \in \Gamma$.

The quantity $S_\Gamma(f)(z)$ is called the *Cauchy singular integral* of $f(\cdot)$ at $z \in \Gamma$. According to the Privalov theorem [22, p. 431], if one of the functions $f^+(\cdot)$ or $f^-(\cdot)$ has non-tangential limits *a.e.* on Γ, then $S_\Gamma(f)(z)$ exists *a.e.* on Γ and also the other one has non-tangential limits *a.e.* on Γ. Conversely, if $S_\Gamma(f)(z)$ exists *a.e.* on Γ, then the functions $f^+(\cdot)$ and $f^-(\cdot)$ have non-tangential limits *a.e.* on Γ. In both cases, the formulae

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2}f(z), \qquad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2}f(z) \tag{1.9}$$

and hence

$$f(z) = f^+(z) - f^-(z) \tag{1.10}$$

hold *a.e.* on Γ. From the results in [39] , it follows that if Γ is a Dini-smooth curve S_Γ is bounded on $L^{p(\cdot),\lambda(\cdot)}(\Gamma)$. Note that some properties of the Cauchy singular integral in the different spaces were investigated in [8,13,15,20,34–36,38].

Let $f(\cdot) \in L^{p(\cdot),\lambda(\cdot)}(\Gamma)$: Using (1.3), (1.4), (1.7), (1.8) and (1.10) we can associate the Faber-Laurent series

$$f(z) \backsim \sum_{k=0}^{\infty} a_k \Phi_k(z) + \sum_{k=1}^{\infty} b_k F_k\left(\frac{1}{z}\right),$$

where the coefficients a_k and b_k are defined by

$$a_k := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f[\psi(w)]}{w^{k+1}} d\omega, \qquad k = 0, 1, 2, \ldots$$

and

$$b_k := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f[\psi_1(w)]}{w^{k+1}} dw, \qquad k = 0, 1, 2, \ldots.$$

The coefficients a_k and b_k are said to be the Faber-Laurent coefficients of $f(\cdot)$.

If Γ is a Dini-smooth curve, then from the results in [53], it follows that

$$\left.\begin{array}{ll} 0 < c_2 < |\phi'(w)| < c_3 < \infty, & 0 < c_4 < |\phi_1'(w)| < c_5 < \infty \\ 0 < c_6 < |\psi'(w)| < c_7 < \infty, & 0 < c_8 < |\psi_1'(\omega)| < c_9 < \infty \end{array}\right\} \tag{1.11}$$

where the constants c_2, c_3, c_4, c_5 and c_6, c_7, c_8, c_9 are independent of $z \in \bar{G}^-$ and $|w| \geq 1$, respectively.

Let Γ be a Dini-smooth curve and let $f_0(w) := f[\psi(w)]$ for $f(\cdot) \in L^{p(\cdot),\lambda(\cdot)}(\Gamma)$, $p_0(w) := p(\psi(w))$ and let $f_1(w) := f[\psi_1(w)]$ for $f(\cdot) \in L^{p(\cdot),\lambda(\cdot)}(\Gamma)$, $p_1(w) := p(\psi_1(w))$. Then using (1.11) and the method applied for the proof of a similar result in [29, Lem. 1], we obtain $f_0(\cdot) \in L^{p_0(\cdot),\lambda(\cdot)}(\mathbb{T})$ and $f_1(\cdot) \in L^{p_1(\cdot),\lambda(\cdot)}(\mathbb{T})$.

Moreover, $f_0^-(\infty) = f_1^-(\infty) = 0$ and by (1.10)

$$\left.\begin{array}{l} f_0(w) = f_0^+(w) - f_0^-(w) \\ f_1(w) = f_1^+(w) - f_1^-(w) \end{array}\right\} \tag{1.12}$$

a.e. on $\mathbb{T}$.

Note that the density of polynomials is an indispensable condition in approximation problems. Therefore, the polynomials are dense in the spaces $L^{p(\cdot),\lambda(\cdot)}(\Gamma)$, $E^{p(\cdot)\lambda(\cdot)}(G)$ and $E^{p(\cdot)\lambda(\cdot)}(G^-)$.

Using [21, Thm. 6.1] and the method applied for the proof of a similar result in [10] we can prove the following Lemma:

Lemma 1.1 *Let $p(\cdot) : \mathbb{T} \to [1, +\infty]$ and $\lambda(\cdot) : \mathbb{T} \to [0, 1]$ be measurable functions. Let $g(\cdot) \in E^{p(\cdot),\lambda(\cdot)}(D)$ with $p(\cdot) \in \wp^{\log}(\mathbb{T})$, $0 \le \lambda_* \le \lambda^* < 1$. If $\sum_{k=0}^{n} d_k(g)w^k$ is the nth partial sum of the Taylor series of $g(\cdot)$ at the origin, then*

$$\left\| g(w) - \sum_{k=0}^{n} d_k w^k \right\|_{L^{p(\cdot),\lambda(\cdot)}(\mathbb{T})} \le c_{10}(p)\, \Omega\left(g, \frac{1}{n}\right)_{p(\cdot),\lambda(\cdot)}, \qquad \text{for all } n \in \mathbb{N}$$

with some constant $c_{10}(p) > 0$ independent of n.

Lemma 1.2 *Let $p(\cdot) : \mathbb{T} \to [1, +\infty]$ and $\lambda(\cdot) : \mathbb{T} \to [0, 1]$ be measurable functions. Let $g(\cdot) \in L^{p(\cdot),\lambda(\cdot)}(\mathbb{T})$ with $p(\cdot) \in \wp^{\log}(\mathbb{T})$, $0 \le \lambda_* \le \lambda^* < 1$. Then the inequality*

$$\Omega(g^+, \cdot)_{p(\cdot),\lambda(\cdot)} \le c_{11} \Omega(g, \cdot)_{p(\cdot),\lambda(\cdot)} \tag{1.13}$$

holds.

Proof of Lemma 1.2 It is clear that the equality

$$g^+ = S_{\mathbb{T}}(g) + \frac{1}{2} g \tag{1.14}$$

holds *a.e.* on $\mathbb{T}$. Using the method of proof of [10, Lem. 3.3] (see also, [29, Lem. 2] and the boundedness of the singular operator $S_{\mathbb{T}}(g)$ in $L^{p(\cdot),\lambda(\cdot)}(\mathbb{T})$ we can prove that

$$\Omega(S_T(g), \cdot)_{p(\cdot),\lambda(\cdot)} \le c_{12} \Omega(g, \cdot)_{p(\cdot),\lambda(\cdot)}. \tag{1.15}$$

Then using the subadditivity of the modulus of smoothness $\Omega(g^+, \cdot)_{p(\cdot),\lambda(\cdot)}$, (1.14) and (1.15) we obtain inequality (1.13) of Lemma 1.2. □

We set

$$R_n(f, z) := \sum_{k=0}^{n} a_k \Phi_k(z) + \sum_{k=0}^{n} b_k F_k\left(\frac{1}{z}\right).$$

The rational function $R_n(f, z)$ is called the Faber-Laurent rational function of degree n of $f(\cdot)$.

The problems of approximation of the functions in classical Morrey spaces and variable exponent Morrey spaces were investigated in [1,2,9,11,12,21,26,27]. In this work the approximation of the functions by Faber-Laurent rational functions in the variable exponent Morrey classes defined on the Dini-smooth curve are investigated. Similar problems of approximation of the functions by Faber-Laurent rational functions in different spaces were studied in [6,7,10,23,25,28,29,31–33,43,44,55].

Our main results are as follows.

Theorem 1.1 *Let Γ be a Dini-smooth curve. Let $p(\cdot)\colon \Gamma \to [1, +\infty]$ and $\lambda(\cdot)\colon \Gamma \to [0, 1]$ be measurable functions. If $p(\cdot) \in \wp^{\log}(\Gamma)$, $0 \le \lambda_* \le \lambda^* < 1$ and $f(\cdot) \in L^{p(\cdot)\lambda(\cdot)}(\Gamma)$, then for every natural number n there are a constant $c_{10} > 0$ and rational function*

$$R_n(z, f) := \sum_{k=-n}^{n} a_k^{(n)} z^k$$

such that

$$\|f - R_n(\cdot, f)\|_{L^{p(\cdot),\lambda(\cdot)}(\Gamma)} \le c_{13}\left[\Omega\left(f_0, \frac{1}{n}\right)_{p_0(\cdot),\lambda(\cdot)} + \Omega\left(f_1, \frac{1}{n}\right)_{p_1(\cdot),\lambda(\cdot)}\right],$$

where $R_n(\cdot, f)$ is the n-th partial sum of the Faber-Laurent series of $f(\cdot)$.

Theorem 1.2 *Let Γ be a Dini-smooth curve. Let $p(\cdot)\colon \Gamma \to [1, +\infty]$ and $\lambda(\cdot)\colon \Gamma \to [0, 1]$ be measurable functions. If $p(\cdot) \in \wp^{\log}(\Gamma)$, $0 \le \lambda_* \le \lambda^* < 1$ and $f(\cdot) \in E^{p(\cdot)\lambda\cdot)}(G)$, then for every natural number n the inequality*

$$\left\| f(z) - \sum_{k=0}^{n} a_k \Phi_k(z) \right\|_{L^{p(\cdot),\lambda(\cdot)}(\Gamma)} \le c_{14}\Omega\left(f_0, \frac{1}{n}\right)_{p_0(\cdot),\lambda(\cdot)} \tag{1.16}$$

holds with a constant $c_{14} > 0$ independent of n.

Note that the order of polynomial approximation in $E^p(G)$, $p \ge 1$ has been investigated by several authors. In [52] Walsh an Rusel gave results when Γ is an analytic curve. When Γ is a Dini-smooth curve direct and inverse theorems were proved by S. Y. Alper [4], These results were later extended to domains with regular boundary for $p > 1$ by Kokilashvili [37] and for $p \ge 1$ by Andersson [5]. For domains with a regular boundary the approximation directly as the nth partial sums of p-Faber polynomial of $f(\cdot) \in E^p(G)$ have been constructed in [23]. The approximation properties of the p-Faber series expansions in the ω-weighted Smirnov class $E^p(G, \omega)$ of analytic functions in G whose boundary is a regular Jordan curve are investigated in [24].

Theorem 1.3 *Let Γ be a Dini-smooth curve. Let $p(\cdot)\colon \Gamma \to [1,+\infty]$ and $\lambda(\cdot)\colon \Gamma \to [0,1]$ be measurable functions. If $p(\cdot) \in \wp^{\log}(\Gamma)$, $0 \le \lambda_* \le \lambda^* < 1$ and $f(\cdot) \in E^{p(\cdot)\lambda(\cdot)}(G^-)$, then for every natural number n the inequality*

$$\left\| f - f(\infty) - \sum_{k=0}^{n} -b_k F_k\left(\frac{1}{z}\right)\right\|_{L^{p(\cdot),\lambda(\cdot)}(\Gamma)} \le c_{15}\Omega\left(f_1, \frac{1}{n}\right)_{p_1(\cdot),\lambda(\cdot)} \quad (1.17)$$

holds, with a constant $c_{15} > 0$ independent of n.

2 Proof of the Main Result

Proof of Theorem 1.1 Let $f(\cdot) \in L^{p(\cdot),\lambda(\cdot)}(\Gamma)$. Then from (1.11), we have $f_0(\cdot) \in L^{p_0(\cdot),\lambda(\cdot)}(\mathbb{T})$, $f_1(\cdot) \in L^{p_1(\cdot),\lambda(\cdot)}(\mathbb{T})$. According to(1.12) we obtain that

$$f(\zeta) = f_0^+(\phi(\zeta)) - f_0^-(\phi(\zeta)), \qquad f(\xi) = f_1^+(\phi_1(\xi)) - f_1^-(\phi_1(\xi)). \quad (2.1)$$

$a.e.$ on Γ.

We prove that the rational function

$$f(z) = \sum_{k=0}^{n} a_k\Phi_k(z) + \sum_{k=1}^{n} b_k F_k\left(\frac{1}{z}\right)$$

satisfies the condition of Theorem 1.1.

Let $z^* \in G^-$. Using the method of proof in [28], we can prove that $f_0^-(\phi(\zeta)) \in E^{p(\cdot),\lambda(\cdot)}(G^-) \in E^1(G^-)$. Then it is clear that

$$\frac{1}{2\pi i}\int_\Gamma \frac{f_0^-(\phi(\zeta))}{\zeta - z^*}d\zeta = -f_0^-(\phi(z^*)).$$

Then from last equality, (1.5) and (2.1) we have

$$\begin{aligned}\sum_{k=0}^{n} a_k\Phi_k(z^*) &= \sum_{k=0}^{n} a_k[\phi(z^*)]^k + \frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta - z^*}\sum_{k=0}^{n} a_k[\phi(\zeta)]^k d\zeta \\ &= \sum_{k=0}^{n} a_k[\phi(z^*)]^k + \frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta - z^*}\sum_{k=0}^{n} a_k[\phi(\zeta)]^k - f_0^+[\phi(\zeta)]d\zeta \\ &\quad + \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z^*}d\zeta - f_0^-[\phi(z^*)]. \end{aligned} \quad (2.2)$$

Use of(1.8) and (2.2) gives us

$$\begin{aligned}\sum_{k=0}^{n} a_k \Phi_k(z^*) &= \sum_{k=0}^{n} a_k[\phi(z^*)]^k + \frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta - z^*}\sum_{k=0}^{n} a_k[\phi(\zeta)]^k d\zeta \\ &= \sum_{k=0}^{n} a_k[\phi(z^*)]^k + \frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta - z^*}\sum_{k=0}^{n} a_k[\phi(\zeta)]^k - f_0^+[\phi(\zeta)]d\zeta \\ &\quad + f^-(z^*) - f_0^-[\phi(z^*)]. \end{aligned} \tag{2.3}$$

Taking the limit as $z^* \to z \in \Gamma$ along all non-tangential paths outside Γ and considering (1.9), (1.10), (2.1) and (2.3) we obtain

$$\begin{aligned} f^+(z) - \sum_{k=0}^{n} a_k \Phi_k(z^*) &= \frac{1}{2}\left[f_0^+[\phi(z^*)] - \sum_{k=0}^{n} a_k[\phi(z^*)]^k \right] \\ &\quad + S_\Gamma\left(\left[f_0^+[\phi(z^*)] - \sum_{k=0}^{n} a_k[\phi(z^*)]^k \right]\right). \end{aligned} \tag{2.4}$$

According to [39] the singular operator $S_\Gamma : L^{p(\cdot),\lambda(\cdot)}(\Gamma) \to L^{p(\cdot),\lambda(\cdot)}(\Gamma)$ is bounded. Then using (2.1), Minkowski's inequality, Lemma 1.1 and 1.2 we reach

$$\begin{aligned} &\left\| f^+(z) - \sum_{k=0}^{n} a_k \Phi_k(z^*) \right\|_{L^{p(\cdot),\lambda(\cdot)}(\Gamma)} \\ &\le \frac{1}{2}\left\| f_0^+[\phi(z^*)] - \sum_{k=0}^{n} a_k[\phi(z^*)]^k \right\|_{L^{p(\cdot),\lambda(\cdot)}(\Gamma)} \\ &\quad + \left\| S_\Gamma\left(\left[f_0^+[\phi(z^*)] - \sum_{k=0}^{n} a_k[\phi(z^*)]^k \right]\right) \right\|_{L^{p(\cdot),\lambda(\cdot)}(\Gamma)} \\ &\le \frac{1}{2}\left\| f_0^+(w) - \sum_{k=0}^{n} a_k w^k \right\|_{L^{p_0(\cdot),\lambda(\cdot)}(\mathbb{T})} + c_{16}\left\| f_0^+(w) - \sum_{k=0}^{n} a_k w^k \right\|_{L^{p_0(\cdot),\lambda(\cdot)}(\mathbb{T})} \\ &\le c_{17}\left\| f_0^+(w) - \sum_{k=0}^{n} a_k w^k \right\|_{L^{p_0(\cdot),\lambda(\cdot)}(T)} \\ &\le c_{18}\left\| f_0^+(w) - \sum_{k=0}^{n} \alpha_k(f_0^+) w^k \right\|_{L^{p_0(\cdot),\lambda(\cdot)}(T)} \\ &\le c_{19}\Omega\left(f_0^+, \frac{1}{n}\right)_{p_0(\cdot),\lambda(\cdot)} \le \Omega_{20}\left(f_0, \frac{1}{n}\right)_{p_0(\cdot),\lambda(\cdot)}. \end{aligned} \tag{2.5}$$

Let $z^* \in G$. Using the method of proof in [28] we can prove that $f_1^-(\phi_1(\zeta)) \in E^{pt(\cdot),\lambda(\cdot)}(G^-) \in E^1(G^-)$. Therefore,

$$\frac{1}{2\pi i}\int_\Gamma \frac{f_1^-(\phi_1(\zeta))}{\zeta - z^*}d\zeta = f_1^-(\phi_1(z^*)).$$

Then, using the last equality, (1.6) and (2.1) we have

$$\begin{aligned}
&\sum_{k=1}^n b_k F_k\left(\frac{1}{z^*}\right)\\
&\quad= \sum_{k=1}^n b_k[\phi_1(z^*)]^k - \frac{1}{2\pi i}\int_\Gamma \frac{1}{\xi - z^*}\sum_{k=1}^n b_k[\phi_1(\xi)]^k d\xi\\
&\quad= \sum_{k=1}^n b_k[\phi_1(z^*)]^k - \frac{1}{2\pi i}\int_\Gamma \frac{1}{\xi - z^*}\left(\sum_{k=1}^n b_k[\phi_1(\xi)]^k - f_1^+[\phi_1(\xi)]\right)d\xi\\
&\qquad -\frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z^*}d\xi - \frac{1}{2\pi i}\int_\Gamma \frac{f_1^-(\phi_1(\zeta))}{\zeta - z^*}d\zeta\\
&\quad= \sum_{k=1}^n b_k[\phi_1(z^*)]^k - \frac{1}{2\pi i}\int_\Gamma \frac{1}{\xi - z^*}\left(\sum_{k=1}^n b_k[\phi_1(\xi)]^k - f_1^+[\phi_1(\xi)]\right)d\xi\\
&\qquad - f^+(z^*) - f_1^-[\phi_1(z^*)].
\end{aligned}$$

Taking the limit as $z^* \to z$ along all non-tangential paths inside of Γ we have

$$\begin{aligned}
&\sum_{k=1}^n b_k F_k\left(\frac{1}{z}\right)\\
&\quad= \sum_{k=1}^n b_k[\phi_1(z)]^k - \frac{1}{2}\left(\sum_{k=1}^n b_k[\phi_1(z)]^k - f_1^+[\phi_1(z)]\right)\\
&\qquad - S_\Gamma\left(\sum_{k=1}^n b_k[\phi_1(z)]^k - f_1^+[\phi_1(z)]\right) - f^+(z) - f_1^-[\phi_1(z)]
\end{aligned}$$

a.e. on Γ. Use of(1.10) and (2.1) gives

$$\begin{aligned}
&f^-(z) + \sum_{k=1}^n b_k F_k\left(\frac{1}{z}\right)\\
&\quad= \frac{1}{2}\left(\sum_{k=1}^n b_k[\phi_1(z)]^k - f_1^+[\phi_1(z)]\right)\\
&\qquad - S_\Gamma\left(\sum_{k=1}^n b_k[\phi_1(z)]^k - f_1^+[\phi_1(z)]\right).
\end{aligned} \tag{2.6}$$

Consideration of (2.6), Minkowski's inequality and the boundedness of S_Γ in $L^{p(\cdot),\lambda(\cdot)}(\Gamma)$, Lemma 1.1 and 1.2 gives rise to

$$\begin{aligned}
&\left\| f^-(z) + \sum_{k=1}^{n} b_k F_k\left(\frac{1}{z}\right)\right\|_{L^{p(\cdot)\lambda(\cdot)}(\Gamma)} \\
&\leq \left\| \frac{1}{2}\left(\sum_{k=1}^{n} b_k[\phi_1(z)]^k - f_1^+[\phi_1(z)]\right)\right\|_{L^{p(\cdot)\lambda(\cdot)}(\Gamma)} \\
&\quad + \left\| S_\Gamma\left(\sum_{k=1}^{n} b_k[\phi_1(z)]^k - f_1^+[\phi_1(z)]\right)\right\|_{L^{p(\cdot),\lambda(\cdot)}(\Gamma)} \\
&\leq \frac{1}{2}\left\| \sum_{k=1}^{n} b_k w^k - f_1^+(w)\right\|_{L^{p_1(\cdot),\lambda(\cdot)}(\mathbb{T})} \\
&\quad + c_{21}\left\| \sum_{k=1}^{n} b_k w^k - f_1^+(w)\right\|_{L^{p_1(\cdot),\lambda(\cdot)}(\mathbb{T})} \\
&\leq c_{22}\left\| \sum_{k=1}^{n} b_k w^k - f_1^+(w)\right\|_{L^{p_1(\cdot),\lambda(\cdot)}(\mathbb{T})} \\
&= c_{22}\left\| \sum_{k=1}^{n} \beta_k(f_1^+) w^k - f_1^+(w)\right\|_{L^{p_1(\cdot),\lambda(\cdot)}(\mathbb{T})} \\
&\leq c_{23}\Omega\left(f_1^+, \frac{1}{n}\right)_{p_1(\cdot),\lambda(\cdot)} \\
&\leq c_{24}\Omega\left(f_1, \frac{1}{n}\right)_{p_1(\cdot),\lambda(\cdot)}
\end{aligned} \tag{2.7}$$

Now combining (1.9), (2.5) and (2.7) we obtain

$$\|f - R_n(\cdot, f)\|_{L^{p(\cdot)}(\Gamma)} \leq c_{25}(p)\left[\Omega\left(f_0, \frac{1}{n}\right)_{p_0(\cdot),\lambda(\cdot)} + \Omega\left(f_1, \frac{1}{n}\right)_{p_1(\cdot),\lambda(\cdot)}\right].$$

The proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2 Let $z^* \in G^-$. If $f(\cdot) \in E^{p(\cdot),\lambda(\cdot)}(G)$, then $f(\cdot) \in E^p(G)$ and $f(\zeta)/(\zeta - z^*) \in E^p(G)$. Therefore, $\int_\Gamma f(\zeta)/(\zeta - z^*)d\zeta = 0$. That is $f^-(z) = 0$ *a.e.* on Γ. Then taking into account (1.10),

$$\left\| f_0^+(w) - \sum_{k=0}^{n} a_k w^k\right\|_{L^{p(\cdot),\lambda(\cdot)}(\mathbb{T})} \leq c_{26}(p)\Omega\left(f_0, \frac{1}{n}\right)_{p(\cdot),\lambda(\cdot)} \quad \text{for all } n \in \mathbb{N},$$

$$\left\| f_0^+(z) - \sum_{k=0}^{n} a_k \Phi_k(z)\right\|_{L^{p(\cdot),\lambda(\cdot)}(\Gamma)} \leq c_{27}\left\| f_0^+(w) - \sum_{k=0}^{n} a_k w^k\right\|_{L^{p(\cdot),\lambda(\cdot)}(\mathbb{T})}$$

we have the inequality (1.16) of Theorem 1.2. □

Proof of Theorem 1.3 Let $z^* \in G$ and $f(\cdot) \in E^{p(\cdot),\lambda(\cdot)}(G^-)$. It is clear that $\int_\Gamma f(\zeta)/(\zeta - z^*) = f(\infty)$. Then we have $f^+(z) = f(\infty)$ *a.e.* on Γ. Now combining (1.10),

$$\left\| f_1^+(w) - \sum_{k=0}^{n} b_k w^k \right\|_{L^{p(\cdot),\lambda(\cdot)}(\mathbb{T})} \leq c_{28}(p)\Omega\left(f_1, \frac{1}{n}\right)_{p(\cdot),\lambda(\cdot)} \quad \text{for all } n \in \mathbb{N},$$

$$\left\| f^-(z) - \sum_{k=0}^{n} b_k F_k\left(\frac{1}{z}\right) \right\|_{L^{p(\cdot),\lambda(\cdot)}(\Gamma)} \leq c_{29} \left\| f_1^+(w) - \sum_{k=0}^{n} b_k w^k \right\|_{L^{p(\cdot),\lambda(\cdot)}(\mathbb{T})}$$

we obtain the inequality (1.17) of Theorem 1.3. □

Acknowledgements The author would like to thank the referees for all their valuable advice and helpful remarks.

References

1. Almeida, A., Samko, S.: Approximation in Morrey spaces. J. Funct. Anal. **272**(6), 2392–2411 (2007)
2. Almeida, A., Samko, S.: Approximation in generalized Morrey spaces. Georgian Math. J. **25**(2), 155–168 (2018)
3. Almeida, A., Hasanov, J., Samko, S.: Maximal and potential operators in variable exponent Morrey spaces. Georgian Math. J. **15**(2), 195–2008 (2008)
4. Alper, S.Y.: Approximation in the mean of analytic functions of class E^p, In: Investigations on the Modern Problems of the Function Theory of a Complex Variable. Gos. Izdat. Fiz.-Mat. Lit., Moscow, pp. 272–286 (1960) (in Russian)
5. Andersson, J.E.: On the degree of polynomial approximation in $E^p(D)$. J. Approx. Theory **19**, 61–68 (1977)
6. Andrievskii, V. V., Israfilov, D. M.: Approximations of functions on quasiconformal curves by rational functions. *Izv. Akad. Nauk Azerb. SSR Ser. Fiz.-Tekhn. Math.* **36**(4) (1980) (in Russian)
7. Andrievskii, V.V.: Jackon's approximation theorem for biharmonic functions in a multiply connected domain. East J. Approx. **6**(2), 229–239 (2000)
8. Böttcher, A., Karlovich, Y.I.: Carleson Curves, Muckenhoupt Weights and Teoplitz Operators. Birkhauser, Boston (1997)
9. Burenkov, V. I., Laza De Crutofors, M., Kydyrmina, N.A.: Approximation by C^∞ functions in Morrey spaces. Math. J. **14**(52), 66–75 (2014)
10. Cavus, A., Israfilov, D.M.: Approximation by Faber–Laurent rational functions in the mean of functions of the class $L_p(\Gamma)$ with $1 < p < \infty$. Approx. Theory Appl. **11**(1), 105–118 (1995)
11. Cakir, Z., Aykol, C., Soylemez, D., Serbetci, A.: Approximation by trigonometric polynomials in Morrey spaces. Trans. Atl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Math. **39**(1), 24–37 (2019)
12. Cakir, Z., Aykol, C., Soylemez, D., Serbetci, A.: Approximation by trigonometric polynomials in weighted Morrey spaces. Tbilisi Math. J. **13**(1), 123–138 (2020)
13. David, G.: Operateurs integraux singulers sur certains courbes du plan complexe. Ann. Sci. Ecol. Norm. Super. **4**, 157–189 (1984)
14. Duren, P.L.: Theory of H^p Spaces. Academic Press, Newyork (1970)
15. Dyn'kin, E. M., Osilenker, B.P.: Weighted estimates for singular integrals and their applications, In: Mathematical Analysis, Vol. 21. Akad. Nauk. SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, pp. 42–129 (1983)
16. Fan, X.: The regularity of Lagrangians $f(x,\xi) = \|\xi\|_{\alpha(x)}$ with Höder exponents $\alpha(x)$. Acta Math. Sin. (N:S.) **12**(3), 254–261 (1996)

17. Fan, D., Lu, S., Yang, D.: Regularity in Morrey spaces of strange solitions to nondivergence elliptic equations with VMO coefficients. Georgian Math. J. **5**, 425–440 (1998)
18. Giaquinta, M.: Multiple Integrals in the Calculus of Variations and Non-linear Elliptic Systems. Princeton University Press, Princeton (1983)
19. Giga, Y., Miyakawa, T.: Navier–Stokes flow in R^3 with measure as initial vorticity and Morrey spaces. Commun. Part. Differ. Equ. **14**(5), 577–618 (1989)
20. Guliyev, V.S., Hasanov, J., Samko, S.: Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces. Math. Scand. **107**, 285–304 (2010)
21. Guliyev, V.S., Ghorbanalizadeh, A., Sawano, Y.: Approximation by trigonometric polynomials in variable exponent Morrey spaces. Anal. Math. Phys. **9**(3), 1265–1286 (2019)
22. Goluzin, G.M.: Geometric Theory of Functions of a Complex Variable, Translation of Mathematical Monographs, 26th edn. AMS, Providence (1968)
23. Israfilov, D.M.: Approximate properties of the generalized Faber series in an integral metric. Izv. Akad. Nauk. Az. SSR Ser. Fiz.-Tekh. Math. Nauk. **2**, 10–14 (1987). ((**in Russian**))
24. Israfilov, D.M.: Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G, w)$ and the Bieberbach polynomials. Constr. Approx. **17**, 335–351 (2001)
25. Israfilov, D.M.: Approximation by p-Faber-Laurent rational functions in weighted Lebesgue spaces. Chechoslovak. Math. J. **54**(129), 751–765 (2004)
26. Israfilov, D.M., Tozman, N.P.: Approximation by polynomials in Morrey-Smirnov classes. East J. Approx. **14**(3), 255–269 (2008)
27. Israfilov, D.M., Tozman, N.P.: Approximation in Morrey-Smirnov classes. Azerbaijan J. Math. **1**(2), 99–113 (2011)
28. Israfilov, D.M., Testici, A.: Approximation in Smirnov classes with variable exponent. Complex Var. Elliptic Equ. **60**(9), 1243–1253 (2015)
29. Israfilov, D.M., Testici, A.: Approximation by Faber-Laurent rational functions in Lebesgue spaces with variable exponent. Indag. Math. **27**, 914–922 (2016)
30. Izuki, M., Nakai, E., Sawano, S.: Function spaces with variable exponents—an introduction. Sci. Math. Jpn. **77**(2), 187–315 (2014)
31. Jafarov, S.Z.: Approximation by rational functions in Smirnov-Orlicz classes. J. Math. Anal. Appl. **379**, 870–877 (2011)
32. Jafarov, S.Z.: Approximation by polynomials and rational functions in Orlicz spaces. J. Comput. Anal. Appl. (JoCAAA) **13**(5), 953–962 (2011)
33. Jafarov, S.Z.: Approximation of function by p-Faber-Laurent functions. Complex Variables Elliptic Equ. **60**(3), 416–428 (2015)
34. Karlovich, A.Yu.: Algebras of singular integral operators with piecewise continuous coefficients on relexive Orlicz spaces. Math. Nachr. **170**, 187–222 (1996)
35. Kokilashvili, V.M., Paatasvili, V., Samko, S.: Boundary value problems for analytic functions in the class of Cauchy type integrals with density in $L^{p(\cdot)}(\Gamma)$. Bound. Value Probl. (Hindawi Publ. Corp.) **2005**, 43–71 (2005)
36. Kokilashvili, V.M., Samko, S.: Weighted boundedness in Lebesgue spaces with variable exponents of classical operators on Carleson curves. Proc. A. Razmadze Math. Inst. **138**, 106–110 (2005)
37. Kokilashvili, V.M.: A direct theorem on mean approximation of analytic functions by polynomials. Soviet Math. Dokl. **10**, 411–414 (1969)
38. Kokilashvili, V.M., Samko, S.: Singular integrals and potentials in some Banach function spaces with variable exponent. J. Funct. Spaces Appl. **1**(1), 45–59 (2003)
39. Kokilashvili, V.M., Meskhi, A.: Boundedness of maximal and singular operators in Morrey spaces with variable exponent. Armen. J. Math. **1**(1), 18–28 (2008)
40. Kufner, A., John, O., Fucik, S.: Function Spaces, p. 454+XV. Noordhoff International Publishing, London (1977)
41. Markushevich, A.I.: Analytic Function Theory: Vols. I, II. Nauka, Moscow (1968)
42. Ohno, T.: Continuity properties for logarithmic potentials of functions in Morrey spaces of variable exponent. Hiroshima Math. J. **38**(3), 363–383 (2008)
43. Pekarski, A.A.: Rational approximation of absolutely continuous functions with derivatives in an Orlizc space. Math. Sb. **45**, 121–137 (1983)
44. Pekarskli, A.A.: Bernstein type inequalities for the derivatives of rational functions and converse theorems for rational approximation. Math. Sb. **124**(166), 571–588 (1984)
45. Pommerenke, Ch.: Boundary Behavior of Conformal Maps. Springer, Berlin (1992)

46. Raferio, H., Samko, N., Samko, S.: Morrey-Campanato spaces: an overview, (English summary) Operator theory, pseudo-differential equations, and mathermatical physics. Oper. Theory Adv. Appl. **228**, 293–323 (2013)
47. Ruzicka, M.: Elektrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics, vol. 1748. Springer, Berlin (2000)
48. Sawano, Y., Tanako, H.: The Fatou property of block spaces. J. Math. Sci. Univ. Tokyo **22**, 663–683 (2015)
49. Suetin, P.K.: Series of Faber Polynomials. Gordon and Breach Science Publishers, London (1998)
50. Taylor, M.E.: Tools for PDE. In: Mathematical Surveys and Monographs, Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials, vol. 81. American Mathematical Society, Providence (2000)
51. Tribel, H.: Hybrid Function Spaces, Heat and Navier-Stokes Equations, Tract in Mathematics, vol. 24, p. 185+x. European Mathematical Society (EMS), Zürich (2014)
52. Walsh, J.L., Russel, H.G.: Integrated continuity conditions and degree of approximation by polynomials or by bounded analytic functions. Trans. Am. Math. Soc. **92**, 355–370 (1959)
53. Warschawskii, S.E.: Über das Randverhalten der Ableitung der Abbildungsfunktionen bei konformer Abbildung. Math. Z. **35**, 321–456 (1932)
54. Yangi, D., Yuan, W.: Dual properties of Tribel-Lizorkin type spaces and their applications. Z. Anal. Anwend. **30**, 29–58 (2011)
55. Yurt, H., Guven, A.: Approximation by Faber-Laurent rational functions on doubly connected domains. N. Z. J. Math. **44**, 113–124 (2014)