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Difference of Composition Operators on Weighted Bergman Spaces with Doubling Weights

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Abstract

In this paper, some characterizations for the compact difference of composition operators on weighted Bergman spaces A^p_ω with doubling weights are given, which extend Moorhouse's characterization for the difference of composition operators on the weighted Bergman space A^2_α.

Keywords Weighted Bergman space · Composition operator · Difference

Mathematics Subject Classification 32A36 · 47B33

1 Introduction

Let $\mathbb{D}$ be the the unit disc and $H(\mathbb{D})$ be the class of analytic functions on $\mathbb{D}$. A function $\omega : \mathbb{D} \to [0, \infty)$, integrable over $\mathbb{D}$, is called a weight. It is radial if $\omega(z) = \omega(|z|)$

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for all $z \in \mathbb{D}$. For $0 < p < \infty$ and a radial weight ω, the weighted Bergman space A^p_ω is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty,$$

where dA is the normalized Lebesgue measure on $\mathbb{D}$. As usual, A^p_α stands for the classical weighted Bergman space induced by the standard radial weight $\omega(z) = (1-|z|^2)^\alpha$, where $-1 < \alpha < \infty$. A^p_ω equipped with the norm $\|\cdot\|_{A^p_\omega}$ is a Banach space for $1 \le p < \infty$ and a complete metric space for $0 < p < 1$ with respect to the translation-invariant metric $(f, g) \mapsto \|f - g\|_{A^p_\omega}$.

For a radial weight ω, we assume throughout the paper that $\widehat{\omega}(r) = \int_r^1 \omega(s)ds$ for all $0 \le r < 1$. We say that ω is a doubling weight, denoted by $\omega \in \widehat{\mathcal{D}}$, if there exists a constant $C \ge 1$ such that $\widehat{\omega}(r) \le C\widehat{\omega}((1+r)/2)$ when $0 \le r < 1$. If there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $\widehat{\omega}(r) \ge C\widehat{\omega}(1-(1-r)/K)$, $0 \le r < 1$, we say that ω is a reverse doubling weight, denoted by $\omega \in \check{\mathcal{D}}$. We write $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$. For some properties of these classes of weights, see [13–19] and the references therein.

Let φ be an analytic self-map of $\mathbb{D}$. The map φ induces the composition operator C_φ on $H(\mathbb{D})$, which is defined by $C_\varphi f = f \circ \varphi$. We refer to [4,22] for various aspects of the theory of composition operators acting on analytic function spaces. Efforts to understand the topological structure of the space of composition operators in the operator norm topology have led to the study of the difference operator $C_\varphi - C_\psi$ of two composition operators induced by analytic self-maps φ and ψ of $\mathbb{D}$. By Littlewood's subordination principle, all composition operators, and hence all differences of two composition operators, are bounded on all Hardy spaces H^p and weighted Bergman spaces A^p_α. Thus the question of when the operator $C_\varphi - C_\psi$ is compact naturally arises. Shapiro and Sundberg [23] raised and studied such a question on Hardy spaces, motivated by the isolation phenomenon observed by Berkson [1]. After that, such related problems have been studied between several spaces of analytic functions by many authors. See, for example, [6,12,24] on Hardy spaces and [2,3,7,9,11,20,21,25] on weighted Bergman spaces.

In 2005, Moorhouse [11] characterized the compact difference of composition operators on weighted Bergman spaces A^2_α with the angular derivative cancellation property. More precisely, she showed that $C_\varphi - C_\psi$ is compact on A^2_α if and only if

$$\lim_{|z|\to 1} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\psi(z)|^2} \right) \rho(\varphi(z), \psi(z)) = 0. \tag{1}$$

We remark here that this characterization has been extended not only to higher dimensional balls and polydisks, but also to a general parameter p, see [2,3,9].

It is known that all composition operators and hence all differences of two composition operators, are bounded on A^p_ω for $\omega \in \widehat{\mathcal{D}}$ (see [16]). In this paper, we extend Moorhouse's characterization to A^p_ω whenever $\omega \in \mathcal{D}$. Our main result (Theorem 12) is a characterization of compact combinations of two composition operators. As a corollary, we obtain that Moorhouse's characterization for compact difference (1) remains valid when $0 < p < \infty$ and $\omega \in \mathcal{D}$. According to this result, the compactness of $C_\varphi - C_\psi : A^p_\omega \to A^p_\omega$ depends neither on p nor ω.

The present paper is organized as follows. In Sect. 2, we give some notation and preliminary results which will be used later. Section 3 is devoted to the question of when a given finite linear combination of composition operators is compact. In Sect. 4 we show that Moorhouse's characterization for compact difference remains valid when $0 < p < \infty$ and $\omega \in \mathcal{D}$. We also obtain a characterization for a composition operator to be equal modulo compact operators to a linear combination of composition operators (see Theorem 14).

For two quantities A and B, we use the abbreviation $A \lesssim B$ whenever there is a positive constant C (independent of the associated variables) such that $A \le CB$. We write $A \asymp B$, if $A \lesssim B \lesssim A$.

2 Prerequisites

In this section we provide some basic tools for the proofs of the main results in this paper.

2.1 Pseudo-Hyperbolic Distance

We denote by σ_z the Möbius transformation on $\mathbb{D}$ that interchanges the points 0 and z. More explicitly, $\sigma_z(w) = (z - w)/(1 - \overline{w}z)$, $w \in \mathbb{D}$. It is well known that σ_z satisfies the following properties: $\sigma_z \circ \sigma_z(w) = w$, and

$$1 - |\sigma_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{w}z|^2}, \quad z, w \in \mathbb{D}.$$

For $z, w \in \mathbb{D}$, the pseudo-hyperbolic distance between z and w is defined by $\rho(z, w) = |\sigma_z(w)|$. For $z \in \mathbb{D}$ and $r > 0$, the pseudo-hyperbolic disk at z with radius $r \in (0, 1)$ is given by $\triangle(z, r) = \{w \in \mathbb{D} : \rho(z, w) < r\}$. Note that $\triangle(z, r)$ is an open Euclidean disk with center and radius given by

$$c = \frac{(1 - r^2)z}{1 - r^2|z|^2} \quad \text{and} \quad t = \frac{1 - |z|^2}{1 - r^2|z|^2} r,$$

respectively. For $w \in \triangle(z, r)$, it is geometrically clear that $|c| - t \le |w| \le |c| + t$. Therefore,

$$\frac{(1 - |z|)(1 - r|z|)(1 - r)}{1 - r^2|z|^2} \le 1 - |w| \le \frac{(1 - |z|)(1 + r|z|)(1 + r)}{1 - r^2|z|^2},$$

and $|w| \to 1$ uniformly as $|z| \to 1$.

2.2 Basic Properties of Weights

The following two lemmas contain some basic properties of weights in the class $\widehat{\mathcal{D}}$ and $\check{\mathcal{D}}$ and will be frequently used in the sequel. For a proof of the first lemma, see [13, Lem. 2]. The second one can be proved by similar arguments.

Lemma A *Let ω be a radial weight. Then the following statements are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;
(ii) *There exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that*

$$\widehat{\omega}(r) \leq C\left(\frac{1-r}{1-t}\right)^{\beta}\widehat{\omega}(t), \quad 0 \leq r \leq t < 1;$$

(iii) *There exists $\gamma = \gamma(\omega) > 0$ such that*

$$\int_{\mathbb{D}} \frac{dA(z)}{|1-\overline{\zeta}z|^{\gamma+1}} \asymp \frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^{\gamma}}, \quad \zeta \in \mathbb{D}.$$

Lemma B *Let ω be a radial weight. Then $\omega \in \check{\mathcal{D}}$ if and only if there exist $C = C(\omega) > 0$ and $\alpha = \alpha(\omega) > 0$ such that*

$$\widehat{\omega}(t) \leq C\left(\frac{1-t}{1-r}\right)^{\alpha}\widehat{\omega}(r), \quad 0 \leq r \leq t < 1.$$

Lemma C [18, Lem. 5] *Let $0 < p < \infty$, $\omega \in \mathcal{D}$ and $-\alpha < \gamma < \infty$, where $\alpha = \alpha(\omega) > 0$ is that of Lemma B. Then*

$$\int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^{\gamma}\omega(z)dA(z) \asymp \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^{\gamma-1}\widehat{\omega}(z)dA(z), \quad f \in H(\mathbb{D}).$$

The following estimate plays an important role in this paper and will be frequently used.

Lemma 1 *Let φ be an analytic self-map of $\mathbb{D}$ and $\omega \in \mathcal{D}$. Then*

$$\left(\frac{1-|z|}{1-|\varphi(z)|}\right)^{\beta+1} \lesssim \frac{\omega(S(z))}{\omega(S(\varphi(z)))} \lesssim \left(\frac{1-|z|}{1-|\varphi(z)|}\right)^{\alpha+1},$$

where $\alpha = \alpha(\omega)$ and $\beta = \beta(\omega)$ are that of Lemmas B and A, respectively.

Remark It is worth noticing that the right hand inequality is valid for all $\omega \in \widehat{\mathcal{D}}$.

Proof An application of Lemma A shows that

$$\omega(S(z)) \asymp \widehat{\omega}(z)(1-|z|) \quad \text{and} \quad \omega(S(\varphi(z))) \asymp \widehat{\omega}(\varphi(z))(1-|\varphi(z)|).$$

By Schwarz's Lemma, we have

$$|\varphi(z)| \leq \frac{c-1}{c} + \frac{|z|}{c}, \quad \text{where } c = \frac{1+|\varphi(0)|}{1-|\varphi(0)|}.$$

By Lemmas A and B, we get

$$\begin{aligned}\frac{\widehat{\omega}(z)}{\widehat{\omega}(\varphi(z))} &= \frac{\widehat{\omega}(z)}{\widehat{\omega}\left(\frac{c-1}{c}+\frac{|z|}{c}\right)} \cdot \frac{\widehat{\omega}\left(\frac{c-1}{c}+\frac{|z|}{c}\right)}{\widehat{\omega}(\varphi(z))} \\ &\gtrsim \left(\frac{1-|z|}{1-\left(\frac{c-1}{c}+\frac{|z|}{c}\right)}\right)^{\alpha} \left(\frac{1-\left(\frac{c-1}{c}+\frac{|z|}{c}\right)}{1-|\varphi(z)|}\right)^{\beta} \asymp \left(\frac{1-|z|}{1-|\varphi(z)|}\right)^{\beta}\end{aligned}$$

and

$$\begin{aligned}\frac{\widehat{\omega}(z)}{\widehat{\omega}(\varphi(z))} &= \frac{\widehat{\omega}(z)}{\widehat{\omega}\left(\frac{c-1}{c}+\frac{|z|}{c}\right)} \cdot \frac{\widehat{\omega}(\frac{c-1}{c}+\frac{|z|}{c})}{\widehat{\omega}(\varphi(z))} \\ &\lesssim \left(\frac{1-|z|}{1-(\frac{c-1}{c}+\frac{|z|}{c})}\right)^{\beta} \left(\frac{1-\left(\frac{c-1}{c}+\frac{|z|}{c}\right)}{1-|\varphi(z)|}\right)^{\alpha} \asymp \left(\frac{1-|z|}{1-|\varphi(z)|}\right)^{\alpha}.\end{aligned}$$

The proof is complete. □

Lemma 2 *Let $\omega \in \mathcal{D}$. If $0 < \lambda < \alpha(\omega)$, where $\alpha(\omega)$ is that of Lemma B, then $\omega_\lambda(\cdot) := \omega(\cdot)/(1-|\cdot|)^{\lambda} \in \mathcal{D}$ and*

$$\widehat{\omega_\lambda}(z) \asymp \frac{\widehat{\omega}(z)}{(1-|z|)^{\lambda}}, \quad \textit{for all } z \in \mathbb{D}.$$

Proof By the trivial estimate on the denominator,

$$\widehat{\omega_\lambda}(r) = \int_r^1 \frac{\omega(t)}{(1-t)^{\lambda}}dt \gtrsim \frac{\widehat{\omega}(r)}{(1-r)^{\lambda}}.$$

An integration by parts shows that

$$\widehat{\omega_\lambda}(r) = \frac{\widehat{\omega}(r)}{(1-r)^{\lambda}} + \lambda \int_r^1 \widehat{\omega}(t)(1-t)^{-1-\lambda}dt.$$

Therefore, by Lemma B, we have

$$\widehat{\omega_\lambda}(r) \lesssim \frac{\widehat{\omega}(r)}{(1-r)^{\lambda}} + \lambda \frac{\widehat{\omega}(r)}{(1-r)^{\alpha}} \int_r^1 (1-t)^{\alpha-1-\lambda}dt \lesssim \frac{\widehat{\omega}(r)}{(1-r)^{\lambda}}.$$

Thus, $\widehat{\omega_\lambda}(z) \asymp \widehat{\omega}(z)/(1-|z|)^{\lambda}$ for all $\in \mathbb{D}$. By Lemmas A and B, $\omega_\lambda \in \mathcal{D}$. □

2.3 Local Estimates and Test Functions

The following lemmas are crucial in our work and will be used in this paper. The following lemma can be found in [10, Lem. 1].

Lemma 3 *Let* $0 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$ *and* $r_1 \in (0,1)$ *be arbitrary. Set* $\widetilde{\omega}(\cdot) = \widehat{\omega}(\cdot)/(1-|\cdot|)$. *Then there exist* $r_2 \in (0,1)$ *and a constant* $C = C(p, \omega, r_1, r_2) > 0$ *such that*

$$|f(z) - f(a)|^p \le C\rho(z,a)^p \frac{\int_{\triangle(z,r_2)} |f(\zeta)|^p \widetilde{\omega}(\zeta) dA(\zeta)}{\omega(S(z))}$$

for all $a \in \mathbb{D}$, $z \in \triangle(a, r_1)$ *and* $f \in A^p_\omega$.

By [26, Lem. 4.30], for all $a, z, w \in \mathbb{D}$ with $\rho(z,w) < r$ and any real s, we have

$$\left|1 - \left(\frac{1-\overline{a}z}{1-\overline{a}w}\right)^s\right| \le C(s,r)\rho(z,w),$$

and therefore, for all $w, z, a \in \mathbb{D}$ with $z \in \triangle(a,r)$ and any $s > 0$,

$$\left|\frac{1}{(1-\overline{a}z)^s} - \frac{1}{(1-\overline{a}w)^s}\right| \le C(s,r)\rho(z,w)\left|\frac{1}{(1-\overline{a}z)^s}\right|.$$

Although the reverse inequality does not hold, we have the following partial reiverse inequality (see [7, Thm. 2.8] or [25, Lem. 2.3]), which is crucial in the proof of the necessity part of Theorems 12 and 14.

Lemma D *Suppose* $s > 1$ *and* $0 < r_0 < 1$. *Then there exist* $N = N(r_0) > 1$ *and* $C = C(s, r_0) > 0$ *such that*

$$\begin{aligned} &\left|\frac{1}{(1-\overline{a}z)^s} - \frac{1}{(1-\overline{a}w)^s}\right| + \left|\frac{1}{(1-t_N\overline{a}z)^s} - \frac{1}{(1-t_N\overline{a}w)^s}\right| \\ &\quad \ge C\rho(z,w)\left|\frac{1}{(1-\overline{a}z)^s}\right|, \end{aligned}$$

for all $z \in \triangle(a, r_0)$ *with* $1 - |a| < 1/(2N)$, $t_N = 1 - N(1-|a|)$ *and* $w \in \mathbb{D}$.

2.4 Carleson Measure

Let μ be a finite positive Borel measure on $\mathbb{D}$. μ is called a p-Carleson measure for A^p_ω if the identity operator $I_d : A^p_\omega \to L^p(d\mu)$ is bounded, i.e. there is a positive constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \le C\|f\|^p_{A^p_\omega}$$

for any $f \in A^p_\omega$. Also, μ is called a vanishing p-Carleson measure for A^p_ω if the identity operator $I_d : A^p_\omega \to L^p(d\mu)$ is compact.

The characterization of a (vanishing) p-Carleson measure for A^p_ω has been solved for $\omega \in \widehat{\mathcal{D}}$ [14,19]. It is worth mentioning that the pseudo-hyperbolic disk is not the right one to describe the Carleson measure for A^p_ω when $\omega \in \widehat{\mathcal{D}}$, since for a fixed $r > 0$, the quantity $\omega(\triangle(a, r))$ may equal to zero for some a close to the boundary (see [15]). However, if $\omega \in \mathcal{D}$, we have the following characterization. The proof is similar to the proof of [14, Thm. 2.1]. For a proof, see [10, Thm.2].

Theorem 4 *Let μ be a positive Borel measure on $\mathbb{D}$, $0 < p < \infty$, $\omega \in \mathcal{D}$ and $0 < r < 1$. Then the following assertions hold:*

(i) *μ is a p-Carleson measure for A^p_ω if and only if*

$$\sup_{a\in\mathbb{D}} \frac{\mu(\triangle(a, r))}{\omega(S(a))} < \infty. \tag{2}$$

(ii) *μ is a vanishing p-Carleson measure for A^p_ω if and only if*

$$\lim_{|a|\to 1} \frac{\mu(\triangle(a, r))}{\omega(S(a))} = 0. \tag{3}$$

Remark In the above, $\omega(S(a))$ can be replaced by $\omega(\triangle(a, r)))$ for any fixed $r \in (0, 1)$ large enough.

The connection between composition operators and the Carleson measure comes from the following standard identity.

$$\int_{\mathbb{D}} (f \circ \varphi)(z)\omega(z)dA(z) = \int_{\mathbb{D}} f(z)d\nu(z),$$

where ν denotes the pullback measure defined by $\nu(E) = \int_{\varphi^{-1}(E)} \omega(z)dA(z)$, for all Borel sets $E \subset \mathbb{D}$. One can easily see from the above equality that $C_\varphi : A^p_\omega \to A^p_\omega$ is bounded (compact) on A^p_ω if and only if ν is a (vanishing p-Carleson measure) p-Carleson measure for A^p_ω.

The following result plays a fundamental role in this study. It is proved by employing the method used by Moorhouse [11].

Lemma 5 *Let φ be an analytic self-map of $\mathbb{D}$, $\omega \in \mathcal{D}$, and u be a non-negative, bounded, measurable function on $\mathbb{D}$. Define the measure $\nu(E) = \int_E u(z)\omega(z)dA(z)$ on each Borel subset E of $\mathbb{D}$. If $\lim_{|z|\to 1} u(z)(1 - |z|)/(1 - |\varphi(z)|) = 0$, then $\nu \circ \varphi^{-1}$ is a vanishing p-Carleson measure for A^p_ω and hence the inclusion map $I_{p,\omega} : A^p_\omega \to L^p(\nu \circ \varphi^{-1})$ is compact.*

Proof Fix $r \in (0, 1)$. For $a \in \mathbb{D}$, set

$$\epsilon := \epsilon(a) = \sup_{z\in\varphi^{-1}(\triangle(a,r))} u(z)\frac{1 - |z|}{1 - |\varphi(z)|}.$$

Using the Schwarz-Pick Lemma, we get

$$\frac{1-|z|}{1-|\varphi(z)|} \leq \frac{1+|\varphi(0)|}{1-|\varphi(0)|} = C < \infty.$$

If $\varphi(z) \in \triangle(a, r)$, then

$$1-|z| \leq C(1-|\varphi(z)|) \leq C\frac{(1-|a|)(1-r|a|)(1+r)}{1-r^2|a|^2}.$$

This implies that $|z| \to 1$ uniformly in $z \in \varphi^{-1}(\triangle(a, r))$ as $|a| \to 1$. Therefore, by the hypothesis $\epsilon(a) \to 0$ as $|a| \to 1$.

Now, fix $0 < \lambda < \min\{1, \alpha(\omega)\}$. Taking M to be an upper bound of u, we have

$$\begin{aligned}\nu \circ \varphi^{-1}(\triangle(a, r)) &= \int_{\varphi^{-1}(\triangle(a,r))} u(z)\omega(z)dA(z)\\ &\lesssim \int_{\varphi^{-1}(\triangle(a,r))} \frac{\epsilon^{\lambda}(1-|\varphi(z)|)^{\lambda}}{(1-|z|)^{\lambda}} u(z)^{1-\lambda}\omega(z)dA(z)\\ &\lesssim \epsilon^{\lambda}M^{1-\lambda}(1-|a|)^{\lambda}\int_{\varphi^{-1}(\triangle(a,r))} \frac{\omega(z)}{(1-|z|)^{\lambda}}dA(z).\end{aligned}$$

Denote $\omega_\lambda(z) = \omega(z)/(1-|z|)^{\lambda}$. By Lemma 2, we get $\omega_\lambda \in \mathcal{D}$. Therefore, $C_\varphi : A^p_{\omega_\lambda} \to A^p_{\omega_\lambda}$ is bounded, that is

$$\begin{aligned}(1-|a|)^{\lambda}\int_{\varphi^{-1}(\triangle(a,r))} \frac{\omega(z)}{(1-|z|)^{\lambda}}dA(z) &\leq (1-|a|)^{\lambda}\omega_\lambda(\triangle(a, r))\\ &\asymp \widehat{\omega_\lambda}(a)(1-|a|)^{1+\lambda}\\ &\asymp \widehat{\omega}(a)(1-|a|)\\ &\asymp \omega(\triangle(a, r)).\end{aligned}$$

Therefore

$$\frac{\nu \circ \varphi^{-1}(\triangle(a, r))}{\omega(\triangle(a, r))} \lesssim \epsilon(a)^{\lambda}$$

for all $a \in \mathbb{D}$. Hence $\nu \circ \varphi^{-1}$ is a vanishing p-Carleson measure for A^p_ω. The proof is complete. □

2.5 Angular Derivative

Let φ be an analytic self-map of $\mathbb{D}$. We say that φ has a finite angular derivative, denoted by $\varphi'(\zeta) \in \mathbb{C}$, at $\zeta \in \partial\mathbb{D}$ if there exists $\eta \in \partial\mathbb{D}$ such that $\angle\lim_{z\to\zeta}(\varphi(z)-\eta)/(z-\zeta) = \varphi'(\zeta)$, where $\angle\lim$ stands for the non-tangential limit. We denote by $F(\varphi)$ the set of all boundary points at which φ has finite angular derivatives. Note from the Julia-Carathéodory Theorem (see [4, Thm. 2.44]) that

$$F(\varphi) = \left\{\zeta \in \partial\mathbb{D} : d_\varphi(\zeta) := \liminf_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} < \infty\right\}.$$

For $\zeta \in F(\varphi)$, we call the vector $\mathcal{D}(\varphi,\zeta) := (\varphi(\zeta), d_\varphi(\zeta)) \in \partial\mathbb{D}\times\mathbb{R}^+$ the first-order data of φ at ζ.

If φ and ψ are two analytic self-maps of the disk with finite angular derivative at $\mathbb{D}$, we say that φ and ψ have the same first-order data at ζ if $\mathcal{D}(\varphi,\zeta) = \mathcal{D}(\psi,\zeta)$.

3 Linear Combination of Composition Operators

For a linear operator $T : X \to Y$, the essential norm of T, denoted by $\|T\|_{e,X\to Y}$, is defined by $\|T\|_{e,X\to Y} = \inf\{\|T-K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\}$. It is obvious that the operator T is compact if and only if $\|T\|_{e,X\to Y} = 0$.

We have the following lower estimate for the essential norm of a linear combination of composition operators acting on A^p_ω.

Lemma 6 *Let $0 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$. Let $\varphi_1,\ldots,\varphi_n$ be finitely many analytic self-maps of $\mathbb{D}$. Then there are constants $C > 0$ and $\gamma = \gamma(\omega) > 0$ such that*

$$\left\|\sum_{j=1}^n \lambda_j C_{\varphi_j}\right\|^p_{e,A^p_\omega} \geq C \limsup_{|a|\to 1} \left\|\left(\sum_{j=1}^n \lambda_j C_{\varphi_j}\right) f_a\right\|^p_{A^p_\omega},$$

where

$$f_a(z) = \left(\frac{1-|a|^2}{1-\overline{a}z}\right)^{(\gamma+1)/p} \omega(S(a))^{-1/p}.$$

Proof Let K be a compact operator on A^p_ω. Consider the operator on $H(\mathbb{D})$ defined by $K_m(f)(z) = f(mz/(m+1))$, $m \in \mathbb{N}$. Denote $R_m = I - K_m$. It is easy to see that K_m is compact on A^p_ω (see [16, Thm. 15]) and $\|K_m\|_{A^p_\omega} \leq 1$, $\|R_m\|_{A^p_\omega} \leq 2$ for any positive integer m. For simplicity of notation we set $T = \sum_{j=1}^n \lambda_j C_{\varphi_j}$. Then we have

$$2\|T-K\|_{A^p_\omega} \geq \left\|R_m \circ \Big(T-K\Big)\right\|_{A^p_\omega} \gtrsim \sup_{a\in\mathbb{D}} \left\|R_m \circ \Big(T-K\Big)(f_a)\right\|_{A^p_\omega}.$$

Since K is compact, we can extract a sequence $\{a_i\} \subset \mathbb{D}$ such that $|a_i| \to 1$ and $K f_{a_i}$ converges to some $f \in A^p_\omega$. So,

$$\begin{aligned}&\left\|R_m\circ(T-K)(f_{a_i})\right\|^p_{A^p_\omega}\\ &\quad\gtrsim\left\|R_m\circ T(f_{a_i})\right\|^p_{A^p_\omega}-\left\|R_m\circ K(f_{a_i})\right\|^p_{A^p_\omega}\\ &\quad\gtrsim\left\|T(f_{a_i})\right\|^p_{A^p_\omega}-\left\|K_m\circ T(f_{a_i})\right\|^p_{A^p_\omega}-\left\|R_m(K(f_{a_i})-f)\right\|^p_{A^p_\omega}-\left\|R_m(f)\right\|^p_{A^p_\omega}.\end{aligned}\tag{4}$$

Since K_m is compact and T is bounded on A^p_ω, we have $K_m \circ T$ is compact on A^p_ω. Therefore, letting $i \to \infty$ and then using Fatou's Lemma as $m \to \infty$ in (4), we have $\|T - K\|_{A^p_\omega} \gtrsim \limsup_{i\to\infty} \|T(f_{a_i})\|_{A^p_\omega}$. Therefore,

$$\|T\|^p_{e,A^p_\omega} \geq C \limsup_{|a|\to 1} \|T f_a\|^p_{A^p_\omega}.$$

The proof is complete. □

For $M > 1$ and $\zeta \in \partial\mathbb{D}$, we denote by $\Gamma_{M,\zeta}$ the ζ-curve consisting of points $|z-\zeta| = M(1-|z|^2)$, the boundary of a non-tangential approach region with vertex at ζ. We will use the notation "$\lim''_{\Gamma_{M,\zeta}}$ to indicate a limit taken as $z \to \zeta$ along the starboard leg of $\Gamma_{M,\zeta}$. The following result can be found in [8].

Lemma E *Let φ and ψ be analytic self-maps of $\mathbb{D}$. Then the following equality*

$$\lim_{M\to\infty} \lim_{\substack{z\to\zeta\\ z\in\Gamma_{M,\zeta}}} \frac{1-|\varphi(z)|^2}{1-\overline{\varphi(z)}\psi(z)} = \begin{cases} 1, & \text{if } \zeta \in F(\varphi) \text{ and } \mathcal{D}(\varphi,\zeta) = \mathcal{D}(\psi,\zeta) \\ 0, & \text{otherwise} \end{cases}\tag{5}$$

holds for $\zeta \in F(\varphi)$.

We are now ready to establish a lower estimate for the essential norm of a general linear combination of composition operators acting on A^p_ω when $\omega \in \widehat{\mathcal{D}}$. Let $\varphi_1, \ldots, \varphi_n$ be finitely many analytic self-maps of $\mathbb{D}$. For $\varphi \in F(\varphi_i)$, we denote by $J_\zeta(i)$ the set of all indices j for which $\zeta \in F(\varphi)$ and φ_i and φ have the same first-order data at ζ.

Theorem 8 *Let $0 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$. Let $\varphi_1, \ldots, \varphi_n$ be finitely many analytic self-maps of $\mathbb{D}$. Then there is a constant $C(p,\omega) > 0$ such that*

$$\left\|\sum_{j=1}^n \lambda_j C_{\varphi_j}\right\|^p_{e,A^p_\omega} \geq C \max_{1\leq i\leq n}\left(\left|\sum_{j\in J_\zeta(i)} \lambda_j\right|^p \frac{1}{d_{\varphi_i}(\zeta)^{\beta+1}}\right)\tag{6}$$

for all $\zeta \in \partial\mathbb{D}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. In case $\zeta \notin F(\varphi_i)$ the quantity inside the parentheses above is to be understood as 0.

Proof Set

$$f_a(z) = \left(\frac{1-|a|^2}{1-\bar{a}z}\right)^{(\gamma+1)/p} \omega(S(a))^{-1/p},$$

for $a \in \mathbb{D}$ and γ is that of Lemma A. Fix any index i such that $\zeta \in F(\varphi_i)$. We have $|\varphi_i(z)| \to 1$ as $z \to \zeta$ along any $\Gamma_{M,\zeta}$. So, by Lemma 6, we obtain

$$\left\| \sum_{j=1}^{n} \lambda_j C_{\varphi_j} \right\|_{e, A^p_\omega} \gtrsim \sup_M \left(\lim_{\substack{z \to \zeta \\ z \in \Gamma_{M,\zeta}}} \| \sum_{j=1}^{n} \lambda_j C_{\varphi_j} f_{\varphi_i(z)} \|^p_{A^p_\omega} \right).$$

Meanwhile, note that

$$\begin{aligned} \left\| \sum_{j=1}^{n} \lambda_j C_{\varphi_j} f_{\varphi_i(z)} \right\|^p_{A^p_\omega} &\geq \left| \sum_{j=1}^{n} \lambda_j C_{\varphi_j} f_{\varphi_i(z)}(z) \right|^p \omega(S(z)) \\ &= \left| \sum_{j=1}^{n} \lambda_j \left(\frac{1 - |\varphi_i(z)|^2}{1 - \overline{\varphi_i(z)} \varphi_j(z)} \right)^{(\gamma+1)/p} \right|^p \frac{\omega(S(z))}{\omega(S(\varphi_i(z)))}. \end{aligned}$$

Thus, applying Lemma E and the remark of Lemma 1, we get the desired result. □

From Theorem 8 we immediately derive the following three corollaries for the compactness of linear combinations.

Corollary 9 *Let $0 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$. Let $\varphi_1, \ldots, \varphi_n$ be finitely many analytic self-maps of $\mathbb{D}$. If $\sum_{j=1}^{n} \lambda_j C_{\varphi_j}$ is compact on A^p_ω, then*

$$\sum_{\substack{\zeta \in F(\varphi_j) \\ \mathcal{D}(\varphi_j, \zeta) = (\eta, s)}} \lambda_j = 0$$

for all $\zeta \in \partial\mathbb{D}$ and $(\zeta, s) \in \partial\mathbb{D} \times \mathbb{R}_+$.

Corollary 10 *Let $0 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$. Let φ, ψ be analytic self-maps of $\mathbb{D}$. Suppose both C_φ and C_ψ are not compact on A^p_ω. If $aC_\varphi + bC_\psi$ is compact on A^p_ω, then the following statements hold:*

(i) $a + b = 0$;
(ii) $F(\varphi) = F(\psi)$;
(iii) $\mathcal{D}(\varphi, \zeta) = \mathcal{D}(\psi, \zeta)$ *for each* $\zeta \in F(\varphi)$.

Corollary 11 *Let $0 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$. Let $\varphi, \varphi_1, \ldots, \varphi_n$ be finitely many analytic self-maps of $\mathbb{D}$. If $C_\varphi - C_{\varphi_1} - C_{\varphi_2} - \cdots - C_{\varphi_n}$ is compact on A^p_ω, then the following statements hold:*

(i) $F(\varphi_1), \ldots, F(\varphi_n)$ *are pairwise disjoint and* $F(\varphi) = \cup_{j=1}^{n} F(\varphi_j)$
(ii) $\mathcal{D}(\varphi, \zeta) = \mathcal{D}(\varphi_j, \zeta)$ *at each* $\zeta \in F(\varphi_j)$ *for* $j = 1, \ldots, n$.

4 Compact Difference and Further Related Results

We have the following characterization for compact linear combinations of two composition operators.

Theorem 12 *Let* $0 < p < \infty$ *and* $\omega \in \mathcal{D}$. *Let* φ *and* ψ *be analytic self-maps of* $\mathbb{D}$. *Then* $\lambda_1 C_\varphi + \lambda_2 C_\psi$ *is compact on* A^p_ω *if and only if either one of the following two conditions holds:*

(i) *Both* C_φ *and* C_ψ *are compact;*
(ii) $\lambda_1 + \lambda_2 = 0$ *and*

$$\lim_{|z|\to 1}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}+\frac{1-|z|^2}{1-|\psi(z)|^2}\right)\rho(\varphi(z),\psi(z))=0. \tag{7}$$

Remark Theorem 12 is a special case of of Theorem 1 in the recent paper [10]. Take $\nu=\omega$, $u(z)=\lambda_1$, $v(z)=-\lambda_2$, $p=q$ in [10, Thm. 1]. Since

$$\left(\frac{1-|z|}{1-|\varphi(z)|}\right)^{(\alpha+1)/p}\gtrsim\left(\frac{\widehat{\omega}(z)(1-|z|)}{\widehat{\omega}(\varphi(z))(1-|\varphi(z)|)}\right)^{1/p}\gtrsim\left(\frac{1-|z|}{1-|\varphi(z)|}\right)^{(\beta+1)/p}$$

and

$$\left(\frac{1-|z|}{1-|\psi(z)|}\right)^{(\alpha+1)/p}\gtrsim\left(\frac{\widehat{\omega}(z)(1-|z|)}{\widehat{\omega}(\psi(z))(1-|\psi(z)|)}\right)^{1/p}\gtrsim\left(\frac{1-|z|}{1-|\psi(z)|}\right)^{(\beta+1)/p},$$

we find that [10, (6)] is equivalent to (7). Here α and β are that of Lemma B and Lemma A, respectively. On the other hand, it is obvious that $\lambda_1+\lambda_2=0$ implies [10, (6)]. Since

$$1-\overline{\varphi(z)}\delta_1(z)=\frac{1-|\varphi(z)|^2}{1-\overline{\varphi(z)}\psi(z)}\quad\text{and}\quad 1-\overline{\psi(z)}\delta_2(z)=\frac{1-|\psi(z)|^2}{1-\overline{\psi(z)}\varphi(z)},$$

by combining with Lemma E and [16, Thm. 20], we see that [10, (6)] implies $\lambda_1+\lambda_2=0$ in Theorem 12. The proofs we provide below are definitely different, though they contain some similar elements.

Proof Suppose that $\lambda_1 C_\varphi+\lambda_2 C_\psi$ is compact on A^p_ω. Note that if (i) fails, then at least one of C_φ and C_ψ is not compact on A^p_ω. We may assume that both C_φ and C_ψ are not compact on A^p_ω and show (ii). By Corollary 10, we have $\lambda_1+\lambda_2=0$ and hence we may assume that $\lambda_1=1$ and $\lambda_2=-1$. Let's prove it by contradiction. We assume that (7) does not hold. Then there exists a sequence $\{z_n\}\subset\mathbb{D}$ with $|z_n|\to 1$ such that either

$$a_n:=\frac{1-|z_n|}{1-|\varphi(z_n)|}\rho(\varphi(z_n),\psi(z_n))$$

or

$$b_n := \frac{1-|z_n|}{1-|\psi(z_n)|}\rho(\varphi(z_n), \psi(z_n))$$

does not converge to zero. By passing to a subsequence, we may assume that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ exist and that one of them is non-zero. Without loss of generality we may further assume that $a \neq 0$. Again by passing to a subsequence, we may assume that $c = \lim_{n\to\infty} |\varphi(z_n)|$ exists. Since $a \neq 0$, we have $c = 1$. Thus, we may assume that $|z_n| \to 1$, $|\varphi(z_n)| \to 1$ and $a \neq 0$. For $u \in \mathbb{D}$, consider the test functions

$$\begin{aligned} g_u(z) &= \left(\frac{1-|u|^2}{1-\bar{u}z}\right)^{(\gamma+1)/p} \omega(S(u))^{-1/p} \quad \text{and} \\ h_u(z) &= \left(\frac{1-|u|^2}{1-t_N\bar{u}z}\right)^{(\gamma+1)/p} \omega(S(u))^{-1/p}, \end{aligned}$$

where t_N is that of Lemma D. It is easy to see that $\|g_u\|_{A^p_\omega} \asymp \|h_u\|_{A^p_\omega} \asymp 1$ and $g_u \to 0$, $h_u \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|u| \to 1$. Therefore,

$$\lim_{n\to\infty} \|(C_\varphi - C_\psi)g_{\varphi(z_n)}\|^p_{A^p_\omega} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(C_\varphi - C_\psi)h_{\varphi(z_n)}\|^p_{A^p_\omega} = 0.$$

Since $\omega(S(z))|f(z)|^p \lesssim \|f\|^p_{A^p_\omega}$ for all $f \in A^p_\omega$ (see [10,15]), we have

$$\begin{aligned} \lim_{n\to\infty} \omega(S(z_n)) \Big(&\left|g_{\varphi(z_n)}(\varphi(z_n)) - g_{\varphi(z_n)}(\psi(z_n))\right|^p \\ &+ \left|h_{\varphi(z_n)}(\varphi(z_n)) - h_{\varphi(z_n)}(\psi(z_n))\right|^p \Big) = 0. \end{aligned}$$

Then Lemma D yields

$$\lim_{n\to\infty} \frac{\omega(S(z_n))}{\omega(S(\varphi(z_n)))}\rho(\varphi(z_n), \psi(z_n))^p = 0.$$

Therefore, by Lemma 1, we obtain that

$$\lim_{n\to\infty} \left(\frac{1-|z_n|}{1-|\varphi(z_n)|}\right)^{\beta+1} \rho(\varphi(z_n), \psi(z_n))^p = 0.$$

Since the two sequences $\{(1-|z_n|)/(1-|\varphi(z_n)|)\}$ and $\{\rho(\varphi(z_n), \psi(z_n))\}$ are both bounded, we obtain

$$a = \lim_{n\to\infty} \left(\frac{1-|z_n|}{1-|\varphi(z_n)|}\right) \rho(\varphi(z_n), \psi(z_n)) = 0,$$

which is a desired contradiction.

Conversely, we only have to prove (10) implies that $C_\varphi - C_\psi$ is compact. Let $\{f_k\}$ be an arbitrary bounded sequence in A^p_ω such that $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. It suffices to show that $\|(C_\varphi - C_\psi)f_k\|_{A^p_\omega} \to 0$, as $k \to \infty$. In order to prove this, given $0 < r < 1$, we put

$$E := \{z \in \mathbb{D} : \rho(\varphi(z), \psi(z)) < r\} \quad \text{and} \quad F := \mathbb{D} \backslash E.$$

Then for each k,

$$\begin{aligned}\|(C_\varphi - C_\psi)f_k\|^p_{A^p_\omega} &= \int_{\mathbb{D}} |f_k(\varphi(z)) - f_k(\psi(z))|^p \omega(z)dA(z) \\ &= \int_E |f_k(\varphi(z)) - f_k(\psi(z))|^p \omega(z)dA(z) + \int_F |f_k(\varphi(z)) - f_k(\psi(z))|^p \omega(z)dA(z).\end{aligned} \tag{8}$$

We first estimate the second term in the right-hand side of the equality (8). Let χ_F denote the characteristic function of F. Since $r\chi_F \le \rho(\varphi, \psi)$, by (7), we get

$$\lim_{|z|\to 1} \chi_F(z) \left(\frac{1-|z|}{1-|\varphi(z)|} + \frac{1-|z|}{1-|\psi(z)|} \right) = 0.$$

This, together with Lemma 5, yields

$$\begin{aligned}&\int_F |f_k(\varphi(z)) - f_k(\psi(z))|^p \omega(z)dA(z) \\ &\lesssim \int_{\mathbb{D}} |f_k(\varphi(z))|^p \chi_F(z)\omega(z)dA(z) + \int_{\mathbb{D}} |f_k(\psi(z))|^p \chi_F(z)\omega(z)dA(z) \\ &:= \int_{\mathbb{D}} |f_k(z)|^p d\nu_1(z) + \int_{\mathbb{D}} |f_k(z)|^p d\nu_2(z) \to 0,\end{aligned}$$

as $k \to \infty$, where

$$\nu_1(K) = \int_{\varphi^{-1}(K)} \chi_F(z)\omega(z)dA(z) \quad \text{and} \quad \nu_2(K) = \int_{\psi^{-1}(K)} \chi_F(z)\omega(z)dA(z),$$

for all Borel sets $K \subset \mathbb{D}$.

Next, we estimate the first term on the right-hand side of the equality (8). Using Lemma 3, Fubini's Theorem, $\omega(S(a)) \asymp \omega(S(\zeta))$ for $\zeta \in \triangle(a, r_2)$, Theorem 4 and Lemma C, we have

$$\begin{aligned}&\int_E |f_k(\varphi(z)) - f_k(\psi(z))|^p \omega(z)dA(z) \\ &\lesssim \int_E \rho(\varphi(z), \psi(z))^p \frac{\int_{\triangle(\varphi(z), r_2)} |f_k(\zeta)|^p \widetilde{\omega}(\zeta)dA(\zeta)}{\omega(S(\varphi(z)))} \omega(z)dA(z)\end{aligned}$$

$$\lesssim r^p \int_{\mathbb{D}} |f_k(\zeta)|^p \frac{\int_{\varphi^{-1}(\triangle(\zeta,r_2))} \omega(z)dA(z)}{\omega(S(\zeta))} \widetilde{\omega}(\zeta)dA(\zeta)$$
$$\lesssim r^p \|f_k\|^p_{A^p_\omega} \|C_\varphi\| \lesssim r^p.$$

Letting $r \to 0$, we get $\|(C_\varphi - C_\psi)f_k\|_{A^p_\omega} \to 0$ as $k \to \infty$. The proof is complete. □

As a corollary, we obtain the following characterization for the operator $C_\varphi - C_\psi : A^p_\omega \to A^p_\omega$. The compactness of $C_\varphi - C_\psi$ on A^p_ω is independent of p and ω, whenever $\omega \in \mathcal{D}$.

Corollary 13 *Let $0 < p < \infty$ and $\omega \in \mathcal{D}$. Suppose φ and ψ are analytic self-maps of $\mathbb{D}$. Then the operator $C_\varphi - C_\psi : A^p_\omega \to A^p_\omega$ is compact if and only if*

$$\lim_{|z|\to 1} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\psi(z)|^2} \right) \rho(\varphi(z), \psi(z)) = 0.$$

In the rest of this section we assume that $\varphi_i : \mathbb{D} \to \mathbb{D}$ is analytic and $\varphi_i \neq \varphi_j$ if $i \neq j$. We define $F_i := \{\zeta \in \partial\mathbb{D} : \varphi_i$ has a finite angular derivative at $\zeta\}$ and

$$\rho_{ij}(z) := \left| \frac{\varphi_i(z) - \varphi_j(z)}{1 - \overline{\varphi_i(z)}\varphi_j(z)} \right|.$$

The proof of the following Theorem will be quite similar to the proof of Theorem 12, with a few added complications.

Theorem 14 *Let $0 < p < \infty$ and $\omega \in \mathcal{D}$. Let $\varphi, \varphi_1, \ldots, \varphi_n$ be finitely many analytic self-maps of $\mathbb{D}$. Suppose that $C_\varphi, C_{\varphi_1}, \ldots, C_{\varphi_n}$ are not compact on A^p_ω. Then the operator $C_\varphi - C_{\varphi_1} - \cdots - C_{\varphi_n} : A^p_\omega \to A^p_\omega$ is compact if and only if the following two conditions hold.*

(i) *$F = \cup_{j=1}^n F_j$ and $F_i \cap F_j = \emptyset$ if $i \neq j$ with $i, j \geq 1$;*
(ii)

$$\lim_{z\to\zeta} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\varphi_j(z)|^2} \right) \rho(\varphi(z), \varphi_j(z)) = 0$$

for all $\zeta \in F(\varphi_j)$ for $j = 1, 2, \ldots, n$.

Proof If $C_\varphi - \sum_{j=1}^n C_{\varphi_j}$ is compact on A^p_ω, then by Corollary 11, (i) holds. Now, assume that (ii) fails. We will derive a contradiction.

Since (ii) fails, there exist $\zeta \in F(\varphi_j)$ for some j and a sequence $\{z_k\} \subset \mathbb{D}$ such that $z_k \to \zeta$ and

$$\lim_{k\to\infty} \rho(\varphi(z_k), \varphi_j(z_k)) \left(\frac{1-|z_k|^2}{1-|\varphi(z_k)|^2} + \frac{1-|z_k|^2}{1-|\varphi_j(z_k)|^2} \right) > 0.$$

By passing to a subsequence, we may assume that

$$a_k := \rho(\varphi(z_k), \varphi_j(z_k)) \frac{1 - |z_k|^2}{1 - |\varphi(z_k)|^2}$$

or

$$b_k := \rho(\varphi(z_k), \varphi_j(z_k)) \frac{1 - |z_k|^2}{1 - |\varphi_j(z_k)|^2}$$

does not converge to zero.

Without loss of generality, we assume that a_k does not converge to zero. We take $g_k := g_{\varphi(z_k)}$ and $h_k := h_{\varphi(z_k)}$ for each k. Note that the two sequences $\{\rho(\varphi(z_k), \varphi_j(z_k))\}$ and $\{(1-|z_k|^2)/1-|\varphi(z_k)|^2)\}$ both are bounded. Thus, by passing to another subsequence if necessary, we may further assume that

$$\lim_{k\to\infty} \rho(\varphi(z_k), \varphi_j(z_k)) = c_1 \quad \text{and} \quad \lim_{k\to\infty} \frac{1 - |z_k|^2}{1 - |\varphi(z_k)|^2} = c_2,$$

for some constant $c_1, c_2 > 0$ with $c_1 \leq 1$.

Also, note that $\zeta \notin F(\varphi_i)$ for $i \neq j$. By the Julia–Caratheodory Theorem, we have

$$\begin{aligned}
&\lim_{k\to\infty} \frac{1 - |z_k|}{1 - |\varphi_i(z_k)|} = 0, \ i \neq j, \\
&\lim_{k\to\infty} \omega(S(z_k))|g_k(\varphi_i(z_k))|^p \\
&\quad = \lim_{k\to\infty} \frac{\omega(S(z_k))}{\omega(S(\varphi_i(z_k)))} \left| \frac{1 - |\varphi(z_k)|^2}{1 - \overline{\varphi(z_k)}\varphi_i(z_k)} \right|^{\gamma+1} \\
&\quad \lesssim \lim_{k\to\infty} \left(\frac{1 - |z_k|}{1 - |\varphi_i(z_k)|} \right)^{\alpha+\gamma+2} = 0, \\
&\lim_{k\to\infty} \omega(S(z_k))|h_k(\varphi_i(z_k))|^p \\
&\quad = \lim_{k\to\infty} \frac{\omega(S(z_k))}{\omega(S(\varphi_i(z_k)))} \left| \frac{1 - |\varphi(z_k)|^2}{1 - t_N\overline{\varphi(z_k)}\varphi_i(z_k)} \right|^{\gamma+1} \\
&\quad \lesssim \lim_{k\to\infty} \left(\frac{1 - |z_k|}{1 - |\varphi_i(z_k)|} \right)^{\alpha+1} \left(\frac{1 - |z_k|}{1 - t_N|\varphi_i(z_k)|} \right)^{\gamma+1} \\
&\quad \lesssim \lim_{k\to\infty} \left(\frac{1 - |z_k|}{1 - |\varphi_i(z_k)|} \right)^{\alpha+\gamma+2} = 0.
\end{aligned}$$

The same argument as in the proof of Theorem 12 yields

$$\lim_{k\to\infty}\omega(S(z_k))\left(\left|g_k(\varphi(z_k))-\left(\sum_{j=1}^{n}C_{\varphi_j}g_k\right)(z_k)\right|^p + \left|h_k(\varphi(z_k))-\left(\sum_{j=1}^{n}C_{\varphi_j}h_k\right)(z_k)\right|^p\right)=0.$$

Thus, similar to the proof of Theorem 12 we get

$$\lim_{k\to\infty}\left(\frac{1-|z_k|}{1-|\varphi(z_k)|}\right)\rho(\varphi(z_k),\varphi_j(z_k))=0,$$

which is a desired contradiction.

Next, assume that both (i) and (ii) hold. We will prove that $C_\varphi-\sum_{j=1}^{n}C_{\varphi_j}$ is compact. The proof will be quite similar to the proof of Theorem 12. Define

$$D_i:=\left\{z\in\mathbb{D}:\frac{1-|z|^2}{1-|\varphi_i(z)|^2}\geq\frac{1-|z|^2}{1-|\varphi_j(z)|^2},\ \text{for all } j\neq i\right\}$$

for $i=1,\ldots,N$. Fix $0<r<1$ and define

$$E_i:=\{z\in D_i:\rho(\varphi(z),\varphi_i(z))<r\}\quad\text{and}\quad E_i':=D_i\backslash E_i.$$

By the proof of [11, Thm. 5], we get

$$\lim_{|z|\to1}\chi_{E_i'}(z)\left(\frac{1-|z|}{1-|\varphi(z)|}+\frac{1-|z|}{1-|\varphi_j(z)|}\right)=0,\quad\text{for all } i,j, \tag{9}$$

and

$$\lim_{|z|\to1}\chi_{E_i}(z)\frac{1-|z|}{1-|\varphi_j(z)|}=0,\quad\text{whenever } i\neq j. \tag{10}$$

Now, let $\{f_k\}$ be a bounded sequence in A^p_ω such that $f_k\to0$ uniformly on compact subset of $\mathbb{D}$. Since $\mathbb{D}=\cup_{i=1}^{n}D_i$, we have

$$\left\|(C_\varphi-\sum_{j=1}^{n}C_{\varphi_j})f_k\right\|^p_{A^p_\omega}=\int_{\mathbb{D}}|f_k\circ\varphi-\sum_{i=1}^{n}f_k\circ\varphi_i|^p\omega dA\leq\sum_{i=1}^{n}\int_{E_i}+\sum_{i=1}^{n}\int_{E_i'}.$$

Note, as in the proof of Theorem 12, that the second sum of the above tends to 0 as $k \to \infty$, by equality (9) and Lemma 5. For the i-th term of the first sum, we have

$$\int_{E_i} \lesssim \int_{E_i} |f_k \circ \varphi - f_k \circ \varphi_i|^p \omega dA + \sum_{j \neq i} \int_{E_i} |f_k \circ \varphi_j|^p \omega dA.$$

Note from equality (10) and Lemma 5 that the second term of the above tends to 0 as $k \to \infty$. Finally, from the proof of Theorem 12 we see that the first term of the above is dominated by r^p. So, we conclude that $\limsup_{k\to\infty} \|(C_\varphi - \sum_{j=1}^n C_{\varphi_j}) f_k\|_{A^p_\omega}^p \lesssim r^p$. Letting $r \to 0$, we obtain

$$\limsup_{k\to\infty} \left\| (C_\varphi - \sum_{j=1}^n C_{\varphi_j}) f_k \right\|_{A^p_\omega}^p = 0.$$

The proof is complete. □

Theorem 14 and Corollary 9 immediately yield the following characterization for a composition operator to be equal module compact operators to a linear combination of composition operators.

Theorem 15 *Let $0 < p < \infty$ and $\omega \in \mathcal{D}$. Let $\varphi, \varphi_1, \ldots, \varphi_n$ be finitely many analytic self-maps of $\mathbb{D}$. Suppose that C_φ, $C_{\varphi_1}, \ldots, C_{\varphi_n}$ are not compact on A^p_ω. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}\backslash\{0\}$. Then the operator $C_\varphi - \sum_{j=1}^n \lambda_j C_{\varphi_j} : A^p_\omega \to A^p_\omega$ is compact if and only if the following three conditions holds:*

(i) $\lambda_1 = \cdots = \lambda_n = 1$;
(ii) $F = \cup_{j=1}^n F_j$ *and* $F_i \cap F_j = \emptyset$ *if* $i \neq j$ *with* $i, j \geq 1$;
(iii)

$$\lim_{z\to\zeta} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\varphi_j(z)|^2} \right) \rho(\varphi(z), \varphi_j(z)) = 0$$

for all $\zeta \in F_j$ *for* $j = 1, 2, \ldots, n$.

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