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Natural Boundary and Zero Distribution of Random Polynomials in Smooth Domains

Igor Pritsker[1] · **Koushik Ramachandran**[2]

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Abstract
We consider the zero distribution of random polynomials of the form $P_n(z) = \sum_{k=0}^{n} a_k B_k(z)$, where $\{a_k\}_{k=0}^{\infty}$ are non-trivial i.i.d. complex random variables with mean 0 and finite variance. Polynomials $\{B_k\}_{k=0}^{\infty}$ are selected from a standard basis such as Szegő, Bergman, or Faber polynomials associated with a Jordan domain G whose boundary is $C^{2,\alpha}$ smooth. We show that the zero counting measures of P_n converge almost surely to the equilibrium measure on the boundary of G. We also show that if $\{a_k\}_{k=0}^{\infty}$ are i.i.d. random variables, and the domain G has analytic boundary, then for a random series of the form $f(z) = \sum_{k=0}^{\infty} a_k B_k(z)$, ∂G is almost surely the natural boundary for $f(z)$.

Keywords Random polynomials · Orthogonal polynomials · Zero distribution · Natural boundary

Mathematics Subject Classification MSC 60F05 · 31A15 · 30B20 · 30B30

1 Introduction

This work is a sequel to [8] where we showed that zeros of a sequence of random polynomials $\{P_n\}_n$ (spanned by an appropriate basis) associated to a Jordan domain

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Dedicated to Prof. R. S. Varga on his 90th birthday.

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✉ Koushik Ramachandran
koushik@tifrbng.res.in

Igor Pritsker
igor@math.okstate.edu

1 Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA

2 Present Address: Centre for Applicable Mathematics, Tata Institute of Fundamental Research, Bengaluru 560065, India

G with analytic boundary L, *equidistributed* near L, i.e., distribute according to the equilibrium measure of L. We refer the reader to [8] for references to the literature on random polynomials. In this note, we extend the above result to Jordan domains with lesser regularity, namely domains with $C^{2,\alpha}$ boundary, see Theorem 1.1 below.

To state our results we need to set up some notation. Let $G \subset \mathbb{C}$ be a Jordan domain. We set $\Omega = \overline{\mathbb{C}} \setminus \overline{G}$, the exterior of $\overline{G}$ and Δ the exterior of the closed unit disc. By the Riemann mapping theorem there is a unique conformal mapping $\Phi : \Omega \to \Delta$, $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$. We denote the equilibrium measure of $E = \overline{G}$ by μ_E. For a polynomial P_n of degree n, with zeros at $\{Z_{k,n}\}_{k=1}^n$, let $\tau_n = \frac{1}{n} \sum_{k=1}^n \delta_{Z_{k,n}}$ denote its normalized zero counting measure. For a sequence of positive measures $\{\mu_n\}_{n=1}^\infty$, we write $\mu_n \xrightarrow{w} \mu$ to denote weak convergence of these measures to μ. A random variable X is called non-trivial if $\mathbb{P}(X = 0) < 1$.

Theorem 1.1 *Let G be a Jordan domain in $\mathbb{C}$ whose boundary L is $C^{2,\alpha}$ smooth for some $0 < \alpha < 1$. Consider a sequence of random polynomials $\{P_n\}_{n=0}^\infty$ defined by $P_n(z) = \sum_{k=0}^n a_k B_k(z)$, where the $\{a_i\}_{i=0}^\infty$ are non-trivial i.i.d. random variables with mean 0 and finite variance, with the basis $\{B_n\}_{n=0}^\infty$ being given either by Szegő, or by Bergman, or by Faber polynomials. Then, $\tau_n \xrightarrow{w} \mu_E$ a.s.*

We summarize some useful facts obtained in the proof of Theorem 1.1 below.

Corollary 1.2 *Suppose that E is the closure of a Jordan domain G with $C^{2,\alpha}$ boundary L, and that the basis $\{B_k\}_{k=0}^\infty$ is given either by Szegő, or by Bergman, or by Faber polynomials. If $\{a_k\}_{k=0}^\infty$ are non-trivial i.i.d. complex random variables with mean 0 and finite variance, then the random polynomials $P_n(z) = \sum_{k=0}^n a_k B_k(z)$ converge almost surely to a random analytic function f that is not identically zero. Moreover,*

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = |\Phi(z)|, \quad z \in \Omega, \tag{1.1}$$

holds with probability one.

As a consequence of Theorem 1.1, we show that the zeros of the sequence of derivatives $\{P_n'\}_{n=0}^\infty$ also equidistribute.

Corollary 1.3 *Let G, $\{a_i\}_{i=0}^\infty$ and P_n be as in Theorem 1.1. Let τ_n' denote the zero counting measures of P_n'. Then, $\tau_n' \xrightarrow{w} \mu_E$ a.s.*

The natural boundary for a random power series of the form $\sum_{k=0}^\infty a_k z^k$ where $\{a_k\}_{k=0}^\infty$ are i.i.d. random variables has been investigated by quite a few authors. We refer especially to [2], but see also [6] and the references therein. The result there is that for such a random series, the circle of convergence is a.s. the natural boundary. Some extensions are possible when the $\{a_k\}_{k=0}^\infty$ are merely independent. Therefore, it seems reasonable to ask if such a result holds when the random series is formed by other polynomial bases. In [8], we remarked (without proof) that the random series formed by the basis $\{B_k\}_{k=0}^\infty$, has natural boundary L. We prove that result here.

Theorem 1.4 *Suppose that E is the closure of a Jordan domain G with analytic boundary L, and that the basis $\{B_k\}_{k=0}^{\infty}$ is given either by Szegő, or by Bergman, or by Faber polynomials. Assume that the random coefficients $\{a_k\}_{k=0}^{\infty}$ are i.i.d. complex random variables which are non-constant almost surely, and furthermore satisfy $\mathbb{E}[\log^+|a_0|] < \infty$. Then the series*

$$\sum_{k=0}^{\infty} a_k B_k(z)$$

converges a.s. to a random analytic function $f \not\equiv 0$ in G, and moreover, with probability one, $\partial G = L$ is the natural boundary for f.

We believe that Theorem 1.4 holds for any bounded Jordan domain. It would be interesting to see a proof of such a result, which we expect will need rather different techniques from the ones we use.

2 Proofs

Proof of Theorem 1.1 and of Corollary 1.2 We closely follow the ideas in [8]. The proof consists of two probabilistic lemmas followed by the use of a deterministic theorem in potential theory. The first lemma below follows from a standard application of the Borel–Cantelli lemma. □

Lemma 2.1 *If $\{a_k\}_{k=0}^{\infty}$ are non-trivial, independent and identically distributed complex random variables that satisfy $\mathbb{E}[\log^+|a_0|] < \infty$, then*

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1 \quad a.s., \tag{2.1}$$

and

$$\limsup_{n\to\infty} \left(\max_{0\le k\le n} |a_k|\right)^{1/n} = 1 \quad a.s. \tag{2.2}$$

A slightly more delicate application of Borel–Cantelli gives the following result. For the proof, we refer to [8].

Lemma 2.2 *If $\{a_k\}_{k=0}^{\infty}$ are non-trivial i.i.d. complex random variables, then there is a value $b > 0$ such that*

$$\liminf_{n\to\infty} \left(\max_{n-b\log n<k\le n} |a_k|\right)^{1/n} \ge 1 \quad a.s. \tag{2.3}$$

We use the following theorem of Grothmann [4] which describes the zero distribution of deterministic polynomials.

Let $E \subset \mathbb{C}$ be a compact set of positive capacity such that $\Omega = \overline{\mathbb{C}} \setminus E$ is connected and regular. The Green function of Ω with pole at ∞ is denoted by $g_\Omega(z, \infty)$. We use $\| \cdot \|_K$ for the supremum norm on a compact set K.

Theorem G. *If a sequence of polynomials* $P_n(z)$, $\deg(P_n) \leq n \in \mathbb{N}$, *satisfies*

$$\limsup_{n\to\infty} \|P_n\|_E^{1/n} \leq 1, \tag{2.4}$$

for any closed set $K \subset E^\circ$

$$\lim_{n\to\infty} \tau_n(K) = 0, \tag{2.5}$$

and there is a compact set $S \subset \Omega$ *such that*

$$\liminf_{n\to\infty} \max_{z\in S} \left(\frac{1}{n} \log |P_n(z)| - g_\Omega(z, \infty) \right) \geq 0, \tag{2.6}$$

then the zero counting measures τ_n *of* P_n *converge weakly to* μ_E *as* $n \to \infty$.

The idea now is to check that with probability 1, our sequence of polynomials satisfies the hypothesis in Grothmann's theorem. Recall that in our setting $g_\Omega(z, \infty) = \log |\Phi(z)|$, $z \in \Omega$.

Note that (2.4) is satisfied for E almost surely by (2.2), and the estimate

$$\|P_n\|_E \leq \sum_{k=0}^{n} |a_k| \|B_k\|_E \leq (n+1) \max_{0\leq k\leq n} |a_k| \max_{0\leq k\leq n} \|B_k\|_E,$$

as

$$\limsup_{n\to\infty} \left(\max_{0\leq k\leq n} \|B_k\|_E \right)^{1/n} \leq 1.$$

This last fact follows from the well known result that in all three cases of polynomial bases we consider in this theorem, we have

$$\lim_{n\to\infty} |B_n(z)|^{1/n} = |\Phi(z)| \tag{2.7}$$

holds uniformly on compact subsets of Ω, see [9]. To check that (2.5) holds, we use the following lemma from [5].

Lemma 2.3 *Let* ψ_n *be holomorphic functions on a domain* Λ. *Assume that* $\sum_{n=0}^{\infty} |\psi_n|^2$ *converges uniformly on compact sets of* Λ. *Let* a_n *be i.i.d. random variables with zero mean and finite variance. Then, almost surely,* $\sum_{n=0}^{\infty} a_n \psi_n(z)$ *converges uniformly on compact subsets of* Λ *and hence defines a random analytic function.*

It is well known that $\sum_{n=0}^{\infty}|B_n(z)|^2 = K(z,z)$, where $K(z,w)$ denotes the Bergman (or correspondingly Szegő) kernel of the domain G when $\{B_i\}_{i=0}^{\infty}$ denotes the Bergman or Szegő basis respectively. For the case of the Faber polynomials, the convergence follows from the estimates of the sup norm $||P_n||_K$ on any compact $K \subset G$, see [11, Ch. 1, Sec. 5]. With this knowledge, taking $\psi_n = B_n$ and $\Lambda = G$ in Lemma 2.3, we obtain that almost surely, $\sum_{n=0}^{\infty} a_n B_n(z)$ converges uniformly on compact subsets of G and hence defines a random analytic function f. The uniqueness of series expansions of these polynomial basis ensures that f is not identically 0. Since $P_n \to f$, an application of Hurwitz's theorem from basic complex analysis now proves (2.5). Incidentally this also proves the corresponding part of Corollary 1.2.

If τ_n do not converge to μ_E a.s., then (2.6) cannot hold a.s. for any compact set S in Ω. We choose $S = L_R = \{z : g_\Omega(z,\infty) = \log R\}$, with $R > 1$, and find a subsequence $n_m,\ m \in \mathbb{N}$, such that

$$\limsup_{m\to\infty} \| P_{n_m} \|_{L_R}^{1/n_m} < R, \tag{2.8}$$

holds with positive probability. It follows from a result of Suetin [11, Ch. 1], that for Bergman polynomials,

$$B_n(z) = \sqrt{\frac{n+1}{\pi}}\,\Phi^n(z)\Phi'(z)\,(1 + A_n(z))\,, \tag{2.9}$$

holds locally uniformly in Ω where we recall that Φ is the exterior conformal map, $\Phi : \Omega \to \Delta,\ \Phi(\infty) = \infty,\ \Phi'(\infty) > 0$, and

$$|A_n(z)| \le c\frac{\log(n)}{n^2}. \tag{2.10}$$

Similar asymptotic formulas as (2.9) are valid for Szegő and Faber polynomials but without the factor $\sqrt{n+1}$. The proofs for these bases have to be accordingly modified. Equation (2.9) implies that all zeros of B_n are contained inside L_R for all large n. This allows us to write an integral representation

$$a_n = \frac{1}{2\pi i}\int_{L_R} \frac{P_n(z)\,dz}{zB_n(z)}, \tag{2.11}$$

which is valid for all large $n \in \mathbb{N}$ because $P_n(z)/(zB_n(z)) = a_n/z + O(1/z^2)$ for $z \to \infty$. The asymptotic on B_n from (2.9) implies that there are positive constants c_1 and c_2 that do not depend on n and z, such that

$$c_2\sqrt{n}\,\rho^n \le |B_n(z)| \le c_1\sqrt{n}\,\rho^n, \quad z \in L_\rho,\ \rho > 1,\ n \in \mathbb{N}. \tag{2.12}$$

We estimate from (2.11) and (2.12) with $\rho = R$ that

$$|a_n| \le \frac{|L_R|}{2\pi d}\frac{\|P_n\|_{L_R}}{c_2\sqrt{n}\,R^n},$$

where $|L_R|$ is the length of L_R and $d := \min_{z\in L_R} |z|$. It follows that

$$\begin{aligned}\|P_{n-1}\|_{L_R} &\le \|P_n\|_{L_R} + |a_n|\|B_n\|_{L_R}\\ &\le \|P_n\|_{L_R}\left(1+\frac{|L_R|}{2\pi d}\frac{c_1}{c_2}\right) =: C\,\|P_n\|_{L_R}, \quad n\in\mathbb{N}.\end{aligned}$$

Applying this estimate repeatedly, we obtain that

$$\|P_{n-k}\|_{L_R} \le C^k\,\|P_n\|_{L_R}, \quad k \le n,$$

so that (2.11) yields

$$|a_{n-k}| \le \frac{|L_R|}{2\pi d}\frac{\|P_{n-k}\|_{L_R}}{c_2\sqrt{n-k}\,R^{n-k}} \le \frac{|L_R|}{2\pi d}\frac{C^k\,\|P_n\|_{L_R}}{c_2\sqrt{n-k}\,R^{n-k}}.$$

Choosing sufficiently small $\varepsilon > 0$ and using (2.8), we deduce from the previous inequality that

$$|a_{n_m-k}| \le q^{n_m}, \quad 0 \le k \le \varepsilon n_m,$$

for some $q \in (0, 1)$ and all sufficiently large n_m, with positive probability. The latter estimate clearly contradicts (2.3) of Lemma 2.2. Hence (2.6) holds for $S = L_R$, with any $R > 1$, and τ_n converge weakly to μ_E with probability one. Note that (2.6) for $S = L_R$, with $R > 1$, is equivalent to (1.1). Indeed, we have equality in (2.6), with lim instead of lim inf, by Bernstein–Walsh inequality and (2.4), see [1, p. 51, Remark 1.2] for more details. This concludes the proof of Theorem 1.1 as well as the proof of Corollary 1.2.

Proof of Corollary 1.3 The method of proof is similar to that of Theorem 1.1, namely check that the conditions in Grothmann's result hold almost surely. First, we use a Markov–Bernstein result (cf. [7] and the references therein) to bound the sup norm of P_n' on E.

$$||P_n'||_E \le c(E)n^2||P_n||_E. \tag{2.13}$$

Therefore, with probability one,

$$\limsup_{n\to\infty} \|P_n'\|_E^{1/(n-1)} \le \limsup_{n\to\infty}\Big(c(E)n^2\|P_n\|_E\Big)^{1/(n-1)} \le 1.$$

This shows that (2.4) holds for P_n'. Next, we know from the proof of Theorem 1.1 that with probability one, $P_n \to f$ uniformly on compacts, where f is a non-zero random analytic function. From this we obtain that $P_n' \to f'$ also uniformly on compacts. The function f' is not identically 0, for if it were, $f \equiv c$ for some constant c, and by the uniqueness of series expansion for the polynomial basis under consideration, this would imply that $a_i = 0$ for $i \ge 1$. This contradicts Lemma 2.2. From here, an application of Hurwitz's theorem now yields that $\tau_n'(K) \to 0$ for every compact set $K \subset G$. This proves Equation (2.5) for P_n'. Finally, recall that

$$B_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\,(1 + A_n(z))$$

where $A_n(z)$ satisfies the estimate (2.10). Differentiating this, we obtain bounds for B_n' on L_R. Namely

$$c_4 n^{\frac{3}{2}} R^{n-1} \leq |B_n'(z)| \leq c_5 n^{\frac{3}{2}} R^{n-1}. \tag{2.14}$$

To obtain this asymptotic, we have used a local Cauchy integral to estimate $A_n'(z)$:

$$A_n'(z) = \frac{1}{2\pi i}\int_{\partial B_\delta(z)} \frac{A_n(w)}{(z-w)^2} dw$$

for $z \in L_R$ with $\delta > 0$ being chosen so that the ball $B_\delta(z)$ stays away from the boundary, say $\delta = \frac{1}{5}d(L_R, L)$. Using the uniform bound (2.10) in the above integral shows that an analogous estimate holds for $A_n'(z)$. Once we obtain (2.14), we note that the proof for (2.6) for P_n' follows as in Theorem 1.1. All the conditions in Grothmann's theorem are satisfied and hence we have the required convergence. □

Remark Although Theorem 1.1 and Corollary 1.3 have been stated for Jordan domains with $C^{2,\alpha}$ boundary, it is easy to see that the same proof goes through if for instance G is a Jordan domain whose boundary is piecewise analytic (with angles at the corners satisfying certain conditions, see [10, Theorem 1.2]). The asymptotic Eq. (2.9) will then have to be replaced by an analogous one for piecewise analytic boundary.

Proof of Theorem 1.4 We have that E is the closure of a Jordan domain G bounded by an analytic curve L with exterior Ω. It is well known that the conformal mapping $\Phi : \Omega \to \Delta,\ \Phi(\infty) = \infty,\ \Phi'(\infty) > 0$, extends through L into G, so that Φ maps a domain Ω_r containing $\overline{\Omega}$ conformally onto $\{|z| > r\}$ for some $r \in (0, 1)$. In particular, the level curves of Φ denoted by L_ρ are contained in G for all $\rho \in (r, 1)$, $L_1 = L$ and $L_\rho \subset \Omega$ for $\rho > 1$.

For the proof that the series $\sum_{k=0}^{\infty} a_k B_k(z)$ converges a.s. to an analytic function f, we refer the reader to [8, Corollary 2.2].

We now show the result about L being the natural boundary of f. We will give the proof for the basis of Faber and Bergman polynomials. The proof for the Szegő polynomials is similar to the Bergman case but simpler.

Let

$$\Phi(z) = \frac{z}{\text{cap}(E)} + \sum_{k=1}^{\infty} \frac{c_k}{z^k},$$

for z in a neighborhood of infinity. Let F_n be the nth Faber polynomial. By definition, F_n is the polynomial part of the Laurent expansion of Φ^n at infinity,

$$\Phi^n(z) = F_n(z) + E_n(z), \quad z \in \Omega_r, \tag{2.15}$$

where E_n is analytic, consisting of all the negative powers of z in the expansion of Φ^n. Fix $\epsilon > 0$ such that $r + \epsilon < 1$. It follows that

$$E_n(z) = \frac{1}{2\pi i}\int_{\Gamma_{r+\epsilon}} \frac{\Phi^n(t)}{t-z}dt, \;\; z \in \Omega_\rho,$$

for $r + \epsilon < \rho$. From the above integral representation it is clear that

$$|E_n(z)| \leq \frac{|\Gamma_{r+\epsilon}|(r+\epsilon)^n}{2\pi d(\Gamma_{r+\epsilon}, \Gamma_\rho)} \tag{2.16}$$

for $z \in \Omega_\rho$. Here $d(\Gamma_{r+\epsilon}, \Gamma_\rho)$ denotes the distance between $\Gamma_{r+\epsilon}$ and Γ_ρ. Using (2.16) and the fact that $\limsup |a_n|^{\frac{1}{n}} = 1$ a.s. (see Eq. (2.1)), we deduce that the series $\sum_{k=0}^{\infty} a_k E_k(z)$ converges a.s. in Ω_ρ and defines a random analytic function there. From Eq. (2.15) we know that for $z \in G \cap \Omega_\rho$, $r + \epsilon < \rho < 1$,

$$\sum_{k=0}^{\infty} a_k F_k(z) = \sum_{k=0}^{\infty} a_k \Phi^k(z) - \sum_{k=1}^{\infty} a_k E_k(z).$$

Now suppose that the series $f = \sum_{k=0}^{\infty} a_k F_k(z)$ has an analytic continuation across $L = L_1$. Then, together with the fact that the second series on the right defines an analytic function in Ω_ρ, this implies that $\sum_{k=0}^{\infty} a_k w^k$ has an analytic continuation across $|w| = 1$, where $w = \Phi(z)$. But this contradicts [2, Satz 8].

If $\{B_k\}_{k=0}^{\infty}$ denotes the Bergman basis, then Carleman's asymptotic formula (see [3, Ch. 1]), yields

$$B_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\,(1 + e_n(z)) \tag{2.17}$$

where

$$e_n(z) = \begin{cases} O(\sqrt{n})r^n, & z \in L_\rho, \;\; \rho > 1 \\ O\left(\frac{1}{\sqrt{n}}\right)\left(\frac{r}{\rho}\right)^n & z \in L_\rho, \;\; r < \rho < 1. \end{cases} \tag{2.18}$$

Using $\limsup |a_n|^{\frac{1}{n}} = 1$ a.s. and estimates (2.18), we observe that the series

$$\sum_{n=0}^{\infty} a_n \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)e_n(z)$$

converges a.s. in a neighborhood of the boundary L, and defines an analytic function there. Now from (2.17), we have

$$\sum_{n=0}^{\infty} a_n B_n(z) = \sum_{n=0}^{\infty} a_n \sqrt{\frac{n+1}{\pi}} \Phi^n(z)\Phi'(z) + \sum_{n=0}^{\infty} a_n \sqrt{\frac{n+1}{\pi}} \Phi^n(z)\Phi'(z) e_n(z)$$

for $z \in G \cap \Omega_\rho$, $r < \rho < 1$. If the series $\sum_{n=0}^{\infty} a_n B_n(z)$ has an analytic continuation across L, then combined with the fact that the second series on the right defines an analytic function near L, we would obtain that $\sum_{n=0}^{\infty} a_n \sqrt{\frac{n+1}{\pi}} \Phi^n(z)\Phi'(z)$ and hence $\sum_{n=0}^{\infty} a_n \sqrt{\frac{n+1}{\pi}} \Phi^n(z)$ has an analytic continuation across L. In other words, taking $w = \Phi(z)$ the series $\sum_{n=0}^{\infty} a_n \sqrt{\frac{n+1}{\pi}} w^n$ has an analytic continuation across $|w| = 1$. This contradicts known results on analytic continuation of power series stating that the unit circle must be the natural boundary for the latter power series with probability one, see [2, Satz 12]. Since the latter reference is not readily available, we give a statement and a brief proof of the claimed fact in the concluding lemma below. □

Lemma 2.4 *Let $\{c_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that*

$$\lim_{n\to\infty} |c_n|^{1/n} = 1. \tag{2.19}$$

If $\{a_n\}_{n=0}^{\infty}$ are i.i.d. complex random variables that are not a.s. constant, and that satisfy $\mathbb{E}[\log^+ |a_0|] < \infty$, then the random power series

$$\sum_{n=0}^{\infty} a_n c_n w^n \tag{2.20}$$

has the unit circle as its natural boundary with probability one.

Proof We follow, in part, the argument of Kahane [6, p. 41]. Let Ω be the common probability space in which all the random variables a_n are defined. Consider the "symmetrized" random power series

$$\begin{aligned} F(w) &:= \sum_{n=0}^{\infty} (a_n(\omega_1) - a_n(\omega_2))\, c_n w^n \\ &= \sum_{n=0}^{\infty} a_n(\omega_1) c_n w^n - \sum_{n=0}^{\infty} a_n(\omega_2) c_n w^n =: f_1(w) - f_2(w), \end{aligned}$$

where $(\omega_1, \omega_2) \in \Omega \times \Omega$. Note that the random variables $(a_n(\omega_1) - a_n(\omega_2))\, c_n$, $n = 0, 1, 2, \ldots$, are symmetric (a random variable X is symmetric if $-X$ has the same distribution). Moreover, $(a_n(\omega_1) - a_n(\omega_2))$ random variables are non-trivial i.i.d., because a_n are i.i.d., and are not almost surely constant. Since

$$\mathbb{E}[\log^+ |a_n(\omega_1) - a_n(\omega_2)|] \le 2\mathbb{E}[\log^+ |a_0|] < \infty,$$

we obtain from Lemma 2.1 that

$$\limsup_{n\to\infty} |a_n(\omega_1) - a_n(\omega_2)|^{1/n} = 1 \quad a.s.$$

Combining this fact with (2.19), we conclude that the random power series F has radius of convergence equal to 1 a.s. It is also clear that the same conclusion holds by Lemma 2.1 for the series f_1 and f_2 that represent the original series (2.20) for different events from Ω. Suppose to the contrary that the series (2.20) can be continued beyond the unit circle with positive probability, which then holds almost surely by a zero-one law as explained in [6, p. 39]. This means that there is an arc J of the unit circle such that both series f_1 and f_2 can be continued analytically beyond J with probability one. Hence F can be continued analytically beyond J with probability one, which is in direct contradiction with [6, p. 40, Theorem 1]. □

Remark Since the submission of this article, it has come to the authors' notice that Theorem 1.1 has recently been shown in full generality in the preprint https://arxiv.org/pdf/1901.07614.pdf.

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