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Entire Solutions of Certain Type of Non-Linear Difference Equations

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Abstract
In this paper, we study the existence of entire solutions of finite-order of non-linear difference equations of the form

$$f^n(z) + q(z)\Delta_c f(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \quad n \geq 2$$

and

$$f^n(z) + q(z)e^{Q(z)} f(z+c) = p_1 e^{\lambda z} + p_2 e^{-\lambda z}, \quad n \geq 3$$

where q, Q are non-zero polynomials, $c, \lambda, p_i, \alpha_i (i = 1, 2)$ are non-zero constants such that $\alpha_1 \neq \alpha_2$ and $\Delta_c f(z) = f(z+c) - f(z) \not\equiv 0$. Our results are improvements and complements of Wen et al. (Acta Math Sin 28:1295–1306, 2012), Yang and Laine (Proc Jpn Acad Ser A Math Sci 86:10–14, 2010) and Zinelâabidine (Mediterr J Math 14:1–16, 2017).

Keywords Entire solutions · Non-linear difference equations · Exponential polynomial · Nevanlinna theory

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1 Introduction and Main Results

In this paper, we assume that the reader is familiar with the fundamental results and standard notation of Nevanlinna theory [5,7,14]. In addition, we use $\rho(f)$ to denote the order of growth of f and $\lambda(f)$ to denote the exponent of convergence of zeros sequence of f. For simplicity, we denote by $S(r, f)$ any quantify satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$, outside of a possible exceptional set of finite logarithmic measure, we use $S(f)$ to denote the family of all small functions with respect to f.

Recently, many scholars have investigated solvability and existence of solutions of non-linear differential equations or difference equations, see [3,4,8–12,16].

Exponential polynomials are important in complex analysis as they have many interesting properties as mentioned, for example, in the paper [13] due to Wen, Heittokangas and Laine. In this paper, we mainly give exact expressions of exponential polynomial solutions of certain class of non-linear difference equations.

In [16], Yang and Laine proved the following result:

Theorem A *A non-linear difference equation*

$$f^3(z) + q(z)f(z+1) = c\sin bz, \tag{1.1}$$

where q is a non-constant polynomial and b, c are non-zero complex constants, Eq. (1.1) does not admit entire solutions of finite order. If q is a non-zero constant, then Eq. (1.1) possesses three district entire solutions of finite order, provided $b = 3\pi n$ and $q^3 = (-1)^{n+1}\frac{27}{4}c^2$ for a non-zero integer n.

Given Theorem A, it is natural to ask about the solutions of the following more general form

$$f^n(z) + q(z)\Delta_c f(z) = p_1 e^{\lambda z} + p_2 e^{-\lambda z}, \tag{1.2}$$

where q is a non-zero polynomial, $c, \lambda, p_i (i = 1, 2)$ are non-zero constants such that $\Delta_c f(z) = f(z+c) - f(z) \not\equiv 0$ and $n \geq 2$ is an integer.

In this paper, we study this problem and obtain the following result.

Theorem 1.1 *Let $n \geq 2$ be an integer, q be a non-zero polynomial, c, λ, p_1, p_2 be non-zero constants. If there exists some entire solution f of finite order to Eq. (1.2), such that $\Delta_c f(z) = f(z+c) - f(z) \not\equiv 0$, then q is a constant, and $n = 2$ or $n = 3$. When $n = 2$, then*

$$f(z) = q + c_1 e^{\frac{\lambda}{2}z} + c_2 e^{-\frac{\lambda}{2}z},$$

where $q^4 = 4p_1p_2$, $c_1^2 = p_1$, $c_2^2 = p_2$, $\lambda c = 2k\pi i$, $k \in \mathbb{Z}$ and k is an odd. When $n = 3$, then

$$f(z) = c_1 e^{\frac{\lambda}{3}z} + c_2 e^{-\frac{\lambda}{3}z},$$

where $q^3 = \frac{27}{8} p_1 p_2$, $c_1^3 = p_1$, $c_2^3 = p_2$, $\lambda c = 3k\pi i$, $k \in \mathbb{Z}$ *and* k *is an odd.*

More recently, Zinelâabidine showed in [17]:

Theorem B *Let* q *be a polynomial,* $p_i, \alpha_i (i = 1, 2)$ *be non-zero constants such that* $\alpha_1 \pm \alpha_2 \neq 0$. *If* f *is an entire solution of finite order of equation*

$$f^3(z) + q(z)\Delta f(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{1.3}$$

such that $\Delta f(z) = f(z+1) - f(z) \not\equiv 0$, *then* q *is a constant, and one of the following relations holds:*

1. $f(z) = c_1 e^{\frac{\alpha_1}{3} z}$ *and* $c_1(e^{\frac{\alpha_1}{3}} - 1)q = p_2$, $\alpha_1 = 3\alpha_2$,
2. $f(z) = c_2 e^{\frac{\alpha_2}{3} z}$ *and* $c_2(e^{\frac{\alpha_2}{3}} - 1)q = p_1$, $\alpha_2 = 3\alpha_1$, *where* c_1, c_2 *are non-zero constants satisfying* $c_1^3 = p_1$, $c_2^3 = p_2$.

The aim of this paper is to study the difference equation

$$f^2(z) + q(z)\Delta_c f(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{1.4}$$

where q is a non-zero polynomial, c, $p_i, \alpha_i (i = 1, 2)$ are non-zero constants such that $\alpha_1 \pm \alpha_2 \neq 0$ and $\Delta_c f(z) = f(z+c) - f(z) \not\equiv 0$. In fact, we prove the following result.

Theorem 1.2 *Let* q *be a non-zero polynomial,* c, $p_i, \alpha_i (i = 1, 2)$ *be non-zero constants such that* $\alpha_1 \pm \alpha_2 \neq 0$. *If* f *is an entire solution of finite order of Eq. (1.4), such that* $\Delta_c f(z) = f(z+c) - f(z) \not\equiv 0$, *then* q *is a constant,* $\rho(f) = 1$ *and one of the following conclusions holds:*

1. $f(z) = c_1 e^{\frac{\alpha_1}{2} z}$, *and* $c_1(e^{\frac{\alpha_1}{2} c} - 1)q = p_2$, $\alpha_1 = 2\alpha_2$;
2. $f(z) = c_2 e^{\frac{\alpha_2}{2} z}$, *and* $c_2(e^{\frac{\alpha_2}{2} c} - 1)q = p_1$, $\alpha_2 = 2\alpha_1$, *where* c_1, c_2 *are non-zero constants satisfying* $c_1^2 = p_1$, $c_2^2 = p_2$;
3. $$T(r, \varphi) + S(r, f) = \kappa T(r, f), \ 0 < \kappa \leq 1, \ \textit{and}$$
   $$N\left(r, \frac{1}{f}\right) + S(r, f) = \iota T(r, f), \ 1 - \frac{\kappa}{2} \leq \iota \leq 1,$$
   where $\varphi = \alpha_1\alpha_2 f^2 - 2(\alpha_1 + \alpha_2) f f' + 2(f')^2 + 2 f f''$.

Wen, Heittokangas and Laine [13] studied and classified the finite order entire solutions f of equation

$$f^n(z) + q(z)e^{Q(z)} f(z+c) = P(z), \tag{1.5}$$

where q, Q, P are polynomials, $n \geq 2$ is an integer and $c \in \mathbb{C}\backslash\{0\}$, and obtained the following Theorem C.

Recall that a function f of the form

$$f(z)=P_1(z)e^{Q_1(z)}+\cdots+P_k(z)e^{Q_k(z)}, \tag{1.6}$$

where $P_j's$ and $Q_j's$ are polynomials in z is called an exponential polynomial. Furthermore, let

$$\Gamma_1=\{e^{\alpha(z)}+d : d\in\mathbb{C} \text{ and } \alpha \text{ is a non-constant polynomial}\},$$
$$\Gamma_0=\{e^{\alpha(z)} : \alpha \text{ is a non-constant polynomial}\}.$$

Theorem C (See [13]) *Let $n\geq 2$ be an integer, let $c\in\mathbb{C}\backslash\{0\}$, and let q, Q, P be polynomials such that Q is not a constant and $q\not\equiv 0$. Then, we identify the finite order entire solutions f of equation (1.5) as follows:*

(a) *Every solution f satisfies $\rho(f)=\deg Q$ and is of mean type.*
(b) *Every solution f satisfies $\lambda(f)=\rho(f)$ if and only if $P\not\equiv 0$.*
(c) *A solution belongs to Γ_0 if and only if $P\equiv 0$. In particular, this is the case if $n\geq 3$.*
(d) *If a solution f belongs to Γ_0 and if g is any other finite-order entire solution to (1.5), then $f=\eta g$, where $\eta^{n-1}=1$.*
(e) *If f is an exponential polynomial solution of the form (1.6), then $f\in\Gamma_1$. Moreover, if $f\in\Gamma_1\backslash\Gamma_0$, then $\rho(f)=1$.*

A natural question to ask is about $P(z)=p_1e^{\lambda z}+p_2e^{-\lambda z}$ in (1.5), where $\lambda, p_1, p_2\in\mathbb{C}\backslash\{0\}$ are constants. We consider this question and obtain the following result.

Theorem 1.3 *Let $n\geq 3$ be an integer, let $c, \lambda, p_1, p_2\in\mathbb{C}\backslash\{0\}$ be constants and let q, Q be polynomials such that Q is not a constant and $q\not\equiv 0$. If f is an entire solution of finite order of the equation*

$$f^n(z)+q(z)e^{Q(z)}f(z+c)=p_1e^{\lambda z}+p_2e^{-\lambda z}, \tag{1.7}$$

then the following conclusions hold.

1. *Every solution f satisfies $\rho(f)=\deg Q=1$.*
2. *If a solution f belongs to Γ_0, then $f(z)=e^{\frac{\lambda}{n}z+B}$, $Q(z)=-\frac{n+1}{n}\lambda z+b$ or $f(z)=e^{-\frac{\lambda}{n}z+B}$, $Q(z)=\frac{n+1}{n}\lambda z+b$, where $b, B\in\mathbb{C}$.*

Remark 1.1 We conjecture that if $n=2$, the conclusions of Theorem 1.3 are still valid, although we have not found a suitable method of proof yet. For example, $f(z)=e^z$ is an entire solution of finite order of the difference equation

$$f^2(z)+2e^{-3z}f(z-\log 2)=e^{2z}+e^{-2z},$$

and $f(z)=e^z+e^{-z}$ is an entire solution of finite order of the difference equation

$$f^2(z)+2e^zf(z+\pi i)=-e^{2z}+e^{-2z}.$$

The following example shows that our estimates in Theorem 1.3 are accurate.

Example 1.1 If $f(z) = \mathrm{e}^z$ is an entire solution of finite order of the difference equation

$$f^3(z) + \frac{1}{2}\mathrm{e}^{-4z} f(z + \log 2) = \mathrm{e}^{3z} + \mathrm{e}^{-3z},$$

where $\lambda = 3,\ n = 3,\ b = B = 0$, then $f(z) = \mathrm{e}^{\frac{\lambda}{n}z+B} = \mathrm{e}^z$, $Q(z) = -\frac{n+1}{n}\lambda z + b = -4z$ and $\rho(f) = \deg Q = 1$.

2 Some Lemmas

Lemma 2.1 $(Clunie's\ Lemma)$ (See [1], [7, Lem. 2.4.2]) *Let f be a transcendental meromorphic solution of*

$$f^n(z)P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are polynomials in f and its derivatives with meromorphic coefficients, say $\{a_\lambda | \lambda \in I\}$, such that $m(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. If the total degree of $Q(z, f)$ as a polynomial in f and its derivatives is at most n, then

$$m(r, P(z, f)) = S(r, f). \tag{2.1}$$

Lemma 2.2 (See [6, Cor. 3.3]) *Let f be a non-constant finite order meromorphic solution of*

$$f^n(z)P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are difference polynomials in f with small meromorphic coefficients, and let $c \in \mathbb{C}$, $\delta < 1$. If the total degree of $Q(z, f)$ as a polynomial in f and its shifts at most n, then

$$m(r, P(z, f)) = o\left(\frac{T(r + |c|, f)}{r^\delta}\right) + o(T(r, f)) \tag{2.2}$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Remark 2.1 In Lemma 2.2, if f is a transcendental meromorphic function with finite order ρ, and $P(z, f)$, $Q(z, f)$ are differential-difference polynomials in f, then by the same reasoning as in the proof of Lemma 2.1, we also obtain the conclusion (2.2). Furthermore, if the coefficients of $P(z, f)$ and $Q(z, f)$ are polynomials $A_j,\ j = 1, \ldots, n$, for each $\varepsilon > 0$, then (2.2) can be written as:

$$m(r, P(z, f)) = O(r^{\rho-1+\varepsilon}) + O\left(\sum_{j=1}^{n} m(r, A_j)\right), \tag{2.3}$$

where r is sufficiently large.

Lemma 2.3 (See [15, Thm. 1.51]) *Suppose that* $f_1, f_2, \ldots, f_n (n \geq 2)$ *are meromorphic functions and* $g_1, g_2, \ldots, g_n$ *are entire functions satisfying the following conditions:*

1. $\sum_{j=1}^{n} f_j e^{g_j} \equiv 0.$
2. $g_j - g_k$ *are not constants for* $1 \leq j < k \leq n$.
3. *For* $1 \leq j \leq n, 1 \leq h < k \leq n$,

$$T(r, f_j) = o(T(r, e^{g_h - g_k})) \ (r \to \infty, r \notin E),$$

where $E \subset [1, \infty)$ *is finite linear measure or finite logarithmic measure. Then* $f_j \equiv 0 \ (j = 1, \ldots, n)$.

Lemma 2.4 (See [11, Lem. 6]) *Suppose that* f *is a transcendental meromorphic function,* a, b, c, d *are small functions with respect to* f *and* $acd \not\equiv 0$. *If*

$$af^2 + bff' + c(f')^2 = d, \tag{2.4}$$

then

$$c(b^2 - 4ac)\frac{d'}{d} + b(b^2 - 4ac) - c(b^2 - 4ac)' + (b^2 - 4ac)c' = 0. \tag{2.5}$$

Lemma 2.5 (See [2, Cor. 2.6]) *Let* η_1, η_2 *be two complex numbers such that* $\eta_1 \neq \eta_2$ *and let* f *be a finite order meromorphic function. Let* ρ *be the order of* f. *Then for each* $\varepsilon > 0$, *we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\rho - 1 + \varepsilon}). \tag{2.6}$$

Using an argument similar to that used in [3, Lem. 2.3], we get the following result.

Lemma 2.6 *Let* $n \geq 1$ *be an integer,* λ *be a non-zero constant and* H *be a non-zero polynomial. Then, the differential equation*

$$\lambda^2 f - n^2 f'' = H \tag{2.7}$$

has a special solution c_0 *which is a non-zero polynomial.*

3 Proof of Theorem 1.1

Denote $P = q\Delta_c f$. Suppose that f is a transcendental entire solution of finite order of Eq. (1.2). Differentiating (1.2), we have

$$nf^{n-1}f' + P' = \lambda(p_1 e^{\lambda z} - p_2 e^{-\lambda z}). \tag{3.1}$$

Differentiating (3.1) yields

$$n(n-1)f^{n-2}(f')^2 + nf^{n-1}f'' + P'' = \lambda^2(p_1 e^{\lambda z} + p_2 e^{-\lambda z}). \tag{3.2}$$

It follows from (1.2) and (3.1) that

$$\begin{aligned}&\lambda^2 f^{2n} - n^2 f^{2(n-1)}(f')^2\\&\quad +2\lambda^2 P f^n - 2nP'f^{n-1}f' + \lambda^2 P^2 - (P')^2 - 4\lambda^2 p_1 p_2 = 0.\end{aligned} \tag{3.3}$$

It follows from (1.2) and (3.2) that

$$\lambda^2 f^n - n(n-1)f^{n-2}(f')^2 - nf^{n-1}f'' + \lambda^2 P - P'' = 0. \tag{3.4}$$

Eliminating $(f')^2$ from (3.3) and (3.4) yields

$$f^{2n-1}\varphi = Q(z, f), \tag{3.5}$$

where

$$\varphi = \lambda^2 f - n^2 f'' \tag{3.6}$$

and

$$\begin{aligned}Q(z, f) = {} & [(n-2)\lambda^2 P + nP'']f^n - 2n(n-1)P'f^{n-1}f'\\&+(n-1)[\lambda^2 P^2 - (P')^2 - 4\lambda^2 p_1 p_2],\end{aligned} \tag{3.7}$$

$Q(z, f)$ is a differential-difference polynomial in f and the total degree is at most $n+1$. Note that when $n \geq 2$, then $2n-1 \geq n+1$, by Lemma 2.2 and Remark 2.1, we have $m(r, \varphi) = S(r, f)$; therefore, $T(r, \varphi) = S(r, f)$. We distinguish two cases below:

Case 1 If $\varphi \equiv 0$, i.e. $\lambda^2 f - n^2 f'' \equiv 0$. Every entire solution $f(\not\equiv 0)$ of this equation can be expressed as:

$$f(z) = c_1 e^{\frac{\lambda}{n}z} + c_2 e^{-\frac{\lambda}{n}z}, \tag{3.8}$$

where c_1, c_2 are non-zero constants. Otherwise, if one of c_1, c_2 is equal to zero, substituting (3.8) into (1.2) and using Lemma 2.3, we obtain a contradiction.

When $n = 3$, then $f(z) = c_1 e^{\frac{\lambda}{3}z} + c_2 e^{-\frac{\lambda}{3}z}$, substituting this into (1.2) yields

$$\begin{aligned}&(c_1^3 - p_1)e^{\lambda z} + (c_2^3 - p_2)e^{-\lambda z} + c_1\left[q(z)\left(e^{\frac{\lambda}{3}c} - 1\right) + 3c_1c_2\right]e^{\frac{\lambda}{3}z}\\&\quad + c_2\left[q(z)\left(e^{-\frac{\lambda}{3}c} - 1\right) + 3c_1c_2\right]e^{-\frac{\lambda}{3}z} = 0.\end{aligned} \tag{3.9}$$

It follows from (3.9) and Lemma 2.3 that

$$\begin{cases} c_1^3 = p_1, c_2^3 = p_2; \\ c_1\left[q(z)\left(\mathrm{e}^{\frac{\lambda}{3}c}-1\right)+3c_1c_2\right]\equiv 0; \\ c_2\left[q(z)\left(\mathrm{e}^{-\frac{\lambda}{3}c}-1\right)+3c_1c_2\right]\equiv 0. \end{cases}$$

Note that $c_1, c_2 \in \mathbb{C}\backslash\{0\}$, then $q(z)(\mathrm{e}^{\frac{\lambda}{3}c}-1) = q(z)(\mathrm{e}^{-\frac{\lambda}{3}c}-1) = -3c_1c_2 \neq 0$, $q(z)$ reduces to a constant q and $\lambda c = 3k\pi i$, $k \in \mathbb{Z}\backslash\{0\}$ and k is an odd, $q^3 = \frac{27}{8}p_1p_2$.

When $n = 2l (l \geq 1)$ is even, $f(z) = c_1\mathrm{e}^{\frac{\lambda}{2l}z} + c_2\mathrm{e}^{-\frac{\lambda}{2l}z}$. Substituting this into (1.2) yields

$$\begin{aligned} &(c_1^{2l}-p_1)\mathrm{e}^{\lambda z} + (c_2^{2l}-p_2)\mathrm{e}^{-\lambda z} + \sum_{k=1}^{2l-1}\binom{k}{2l}c_1^{2l-k}c_2^k\mathrm{e}^{\frac{2l-2k}{2l}\lambda z} \\ &\quad + c_1q(z)(\mathrm{e}^{\frac{\lambda c}{2l}}-1)\mathrm{e}^{\frac{\lambda z}{2l}} + c_2q(z)\left(\mathrm{e}^{-\frac{\lambda c}{2l}}-1\right)\mathrm{e}^{-\frac{\lambda z}{2l}} = 0. \end{aligned} \tag{3.10}$$

If $k = l$, then $\sum_{k=1}^{2l-1}\binom{k}{2l}c_1^{2l-k}c_2^k e^{\frac{2l-2k}{2l}\lambda z}$ must have a constant term. That is $\binom{l}{2l}c_1^l c_2^l = \frac{(2l)!}{l!l!}(c_1c_2)^l$. By Lemma 2.3, we obtain $c_1c_2 = 0$, a contradiction.

When $n = 2l + 1 (l \geq 2)$ is odd, $f(z) = c_1\mathrm{e}^{\frac{\lambda}{2l+1}z} + c_2\mathrm{e}^{-\frac{\lambda}{2l+1}z}$. Substituting this into (1.2) yields

$$\begin{aligned} &\sum_{k=0}^{2l+1}\binom{k}{2l+1}c_1^{2l+1-k}c_2^k\mathrm{e}^{\frac{2l+1-2k}{2l+1}\lambda z} + c_1q(z)\left(\mathrm{e}^{\frac{\lambda c}{2l+1}}-1\right)\mathrm{e}^{\frac{\lambda z}{2l+1}} \\ &\quad + c_2q(z)\left(\mathrm{e}^{-\frac{\lambda c}{2l+1}}-1\right)\mathrm{e}^{-\frac{\lambda z}{2l+1}} = p_1\mathrm{e}^{\lambda z} + p_2\mathrm{e}^{-\lambda z}, \end{aligned}$$

i.e.

$$\begin{aligned} &\left(c_1^{2l+1}-p_1\right)\mathrm{e}^{\lambda z} + \left(c_2^{2l+1}-p_2\right)\mathrm{e}^{-\lambda z} + \sum_{\substack{k=1 \\ k\neq l,l+1}}^{2l}\binom{k}{2l+1}c_1^{2l+1-k}c_2^k\mathrm{e}^{\frac{2l+1-2k}{2l+1}\lambda z} \\ &+ \left[c_1q(z)\left(\mathrm{e}^{\frac{\lambda c}{2l+1}}-1\right) + \binom{l}{2l+1}c_1^{l+1}c_2^l\right]\mathrm{e}^{\frac{\lambda z}{2l+1}} \\ &+ \left[c_2q(z)\left(\mathrm{e}^{-\frac{\lambda c}{2l+1}}-1\right) + \binom{l+1}{2l+1}c_1^l c_2^{l+1}\right]\mathrm{e}^{-\frac{\lambda z}{2l+1}} = 0. \end{aligned} \tag{3.11}$$

Since $l \geq 2$, then $\sum_{\substack{k=1 \\ k\neq l,l+1}}^{2l}\binom{k}{2l+1}c_1^{2l+1-k}c_2^k\mathrm{e}^{\frac{2l+1-2k}{2l+1}\lambda z}$ contains at least two terms. By Lemma 2.3, we have $\binom{k}{2l+1}c_1^{2l+1-k}c_2^k = 0$, $k \neq l, l+1$, $k = 1, \ldots, 2l$. Then, $c_1c_2 = 0$, a contradiction.

Case 2 As in the beginning of the proof of Theorem 1.2 below, we obtain $\rho(f) = 1$. If $\varphi \not\equiv 0$, since f is a transcendental entire function with order $\rho(f) = 1$, we see that

(3.5) satisfies the conditions of Lemma 2.2 and Remark 2.1. Thus, we have

$$m(r, \lambda^2 f - n^2 f'') = S(r, f) + O(m(r, q)) = O(\log r),$$

which implies that $\lambda^2 f - n^2 f''$ is a polynomial. Denote

$$\lambda^2 f - n^2 f'' = H, \tag{3.12}$$

where H is a non-zero polynomial. By Lemma 2.6, we see that (3.12) must have a non-zero polynomial solution, say, $c_0(z)$. Since the differential equation

$$\lambda^2 f - n^2 f'' = 0,$$

has two fundamental solutions

$$f_1(z) = \mathrm{e}^{\frac{\lambda}{n} z}, \quad f_2(z) = \mathrm{e}^{-\frac{\lambda}{n} z},$$

the general entire solution $f(\not\equiv 0)$ of (3.12) can be expressed as:

$$f(z) = c_0(z) + c_1 \mathrm{e}^{\frac{\lambda}{n} z} + c_2 \mathrm{e}^{-\frac{\lambda}{n} z}, \tag{3.13}$$

where c_1, c_2 are non-zero constants, $c_0(z)$ is a non-zero polynomial. Otherwise, if one of c_1, c_2 is equal to zero, substituting (3.13) into (1.2) and using Lemma 2.3, we obtain a contradiction.

When $n = 2$, then $f(z) = c_0(z) + c_1 \mathrm{e}^{\frac{\lambda}{2} z} + c_2 \mathrm{e}^{-\frac{\lambda}{2} z}$. Substituting this into (1.2) yields

$$\begin{aligned}
&(c_1^2 - p_1)\mathrm{e}^{\lambda z} + (c_2^2 - p_2)\mathrm{e}^{-\lambda z} + c_0^2(z) + q(z)[c_0(z+c) - c_0(z)] + 2c_1c_2 \\
&\quad + c_1[q(z)(\mathrm{e}^{\frac{\lambda c}{2}} - 1) + 2c_0(z)]\mathrm{e}^{\frac{\lambda}{2} z} + c_2\left[q(z)\left(\mathrm{e}^{-\frac{\lambda c}{2}} - 1\right) + 2c_0(z)\right]\mathrm{e}^{-\frac{\lambda}{2} z} = 0.
\end{aligned} \tag{3.14}$$

It follows from (3.14) and Lemma 2.3 that

$$\begin{cases}
c_1^2 = p_1, c_2^2 = p_2; \\
c_0^2(z) + q(z)[c_0(z+c) - c_0(z)] + 2c_1c_2 \equiv 0; \\
c_1[q(z)(\mathrm{e}^{\frac{\lambda c}{2}} - 1) + 2c_0(z)] \equiv 0; \\
c_2[q(z)(\mathrm{e}^{-\frac{\lambda c}{2}} - 1) + 2c_0(z)] \equiv 0.
\end{cases}$$

Note that $c_1, c_2 \in \mathbb{C}\backslash\{0\}$, then $q(z)(\mathrm{e}^{\frac{\lambda c}{2}} - 1) \equiv q(z)(\mathrm{e}^{-\frac{\lambda c}{2}} - 1) \equiv -2c_0(z) \not\equiv 0$, that is $q(z) \equiv c_0(z)$ and $\lambda c = 2k\pi i$, $k \in \mathbb{Z}\backslash\{0\}$ and k is odd. Substituting $q(z) \equiv c_0(z)$ into $c_0^2(z) + q(z)[c_0(z+c) - c_0(z)] + 2c_1c_2 \equiv 0$ yields $c_0(z+c)c_0(z) + 2c_1c_2 \equiv 0$. Then, $c_0(z)$ must be a non-zero constant. Therefore, $c_0^4 = q^4 = 4c_1^2c_2^2 = 4p_1p_2$.

When $n \geq 3$, then $2n-2 \geq n+1$, it follows from (3.5) that $f^{2n-2}(f\varphi) = Q(z, f)$. By Lemma 2.2 and Remark 2.1, we have $m(r, f\varphi) = S(r, f)$. Therefore, $T(r, f\varphi) = S(r, f)$. Since $\varphi \not\equiv 0$, then $T(r, f) = m(r, f) \leq m(r, f\varphi) + m\left(r, \frac{1}{\varphi}\right) \leq T(r, \varphi) + S(r, f) = S(r, f)$, which gives a contradiction.

This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Clearly, $\rho(p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) = 1$, where $\alpha_1 \pm \alpha_2 \neq 0$. From (1.4) and Lemma 2.5, we have

$$\begin{aligned}T(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) &= T(r, f^2(z) + q(z)\Delta_c f(z)) \\ &\leq T(r, f^2) + m\left(r, \frac{q(z)\Delta_c f(z)}{f}\right) + m(r, f) + O(1) \\ &\leq 3T(r, f) + S(r, f),\end{aligned}$$

and

$$\begin{aligned}T(r, f^2(z) + q(z)\Delta_c f(z)) &\geq T(r, f^2) - T(r, q(z)\Delta_c f(z)) + O(1) \\ &\geq 2T(r, f) - \left[m\left(r, \frac{q(z)\Delta_c f(z)}{f}\right) + m(r, f)\right] + O(1) \\ &\geq 2T(r, f) - T(r, f) + S(r, f) = T(r, f) + S(r, f),\end{aligned}$$

i.e.

$$T(r, f) + S(r, f) \leq T(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) \leq 3T(r, f) + S(r, f),$$

thus, $\rho(f) = 1$. Denote $P = q\Delta_c f$. Suppose that f is a transcendental entire solution of finite order of equation (1.4). By differentiating (1.4), we have

$$2ff' + P' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \tag{4.1}$$

Eliminating $e^{\alpha_2 z}$ from (1.4) and (4.1), we have

$$\alpha_2 f^2 - 2ff' + \alpha_2 P - P' = (\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \tag{4.2}$$

Differentiating (4.2) yields

$$2\alpha_2 ff' - 2(f')^2 - 2ff'' + \alpha_2 P' - P'' = \alpha_1(\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\varphi(z) = Q(z, f), \tag{4.4}$$

where

$$\varphi(z) = \alpha_1\alpha_2 f^2 - 2(\alpha_1+\alpha_2)ff' + 2(f')^2 + 2ff'' \tag{4.5}$$

and

$$Q(z, f) = -\alpha_1\alpha_2 P + (\alpha_1+\alpha_2)P' - P''. \tag{4.6}$$

We distinguish two cases below:

Case 1 If $\varphi \equiv 0$, then $Q(z, f) \equiv 0$. Since $\alpha_1 \neq \alpha_2$, we see that $\alpha_1 P - P' \equiv 0$ and $\alpha_2 P - P' \equiv 0$ cannot hold simultaneously. Suppose that $\alpha_2 P - P' \not\equiv 0$. By (4.6), we have

$$\alpha_2 P - P' = Ae^{\alpha_1 z}, \tag{4.7}$$

where A is a non-zero constant. Substituting (4.7) into (4.2), we have

$$f(\alpha_2 f - 2f') = \frac{[(\alpha_2-\alpha_1)p_1 - A]\alpha_2}{A}P - \frac{(\alpha_2-\alpha_1)p_1 - A}{A}P'. \tag{4.8}$$

Since the right-hand side of (4.8) is a differential-difference polynomial in f of degree at most 1, by Lemma 2.2 and Remark 2.1, we have

$$m(r, \alpha_2 f - 2f') = S(r, f).$$

Denote $\psi = \alpha_2 f - 2f'$. We consider two subcases as follows.

Subcase 1.1 If $\psi \equiv 0$, that is $\alpha_2 f - 2f' \equiv 0$, then $f^2 = \widetilde{p}_2 e^{\alpha_2 z}$, $\widetilde{p}_2 \in \mathbb{C}\backslash\{0\}$. Substituting this and (4.7) into (1.4) yields

$$\left(1 - \frac{p_2}{\widetilde{p}_2}\right) f^2 = \frac{\alpha_2 p_1 - A}{A}P - \frac{p_1}{A}P'.$$

If $p_2 \neq \widetilde{p}_2$, by Lemma 2.2 and Remark 2.1, we have $T(r, f) = m(r, f) = S(r, f)$, a contradiction. Therefore, $p_2 = \widetilde{p}_2$, $f(z) = c_2 e^{\frac{\alpha_2}{2}z}$, $c_2(e^{\frac{\alpha_2}{2}c} - 1)q = p_1$, $c_2^2 = p_2$, $\alpha_2 = 2\alpha_1$.

Subcase 1.2 If $\psi \not\equiv 0$, then $\psi' = \alpha_2 f' - 2f''$, $f' = \frac{\alpha_2}{2}f - \frac{\psi}{2}$, $f'' = \frac{\alpha_2^2}{4}f - \frac{\alpha_2}{4}\psi - \frac{\psi'}{2}$. Note that $\varphi \equiv 0$ and substitute this into (4.5) gives

$$\left[\left(\alpha_1 - \frac{\alpha_2}{2}\right)\psi - \psi'\right] f = -\frac{\psi^2}{2}.$$

Since $\psi \not\equiv 0$, then $\left(\alpha_1 - \frac{\alpha_2}{2}\right)\psi - \psi' \not\equiv 0$. From the above equality, we have $T(r, f) = S(r, f)$, which implies a contradiction.

Similarly, if $\alpha_1 P - P' \not\equiv 0$, then we obtain $f(z) = c_1 e^{\frac{\alpha_1}{2}z}$, $c_1(e^{\frac{\alpha_1}{2}c} - 1)q = p_2$, $c_1^2 = p_1$, $\alpha_1 = 2\alpha_2$.

Case 2 If $\varphi \not\equiv 0$, by applying the lemma of logarithmic derivative and Lemma 2.2, from (4.4)–(4.6), we have

$$m\left(r, \frac{\varphi}{f}\right) = m\left(r, \frac{Q}{f}\right) = S(r, f) \quad and \quad m\left(r, \frac{\varphi}{f^2}\right) = S(r, f).$$

Then

$$\begin{aligned} 2m\left(r, \frac{1}{f}\right) = m\left(r, \frac{1}{f^2}\right) &\le m\left(r, \frac{\varphi}{f^2}\right) + m\left(r, \frac{1}{\varphi}\right) \\ &\le T(r, \varphi) + S(r, f) = T(r, Q) + S(r, f) \\ &\le m\left(r, \frac{Q}{f}\right) + m(r, f) + S(r, f) \\ &\le T(r, f) + S(r, f). \end{aligned} \tag{4.9}$$

Suppose that there exist $\kappa, \iota \ge 0$ such that

$$T(r, \varphi) + S(r, f) = \kappa T(r, f), \quad N\left(r, \frac{1}{f}\right) + S(r, f) = \iota T(r, f),$$

where $0 \le \kappa, \ \iota \le 1$. It follows from (4.9) that

$$2m\left(r, \frac{1}{f}\right) \le T(r, \varphi) + S(r, f) = \kappa T(r, f)$$

and

$$\begin{aligned} T(r, f) &\ge N\left(r, \frac{1}{f}\right) + O(1) = T\left(r, \frac{1}{f}\right) - m\left(r, \frac{1}{f}\right) + O(1) \\ &\ge T(r, f) - \frac{\kappa}{2} T(r, f) + O(1) = \left(1 - \frac{\kappa}{2}\right) T(r, f) + O(1), \end{aligned}$$

then we have $1 - \frac{\kappa}{2} \le \iota \le 1$ and $0 \le \kappa \le 1$.

Next, we deduce that $\kappa \ne 0$.

If $\kappa = 0$, then $T(r, \varphi) = S(r, f)$. It follows from (4.9) that

$$m\left(r, \frac{1}{f}\right) = S(r, f), \quad T(r, f) = N\left(r, \frac{1}{f}\right) + S(r, f).$$

By (4.5), if z_0 is a multiple zero of f, then z_0 must be a zero of φ. Hence, $N_{(2}\left(r, \frac{1}{f}\right) = S(r, f)$. Differentiating (4.5) gives

$$\varphi' = 2\alpha_1\alpha_2 f f' - 2(\alpha_1 + \alpha_2) f f'' - 2(\alpha_1 + \alpha_2)(f')^2 + 6 f' f'' + 2 f f'''. \tag{4.10}$$

If z_0 is a simple zero of f, it follows from (4.5) and (4.10) that z_0 is a zero of $3\varphi f'' - [\varphi' + (\alpha_1 + \alpha_2)\varphi]f'$. Define

$$\alpha := \frac{3\varphi f'' - [\varphi' + (\alpha_1 + \alpha_2)\varphi]f'}{f}, \tag{4.11}$$

then we have $T(r, \alpha) = S(r, f)$. It follows that

$$f'' = \frac{1}{3}\left(\frac{\varphi'}{\varphi} + \alpha_1 + \alpha_2\right) f' + \frac{\alpha}{3\varphi} f. \tag{4.12}$$

Substituting (4.12) into (4.5) yields

$$af^2 + bff' + 2(f')^2 = \varphi, \tag{4.13}$$

where $a = \alpha_1\alpha_2 + \frac{2\alpha}{3\varphi}$, $b = \frac{2}{3}\left[\frac{\varphi'}{\varphi} - 2(\alpha_1 + \alpha_2)\right]$. By Lemma 2.4, we have

$$2(b^2 - 8a)\frac{\varphi'}{\varphi} = 2(b^2 - 8a)' - b(b^2 - 8a). \tag{4.14}$$

Now, we distinguish two subcases below.

Subcase 2.1 Suppose that $b^2 - 8a \not\equiv 0$. It follows from (4.14) that

$$4\frac{\varphi'}{\varphi} = 2(\alpha_1 + \alpha_2) + 3\frac{(b^2 - 8a)'}{b^2 - 8a}. \tag{4.15}$$

By integration, we see that there exists a $B \in \mathbb{C}\backslash\{0\}$ such that

$$\mathrm{e}^{2(\alpha_1+\alpha_2)z} = B\varphi^4(b^2 - 8a)^{-3}, \tag{4.16}$$

which implies $\mathrm{e}^{2(\alpha_1+\alpha_2)z} \in S(f)$, then $\alpha_1 + \alpha_2 = 0$, a contradiction.

Subcase 2.2 Suppose that $b^2 - 8a \equiv 0$. Differentiating (4.13) yields

$$\varphi' = a'f^2 + (2a + b')ff' + b(f')^2 + bff'' + 4f'f''. \tag{4.17}$$

Suppose z_0 is a simple zero of f which is not the zero of a, b. It follows from (4.13) and (4.17) that z_0 is a zero of $2\varphi f'' - \left(\varphi' - \frac{b}{2}\varphi\right) f'$. Putting

$$\beta := \frac{2\varphi f'' - \left(\varphi' - \frac{b}{2}\varphi\right) f'}{f}, \tag{4.18}$$

we have $T(r, \beta) = S(r, f)$. It follows that

$$f'' = \left(\frac{1}{2}\frac{\varphi'}{\varphi} - \frac{b}{4}\right) f' + \frac{\beta}{2\varphi} f. \tag{4.19}$$

Substituting (4.19) into (4.17) yields

$$\varphi' = cf^2 + dff' + 2\frac{\varphi'}{\varphi}(f')^2, \tag{4.20}$$

where $c = a' + \frac{b\beta}{2\varphi}$, $d = 2a + b' + \frac{b}{2}\frac{\varphi'}{\varphi} - \frac{b^2}{4} + \frac{2\beta}{\varphi}$. Eliminating $(f')^2$ from (4.13) and (4.20), we have

$$A(z)f(z) + B(z)f'(z) \equiv 0, \tag{4.21}$$

where

$$A(z) = c - a\frac{\varphi'}{\varphi} = a' + \frac{b\beta}{2\varphi} - a\frac{\varphi'}{\varphi},$$
$$B(z) = d - b\frac{\varphi'}{\varphi} = 2a + b' - \frac{b}{2}\frac{\varphi'}{\varphi} - \frac{b^2}{4} + \frac{2\beta}{\varphi}.$$

Note that $A(z)$ and $B(z)$ are small functions of f. If z_0 is a simple zero of f and not the zero of $B(z)$, it follows from (4.21) that $A(z) = B(z) \equiv 0$. By (4.19), we have

$$f'' = \left(\frac{1}{2}\frac{\varphi'}{\varphi} - \frac{b}{4}\right) f' - \frac{1}{b}\left(a' - a\frac{\varphi'}{\varphi}\right) f, \tag{4.22}$$

where $b = \frac{2}{3}\left[\frac{\varphi'}{\varphi} - 2(\alpha_1 + \alpha_2)\right] \not\equiv 0$. Otherwise, $e^{2(\alpha_1+\alpha_2)z} = C\varphi \in S(f)$, then $\alpha_1 + \alpha_2 = 0$, a contradiction. Substituting $b^2 - 8a \equiv 0$ into (4.22) yields

$$f'' = \frac{1}{3}\left(\frac{\varphi'}{\varphi} + \alpha_1 + \alpha_2\right) f' - \frac{1}{6}\left[\left(\frac{\varphi'}{\varphi}\right)' - \frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)^2 + (\alpha_1 + \alpha_2)\frac{\varphi'}{\varphi}\right] f. \tag{4.23}$$

It follows from (4.12) and (4.23) that

$$\frac{\alpha}{\varphi} = -\frac{1}{2}\left[\left(\frac{\varphi'}{\varphi}\right)' - \frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)^2 + (\alpha_1 + \alpha_2)\frac{\varphi'}{\varphi}\right]. \tag{4.24}$$

We deduce that $\varphi' \not\equiv 0$. Otherwise, we assume that $\varphi' \equiv 0$, then $\frac{\alpha}{\varphi} \equiv 0$. Substituting this into $b^2 - 8a \equiv 0$ yields

$$2\left(\frac{\alpha_1}{\alpha_2}\right)^2 - 5\frac{\alpha_1}{\alpha_2} + 2 = 0,$$

which implies that $\frac{\alpha_1}{\alpha_2} = 2$ or $\frac{\alpha_1}{\alpha_2} = \frac{1}{2}$. By substituting $\varphi' \equiv 0$ into (4.23), we obtain

$$f'' = \frac{1}{3}(\alpha_1 + \alpha_2) f',$$

then

$$f = \frac{3C_1}{\alpha_1 + \alpha_2} \mathrm{e}^{\frac{1}{3}(\alpha_1+\alpha_2)z} + C_2,$$

where $C_1, C_2 \in \mathbb{C}\backslash\{0\}$. Without loss of generality, when $\frac{\alpha_1}{\alpha_2} = 2$, then

$$f = \frac{C_1}{\alpha_2} \mathrm{e}^{\alpha_2 z} + C_2.$$

Substituting this into (1.4) and using Lemma 2.3, we can obtain a contradiction.

Differentiating (4.24) gives

$$\left(\frac{\alpha}{\varphi}\right)' = -\frac{1}{2}\left[\left(\frac{\varphi'}{\varphi}\right)'' - \left(\frac{\varphi'}{\varphi}\right)'\frac{\varphi'}{\varphi} + (\alpha_1 + \alpha_2)\left(\frac{\varphi'}{\varphi}\right)'\right].$$

It follows from $b^2 - 8a \equiv 0$ that $bb' = 4a'$, that is

$$\left(\frac{\alpha}{\varphi}\right)' = \frac{1}{6}\left(\frac{\varphi'}{\varphi}\right)'\left[\frac{\varphi'}{\varphi} - 2(\alpha_1 + \alpha_2)\right].$$

Putting $\gamma := \frac{\varphi'}{\varphi}$ and combining the above two equality yields

$$(\alpha_1 + \alpha_2)\gamma' = 2\gamma\gamma' - 3\gamma''. \tag{4.25}$$

If $\gamma' \equiv 0$, then $\varphi = C_3 e^{C_4 z}$, $C_3, C_4 \in \mathbb{C}$. It follows from $\varphi' \not\equiv 0$ that $C_3, C_4 \neq 0$, which implies that $\varphi \notin S(f)$, a contradiction. If $\gamma' \not\equiv 0$, it follows from (4.25) that

$$e^{(\alpha_1+\alpha_2)z} = C_5\varphi^2\left(\left(\frac{\varphi'}{\varphi}\right)'\right)^{-3}, \; C_5 \in \mathbb{C}\backslash\{0\},$$

which implies that $e^{(\alpha_1+\alpha_2)z} \in S(f)$, then $\alpha_1 + \alpha_2 = 0$, a contradiction.

This completes the proof of Theorem 1.2.

5 Proof of Theorem 1.3

Suppose that f is a transcendental entire solution of finite order of Eq. (1.7). In what follows, we consider three cases.

Case 1 If $\rho(f) < 1$, using Lemma 2.5, from (1.7) we have

$$T(r, e^{Q(z)}) = m(r, \mathrm{e}^{Q(z)}) = m\left(\frac{p_1\mathrm{e}^{\lambda z} + p_2\mathrm{e}^{-\lambda z} - f^n(z)}{q(z)f(z+c)}\right)$$

$$
\begin{aligned}
&\le m\left(r, \frac{1}{q(z)f(z+c)}\right) + m(r, p_1 e^{\lambda z} + p_2 e^{-\lambda z}) \\
&\quad + m(r, f^n(z)) + O(1) \\
&\le m\left(r, \frac{f(z)}{q(z)f(z+c)}\right) + m\left(r, \frac{1}{f(z)}\right) + nT(r, f) \\
&\quad + 2T(r, e^{\lambda z}) + S(r, e^{\lambda z}) \\
&\le 2T(r, e^{\lambda z}) + S(r, e^{\lambda z}) + (n+1)T(r, f) + S(r, f) \\
&\le 2T(r, e^{\lambda z}) + S(r, e^{\lambda z}),
\end{aligned}
$$

then $\deg Q \le 1$, note that $\deg Q \ge 1$, therefore $\deg Q = 1$. Denote $Q(z) = az + b,\ a \in \mathbb{C}\backslash\{0\},\ b \in \mathbb{C}$. Rewriting (1.7) in the following form:

$$
f^n(z) + q(z)e^{az+b} f(z+c) = p_1 e^{\lambda z} + p_2 e^{-\lambda z} \tag{5.1}
$$

and differentiating (5.1) we get

$$
nf^{n-1} f' + A(z)e^{az+b} = \lambda(p_1 e^{\lambda z} - p_2 e^{-\lambda z}), \tag{5.2}
$$

where

$$
A(z) = q'(z)f(z+c) + aq(z)f(z+c) + q(z)f'(z+c).
$$

Eliminating $e^{\lambda z}$ and $e^{-\lambda z}$ from (5.1) and (5.2) yields

$$
B(z)e^{2(az+b)} + C(z)e^{az+b} + D(z) \equiv 0, \tag{5.3}
$$

where

$$
\begin{cases}
B(z) = \lambda^2 q^2(z) f^2(z+c) - A^2(z); \\
C(z) = 2\lambda^2 q(z) f^n(z) f(z+c) - 2nA(z) f^{n-1}(z) f'(z); \\
D(z) = \lambda^2 f^{2n}(z) - n^2 f^{2(n-1)}(z)(f'(z))^2 - 4\lambda^2 p_1 p_2.
\end{cases} \tag{5.4}
$$

Thus, from Lemma 2.3, we have

$$
B(z) \equiv C(z) \equiv D(z) \equiv 0.
$$

It follows from $D(z) \equiv 0$ that

$$
f^{2(n-1)}(z)(\lambda^2 f^2(z) - n^2(f'(z))^2) \equiv 4\lambda^2 p_1 p_2,\ n \ge 3.
$$

Using Lemma 2.1, we have

$$
m(r, \lambda^2 f^2(z) - n^2(f'(z))^2) = S(r, f)
$$

and

$$m(r, f(z)(\lambda^2 f^2(z) - n^2(f'(z))^2)) = S(r, f).$$

We deduce that $\lambda^2 f^2(z) - n^2(f'(z))^2 \not\equiv 0$. Otherwise, $4\lambda^2 p_1 p_2 = 0$, a contradiction. Since f is entire,

$$\begin{aligned} T(r, f) = m(r, f) &\leq m(r, f(z)(\lambda^2 f^2(z) - n^2(f'(z))^2)) \\ &+ m\left(r, \frac{1}{\lambda^2 f^2(z) - n^2(f'(z))^2}\right) \\ &\leq T(r, \lambda^2 f^2(z) - n^2(f'(z))^2) + S(r, f) = S(r, f), \end{aligned}$$

a contradiction.

Case 2 If $\rho(f) > 1$. Denote $P(z) = p_1 e^{\lambda z} + p_2 e^{-\lambda z}$, $H(z) = q(z) f(z+c)$. It is clear that $\rho(P) = 1$, then $T(r, P) = S(r, f)$. Equation (1.7) can be written as:

$$f^n(z) + H(z) e^{Q(z)} = P(z). \tag{5.5}$$

Differentiating (5.5) yields

$$n f^{n-1}(z) f'(z) + L(z) e^{Q(z)} = P'(z), \tag{5.6}$$

where $L(z) = H'(z) + Q'(z) H(z)$. Eliminating $e^{Q(z)}$ from (5.5) and (5.6), we have

$$f^{n-1}(z)(L(z) f(z) - n H(z) f'(z)) = P(z) L(z) - P'(z) H(z). \tag{5.7}$$

Note that $n - 1 \geq 2$ and $P(z) L(z) - P'(z) H(z)$ is a differential-difference polynomial in f and the total degree is at most 1. By Lemma 2.2 and Remark 2.1, we obtain

$$m(r, L(z) f(z) - n H(z) f'(z)) = S(r, f)$$

and

$$m(r, f(z)(L(z) f(z) - n H(z) f'(z))) = S(r, f).$$

If $L(z) f(z) - n H(z) f'(z) \not\equiv 0$, then

$$\begin{aligned} T(r, f) = m(r, f) &\leq m(r, f(z)(L(z) f(z) - n H(z) f'(z))) \\ &+ m\left(r, \frac{1}{L(z) f(z) - n H(z) f'(z)}\right) \\ &\leq T(r, L(z) f(z) - n H(z) f'(z)) + S(r, f) = S(r, f), \end{aligned}$$

which yields a contradiction. If $L(z) f(z) - n H(z) f'(z) \equiv 0$, then

$$\frac{q'(z)}{q(z)} + Q'(z) + \frac{f'(z+c)}{f(z+c)} = n \frac{f'(z)}{f(z)}.$$

By integration, we see that there exists a $C \in \mathbb{C}\backslash\{0\}$ such that

$$q(z)f(z+c)\mathrm{e}^{Q(z)} = Cf^n(z),\ C \in \mathbb{C}\backslash\{0\}. \tag{5.8}$$

Substituting (5.8) into (1.7) gives

$$(1+C)f^n(z) = P.$$

If $1+C \neq 0$, we have $T(r,f) = m(r,f) = S(r,f)$, a contradiction. If $1+C=0$, then $P(z) = p_1\mathrm{e}^{\lambda z} + p_2\mathrm{e}^{-\lambda z} \equiv 0$, a contradiction.

Case 3 If $\rho(f) = 1$, by applying Lemma 2.5, from (1.7) we obtain

$$\begin{aligned}
T(r,\mathrm{e}^{Q(z)}) = m(r,\mathrm{e}^{Q(z)}) &= m\left(\frac{p_1\mathrm{e}^{\lambda z} + p_2\mathrm{e}^{-\lambda z} - f^n(z)}{q(z)f(z+c)}\right)\\
&\leq m\left(r, \frac{1}{q(z)f(z+c)}\right) + m(r, p_1\mathrm{e}^{\lambda z} + p_2\mathrm{e}^{-\lambda z}) + m(r, f^n(z)) + O(1)\\
&\leq m\left(r, \frac{f(z)}{q(z)f(z+c)}\right) + m\left(r, \frac{1}{f(z)}\right) + nT(r,f)\\
&\quad +2T(r,\mathrm{e}^{\lambda z}) + S(r,\mathrm{e}^{\lambda z})\\
&\leq (n+1)T(r,f) + S(r,f) + 2T(r,\mathrm{e}^{\lambda z}) + S(r,\mathrm{e}^{\lambda z}).
\end{aligned}$$

Note that $\deg Q \geq 1$, then $1 \leq \deg Q = \sigma(\mathrm{e}^{Q(z)}) \leq \max\{\rho(\mathrm{e}^{\lambda z}), \rho(f)\} = 1$, that is $\rho(f) = \deg Q = 1$.

If f belongs to Γ_0, and noting that $\rho(f) = \deg Q = 1$, we define $f = \mathrm{e}^{Az+B}$ and $Q(z) = az+b$, where $a, A \in \mathbb{C}\backslash\{0\}$ and $b, B \in \mathbb{C}$. Substituting these into (1.7) yields

$$\mathrm{e}^{nB}\mathrm{e}^{(nA+\lambda)z} + \mathrm{e}^{Ac+b+B}q(z)\mathrm{e}^{(A+a+\lambda)z} - p_1\mathrm{e}^{2\lambda z} - p_2 = 0. \tag{5.9}$$

We distinguish four cases below.

Case 1 If $nA+\lambda = 0$ and $A+a+\lambda = 0$. By Lemma 2.3 and (5.9), we obtain $p_1 = 0$, a contradiction.

Case 2 If $nA+\lambda = 0$ and $A+a+\lambda \neq 0$. If $A+a+\lambda \neq 2\lambda$, by Lemma 2.3 and (5.9), we have $p_1 = q(z) \equiv 0$, a contradiction. If $A+a+\lambda = 2\lambda$, then $A = -\frac{\lambda}{n}$, $a = \frac{n+1}{n}\lambda$, substituting these into (5.9) yields

$$\left(q(z)\mathrm{e}^{Ac+b+B} - p_1\right)\mathrm{e}^{2\lambda z} = p_2 - \mathrm{e}^{nB},$$

by Lemma 2.3, we have $p_2 - \mathrm{e}^{nB} \equiv q(z)\mathrm{e}^{Ac+b+B} - p_1 \equiv 0$, then $q(z)$ reduces to a non-zero constant, and $f(z) = \mathrm{e}^{-\frac{\lambda}{n}z+B}$, $Q(z) = \frac{n+1}{n}\lambda z + b$.

Case 3 If $nA+\lambda \neq 0$ and $A+a+\lambda = 0$. If $nA+\lambda \neq 2\lambda$, it follows from Lemma 2.3 that $p_1 = 0$, a contradiction. If $nA+\lambda = 2\lambda$, then $A = \frac{\lambda}{n}$, $a = -\frac{n+1}{n}\lambda$.

Substituting these into (5.9) yields

$$(e^{nB} - p_1)e^{2\lambda z} = p_2 - q(z)e^{Ac+b+B},$$

by Lemma 2.3, we have $e^{nB} - p_1 \equiv p_2 - q(z)e^{Ac+b+B} \equiv 0$, then $q(z)$ reduces to a non-zero constant, and $f(z) = e^{\frac{\lambda}{n}z+B}$, $Q(z) = -\frac{n+1}{n}\lambda z + b$.

Case 4 If $nA + \lambda \neq 0$ and $A + a + \lambda \neq 0$. If $nA + \lambda$, $A + a + \lambda$ and 2λ are pairwise distinct from each other, by Lemma 2.3 and (5.9), we have $p_1 = p_2 \equiv q(z) \equiv 0$, a contradiction. If only two of $nA + \lambda$, $A + a + \lambda$ and 2λ coincide, without loss of generality, suppose that $nA + \lambda = A + a + \lambda \neq 2\lambda$, then (5.9) can be written as:

$$\left(e^{nB} + e^{Ac+b+B}q(z)\right)e^{(nA+\lambda)z} - p_1e^{2\lambda z} - p_2 = 0.$$

From the above equality and using Lemma 2.3, we have $p_1 = p_2 = 0$, which implies a contradiction. If $nA + \lambda = A + a + \lambda = 2\lambda$, then we write (5.9) as:

$$\left(e^{nB} + e^{Ac+b+B}q(z) - p_1\right)e^{2\lambda z} - p_2 = 0.$$

It follows from Lemma 2.3 that $p_2 = 0$, a contradiction.

This completes the proof of Theorem 1.3.

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