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On the Fourth Coefficient of Functions in the Carathéodory Class

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Abstract In the present paper, a formula for the fourth coefficient of Carathéodory functions was computed.

Keywords Carathéodory functions · Coefficient functionals · Fourth coefficient · Fekete-Szegö problem · Hankel determinants

Mathematics Subject Classification Primary 30C45

1 Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ be its subclass of functions f normalized by $f(0) := 0$, $f'(0) := 1$, i.e., of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1, \ z \in \mathbb{D}. \tag{1.1}$$

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The class of Carathéodory functions $\mathcal{P}$ consists of the functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{1.2}$$

having positive real part in $\mathbb{D}$. If the members of a subclass $\mathcal{F}$ of $\mathcal{A}$ have a representation involving the Carathéodory class $\mathcal{P}$, the coefficients of the corresponding functions in $\mathcal{F}$ can be expressed using the coefficients of functions in $\mathcal{P}$. Starlike and convex univalent functions in $\mathcal{F}$ [6, pp. 41–45] form classical examples of classes having an analytic description in terms of by the class $\mathcal{P}$. Therefore coefficient formulas for functions in $\mathcal{P}$ are a basic tool for the examination of coefficient functionals in the corresponding classes $\mathcal{F}$.

In the study of extremal problems related to early coefficients, the known formula for c_2 (e.g., [15, p.166]) and the formula for c_3 found by Libera and Zlotkiewicz [12,13], both cited here in Lemma 2.2, are particularly efficient. Because of this, the two formulas were often used by various authors to study the Fekete-Szegö functional, Hankel determinants (e.g., [1–4,7,11,14,16,17]), coefficients of inverse functions (e.g., [12,13]), and many others issues.

According to the authors' knowledge, formulas for the coefficients c_n for $n \geq 4$ analogous to the formulas (2.2) and (2.3) where not yet published. In this paper, we provide a formula for c_4 and this is a new result. We think that the formula (2.10) for c_4 can be applied to different extremal coefficient problems in a similar way as the formulas for c_2 and c_3 given in Lemma 2.2 have been used in the past. This was done in the forthcoming papers [9,10], where, for instance, the Hankel determinants of the third kind for starlike and convex functions have been estimated.

2 Main Results

The following two lemmas for functions in the class $\mathcal{P}$ are well known:

Lemma 2.1 ([8], Carathéodory) *The power series for a function p given by* (1.2) *converges in $\mathbb{D}$ to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n := \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ \overline{c}_1 & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \overline{c}_n & \overline{c}_{n-1} & \overline{c}_{n-2} & \cdots & 2 \end{vmatrix}, \quad n \in \mathbb{N},$$

are non-negative. They are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z), \quad z \in \mathbb{D}, \tag{2.1}$$

where $m \in \mathbb{N}$ *and*

$$p_0(z) := \frac{1+z}{1-z}, \quad z \in \mathbb{D},$$

$\rho_k > 0$, $\sum_{k=1}^{m} \rho_k = 1$, $t_k \in [0, 2\pi)$ *and* $t_k \neq t_j$ *for* $k \neq j$*; in this case* $D_n > 0$ *for* $n < m - 1$ *and* $D_n = 0$ *for* $n \geq m$*.*

The following lemma contains the well-known formula for c_2 (e.g., [15, p.166]) and the formula for c_3 found by Libera and Zlotkiewicz [12,13].

Lemma 2.2 *If* $p \in \mathcal{P}$ *is of the form* (1.2) *with* $c_1 \geq 0$*, then*

$$2c_2 = c_1^2 + (4 - c_1^2)\zeta \tag{2.2}$$

and

$$4c_3 = c_1^3 + (4 - c_1^2)c_1\zeta(2 - \zeta) + 2(4 - c_1^2)(1 - |\zeta|^2)\eta \tag{2.3}$$

for some $\zeta, \eta \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$*.*

We now complete the formula (2.2) in case $\zeta \in \mathbb{T} := \partial\mathbb{D}$ and $c_1 \in [0, 2)$. When $c_1 = 2$, then $D_1 = 0$, and by Lemma 2.1 the function p is of the form (2.1) with $m = 1$, $\rho_1 = 1$ and $t_1 = 0$, i.e., $p \equiv p_0$.

Lemma 2.3 *The formula* (2.2) *with* $c_1 \in [0, 2)$ *and* $\zeta \in \mathbb{T}$ *holds only for the functions*

$$p(z) := \frac{1 + \tau(1+\zeta)z + \zeta z^2}{1 - \tau(1-\zeta)z - \zeta z^2}, \quad z \in \mathbb{D}, \tag{2.4}$$

where $\tau \in [0, 1)$*.*

Proof Let $p \in \mathcal{P}$ be of the form (1.2) with $c_1 \in [0, 2)$ and such that (2.2) holds for some $\zeta \in \mathbb{T}$. Clearly, (2.2) is equivalent to

$$\left|2c_2 - c_1^2\right| = 4 - c_1^2. \tag{2.5}$$

Define

$$\varphi(z) := \frac{p(z) - 1}{z(p(z) + 1)}, \quad z \in \mathbb{D}\backslash\{0\}, \quad \varphi(0) := \frac{1}{2}c_1. \tag{2.6}$$

Thus $\varphi \in \mathcal{H}$ and (2.5) can be rewritten as

$$|\varphi'(0)| = 1 - |\varphi(0)|^2. \tag{2.7}$$

Since $\mathbb{D} \ni z \mapsto z\varphi(z)$ is a Schwarz function, i.e., a self-map of $\mathbb{D}$ keeping the origin fixed, by the maximum principle for analytic functions, the function φ is a self-map of $\mathbb{D}$. By the Schwarz-Pick Lemma (e.g., [5, p. 5]) it follows that (2.7) holds if and only if φ is an automorphism of $\mathbb{D}$, i.e.,

$$\varphi(z) := \lambda \frac{z-\alpha}{1-\overline{\alpha}z}, \quad z \in \mathbb{D}, \tag{2.8}$$

for some $\lambda \in \mathbb{T}$ and $\alpha \in \mathbb{D}$. Since $c_1 \in [0, 2)$, by (2.8) and (2.6) we have $-\varphi(0) = \lambda\alpha \in (-1, 0]$. Thus $\lambda\alpha = -|\alpha|$, so that $\alpha = -\tau\overline{\lambda}$, where $\tau := |\alpha| \in [0, 1)$. Hence, by (2.6), and (2.8) it follows that (2.5) holds if and only if

$$p(z) = \frac{1+z\varphi(z)}{1-z\varphi(z)} = \frac{1+\tau(1+\lambda)z+\lambda z^2}{1-\tau(1-\lambda)z-\lambda z^2}, \quad z \in \mathbb{D}. \tag{2.9}$$

Since $c_1 = 2\tau$ and $c_2 = 2(\lambda - \lambda\tau^2 + \tau^2)$, we have

$$\zeta = \frac{2c_2 - c_1^2}{4-c_1^2} = \frac{4(\lambda - \lambda\tau^2 + \tau^2) - 4\tau^2}{4-4\tau^2} = \lambda.$$

This together with (2.9) shows that p is as in (2.4) and completes the proof. □

We will now prove the main result of this paper, i.e., the formula (2.10) for the coefficient c_4. The proof of (2.10) is elementary, but requires tedious algebraic computations. In particular, the proof of equality (2.16), which is the key in Lemma 2.4, requires long arduous algebraic transformations and appropriate grouping of expressions.

Lemma 2.4 *If $p \in \mathcal{P}$ is of the form (1.2) with $c_1 \geq 0$, then*

$$\begin{aligned} 8c_4 = c_1^4 + (4-c_1^2)\zeta\left[c_1^2(\zeta^2 - 3\zeta + 3) + 4\zeta\right] - 4(4-c_1^2)(1-|\zeta|^2) \\ \times\left[c_1(\zeta-1)\eta + \overline{\zeta}\eta^2 - \left(1-|\eta|^2\right)\xi\right] \end{aligned} \tag{2.10}$$

for some $\zeta, \eta, \xi \in \overline{\mathbb{D}}$.

Proof By Lemma 2.1, we have

$$D_4 = 2M_1 - c_1M_2 + c_2M_3 - c_3M_4 + c_4M_5 \geq 0, \tag{2.11}$$

where M_i $(i = 1, 2, 3, 4, 5)$ is the determinant of the minor matrix of the ith entry in the first row in D_4. Then inequality (2.11) is equivalent to

$$\begin{aligned} &- c_1\overline{c}_4M_2^4 + c_2\overline{c}_4M_3^4 - c_3\overline{c}_4M_4^4 - c_1c_4M_5^1 + \overline{c}_2c_4M_5^2 - \overline{c}_3c_4M_5^3 + |c_4|^2M_5^4 \\ &- 2M_1 + c_1^2M_2^1 - c_1\overline{c}_2M_2^2 + c_1\overline{c}_3M_2^3 - c_1c_2M_3^1 + |c_2|^2M_3^2 - c_2\overline{c}_3M_3^3 \\ &+ c_1c_3M_4^1 - \overline{c}_2c_3M_4^2 + |c_3|^2M_4^3 \leq 0, \end{aligned}$$

where M_i^j ($i = 1, 2, 3, 4, 5,\ j = 1, 2, 3, 4$) is the determinant of the minor matrix of the jth entry in the first column in M_i. Since $M_5^1 = \overline{M}_2^4$, $M_5^2 = \overline{M}_3^4$ and $M_5^3 = \overline{M}_4^4$, the latter inequality is equivalent to

$$\begin{aligned}&- c_1\overline{c}_4 M_2^4 + c_2\overline{c}_4 M_3^4 - c_3\overline{c}_4 M_4^4 - c_1 c_4 \overline{M}_2^4 + \overline{c}_2 c_4 \overline{M}_3^4 - \overline{c}_3 c_4 \overline{M}_4^4 + |c_4|^2 M_5^4 \\ &- 2M_1 + c_1^2 M_2^1 - c_1\overline{c}_2 M_2^2 + c_1\overline{c}_3 M_2^3 - c_1 c_2 M_3^1 + |c_2|^2 M_3^2 - c_2\overline{c}_3 M_3^3 \\ &+ c_1 c_3 M_4^1 - \overline{c}_2 c_3 M_4^2 + |c_3|^2 M_4^3 \leq 0.\end{aligned}$$

Multiplying both sides of the above inequality by $M_5^4 = D_2 \geq 0$, we equivalently obtain

$$\begin{aligned}&\left|c_4 M_5^4 - c_1 M_2^4 + c_2 M_3^4 - c_3 M_4^4\right|^2 - \left|-c_1 M_2^4 + c_2 M_3^4 - c_3 M_4^4\right|^2 \\ &\quad \leq M_5^4 \Big(2M_1 - c_1^2 M_2^1 + c_1\overline{c}_2 M_2^2 - c_1\overline{c}_3 M_2^3 + c_1 c_2 M_3^1 - |c_2|^2 M_3^2 + c_2\overline{c}_3 M_3^3 \\ &\qquad - c_1 c_3 M_4^1 + \overline{c}_2 c_3 M_4^2 - |c_3|^2 M_4^3\Big). \qquad (2.12)\end{aligned}$$

Let

$$A := c_1 M_2^4 - c_2 M_3^4 + c_3 M_4^4 \qquad (2.13)$$

and

$$\begin{aligned}B_1 := &\left|-c_1 M_2^4 + c_2 M_3^4 - c_3 M_4^4\right|^2 \\ &+ M_5^4 \Big(2M_1 - c_1^2 M_2^1 + c_1\overline{c}_2 M_2^2 - c_1\overline{c}_3 M_2^3 + c_1 c_2 M_3^1 - |c_2|^2 M_3^2 + c_2\overline{c}_3 M_3^3 \\ &\quad - c_1 c_3 M_4^1 + \overline{c}_2 c_3 M_4^2 - |c_3|^2 M_4^3\Big). \qquad (2.14)\end{aligned}$$

Since

$$M_2^4 = \begin{vmatrix} c_1 & c_2 & c_3 \\ 2 & c_1 & c_2 \\ c_1 & 2 & c_1 \end{vmatrix} = c_1^3 + c_1(-4 + c_2)c_2 + 4c_3 - c_1^2 c_3,$$

$$M_3^4 = \begin{vmatrix} 2 & c_2 & c_3 \\ c_1 & c_1 & c_2 \\ \overline{c}_2 & 2 & c_1 \end{vmatrix} = 2c_1^2 + 2c_1 c_3 + |c_2|^2 c_2 - c_1\overline{c}_2 c_3 - 4c_2 - c_1^2 c_2$$

and

$$M_4^4 = \begin{vmatrix} 2 & c_1 & c_3 \\ c_1 & 2 & c_2 \\ \overline{c}_2 & c_1 & c_1 \end{vmatrix} = 4c_1 + c_1^2 c_3 + c_1|c_2|^2 - 2\overline{c}_2 c_3 - 2c_1 c_2 - c_1^3,$$

we get from (2.13)

$$A = c_1^4 + 4c_2^2 - 2c_1^3 c_3 - 4c_1(c_2-2)c_3 + c_1^2\left(-6c_2 + 2c_2^2 + c_3^2\right) - \left(c_2^3 - 2c_1c_2c_3 + 2c_3^2\right)\overline{c}_2. \tag{2.15}$$

Computing the determinant M_1, each of the determinants M_i^j ($i = 1, 2, 3, 4, 5$, $j = 1, 2, 3, 4$), and performing further elementary but tedious transformations show that the following identity holds:

$$B_1 = B^2, \tag{2.16}$$

where

$$B = 16 - 12c_1^2 + c_1^4 + 4c_1^2 c_2 - c_1^3 c_3 + \left(-2c_1^2(c_2-2) - 8c_2 + 4c_1c_3\right)\overline{c}_2 + \left(c_2^2 - c_1c_3\right)\overline{c}_2^2 - \left(c_1^3 + c_1(c_2-4)c_2 + 4c_3 - c_1^2 c_3\right)\overline{c}_3. \tag{2.17}$$

Using (2.2) and (2.3) in (2.15) and (2.17) we respectively obtain

$$A = -\frac{1}{16}(4-c_1^2)^2(1-|\zeta|^2)C, \tag{2.18}$$

where

$$C := c_1^4(-1+\zeta)^3 - 16\zeta^2 - 4c_1^2\zeta(3-4\zeta+\zeta^2) + 16c_1(-1+\zeta)\eta - 4c_1^3(-1+\zeta)\eta + 4(-4+c_1^2)\left[(c_1(-1+\zeta)\zeta - \eta)\overline{\zeta}\eta + |\zeta|^2\overline{\zeta}\eta^2\right],$$

and

$$B = \frac{1}{4}(4-c_1^2)^3(1-|\zeta|^2)^2(1-|\eta|^2). \tag{2.19}$$

Taking into account (2.13), (2.14) and (2.16) the inequality (2.12) can be written as

$$\left|M_5^4 c_4 - A\right|^2 \le B^2,$$

i.e., using the fact that $B \ge 0$, as

$$M_5^4 c_4 = A + B\xi, \tag{2.20}$$

for some $\xi \in \overline{\mathbb{D}}$. Since by (2.2),

$$M_5^4 = \begin{vmatrix} 2 & c_1 & c_2 \\ c_1 & 2 & c_1 \\ \overline{c}_2 & c_1 & 2 \end{vmatrix} = 8 + 2c_1^2\,\mathrm{Re}(c_2) - 2|c_2|^2 - 4c_1^2 = \frac{1}{2}(4-c_1^2)^2(1-|\zeta|^2),$$

it follows from (2.18) and (2.19) that (2.20) takes the form

$$\frac{1}{2}(4-c_1^2)^2(1-|\zeta|^2)c_4 = -\frac{1}{16}(4-c_1^2)^2(1-|\zeta|^2)C+\frac{1}{4}(4-c_1^2)^3(1-|\zeta|^2)^2(1-|\eta|^2)\xi,$$

or, equivalently,

$$(4-c_1^2)^2(1-|\zeta|^2)\left[8c_4 + C - 4(4-c_1^2)(1-|\zeta|^2)(1-|\eta|^2)\xi\right] = 0$$

with $\zeta,\ \eta,\ \xi \in \overline{\mathbb{D}}$. Thus (2.10) holds in the cases $c_1 \neq 2$ and $|\zeta| \neq 1$.

As previously noted, if $c_1 = 2$, then $p \equiv p_0$. Thus $c_4 = 2$ and (2.10) becomes obvious.

Assume now that $c_1 \in [0, 2)$ and $\zeta \in \mathbb{T}$. Due to Lemma 2.3 it remains to consider functions of the form (2.4) for which

$$c_1 = 2\tau, \quad c_4 = 2\zeta^2 + 6\zeta\tau^2 - 8\zeta^2\tau^2 + 2\zeta^3\tau^2 + 2\tau^4 - 6\zeta\tau^4 + 6\zeta^2\tau^4 - 2\zeta^3\tau^4. \tag{2.21}$$

Therefore it is sufficient to show that (2.10) with $\zeta \in \mathbb{T}$ holds for such functions, i.e., that

$$8c_4 = c_1^4 + (4-c_1^2)\zeta\left[c_1^2(\zeta^2 - 3\zeta + 3) + 4\zeta\right]$$

is true for c_1 and c_4 given by (2.21). Since this can be easily confirmed the proof of the lemma is complete. □

Remark 2.5 Let us announce that in the forthcoming papers [9,10], by using Lemmas 2.2 and 2.4, it is shown that for starlike and convex functions f of the form (1.1) (see, e.g., [6, pp. 40–42]) the Hankel determinant

$$H_{3,1}(f) := \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

is bounded by $8/9$ and by $4/135$, respectively. In the convex case the result is sharp.

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