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Bohr Inequality for Odd Analytic Functions

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Abstract We determine the Bohr radius for the class of odd functions f satisfying $|f(z)| \leq 1$ for all $|z| < 1$, solving the recent problem of Ali et al. (J Math Anal Appl 449(1):154–167, 2017). In fact, we solve this problem in a more general setting. Then we discuss Bohr's radius for the class of analytic functions g, when g is subordinate to a member of the class of odd univalent functions.

Keywords Analytic functions · p-symmetric functions · Bohr's inequality · Schwarz lemma · Subordination and odd univalent functions

Mathematics Subject Classification Primary 30A10 · 30H05 · 30C35; Secondary 30C45

1 Preliminaries and Main Results

Let $\mathcal{A}$ denote the space of all functions analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ equipped with the topology of uniform convergence on compact subsets of $\mathbb{D}$. Then the classical Bohr's inequality [14] states that if a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to $\mathcal{A}$ and $|f(z)| < 1$ for all $z \in \mathbb{D}$, then $M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n \leq 1$ for

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all $|z| = r \leq 1/3$ and the constant $1/3$ cannot be improved. The constant $r_0 = 1/3$ is known as Bohr's radius. Bohr actually obtained the inequality for $r \leq 1/6$, but subsequently later, Wiener, Riesz and Schur, independently established the sharp inequality for $|z| \leq 1/3$. For a detailed account of the development, we refer to the recent survey article on this topic [8] and the references therein.

A variety of results related to Bohr's theorem in several complex variables have appeared recently. See [19] and the references there. For example, Boas and Khavinson [13] obtained some multidimensional generalizations of Bohr's theorem and Aizenberg [4,5] extended it for further studies on the topic. Using Bohr's inequality, Dixon [15] constructed an example of a Banach algebra that satisfies von Neumann's inequality but is not isomorphic to the algebra of bounded operators on a Hilbert space. There has been considerable interest after the appearance of the work of Dixon. Paulsen and Singh extended Bohr's inequality to Banach algebras in [18]. In [12], the Bohr phenomenon for functions in Hardy spaces is discussed. In [11], Balasubramanian *et al.* extended the Bohr inequality to the setting of Dirichlet series. For certain other results on the Bohr phenomenon, we refer to [1–3,6,7].

The present investigation is motivated by the following problem of Ali, Barnard and Solynin [9].

Problem 1 [9] Find the Bohr radius for the class of odd functions f satisfying $|f(z)| \leq 1$ for all $z \in \mathbb{D}$.

In [9, Lem. 2.2], it was shown that for odd f, $M_f(r) \leq 1$ for all $|z| = r \leq r_*$, where r_* is a solution of the equation

$$5r^4 + 4r^3 - 2r^2 - 4r + 1 = 0,$$

which is unique in the interval $1/\sqrt{3} < r < 1$. The value of r_* can be calculated in terms of radicals and it is equal to $0.7313\ldots$.

Moreover, in [9], an example of the form $f(z) = z(z^2 - a)/(1 - az^2)$ was also given to conclude that the Bohr radius for the class of odd functions satisfies the inequalities $r_* \leq r \leq r^* \approx 0.789991$, where

$$r^* = \frac{1}{4}\sqrt{\frac{B-2}{6}} + \frac{1}{2}\sqrt{3\sqrt{\frac{6}{B-2}} - \frac{B}{24} - \frac{1}{6}}, \tag{1}$$

with

$$B = (3601 - 192\sqrt{327})^{\frac{1}{3}} + (3601 + 192\sqrt{327})^{\frac{1}{3}}.$$

One of the aims of this article is to solve this problem in a more general form. Namely, we are going to solve an analogous problem for p-symmetric functions of the form $f(z) = z\sum_{k=0}^{\infty} a_{pk+1} z^{pk}$. We should remark that the p-symmetric property in the case $p > 1$ brings serious difficulties because if we use the sharp inequalities $|a_n| \leq 1 - |a_0|^2$ $(n \geq 1)$ simultaneously (as in the classical case) we will not obtain the sharp result due to the fact that in the extremal case we have $|a_0| < 1$. Also it is important to note that in the classical case there is no extremal function while in our case there is. We now state our main results and their corollaries. The proofs will be presented in Sect. 2.

Theorem 1 *Let $p \in \mathbb{N}$, $f(z)$ be analytic and p-symmetric in $\mathbb{D}$ such that $|f(z)| \leq 1$ in $\mathbb{D}$. Then*

$$M_f(r) \leq 1 \text{ for } r \leq r_p,$$

where r_p is the maximal positive root of the equation

$$-6r^{p-1} + r^{2(p-1)} + 8r^{2p} + 1 = 0$$

in $(0, 1)$. The extremal function has the form $z(z^p - a)/(1 - az^p)$, where

$$a = \left(1 - \frac{\sqrt{1 - {r_p}^{2p}}}{\sqrt{2}}\right) \frac{1}{{r_p}^p}.$$

The result for $p = 1$ is well known with $r_1 = 1/\sqrt{2}$.
The case $p = 2$ has a special interest since it provides a solution to Problem 1.

Corollary 1 *If $f(z)$ is odd analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$, then*

$$M_f(r) \leq 1 \text{ for } r \leq r_2 = 0.789991\ldots,$$

where $r_2 = r^*$ is given by (1). The extremal function has the form $z(z^2 - a)/(1 - az^2)$.

Thus, it turns out that the upper bound r^* found in [9] is sharp and provides the exact value for the Bohr radius for the class of odd functions. In addition, it is worth pointing out that for the case $p = 3$ in Theorem 1, r_3^* gives the value $(\sqrt{7 + \sqrt{17}})/4$.

To state our next result, we need to introduce the notion of subordination. Let $f, g \in \mathcal{A}$. Then g is *subordinate* to f, written $g \prec f$ or $g(z) \prec f(z)$, if there exists a $w \in \mathcal{A}$ satisfying $w(0) = 0$, $|w(z)| < 1$ and $g(z) = f(w(z))$ for $z \in \mathbb{D}$. In the case when f is univalent in $\mathbb{D}$, $g \prec f$ if and only if $g(0) = f(0)$ and $g(\mathbb{D}) \subset f(\mathbb{D})$ (see [10, Ch. 2] and [16, p. 190, p. 253]). By the Schwarz lemma, it follows that

$$|g'(0)| = |f'(w(0))w'(0)| \leq |f'(0)|.$$

For important discussions on the Schwarz lemma and its various consequences, we refer to [10].

Now for a given f, let $S(f) = \{g : g \prec f\}$. In [1, Thm. 1], it was shown that if $f, g \in \mathcal{A}$ such that f is univalent in $\mathbb{D}$ and $g \in S(f)$, then the inequality $M_g(r) \leq 1$ holds with $r_f = 3 - 2\sqrt{2} \approx 0.17157$. The sharpness of r_f is shown by the Koebe function $f(z) = z/(1 - z)^2$. Our next result concerns Bohr's radius for the space of subordinations when the subordinating function is odd and univalent in $\mathbb{D}$. In this case, the Bohr radius is much larger and the proof in this case is completely different. Unlike the earlier case [1, Thm. 1] where the proof requires coefficient estimation, for the odd univalent function $f(z) = \sum_{k=1}^{\infty} a_{2k-1} z^{2k-1}$, the sharp bound for a_{2k-1} is still unknown. Even if we use the known coefficient bound for odd univalent functions, we do not get a better bound for the Bohr radius.

Theorem 2 *If f, g are analytic in $\mathbb{D}$ such that $f(z) = z + \sum_{k=2}^{\infty} a_{2k-1} z^{2k-1}$ is odd univalent in $\mathbb{D}$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n \in S(f)$, then $M_g(r) \leq 1$ holds for $r \leq r_*$, where $r_* = 0.554958...$ is the minimal positive root of the equation*

$$x^2 = (1-x)^2(1+x).$$

If g in Theorem 2 is also odd analytic, then one can easily obtain the sharp value of the Bohr radius in Theorem 2 (see Remark 2). We conclude the section with the following problem.

Problem 2 Find the Bohr radius for the class of odd functions f satisfying $0 < |f(z)| \leq 1$ for all $0 < |z| < 1$.

2 Proofs of Theorems 1 and 2, and Remarks

For the proof of Theorem 1, we need the following lemmas.

Lemma 1 *If r_p is the maximal positive root of the equation*

$$8r^{2p} + r^{2(p-1)} - 6r^{p-1} + 1 = 0,$$

then $2{r_p}^{p+1} \leq 1$.

Proof Let $y = r_p^{p+1}$. Then we have a quadratic equation:

$$\left(8 + 1/r_p^2\right) y^2 - 6y + r_p^2 = 0$$

which has two solutions

$$y = \frac{3 \pm 2\sqrt{2}\sqrt{1-r_p^2}}{8 + 1/r_p^2} \leq \frac{3 + 2\sqrt{2}\sqrt{1-r_p^2}}{8 + 1/r_p^2}.$$

Consequently,

$$2r_p^{p+1} = 2y \leq \frac{6 + 4\sqrt{2}\sqrt{1-r_p^2}}{8 + 1/r_p^2} \leq \sup_{r \in (0,1]} \frac{6 + 4\sqrt{2}\sqrt{1-r^2}}{8 + 1/r^2} = 1,$$

which completes the proof of Lemma 1. ◻

Lemma 2 *Let* $|b_0| < 1$ *and* $0 < R \leq 1$. *If* $g(z) = \sum_{k=0}^{\infty} b_k z^k$ *is analytic and satisfies the inequality* $|g(z)| \leq 1$ *in* $\mathbb{D}$, *then the following sharp inequality holds*:

$$\sum_{k=1}^{\infty} |b_k|^2 R^{pk} \leq R^p \frac{\left(1 - |b_0|^2\right)^2}{1 - |b_0|^2 R^p}. \tag{2}$$

Proof Let $b_0 = a$. Then, it is easy to see that the condition on g can be rewritten in terms of subordination as

$$g(z) = \sum_{k=0}^{\infty} b_k z^k \prec \phi(z), \tag{3}$$

where

$$\phi(z) = \frac{a - z}{1 - \overline{a}z} = a - (1 - |a|^2) \sum_{k=1}^{\infty} (\overline{a})^{k-1} z^k, \quad z \in \mathbb{D}.$$

Note that ϕ is analytic in $\mathbb{D}$ and $|\phi(z)| \leq 1$ for $z \in \mathbb{D}$. The subordination relation (3) gives (see for example Goluzin [17, p. 370–371] and [16, p. 193])

$$\sum_{k=1}^{\infty} |b_k|^2 R^{2k} \leq (1 - |a|^2)^2 \sum_{k=1}^{\infty} |a|^{2(k-1)} R^{2k} = R^2 \frac{(1 - |a|^2)^2}{1 - |a|^2 R^2}$$

from which we arrive at the inequality (2). □

Proof of Theorem 1 Let $r = r_p$ and $f(z) = \sum_{k=0}^{\infty} a_{pk+1} z^{pk+1}$, where $|f(z)| \leq 1$ for $z \in \mathbb{D}$. First, we remark that the function f can be represented as $f(z) = zg(z^p)$, where $|g(z)| \leq 1$ in $\mathbb{D}$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is analytic in $\mathbb{D}$ with $b_k = a_{pk+1}$. Let $|b_0| = a$. Choose any $\rho > 1$ such that $\rho r \leq 1$. Then it follows that

$$\begin{aligned}
\sum_{k=1}^{\infty} |a_{pk+1}| r^{pk} &= \sum_{k=1}^{\infty} |b_k| r^{pk} \\
&\leq \sqrt{\sum_{k=1}^{\infty} |b_k|^2 \rho^{pk} r^{pk}} \sqrt{\sum_{k=1}^{\infty} \rho^{-pk} r^{pk}} \\
&\leq \sqrt{r^p \rho^p \frac{(1 - a^2)^2}{1 - a^2 r^p \rho^p}} \sqrt{\frac{\rho^{-p} r^p}{1 - \rho^{-p} r^p}} \\
&= \frac{r^p (1 - a^2)}{\sqrt{1 - a^2 r^p \rho^p}} \frac{1}{\sqrt{1 - \rho^{-p} r^p}}.
\end{aligned}$$

In the second and the third steps above we have used the classical Cauchy–Schwarz inequality and (2) with $R = \rho r$, respectively. Hence,

$$\sum_{k=1}^{\infty} |a_{pk+1}| r^{pk} \le \frac{r^p(1-a^2)}{\sqrt{1-a^2 r^p \rho^p}} \frac{1}{\sqrt{1-\rho^{-p} r^p}}. \tag{4}$$

We need to consider the cases $a \ge r^p$ and $a < r^p$ separately.
Case 1 $a \ge r^p$. In this case set $\rho = 1/\sqrt[p]{a}$ and obtain

$$\sum_{k=0}^{\infty} |a_{pk+1}| r^{pk+1} \le r\left(a + r^p \frac{(1-a^2)}{1-r^p a}\right). \tag{5}$$

For convenience, we may let $\alpha = r^p$ and consider

$$\psi(x) = x + \alpha \frac{(1-x^2)}{1-\alpha x}, \quad x \in [0, 1].$$

Finally, we just need to maximize $\psi(x)$ over the interval [0, 1]. We see that ψ has two critical points and find that the maximum occurs when

$$x_1 = \left(1 - \frac{\sqrt{1-\alpha^2}}{\sqrt{2}}\right)\frac{1}{\alpha}, \quad \alpha \ge \frac{1}{3},$$

and thus, $\psi(x) \le \psi(x_1)$. Consequently, by (5), we find for the $r = r_p$ defined in Theorem 1 that

$$\sum_{k=0}^{\infty} |a_{pk+1}| r^{pk+1} \le \frac{1}{r^{p-1}} \left(3 - 2\sqrt{2}\sqrt{1-r^{2p}}\right) = 1. \tag{6}$$

Case 2 $a < r^p$. In this case we set $\rho = 1/r$ and apply (4). As a result we get

$$\sum_{k=0}^{\infty} |a_{pk+1}| r^{pk+1} \le r\left(a + r^p \sqrt{1-a^2}/\sqrt{1-r^{2p}}\right) \le 2r^{p+1} \le 1. \tag{7}$$

Here we omitted the critical point $a = \sqrt{1-r^{2p}}$ because it is less than or equal to r^p only in the case $r^{2p} > 1/2$ which contradicts Lemma 1.

The last inequality in (7) follows from Lemma 1.

Now, (6) and (7) complete the proof of the first part of Theorem 1. Now we need to say a few words about the extremal cases. We set $f(z) = z(z^p - a)/(1 - az^p)$ with $a = (1 - \frac{\sqrt{1-r^{2p}}}{\sqrt{2}})/r^p$ and then calculate the Bohr radius for it. It coincides with r.

Certainly, an extremal function is unique up to a rotation of a. To see this we just trace our inequalities and see that equality holds only when $|b_0| = a$. □

Proof of Theorem 2 Let $g \prec f$, where $g(z) = \sum_{n=1}^{\infty} b_n z^n$, and $f(z) = \sum_{k=1}^{\infty} a_{2k-1} z^{2k-1}$ is an odd univalent in $\mathbb{D}$. Here $a_1 = 1$ and thus, by the definition of the subordination, $|b_1| \leq 1$. First we show that

$$\sum_{k=1}^{\infty} |a_{2k-1}| r^{2k-1} \leq \frac{r}{1-r^2} \quad \text{for } |z| = r < 1. \tag{8}$$

To prove this, we use Robertson's inequality for odd univalent functions f (see, for instance, [10, Sec. 2.2]),

$$\sum_{k=1}^{n} |a_{2k-1}|^2 \leq n.$$

Using this, we derive that

$$S_n = \sum_{k=1}^{n} |a_{2k-1}| \leq \sqrt{n} \sqrt{\sum_{k=1}^{n} |a_{2k-1}|^2} \leq n. \tag{9}$$

It follows from (9) that

$$\begin{aligned}\sum_{k=1}^{\infty} |a_{2k-1}| r^{2k-1} &= |a_1|(r - r^3) + \sum_{k=2}^{\infty} S_k (r^{2k-1} - r^{2k+1}) \\ &\leq r - r^3 + \sum_{k=2}^{\infty} k(r^{2k-1} - r^{2k+1}) = \frac{r}{1-r^2}\end{aligned}$$

which proves (8).

Next, as $g \prec f$, we have by (8) that

$$\sum_{k=1}^{\infty} |b_k|^2 r^{2k} \leq \sum_{k=1}^{\infty} |a_{2k-1}|^2 r^{2(2k-1)} \leq \frac{r^2}{1-r^4}$$

which gives

$$\sum_{k=1}^{\infty} |b_k|^2 r^k \leq \frac{r}{1-r^2}. \tag{10}$$

Consequently, by the classical Cauchy–Schwarz inequality, we obtain

$$\sum_{k=1}^{\infty} |b_k| r^k \leq \sqrt{\sum_{k=1}^{\infty} |b_k|^2 r^k} \sqrt{\sum_{k=1}^{\infty} r^k} \leq \sqrt{\frac{r}{1-r^2} \frac{r}{1-r}} = \frac{r}{(1-r)\sqrt{1+r}}$$

which is less than or equal to 1 if $r^3 - 2r^2 - r + 1 \geq 0$. ◻

Remark 1 Now we show a possibility to slightly improve the Bohr inequality in Theorem 2. We see that

$$\begin{aligned}\sum_{k=1}^{\infty}|b_k|^2 r^k &= |b_1|r + |b_2|r^2 + \sum_{k=3}^{\infty}|b_k|r^k \\ &\le |b_1|r + |b_2|r^2 + \sqrt{\sum_{k=3}^{\infty}|b_k|^2 r^k}\sqrt{\sum_{k=3}^{\infty} r^k} \\ &\le \psi(|b_1|, |b_2|),\end{aligned}$$

where we have used (10) and the function

$$\psi(x, y) = rx + r^2 y + \frac{r^2}{\sqrt{1-r}}\sqrt{\frac{1}{1-r^2} - x^2 - ry^2} \tag{11}$$

with $x = |b_1|$ and $y = |b_2|$. Therefore, we have to find

$$\max\{\psi(x, y) : 0 \le x \le 1, \ x^2 + y \le 1\}.$$

First let us consider the case $y < 1 - x^2$. In this case

$$\frac{\partial}{\partial y}\psi(x, y) = 0 \text{ or } y = 0.$$

If $y = 0$, then $\max\{\psi(x, 0) : 0 \le x \le 1\} = 1121\sqrt{7/53}/410 < 1$ for $r = 0.59$ which is too big. Now consider the case

$$\frac{\partial}{\partial y}\psi(x, y) = r^2 - \frac{\sqrt{1/(1-r)}r^3 y}{\sqrt{1/(1-r^2) - x^2 - ry^2}} = 0$$

so that

$$y = \frac{\sqrt{1 - x^2 + r^2 x^2}}{\sqrt{r + r^2}} > 1 - x^2 \text{ for } r \le 0.6.$$

Therefore, we may assume that $y = 1 - x^2$. In this case we have to verify that

$$\max\{\psi(x, 1 - x^2) : 0 \le x \le 1\} \le 1,$$

where $\psi(x, 1 - x^2)$ is obtained from (11) by letting $y = 1 - x^2$. Straightforward and routine computations show that $r = 0.564\ldots$ which is slightly better than the estimate presented in Theorem 2.

We conclude that the Bohr radius in this case cannot be greater than $(\sqrt{5} - 1)/2 = 0.618034\ldots$. This upper bound can be easily obtained from the example $f(z) = z/(1 - z^2)$.

Remark 2 If $g \prec f$, where $g(z) = \sum_{k=1}^{\infty} b_{2k-1}z^{2k-1}$ is also odd, and $f(z) = \sum_{k=1}^{\infty} a_{2k-1}z^{2k-1}$ is an odd univalent in $\mathbb{D}$ (with $a_1 = 1$), then by Rogosinski's theorem [20] and then Robertson's inequality, we obtain that

$$\sum_{k=1}^{n} |b_{2k-1}|^2 \leq \sum_{k=1}^{n} |a_{2k-1}|^2 \leq n$$

which, as in the proof of Theorem 2, implies that

$$\sum_{k=1}^{\infty} |b_{2k-1}|r^{2k-1} \leq \frac{r}{1-r^2}.$$

Note that $\frac{r}{1-r^2} = 1$ gives $r = (\sqrt{5}-1)/2$ and thus, in this case, we have the sharp Bohr radius and the extremal function is $f(z) = z/(1-z^2)$.

While determining the Bohr radius in the case of functions f analytic in the unit disk, one often requires sharp estimates on the Taylor coefficients of f. In the class of *odd* univalent functions, the sharp coefficient estimate is still unknown, unlike for the class of all univalent analytic functions solved by de Branges. In spite of this drawback, it is interesting to state the following sharp result as a corollary of the relation (8).

Corollary 2 *If* $f(z) = \sum_{k=0}^{\infty} a_{2k+1}z^{2k+1}$ *is analytic in* $\mathbb{D}$ *and univalent in* $\mathbb{D}$*, where* $0 < \alpha = |a_1| \leq 1$*, then*

$$\sum_{k=0}^{\infty} |a_{2k+1}|r^{2k+1} \leq 1 \quad \textit{for } r \leq r_\alpha = \frac{-\alpha + \sqrt{4+\alpha^2}}{2}.$$

The extremal function has the form $\alpha z/(1-z^2)$.

Proof By hypothesis, the relation (8) implies that for $|z| = r < 1$,

$$\sum_{k=0}^{\infty} |a_{2k+1}|r^{2k+1} \leq \frac{\alpha r}{1-r^2}$$

which is less than or equal to 1 for $r \leq r_\alpha$. Observe that in the normalized case (i.e. $a_1 = 1$), the radius r_α takes the value $(\sqrt{5}-1)/2 = 0.618034\ldots$. □

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