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# A New Characterization of Differences of Weighted Composition Operators on Weighted-Type Spaces

**Qinghua Hu**[1] · **Songxiao Li**[2] · **Yecheng Shi**[3]

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**Abstract** In this paper, we give a new characterization for the boundedness, compactness and essential norm of differences of weighted composition operators between weighted-type spaces.

**Keywords** Weighted composition operators · Difference · Weighted-type spaces · Essential norm

**Mathematics Subject Classification** 30H99 · 47B38

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✉ Songxiao Li
jyulsx@163.com

Qinghua Hu
hqhmath@sina.com

Yecheng Shi
09ycshi@sina.cn

1 Department of Mathematics, Shantou University, Shantou 515063, Guangdong, People's Republic of China

2 Institute of Systems Engineering, Macau University of Science and Technology, Avenida Wai Long, Taipa, Macau

3 Faculty of Information Technology, Macau University of Science and Technology, Avenida Wai Long, Taipa, Macau

## 1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. Let $\mathbb{N}$ denote the set of all non-negative integers. Let $\sigma_a$ be the Möbius transformation on $\mathbb{D}$ defined by $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$. For $z, w \in \mathbb{D}$, the pseudo-hyperbolic distance between $z$ and $w$ is given by

$$\rho(z, w) = |\sigma_w(z)| = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

It is well known that $\rho(z, w) \leq 1$.

Let $\varphi$ be an analytic self-map of $\mathbb{D}$. The self-map $\varphi$ induces a linear operator $C_\varphi$ which is defined on $H(\mathbb{D})$ by $C_\varphi(f)(z) = f(\varphi(z)), \quad z \in \mathbb{D}$. $C_\varphi$ is called the composition operator. The compactness and essential norm of composition operator on the Bloch space were studied by many authors (see, e.g., [3,8,13,14,17]). Here, the Bloch space, denoted by $\mathcal{B} = \mathcal{B}(\mathbb{D})$, is defined as follows.

$$\mathcal{B} = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty\}.$$

In particular, Wulan et al. [14] proved that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if

$$\lim_{j\to\infty} \|\varphi^j\|_{\mathcal{B}} = 0.$$

Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The weighted composition operator, denoted by $uC_\varphi$, is defined as follows.

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad z \in \mathbb{D}.$$

Let $0 < \alpha < \infty$. An $f \in H(\mathbb{D})$ is said to belong to the weighted-type space, denoted by $H_\alpha^\infty$, if

$$\|f\|_{H_\alpha^\infty} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(z)| < \infty.$$

It is well known that $H_\alpha^\infty$ is a Banach space under the norm $\|\cdot\|_{H_\alpha^\infty}$. For all $z, w \in \mathbb{D}$, we define

$$\flat_\alpha(z, w) = \sup_{\|f\|_{H_\alpha^\infty}\leq 1} |(1-|z|^2)^\alpha f(z) - (1-|w|^2)^\alpha f(w)|.$$

Let $\varphi$ and $\psi$ be analytic self-maps of $\mathbb{D}$, $u, v \in H(\mathbb{D})$. For simplicity, we denote

$$\mathcal{D}_{u,\varphi}(z) = \frac{(1-|z|^2)^\beta u(z)}{(1-|\varphi(z)|^2)^\alpha}, \quad \mathcal{D}_{v,\psi}(z) = \frac{(1-|z|^2)^\beta v(z)}{(1-|\psi(z)|^2)^\alpha}.$$

Recently, many researchers have studied the differences of composition operators, as well as the differences of weighted composition operators on some analytic function spaces. The purpose of the study of the differences of composition operators is to understand the topological structure of the set of composition operators acting on a given function space. This line of research was first started in the setting of Hardy spaces by Berkson and Shapiro and Sundberg (see [1,10]). After that, such related problems have been studied on several analytic function spaces like $H^\infty$, the Bloch space, $H_\alpha^\infty$ and its generalizations (see, e.g., [2,4–7,9,11,12,15,16]).

In [9], Nieminen obtained a characterization of the compactness of differences of weighted composition operators on weighted-type spaces. Among others, he proved the following result.

**Theorem A** *Let $0 < \alpha, \beta < \infty$, $u, v \in H(\mathbb{D})$. Let $\varphi$ and $\psi$ be analytic self-maps of $\mathbb{D}$. Suppose that $uC_\varphi : H_\alpha^\infty \to H_\beta^\infty$ and $vC_\psi : H_\alpha^\infty \to H_\beta^\infty$ are bounded. Then, $uC_\varphi - vC_\psi : H_\alpha^\infty \to H_\beta^\infty$ is compact if and only if*

$$\begin{aligned}\lim_{|\varphi(z)|\to 1} |\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z),\psi(z)) &= \lim_{|\psi(z)|\to 1} |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z))\\ &= \lim_{\substack{|\varphi(z)|\to 1\\ |\psi(z)|\to 1}} |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| = 0. \qquad (1)\end{aligned}$$

Motivated by the results in [14] and Theorem A, we will give a new characterization for the boundedness, compactness and essential norm of the operator $uC_\varphi - vC_\psi : H_\alpha^\infty \to H_\beta^\infty$. More precisely, we show that $uC_\varphi - vC_\psi : H_\alpha^\infty \to H_\beta^\infty$ is bounded (respectively, compact) if and only if the sequence $\left(\frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}\right)_{n=0}^\infty$ is bounded (respectively, convergent to 0 as $n \to \infty$).

For two quantities $A$ and $B$ which may depend on $\varphi$ and $\psi$, we use the abbreviation $A \lesssim B$ whenever there is a positive constant $c$ (independent of $\varphi$ and $\psi$) such that $A \le cB$. We write $A \approx B$, if $A \lesssim B \lesssim A$.

## 2 Boundedness of $uC_\varphi - vC_\psi : H_\alpha^\infty \to H_\beta^\infty$

In this section, we characterize the bounded differences of weighted composition operators from $H_\alpha^\infty$ to $H_\beta^\infty$. For any $a \in \mathbb{D}$, we define the following two families of test functions:

$$f_a(z) = \frac{(1-|a|^2)^\alpha}{(1-\overline{a}z)^{2\alpha}}, \quad g_a(z) = \frac{(1-|a|^2)^\alpha}{(1-\overline{a}z)^{2\alpha}} \cdot \frac{a-z}{1-\overline{a}z}, \quad z \in \mathbb{D}.$$

It is easy to see that $\|g_a\|_{H_\alpha^\infty} \le \|f_a\|_{H_\alpha^\infty} = 1$.

To prove the result in this section, we need the following lemmas.

**Lemma 2.1** *Let $0 < \alpha, \beta < \infty$, $u, v \in H(\mathbb{D})$. Let $\varphi$ and $\psi$ be analytic self-maps of $\mathbb{D}$. Then the following inequalities hold:*

(i)

$$\sup_{z\in\mathbb{D}} |\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z),\psi(z)) \lesssim \sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi)f_a\|_{H^\infty_\beta} + \sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi)g_a\|_{H^\infty_\beta}.$$

(ii)

$$\sup_{z\in\mathbb{D}} |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z)) \lesssim \sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi)f_a\|_{H^\infty_\beta} + \sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi)g_a\|_{H^\infty_\beta}.$$

(iii)

$$\sup_{z\in\mathbb{D}} |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| \lesssim \sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi)f_a\|_{H^\infty_\beta} + \sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi)g_a\|_{H^\infty_\beta}.$$

*Proof* (i) For any $z \in \mathbb{D}$, we have

$$\begin{aligned}
&\|(uC_\varphi - vC_\psi)f_{\varphi(z)}\|_{H^\infty_\beta} \\
&\geq \left|u(z)f_{\varphi(z)}(\varphi(z)) - v(z)f_{\varphi(z)}(\psi(z))\right|(1-|z|^2)^\beta \\
&= \left|\mathcal{D}_{u,\varphi}(z) - \frac{(1-|\varphi(z)|^2)^\alpha(1-|\psi(z)|^2)^\alpha}{(1-\overline{\varphi(z)}\psi(z))^{2\alpha}}\mathcal{D}_{v,\psi}(z)\right| \\
&\geq |\mathcal{D}_{u,\varphi}(z)| - \frac{(1-|\varphi(z)|^2)^\alpha(1-|\psi(z)|^2)^\alpha}{|1-\overline{\varphi(z)}\psi(z)|^{2\alpha}}|\mathcal{D}_{v,\psi}(z)|
\end{aligned}$$

and

$$\begin{aligned}
&\|(uC_\varphi - vC_\psi)g_{\varphi(z)}\|_{H^\infty_\beta} \\
&\geq \left|u(z)g_{\varphi(z)}(\varphi(z)) - v(z)g_{\varphi(z)}(\psi(z))\right|(1-|z|^2)^\beta \\
&= \frac{(1-|\varphi(z)|^2)^\alpha(1-|\psi(z)|^2)^\alpha}{|1-\overline{\varphi(z)}\psi(z)|^{2\alpha}}|\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z)).
\end{aligned}$$

Hence,

$$\begin{aligned}
&|\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z),\psi(z)) \\
&\leq \|(uC_\varphi - vC_\psi)f_{\varphi(z)}\|_{H^\infty_\beta}\rho(\varphi(z),\psi(z)) + \|(uC_\varphi - vC_\psi)g_{\varphi(z)}\|_{H^\infty_\beta} \\
&\leq \|(uC_\varphi - vC_\psi)f_{\varphi(z)}\|_{H^\infty_\beta} + \|(uC_\varphi - vC_\psi)g_{\varphi(z)}\|_{H^\infty_\beta}. \qquad (2)
\end{aligned}$$

Similarly,

$$|\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z)) \le \|(uC_\varphi - vC_\psi)f_{\psi(z)}\|_{H_\beta^\infty} + \|(uC_\varphi - vC_\psi)g_{\psi(z)}\|_{H_\beta^\infty}. \tag{3}$$

Therefore,

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} |\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z),\psi(z)) \\
&\quad\le \sup_{z\in\mathbb{D}} \|(uC_\varphi - vC_\psi)f_{\varphi(z)}\|_{H_\beta^\infty} + \sup_{z\in\mathbb{D}} \|(uC_\varphi - vC_\psi)g_{\varphi(z)}\|_{H_\beta^\infty} \\
&\quad\le \sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi)f_a\|_{H_\beta^\infty} + \sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi)g_a\|_{H_\beta^\infty}.
\end{aligned}$$

(ii) The proof is similar to (i). From (3) we get the desired result.
(iii) By [9, Lem. 2.3],

$$\begin{aligned}
&\|(uC_\varphi - vC_\psi)f_{\varphi(z)}\|_{H_\beta^\infty} \\
&\ge \left|\mathcal{D}_{u,\varphi}(z) - \frac{(1-|\varphi(z)|^2)^\alpha(1-|\psi(z)|^2)^\alpha}{(1-\overline{\varphi(z)}\psi(z))^{2\alpha}}\mathcal{D}_{v,\psi}(z)\right| \\
&\ge |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| - \left|1 - \frac{(1-|\varphi(z)|^2)^\alpha(1-|\psi(z)|^2)^\alpha}{(1-\overline{\varphi(z)}\psi(z))^{2\alpha}}\right| |\mathcal{D}_{v,\psi}(z)| \\
&= |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| - \Big|(1-|\varphi(z)|^2)^\alpha f_{\varphi(z)}(\varphi(z)) \\
&\quad - (1-|\psi(z)|^2)^\alpha f_{\varphi(z)}(\psi(z))\Big| |\mathcal{D}_{v,\psi}(z)| \\
&\ge |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| - \flat_\alpha(\varphi(z),\psi(z))|\mathcal{D}_{v,\psi}(z)| \\
&\gtrsim |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| - |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z)).
\end{aligned}$$

Thus, by (2) we obtain

$$\begin{aligned}
&|\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| \\
&\quad\lesssim \|(uC_\varphi - vC_\psi)f_{\varphi(z)}\|_{H_\beta^\infty} + |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z)) \\
&\quad\lesssim \|(uC_\varphi - vC_\psi)f_{\varphi(z)}\|_{H_\beta^\infty} + \|(uC_\varphi - vC_\psi)f_{\psi(z)}\|_{H_\beta^\infty} \\
&\qquad + \|(uC_\varphi - vC_\psi)g_{\psi(z)}\|_{H_\beta^\infty}.
\end{aligned} \tag{4}$$

Therefore,

$$\sup_{z\in\mathbb{D}} |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| \lesssim \sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi)f_a\|_{H_\beta^\infty} + \sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi)g_a\|_{H_\beta^\infty}.$$

The proof of the lemma is completed. □

**Lemma 2.2** *Let* $0 < \alpha, \beta < \infty$, $u, v \in H(\mathbb{D})$. *Let* $\varphi$ *and* $\psi$ *be analytic self-maps of* $\mathbb{D}$. *Then the following inequalities hold:*

(i)
$$\sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi) f_a\|_{H_\beta^\infty} \lesssim \sup_{n\in\mathbb{N}} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}.$$

(ii)
$$\sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi) g_a\|_{H_\beta^\infty} \lesssim \sup_{n\in\mathbb{N}} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}.$$

*Proof* (i) When $a = 0$, we see that $f_a(z) = 1$. It is clear that

$$\|(uC_\varphi - vC_\psi) f_a\|_{H_\beta^\infty} = \|u - v\|_{H_\beta^\infty} \lesssim \sup_{n\in\mathbb{N}} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}.$$

For any $a \in \mathbb{D}$ with $a \neq 0$, note that

$$f_a(z) = \frac{(1-|a|^2)^\alpha}{(1-\overline{a}z)^{2\alpha}} = (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} \bar{a}^k z^k, \quad z \in \mathbb{D}.$$

After a simple calculation, we see that $n^\alpha \|z^n\|_{H_\alpha^\infty} \approx 1$. By the following well-known formulas,

$$\frac{\Gamma(k+\alpha)}{k!} \approx k^{\alpha-1},\ k \to \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} k^{\alpha-1}|a|^k \approx \frac{1}{(1-|a|)^\alpha}, \quad |a| \to 1,$$

we have

$$\begin{aligned}
&\|(uC_\varphi - vC_\psi) f_a\|_{H_\beta^\infty} \\
&\leq (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} |a|^k \|u\varphi^k - v\psi^k\|_{H_\beta^\infty} \\
&= (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} k^{-\alpha}|a|^k k^\alpha \|u\varphi^k - v\psi^k\|_{H_\beta^\infty} \\
&\leq (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} k^{-\alpha}|a|^k \sup_{n\in\mathbb{N}} n^\alpha \|u\varphi^n - v\psi^n\|_{H_\beta^\infty} \\
&\approx (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} k^{-\alpha}|a|^k \sup_{n\in\mathbb{N}} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}} \\
&\lesssim \sup_{n\in\mathbb{N}} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}.
\end{aligned} \tag{5}$$

By the arbitrariness of $a$, we see that (i) holds.

(ii) When $a = 0$, we see that $g_a(z) = -z$. It is clear that

$$\|(uC_\varphi - vC_\psi)g_a\|_{H_\beta^\infty} = \|u\varphi - v\psi\|_{H_\beta^\infty} \lesssim \sup_{n\in\mathbb{N}} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}.$$

Similarly, for any $a \in \mathbb{D}$ with $a \neq 0$,

$$\begin{aligned}
g_a(z) &= \frac{(1-|a|^2)^\alpha}{(1-\overline{a}z)^{2\alpha}} \cdot \frac{a-z}{1-\overline{a}z} \\
&= (1-|a|^2)^\alpha \left(\sum_{k=0}^\infty \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} \bar{a}^k z^k\right)\left(a - (1-|a|^2)\sum_{k=0}^\infty \bar{a}^k z^{k+1}\right) \\
&= af_a(z) - (1-|a|^2)^{\alpha+1} \left(\sum_{k=0}^\infty \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} \bar{a}^k z^k\right)\left(\sum_{k=0}^\infty \bar{a}^k z^{k+1}\right) \\
&= af_a(z) - (1-|a|^2)^{\alpha+1} \sum_{k=1}^\infty \left(\sum_{l=0}^{k-1} \frac{\Gamma(l+2\alpha)}{l!\Gamma(2\alpha)}\right) \bar{a}^{k-1} z^k.
\end{aligned}$$

By Stirling's formula, we have

$$\sum_{l=0}^{k-1} \frac{\Gamma(l+2\alpha)}{l!\Gamma(2\alpha)} \approx \sum_{l=0}^{k-1} l^{2\alpha-1} \approx k^{2\alpha}, \quad k \to \infty.$$

Therefore,

$$\begin{aligned}
\|(uC_\varphi - vC_\psi)g_a\|_{H_\beta^\infty} &\le \|(uC_\varphi - vC_\psi)f_a\|_{H_\beta^\infty} + (1-|a|^2)^{\alpha+1} \\
&\quad \times \sum_{k=1}^\infty \left(\sum_{l=0}^{k-1} \frac{\Gamma(l+2\alpha)}{l!\Gamma(2\alpha)}\right)|a|^{k-1} \|u\varphi^k - v\psi^k\|_{H_\beta^\infty} \\
&\lesssim \|(uC_\varphi - vC_\psi)f_a\|_{H_\beta^\infty} + (1-|a|^2)^{\alpha+1} \\
&\quad \times \sum_{k=1}^\infty \frac{1}{k^\alpha} k^{2\alpha} |a|^{k-1} \sup_{n\ge 2} n^\alpha \|u\varphi^n - v\psi^n\|_{H_\beta^\infty} \\
&\approx \|(uC_\varphi - vC_\psi)f_a\|_{H_\beta^\infty} + \sup_{n\in\mathbb{N}} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}} \\
&\lesssim \sup_{n\in\mathbb{N}} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}.
\end{aligned}$$

By the arbitrariness of $a$, we see that (ii) holds. The proof of the lemma is completed. □

**Theorem 2.1** *Let* $0 < \alpha, \beta < \infty$, $u, v \in H(\mathbb{D})$. *Let* $\varphi$ *and* $\psi$ *be analytic self-maps of* $\mathbb{D}$. *Then* $uC_\varphi - vC_\psi : H_\alpha^\infty \to H_\beta^\infty$ *is bounded if and only if*

$$\sup_{n\in\mathbb{N}} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}} < \infty. \tag{6}$$

*Proof* First, we assume that $uC_\varphi - vC_\psi : H_\alpha^\infty \to H_\beta^\infty$ is bounded. For any $n \in \mathbb{N}$, let $f_n(z) = z^n/\|z^n\|_{H_\alpha^\infty}$. Then $\|f_n\|_{H_\alpha^\infty} = 1$. Thus, by the boundedness of $uC_\varphi - vC_\psi$, we get

$$\infty > \|uC_\varphi - vC_\psi\|_{H_\alpha^\infty \to H_\beta^\infty} \geq \|(uC_\varphi - vC_\psi)f_n\|_{H_\beta^\infty} = \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}},$$

as desired.

Conversely, assume that (6) holds. For any $f \in H_\alpha^\infty$ with $\|f\|_{H_\alpha^\infty} \leq 1$, by [9, Lem. 2.3], we have

$$\begin{aligned}
&\|(uC_\varphi - vC_\psi)f\|_{H_\beta^\infty} \\
&\quad = \sup_{z\in\mathbb{D}} |u(z)f(\varphi(z)) - v(z)f(\psi(z))|(1-|z|^2)^\beta \\
&\quad \leq \sup_{z\in\mathbb{D}} |f(\varphi(z))(1-|\varphi(z)|^2)^\alpha - f(\psi(z))(1-|\psi(z)|^2)^\alpha||\mathcal{D}_{u,\varphi}(z)| \\
&\qquad + \sup_{z\in\mathbb{D}} |f(\psi(z))|(1-|\psi(z)|^2)^\alpha|\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| \\
&\quad \leq \sup_{z\in\mathbb{D}} \flat_\alpha(\varphi(z), \psi(z))|\mathcal{D}_{u,\varphi}(z)| + \|f\|_{H_\alpha^\infty} \sup_{z\in\mathbb{D}} |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| \\
&\quad \lesssim \sup_{z\in\mathbb{D}} |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| + \sup_{z\in\mathbb{D}} |\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z), \psi(z)).
\end{aligned}$$

Hence, by Lemmas 2.1 and 2.2 we have

$$\begin{aligned}
&\|uC_\varphi - vC_\psi\|_{H_\alpha^\infty \to H_\beta^\infty} \\
&\quad \lesssim \sup_{z\in\mathbb{D}} |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| + \sup_{z\in\mathbb{D}} |\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z), \psi(z)) \\
&\quad \lesssim \sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi)f_a\|_{H_\beta^\infty} + \sup_{a\in\mathbb{D}} \|(uC_\varphi - vC_\psi)g_a\|_{H_\beta^\infty} \\
&\quad \lesssim \sup_{n\in\mathbb{N}} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}} < \infty.
\end{aligned}$$

Therefore, $uC_\varphi - vC_\psi : H_\alpha^\infty \to H_\beta^\infty$ is bounded. This completes the proof of the theorem. □

## 3 Essential norm estimates

In this section, we give an estimate for the essential norm of $uC_\varphi - vC_\psi$ from $H_\alpha^\infty$ to $H_\beta^\infty$. For this purpose, we need some auxiliary results as follows.

**Lemma 3.1** *Let* $0 < \alpha, \beta < \infty$, $u, v \in H(\mathbb{D})$. *Let* $\varphi$ *and* $\psi$ *be analytic self-maps of* $\mathbb{D}$. *Then the following inequalities hold:*

(i)

$$\lim_{s\to 1} \sup_{|\varphi(z)|>s} |\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z), \psi(z)) \lesssim \limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi) f_a\|_{H_\beta^\infty} + \limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi) g_a\|_{H_\beta^\infty}.$$

(ii)

$$\lim_{s\to 1} \sup_{|\psi(z)|>s} |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z), \psi(z)) \lesssim \limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi) f_a\|_{H_\beta^\infty} + \limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi) g_a\|_{H_\beta^\infty}.$$

(iii)

$$\lim_{s\to 1} \sup_{\substack{|\varphi(z)|>s \\ |\psi(z)|>s}} |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| \lesssim \limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi) f_a\|_{H_\beta^\infty} + \limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi) g_a\|_{H_\beta^\infty}.$$

*Proof* For any $z \in \mathbb{D}$, from the Proof of Lemma 2.1, we have

$$|\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z), \psi(z)) \le \|(uC_\varphi - vC_\psi) f_{\varphi(z)}\|_{H_\beta^\infty} + \|(uC_\varphi - vC_\psi) g_{\varphi(z)}\|_{H_\beta^\infty},$$
$$|\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z), \psi(z)) \le \|(uC_\varphi - vC_\psi) f_{\psi(z)}\|_{H_\beta^\infty} + \|(uC_\varphi - vC_\psi) g_{\psi(z)}\|_{H_\beta^\infty},$$

and

$$|\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| \lesssim \|(uC_\varphi - vC_\psi) f_{\varphi(z)}\|_{H_\beta^\infty} + \|(uC_\varphi - vC_\psi) f_{\psi(z)}\|_{H_\beta^\infty} + \|(uC_\varphi - vC_\psi) g_{\psi(z)}\|_{H_\beta^\infty}.$$

From the above inequalities, the assertion follows easily. The proof is completed. □

**Lemma 3.2** *Let* $0 < \alpha, \beta < \infty$, $u, v \in H(\mathbb{D})$. *Let* $\varphi$ *and* $\psi$ *be analytic self-maps of* $\mathbb{D}$. *Suppose that* $uC_\varphi - vC_\psi : H_\alpha^\infty \to H_\beta^\infty$ *is bounded, then the following inequalities hold:*

(i)

$$\limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi) f_a\|_{H_\beta^\infty} \lesssim \limsup_{n\to\infty} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}.$$

(ii)

$$\limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi) g_a\|_{H_\beta^\infty} \lesssim \limsup_{n\to\infty} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}.$$

*Proof* For each $N$ and any $a \in \mathbb{D}$ with $a \neq 0$, from the Proof of Lemma 2.2, we have

$$\begin{aligned}
&\|(uC_\varphi - vC_\psi) f_a\|_{H_\beta^\infty} \\
&\lesssim (1-|a|^2)^\alpha \sum_{k=0}^{N} \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} k^{-\alpha} |a|^k k^\alpha \|u\varphi^k - v\psi^k\|_{H_\beta^\infty} \\
&\quad + (1-|a|^2)^\alpha \sum_{k=N+1}^{\infty} k^{\alpha-1} |a|^k \sup_{n\geq N+1} n^\alpha \|u\varphi^n - v\psi^n\|_{H_\beta^\infty}. \qquad (7)
\end{aligned}$$

From the boundedness of $uC_\varphi - vC_\psi$ we see that $\sup_{n\in\mathbb{N}} n^\alpha \|u\varphi^n - v\psi^n\|_{H_\beta^\infty} < \infty$. Let $|a| \to 1$ in (7). We obtain

$$\begin{aligned}
\limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi) f_a\|_{H_\beta^\infty} &\lesssim \sup_{n\geq N+1} n^\alpha \|u\varphi^n - v\psi^n\|_{H_\beta^\infty} \\
&\approx \sup_{n\geq N+1} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}
\end{aligned}$$

for any positive integer $N$. Hence,

$$\limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi) f_a\|_{H_\beta^\infty} \lesssim \limsup_{n\to\infty} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}.$$

Also for each $N$ and any $a \in \mathbb{D}$ with $a \neq 0$, from the Proof of Lemma 2.2,

$$\begin{aligned}
&\|(uC_\varphi - vC_\psi) g_a\|_{H_\beta^\infty} \\
&\lesssim (1-|a|^2)^{\alpha+1} \sum_{k=1}^{N} k^\alpha |a|^{k-1} \sup_{1\leq n\leq N+1} n^\alpha \|u\varphi^n - v\psi^n\|_{H_\beta^\infty} \\
&\quad + (1-|a|^2)^{\alpha+1} \sum_{k=N+1}^{\infty} k^\alpha |a|^{k-1} \sup_{n\geq N+1} n^\alpha \|u\varphi^n - v\psi^n\|_{H_\beta^\infty} \\
&\quad + \|(uC_\varphi - vC_\psi) f_a\|_{H_\beta^\infty}. \qquad (8)
\end{aligned}$$

Let $|a| \to 1$ in (8). We get

$$\begin{aligned}
&\limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi)g_a\|_{H_\beta^\infty} \\
&\lesssim \limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi)f_a\|_{H_\beta^\infty} + \sup_{n\geq N+1} n^\alpha \|u\varphi^n - v\psi^n\|_{H_\beta^\infty} \\
&\approx \limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi)f_a\|_{H_\beta^\infty} + \sup_{n\geq N+1} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}
\end{aligned}$$

for any positive integer $N$. Thus, by (i) we obtain

$$\limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi)g_a\|_{H_\beta^\infty} \lesssim \limsup_{n\to\infty} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}.$$

The proof is completed. □

**Theorem 3.1** *Let* $0 < \alpha, \beta < \infty$, $u, v \in H(\mathbb{D})$. *Let* $\varphi$ *and* $\psi$ *be analytic self-maps of* $\mathbb{D}$. *Suppose that* $uC_\varphi : H_\alpha^\infty \to H_\beta^\infty$ *and* $vC_\psi : H_\alpha^\infty \to H_\beta^\infty$ *are bounded, then*

$$\|uC_\varphi - vC_\psi\|_{e, H_\alpha^\infty \to H_\beta^\infty} \approx \limsup_{n\to\infty} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}.$$

*Proof* For $r \in [0, 1)$, set $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by

$$(K_r f)(z) = f_r(z) = f(rz), \quad f \in H(\mathbb{D}).$$

It is clear that $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Moreover, $K_r$ is compact on $H_\alpha^\infty$ and $\|K_r\|_{H_\alpha^\infty \to H_\alpha^\infty} \leq 1$. Let $\{r_j\} \subset (0, 1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then for all positive integers $j$, the operator $(uC_\varphi - vC_\psi)K_{r_j} : H_\alpha^\infty \to H_\beta^\infty$ is compact. Hence,

$$\begin{aligned}
&\|uC_\varphi - vC_\psi\|_{e, H_\alpha^\infty \to H_\beta^\infty} \\
&\leq \limsup_{j\to\infty} \|uC_\varphi - vC_\psi - (uC_\varphi - vC_\psi)K_{r_j}\|_{H_\alpha^\infty \to H_\beta^\infty} \\
&= \limsup_{j\to\infty} \|(uC_\varphi - vC_\psi)(I - K_{r_j})\|_{H_\alpha^\infty \to H_\beta^\infty} \\
&= \limsup_{j\to\infty} \sup_{\|f\|_{H_\alpha^\infty}\leq 1} \|(uC_\varphi - vC_\psi)(I - K_{r_j})f\|_{H_\beta^\infty} \\
&= \limsup_{j\to\infty} \sup_{\|f\|_{H_\alpha^\infty}\leq 1} \sup_{z\in\mathbb{D}} \Omega_j^f(z),
\end{aligned}$$

where

$$\Omega_j^f(z) := |u(z)(f - f_{r_j})(\varphi(z)) - v(z)(f - f_{r_j})(\psi(z))|(1 - |z|^2)^\beta.$$

For any $r \in (0, 1)$, define

$$\mathbb{D}_1 := \{z \in \mathbb{D} : |\varphi(z)| \le r, |\psi(z)| \le r\}, \quad \mathbb{D}_2 := \{z \in \mathbb{D} : |\varphi(z)| \le r, |\psi(z)| > r\},$$
$$\mathbb{D}_3 := \{z \in \mathbb{D} : |\varphi(z)| > r, |\psi(z)| \le r\}, \quad \mathbb{D}_4 := \{z \in \mathbb{D} : |\varphi(z)| > r, |\psi(z)| > r\}.$$

Then,

$$\begin{aligned}\limsup_{j\to\infty} \sup_{\|f\|_{H_\alpha^\infty}\le 1} \sup_{z\in\mathbb{D}} \Omega_j^f &= \max_{1\le i\le 4} \limsup_{j\to\infty} \sup_{\|f\|_{H_\alpha^\infty}\le 1} \sup_{z\in\mathbb{D}_i} \Omega_j^f \\ &= \max\left\{\limsup_{j\to\infty} J_1, \limsup_{j\to\infty} J_2, \limsup_{j\to\infty} J_3, \limsup_{j\to\infty} J_4\right\},\end{aligned}$$

where $J_i = \sup_{\|f\|_{H_\alpha^\infty}\le 1} \sup_{z\in\mathbb{D}_i} \Omega_j^f$. Since $uC_\varphi : H_\alpha^\infty \to H_\beta^\infty$ and $vC_\psi : H_\alpha^\infty \to H_\beta^\infty$ are bounded and we see that $u, v \in H_\beta^\infty$. Since $f - f_{r_j} \to 0$ is uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, we have

$$\begin{aligned}\limsup_{j\to\infty} J_1 &= \limsup_{j\to\infty} \sup_{\|f\|_{H_\alpha^\infty}\le 1} \sup_{z\in\mathbb{D}_1} \Omega_j^f \\ &\le \limsup_{j\to\infty} \sup_{\|f\|_{H_\alpha^\infty}\le 1} \sup_{|\varphi(z)|\le r} |u(z)(f - f_{r_j})(\varphi(z))|(1 - |z|^2)^\beta \\ &\quad + \limsup_{j\to\infty} \sup_{\|f\|_{H_\alpha^\infty}\le 1} \sup_{|\psi(z)|\le r} |v(z)(f - f_{r_j})(\psi(z))|(1 - |z|^2)^\beta \\ &= 0.\end{aligned}$$

In addition, we have

$$\begin{aligned}\Omega_j^f(z) &\le |(f - f_{r_j})(\varphi(z))(1 - |\varphi(z)|^2)^\alpha - (f - f_{r_j})(\psi(z))(1 - |\psi(z)|^2)^\alpha| \\ &\quad |\mathcal{D}_{u,\varphi}(z)| + |(f - f_{r_j})(\psi(z))|(1 - |\psi(z)|^2)^\alpha |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| \\ &\le \flat_\alpha(\varphi(z), \psi(z))|\mathcal{D}_{u,\varphi}(z)| \\ &\quad + |(f - f_{r_j})(\psi(z))|(1 - |\psi(z)|^2)^\alpha |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)| \\ &\lesssim |\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z), \psi(z)) \\ &\quad + |(f - f_{r_j})(\psi(z))|(1 - |\psi(z)|^2)^\alpha |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)|.\end{aligned}$$

Similarly,

$$\begin{aligned}\Omega_j^f(z) &\lesssim |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z), \psi(z)) \\ &\quad + |(f - f_{r_j})(\varphi(z))|(1 - |\varphi(z)|^2)^\alpha |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)|.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\limsup_{j\to\infty} J_2 &\lesssim \limsup_{j\to\infty} \sup_{\|f\|_{H_\alpha^\infty}\le 1} \sup_{z\in\mathbb{D}_2} \big(|\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z), \psi(z)) \\ &\quad + |(f - f_{r_j})(\varphi(z))|(1 - |\varphi(z)|^2)^\alpha |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)|\big)\end{aligned}$$

$$\begin{aligned}&\leq \limsup_{j\to\infty} \sup_{\|f\|_{H_\alpha^\infty}\leq 1} \sup_{|\varphi(z)|\leq r} |(f-f_{r_j})(\varphi(z))|(1-|\varphi(z)|^2)^\alpha \\ &\quad \times |\mathcal{D}_{u,\varphi}(z)-\mathcal{D}_{v,\psi}(z)| + \sup_{|\psi(z)|>r} |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z)) \\ &= \sup_{|\psi(z)|>r} |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z)),\end{aligned}$$

where we used the fact that $\sup_{z\in\mathbb{D}} |\mathcal{D}_{u,\varphi}(z)-\mathcal{D}_{v,\psi}(z)| < \infty$, since $uC_\varphi - vC_\psi$ is bounded (see the proof of Theorem 2.1), and $f - f_{r_j} \to 0$ uniformly on compact subset of $\mathbb{D}$ as $j\to\infty$ again in the last inequality. Since $r$ is arbitrary, we have

$$\limsup_{j\to\infty} J_2 \lesssim \lim_{r\to 1} \sup_{|\psi(z)|>r} |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z)).$$

Similarly,

$$\limsup_{j\to\infty} J_3 \lesssim \lim_{r\to 1} \sup_{|\varphi(z)|>r} |\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z),\psi(z)).$$

Next we consider $\limsup_{j\to\infty} J_4$. We have

$$\begin{aligned}\limsup_{j\to\infty} J_4 &\lesssim \limsup_{j\to\infty} \sup_{\|f\|_{H_\alpha^\infty}\leq 1} \sup_{z\in\mathbb{D}_4} \big(|(f-f_{r_j})(\varphi(z))|(1-|\varphi(z)|^2)^\alpha \\ &\quad \times |\mathcal{D}_{u,\varphi}(z)-\mathcal{D}_{v,\psi}(z)| + |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z))\big) \\ &\lesssim \limsup_{j\to\infty} \sup_{\|f\|_{H_\alpha^\infty}\leq 1} \sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}} \|f-f_{r_j}\|_{H_\alpha^\infty} |\mathcal{D}_{u,\varphi}(z)-\mathcal{D}_{v,\psi}(z)| \\ &\quad + \sup_{|\psi(z)|>r} |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z)) \\ &\lesssim \sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}} |\mathcal{D}_{u,\varphi}(z)-\mathcal{D}_{v,\psi}(z)| + \sup_{|\psi(z)|>r} |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z)),\end{aligned}$$

where we used the fact that $\limsup_{j\to\infty} \|f-f_{r_j}\|_{H_\alpha^\infty} \leq 2$ in the last inequality. Thus,

$$\begin{aligned}\limsup_{j\to\infty} J_4 \lesssim &\lim_{r\to 1} \sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}} |\mathcal{D}_{u,\varphi}(z)-\mathcal{D}_{v,\psi}(z)| \\ &+ \lim_{r\to 1} \sup_{|\psi(z)|>r} |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z)).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\limsup_{j\to\infty} \sup_{\|f\|_{H_\alpha^\infty}\leq 1} \sup_{z\in\mathbb{D}} \Omega_j^f(z) \\ &\quad = \max\left\{\limsup_{j\to\infty} J_1, \limsup_{j\to\infty} J_2, \limsup_{j\to\infty} J_3, \limsup_{j\to\infty} J_4\right\}\end{aligned}$$

$$\lesssim \lim_{r\to 1} \sup_{|\varphi(z)|>r} |\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z),\psi(z)) + \lim_{r\to 1} \sup_{|\psi(z)|>r} |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z))$$
$$+ \lim_{r\to 1} \sup_{\substack{|\varphi(z)|>r\\ |\psi(z)|>r}} |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)|.$$

Combining this with Lemmas 3.1 and 3.2, we have

$$\begin{aligned}
&\|uC_\varphi - vC_\psi\|_{e,H_\alpha^\infty\to H_\beta^\infty}\\
&\lesssim \lim_{r\to 1} \sup_{|\varphi(z)|>r} |\mathcal{D}_{u,\varphi}(z)|\rho(\varphi(z),\psi(z)) + \lim_{r\to 1} \sup_{|\psi(z)|>r} |\mathcal{D}_{v,\psi}(z)|\rho(\varphi(z),\psi(z))\\
&\quad + \lim_{r\to 1} \sup_{\substack{|\varphi(z)|>r\\ |\psi(z)|>r}} |\mathcal{D}_{u,\varphi}(z) - \mathcal{D}_{v,\psi}(z)|\\
&\lesssim \limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi)f_a\|_{H_\beta^\infty} + \limsup_{|a|\to 1} \|(uC_\varphi - vC_\psi)g_a\|_{H_\beta^\infty}\\
&\lesssim \limsup_{n\to\infty} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}. 
\end{aligned} \tag{9}$$

Next, we prove that

$$\|uC_\varphi - vC_\psi\|_{e,H_\alpha^\infty\to H_\beta^\infty} \gtrsim \limsup_{n\to\infty} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}.$$

Let $n$ be any non-negative integer. Let $f_n(z) = z^n/\|z^n\|_{H_\alpha^\infty}$. Then, $f_n \in H_\alpha^\infty$ with $\|f_n\|_{H_\alpha^\infty} = 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. If $K$ is any compact operator from $H_\alpha^\infty$ to $H_\beta^\infty$, then $\lim_{n\to\infty} \|Kf_n\|_{H_\beta^\infty} = 0$. Hence,

$$\begin{aligned}
\|uC_\varphi - vC_\psi - K\| &\geq \limsup_{n\to\infty} \|(uC_\varphi - vC_\psi - K)f_n\|_{H_\beta^\infty}\\
&\geq \limsup_{n\to\infty} \|(uC_\varphi - vC_\psi)f_n\|_{H_\beta^\infty}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|uC_\varphi - vC_\psi\|_{e,H_\alpha^\infty\to H_\beta^\infty} &\geq \limsup_{n\to\infty} \|(uC_\varphi - vC_\psi)f_n\|_{H_\beta^\infty}\\
&= \limsup_{n\to\infty} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}}.
\end{aligned} \tag{10}$$

Combining (9) with (10), we immediately get the desired result. The proof of this theorem is complete. □

From Theorem 3.1, we immediately get the following corollary.

**Corollary 3.1** *Let* $0 < \alpha, \beta < \infty$, $u, v \in H(\mathbb{D})$. *Let* $\varphi$ *and* $\psi$ *be analytic self-maps of* $\mathbb{D}$. *Suppose that* $uC_\varphi : H_\alpha^\infty \to H_\beta^\infty$ *and* $vC_\psi : H_\alpha^\infty \to H_\beta^\infty$ *are bounded, then*

$uC_\varphi - vC_\psi : H_\alpha^\infty \to H_\beta^\infty$ *is compact if and only if*

$$\limsup_{n\to\infty} \frac{\|u\varphi^n - v\psi^n\|_{H_\beta^\infty}}{\|z^n\|_{H_\alpha^\infty}} = 0.$$

Assume that $w$ is a continuous, strictly positive and bounded function on $\mathbb{D}$. The weight $w$ is called radial if $w(z) = w(|z|)$ for all $z \in \mathbb{D}$. The weighted space, denoted by $H_w^\infty$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_w^\infty} = \sup_{z\in\mathbb{D}} w(z)|f(z)| < \infty.$$

$H_w^\infty$ is a Banach space with the norm $\|\cdot\|_{H_w^\infty}$.

*Remark* Let $w_1$ and $w_2$ be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Let $u, v \in H(\mathbb{D})$, $\varphi$ and $\psi$ be analytic self-maps of $\mathbb{D}$. We conjecture that the following statements hold:

(a) $uC_\varphi - vC_\psi : H_{w_1}^\infty \to H_{w_2}^\infty$ is bounded if and only if

$$\sup_{n\in\mathbb{N}} \frac{\|u\varphi^n - v\psi^n\|_{H_{w_2}^\infty}}{\|z^n\|_{H_{w_1}^\infty}} < \infty,$$

with the norm comparable to the above supremum.

(b) Suppose $uC_\varphi : H_{w_1}^\infty \to H_{w_2}^\infty$ and $vC_\psi : H_{w_1}^\infty \to H_{w_2}^\infty$ are bounded. Then,

$$\|uC_\varphi - vC_\psi\|_{e,H_{w_1}^\infty \to H_{w_2}^\infty} \approx \limsup_{n\to\infty} \frac{\|u\varphi^n - v\psi^n\|_{H_{w_2}^\infty}}{\|z^n\|_{H_{w_1}^\infty}}.$$

We are not able, at the moment, to prove this conjecture. Hence, we leave the problem to the readers interested in this research area.

## References

1. Berkson, E.: Composition operators isolated in the uniform operator topology. Proc. Am. Math. Soc. **81**, 230–232 (1981)
2. Bonet, J., Lindström, M., Wolf, E.: Differences of composition operators between weighted Banach spaces of holomorphic functions. J. Aust. Math. Soc. **84**, 9–20 (2008)
3. Cowen, C., Maccluer, B.: Composition Operators on Spaces of Analytic Functions. CRC Press, Boca Raton (1995)
4. Dai, J., Ouyang, C.: Differences of weighted composition operators on $H_\alpha^\infty(B_N)$. J. Inequal. Appl. 19 (2009) (Art. ID 127431)
5. Hosokawa, T., Ohno, S.: Differences of composition operators on the Bloch spaces. J. Oper. Theory **57**, 229–242 (2007)
6. Li, S.: Differences of generalized composition operators on the Bloch space. J. Math. Anal. Appl. **394**, 706–711 (2012)
7. Lindström, M., Wolf, E.: Essential norm of the difference of weighted composition operators. Monatsh. Math. **153**, 133–143 (2008)

8. Madigan, K., Matheson, A.: Compact composition operators on the Bloch space. Trans. Am. Math. Soc. **347**, 2679–2687 (1995)
9. Nieminen, P.: Compact differences of composition operators on Bloch and Lipschitz spaces. Comput. Method Funct. Theory **7**, 325–344 (2007)
10. Shapiro, J., Sundberg, C.: Isolation amongst the composition operators. Pac. J. Math. **145**, 117–152 (1990)
11. Stević, S., Wolf, E.: Differences of composition operators between weighted-type spaces of holomorphic functions on the unit ball of $\mathbb{C}^n$. Appl. Math. Comput. **215**, 1752–1760 (2009)
12. Wolf, E.: Differences of weighted composition operators. Collect. Math. **60**, 1–10 (2009)
13. Wu, Y., Wulan, H.: Products of differentiation and composition operators on the Bloch space. Collet. Math. **63**, 93–107 (2012)
14. Wulan, H., Zheng, D., Zhu, K.: Compact composition operators on BMOA and the Bloch space. Proc. Am. Math. Soc. **137**, 3861–3868 (2009)
15. Yang, K., Zhou, Z.: Essential norm of the difference of composition operators on Bloch space. Czechoslov. Math. J. **60**, 1139–1152 (2010)
16. Yang, W., Zhu, X.: Differences of generalized weighted composition operators between growth spaces. Ann. Pol. Math. **112**, 67–84 (2014)
17. Zhao, R.: Essential norms of composition operators between Bloch type spaces. Proc. Am. Math. Soc. **138**, 2537–2546 (2010)
18. Zhu, K.: Operator Theory in Function Spaces. American Mathematical Society, Providence (2007)