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Poisson Summation Formula in Hardy Spaces $H^p(T_\Gamma)$, $p \in (0, 1]$

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Abstract The Poisson summation formula for Hardy spaces $H^p(T_\Gamma)$ in tubes $T_\Gamma \subset \mathbb{C}^n$ for $p \in (0, 1]$ is obtained. Unlike the case of $L^p(\mathbb{R}^n)$ spaces, the formula holds everywhere in T_Γ without any additional assumptions. To the best of our knowledge, the result is new even for the univariate case—Hardy spaces in the upper half-plane.

Keywords Poisson summation formula · Hardy space · Tube area · Open cone · Absolute and locally-uniform convergence

Mathematics Subject Classification 42B30 · 42B05 · 42B08

1 Introduction

The Poisson summation formula is one of the famous results of the Fourier analysis. The classical one (see, e.g., [19, Chapter VII, Section 2, Theorem 2.4]) says that if $f \in L^1(\mathbb{R}^n)$, then the series $\sum_{m\in\Lambda} f(x+m)$ converges in the norm of $L^1(\mathbb{T}^n)$ and the resulting function in $L^1(\mathbb{T}^n)$ has the Fourier expansion

$$\sum_{m\in\Lambda} \widehat{f}(m)\, e^{2\pi i(x,m)}.$$

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Here, and throughout the article, Λ denotes the unit lattice, i.e., the additive group of points in $\mathbb{R}^n$ having integral coordinates; $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^n$.

In other words, the Poisson summation formula states that two approaches to periodization of a function f give the same result. This immediately leads to the question when actually equality

$$\sum_{m\in\Lambda} f(x+m) = \sum_{m\in\Lambda} \widehat{f}(m)\, e^{2\pi i(x,m)} \tag{1}$$

holds. For example, if we have the inversion formula for f, that is:

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(t)\, e^{2\pi i(x,t)}\, \mathrm{d}t,$$

with $|f(x)| \le A(1+|x|)^{-n-\delta}$ and $\left|\widehat{f}(t)\right| \le A(1+|t|)^{-n-\delta}$, $\delta > 0$, then (1) holds for any $x \in \mathbb{R}^n$ (see [19, Chapter VII, Section 2, Corollary 2.6]). There are other results stating that (1) in the univariate case holds under relaxed requirements (see, e.g., [6,10]). Note that having just $f, \widehat{f} \in L^1(\mathbb{R})$ is not enough. For example, Katznelson [11] proved that there exists a function f, such that $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, but $f(0) = 1$, $f(n) = 0$, $n \in \mathbb{Z}\backslash\{0\}$, $\widehat{f}(n) = 0$, $n \in \mathbb{Z}$.

There are several generalizations of the Poisson summation formula. It has many interesting applications, such as periodization and sampling operators [1], uncertainty principle [6], problems of number theory (see, e.g., [14,15]), and many applications in Physics, Signal Processing, Communications, Electromagnetics, etc. (see [8,16], and references therein). In particular, the Poisson summation formula is a useful tool for dealing with various Dyadic Green Functions. There is even its generalization for manifolds with boundary [7]. New results, generalizations, and applications of the formula are still appearing (see, e.g., [3,4,9,12]).

It is interesting that Riemann used the Poisson summation formula to prove one of his results on zeta-function [17]. We refer the reader to the work of Miller and Schmid [14] for an excellent review of related problems.

The Poisson summation formula itself has been obtained for several functional spaces. However, the results usually state (1) almost everywhere only.

For functions of bounded variation on $\mathbb{R}$, the Poisson summation formula valid pointwise is obtained by Trigub [20, Lemma 2] (see also [21, Section 3.1.11]). For functions of bounded variation on $\mathbb{R}^d$, a generalization of Trigub's result was recently obtained by Liflyand and Stadtmüller [13]. It is interesting that the formula holds under mild restrictions, sharpness of which is also discussed in [13].

Recent work of Butzer et al. [2] is devoted to obtaining generalizations of several classical formulas and inequalities in various function spaces. The Poisson summation formula was among them. The idea is that the classical formulas and inequalities usually become invalid in not-so-perfect spaces. To get their analogues, the authors developed a sophisticated and promising technique of the so-called unified distance concept.

We claim that for Hardy spaces $H^p(T_\Gamma)$, $p \in (0, 1]$, the classical formula (1) holds everywhere. No additional assumptions are needed. Moreover, the periodization of f

is analytic in T_Γ. The exact statement is Theorem 1 below. This answers a question asked by Roald M. Trigub (Donetsk National University, Ukraine) several years ago. In his opinion, the inversion formula for the Fourier transform obtained in [22] should result in something like the Poisson summation formula.

2 Definitions and Main Results

Let us recall some notation related to Hardy spaces in tubes, which are a natural multivariate generalization of the upper half-plane in $\mathbb{C}$.

Let B be an open set in $\mathbb{R}^n$, $n \in \mathbb{N}$. Following [19, Chapter III], the tube with base B is

$$T_B = \left\{z \in \mathbb{C}^n, z = x + iy : x \in \mathbb{R}^n, y \in B\right\}.$$

The Hardy space $H^p(T_B)$, $p \in (0, \infty]$, consists of functions f holomorphic in T_B, such that

$$\|f\|_{H^p} := \|f\|_{H^p(T_B)} := \begin{cases} \sup\limits_{y \in B} \left(\int_{\mathbb{R}^n} |f(x+iy)|^p \, \mathrm{d}x\right)^{1/p}, & p \in (0, \infty), \\ \sup\limits_{z \in T_B} |f(z)|, & p = \infty \end{cases}$$

is finite. We will also use the following notation:

$$f_\delta(\cdot) := f(\cdot + i\delta), \quad \delta \in B.$$

This notation is convenient to use in both contexts: $f_\delta(z) = f(z + i\delta)$, $z \in \mathbb{C}^n$, and $f(z) = f(x+iy) = f_y(x)$, $x \in \mathbb{R}^n$. Thus, $\|f\|_{H^p(T_B)} = \sup_{y \in B} \|f_y\|_p$, where $\|\cdot\|_p$ is a standard norm (or pre-norm) in $L^p(\mathbb{R}^n)$.

Since the case of an arbitrary tube T_B is too complicated, the attention is usually restricted to the case, where B is chosen to be an open cone.

A non-empty open set $\Gamma \subset \mathbb{R}^n$ is called an open cone if $0 \notin \Gamma$ and whenever $x, y \in \Gamma$ and $\alpha, \beta > 0$, the linear combination $\alpha x + \beta y \in \Gamma$. In particular, Γ is a convex set. The closure of an open cone is called a closed cone.

For any open cone Γ, the set

$$\Gamma^* = \left\{x \in \mathbb{R}^n : (x, t) \geq 0, \quad \forall t \in \Gamma\right\}$$

is closed. If Γ^* has non-empty interior, then it is a closed cone, and Γ is called a regular cone. The closed cone Γ^* is called the cone dual to Γ.

Let us start with an easy but very useful observation, which follows from the proof of [19, Chapter III, Section 2, Lemma 2.12].

Lemma 1 ([22, Lemma 1]) *Let* Γ *be an open cone in* $\mathbb{R}^n$, $p \in (0, \infty]$, *and* $q \in [p, \infty]$. *If* $f \in H^p(T_\Gamma)$, *then for any* $\delta \in \Gamma$, *we have* $f_\delta \in H^q(T_\Gamma)$ *and*

$$\|f_\delta\|_{H^q} \leq \left(\frac{\Omega_n}{\Omega_{2n}}\right)^{\frac{1}{p}-\frac{1}{q}} D_{\delta,\Gamma}^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{H^p},$$

where Ω_m *is the volume of the unit ball in* R^m, *i.e.,* $\Omega_m = \pi^{m/2}/\Gamma(m/2+1)$, *and* $D_{\delta,\Gamma} = \mathrm{dist}(\delta, \mathbb{R}^n \backslash \Gamma)$.

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i(\xi,t)} \mathrm{d}t, \qquad \xi \in \mathbb{R}^n.$$

It is extended to the case of a function from $L^2(\mathbb{R}^n)$ in the usual way.

Lemma 1 and [22, Theorem1] justify the following definition of the Fourier transform of a function from $H^p(T_\Gamma)$ and $p \in (0, 1]$.

Definition 1 The Fourier transform of a function $f \in H^p(T_\Gamma)$, $p \in (0, 1]$, is defined by

$$\widehat{f}(\xi) = e^{2\pi(\xi,\delta)} \widehat{f_\delta}(\xi), \qquad \xi \in \mathbb{R}^n \quad (\delta \in \Gamma - \mathit{arbitrary}). \tag{2}$$

It is easy to see that the right-hand side of (2) is independent of δ. Let us also note that for $p = 1$, our $\widehat{f}$ coincides with the classical Fourier transform of the limit function $F(x) := \lim_{\zeta \to 0, \zeta \in \Gamma} f_\zeta(x)$, $x \in \mathbb{R}^n$.

Furthermore, if $f \in H^p(T_\Gamma)$ for some $p \in (0, 1]$, then the following inversion formula holds true (see [22])

$$f(z) = \int_{\Gamma^*} \widehat{f}(t) e^{2\pi i(z,t)} \mathrm{d}t, \qquad z \in T_\Gamma. \tag{3}$$

Therefore, for any $p \in (0, 1]$, the space $f \in H^p(T_\Gamma)$ contains non-zero functions if and only if the cone Γ is regular (in fact, this is true for $p \in (0, \infty)$, since $f \in H^p$ implies $(f)^{[p]+1} \in H^s$ with $s = p/([p]+1) \in (0, 1]$, where $[p]$ denotes the integral part of p). This is why we investigate only the case of a regular cone.

The existence of the inversion formula (3) suggested that the Poisson summation formula should hold in $H^p(T_\Gamma)$ and $p \in (0, 1]$, without any additional assumptions. Indeed, the following result holds.

Theorem 1 *Let* Γ *be a regular cone in* $\mathbb{R}^n$, $n \in \mathbb{N}$, *and let* $f \in H^p(T_\Gamma)$ *for some* $p \in (0, 1]$. *Then*

$$\sum_{m \in \Lambda} f(z+m) = \sum_{m \in \Lambda} \widehat{f}(m) e^{2\pi i(z,m)} = \sum_{m \in \Lambda \cap \Gamma^*} \widehat{f}(m) e^{2\pi i(z,m)}, \qquad \forall z \in T_\Gamma. \tag{4}$$

Moreover, both the sides of (4) *are holomorphic in* T_Γ.

Note that Theorem 1 holds not just because of the inversion formula for the Fourier transform. The most important part is that the Fourier transform defined by (2) has a well-controlled growth (see estimate (13) below), which implies absolute and uniform convergence of the series in the right-hand side of (4). This growth estimate and the inversion formula (3) make $H^p(T_\Gamma)$ spaces so special when $p \in (0, 1]$, and allows us to obtain various inequalities, convenient conditions for Fourier multipliers, etc. (see [18,22,23]).

3 Proof

To prove Theorem 1, we need the following lemma, which could be of independent interest.

Lemma 2 *Let* Γ *be a regular cone in* $\mathbb{R}^n$, $n \in \mathbb{N}$, *and let* $f \in H^p(T_\Gamma)$ *for some* $p \in (0, 1]$. *Then, for any* $\delta \in \Gamma$, *the series*

$$\sum_{m\in\Lambda} f(x + m + i\delta)$$

converges absolutely and uniformly on $x \in Q^n$, *where* $Q^n = \left[-\frac{1}{2}, \frac{1}{2}\right)^n$*—the fundamental cube. Moreover, for any* $x \in Q^n$, *we have*

$$\sum_{m\in\Lambda} |f(x + m + i\delta)| \le 3^n D_{\delta,\Gamma}^{-n\left(\frac{1}{p}-1\right)} \left(\min\left\{D_{\delta,\Gamma}, 1\right\}\right)^{-n} \left(\frac{\Omega_n}{\Omega_{2n}}\right)^{\frac{1}{p}} \|f\|_{H^p}$$

$$\le 3^n \left(\min\left\{D_{\delta,\Gamma}, 1\right\}\right)^{-\frac{n}{p}} \left(\frac{\Omega_n}{\Omega_{2n}}\right)^{\frac{1}{p}} \|f\|_{H^p}, \tag{5}$$

where $D_{\delta,\Gamma} = \operatorname{dist}(\delta, \mathbb{R}^n\backslash\Gamma)$, $\Omega_m = \pi^{m/2}/\Gamma(m/2 + 1)$

Proof Take an arbitrary $\alpha \in (0, 1)$, such that the closure of the (n-dimensional) ball centered at δ and of radius α, $\overline{B_{\mathbb{R}^n}(\delta, \alpha)} \subset \Gamma$. Since f is analytic in T_Γ, $|f|$ is subharmonic there. Therefore, taking arbitrary $m = (m_1, \ldots, m_n) \in \Lambda$ and $x \in Q^n$ and considering that the closure of the ($2n$-dimesional) ball $\overline{B_{\mathbb{C}^n}(x + m + i\delta, \alpha)} \subset T_\Gamma$, we arrive at

$$|f(x + m + i\delta)| \le \frac{1}{\mu\left(B_{\mathbb{C}^n}(x + m + i\delta, \alpha)\right)} \int_{B_{\mathbb{C}^n}(x+m+i\delta,\alpha)} |f(z)|\, d\mu(z)$$

$$= \frac{1}{\Omega_{2n}\alpha^{2n}} \int_{B_{\mathbb{C}^n}(x+m+i\delta,\alpha)} |f(z)|\, d\mu(z),$$

where μ is the Lebesgue (volume) measure in $\mathbb{C}^n$ treated as $\mathbb{R}^{2n}$.

Since $\alpha \in (0,1)$, it is easy to see that $B_{\mathbb{C}^n}(x+m+i\delta,\alpha) \subset E_{m,\delta,\alpha}$, where

$$E_{m,\delta,\alpha} := \left\{ x+iy \in \mathbb{C}^n : x \in \prod_{j=1}^{n} \left[m_j - \frac{3}{2}, m_j + \frac{3}{2} \right],\ y \in B_{\mathbb{R}^n}(\delta,\alpha) \right\}.$$

Now, the last estimate implies

$$|f(x+m+i\delta)| \leq \frac{1}{\Omega_{2n}\alpha^{2n}} \int_{E_{m,\delta,\alpha}} |f(z)|\, \mathrm{d}\mu(z). \tag{6}$$

Furthermore,

$$\int_{E_{m,\delta,\alpha}} |f(z)|\, \mathrm{d}\mu(z) = \int_{B_{\mathbb{R}^n}(\delta,\alpha)} \left(\int_{\prod_{j=1}^{n}[m_j-3/2, m_j+3/2]} |f(t+iy)|\, \mathrm{d}t \right) \mathrm{d}y. \tag{7}$$

Since

$$\begin{aligned} \sum_{m\in\Lambda} \int_{\prod_{j=1}^{n}[m_j-3/2, m_j+3/2]} |f(t+iy)|\, \mathrm{d}t &= 3^n \sum_{m\in\Lambda} \int_{\prod_{j=1}^{n}[m_j-1/2, m_j+1/2]} |f(t+iy)|\, \mathrm{d}t \\ &= 3^n \left\| f_y \right\|_{L^1(\mathbb{R}^n)} \leq 3^n \|f\|_{H^1} \\ &\leq 3^n \left(\frac{\Omega_n}{\Omega_{2n}} \right)^{\frac{1}{p}-1} D_{\delta,\Gamma}^{-n\left(\frac{1}{p}-1\right)} \|f\|_{H^p}. \end{aligned} \tag{8}$$

Note that the last inequality follows from Lemma 1.

Thus, from (6), (7), and (8), we easily conclude that any partial sum of the series $\sum_{m\in\Lambda} |f(x+m+i\delta)|$ is bounded above by

$$\begin{aligned} &\frac{1}{\Omega_{2n}\alpha^{2n}} 3^n \left(\frac{\Omega_n}{\Omega_{2n}} \right)^{\frac{1}{p}-1} D_{\delta,\Gamma}^{-n\left(\frac{1}{p}-1\right)} \|f\|_{H^p} \int_{B_{\mathbb{R}^n}(\delta,\alpha)} \mathrm{d}y \\ &= \frac{3^n}{\alpha^n} \left(\frac{\Omega_n}{\Omega_{2n}} \right)^{\frac{1}{p}} D_{\delta,\Gamma}^{-n\left(\frac{1}{p}-1\right)} \|f\|_{H^p}. \end{aligned}$$

Hence, the series $\sum_{m\in\Lambda} f(x+m+i\delta)$ converges absolutely for any $x \in Q^n$ and

$$\sum_{m\in\Lambda} |f(x+m+i\delta)| \leq \frac{3^n}{\alpha^n} \left(\frac{\Omega_n}{\Omega_{2n}} \right)^{\frac{1}{p}} D_{\delta,\Gamma}^{-n\left(\frac{1}{p}-1\right)} \|f\|_{H^p}.$$

Finally, if we pass to the limit as $\alpha \to \min\left\{D_{\delta,\Gamma}, 1\right\}-$ (from the left), we get (5).

To prove that the series $\sum_{m\in\Lambda} |f(x+m+i\delta)|$ converges uniformly on $x \in Q^n$, let us estimate its remainder. For $N \in \mathbb{N}$, from (6) and (7), we have

$$\begin{aligned}\sup_{x\in Q^n} \sum_{|m|\geq N} |f(x+m+i\delta)| \\ \leq \frac{1}{\Omega_{2n}\alpha^{2n}} \sum_{|m|\geq N} \int_{B_{\mathbb{R}^n}(\delta,\alpha)} \left(\int_{\prod_{j=1}^n [m_j-3/2, m_j+3/2]} |f(t+iy)| \, \mathrm{d}t \right) \mathrm{d}y \\ = \frac{1}{\Omega_{2n}\alpha^{2n}} \int_{B_{\mathbb{R}^n}(\delta,\alpha)} \sum_{|m|\geq N} \left(\int_{\prod_{j=1}^n [m_j-3/2, m_j+3/2]} |f(t+iy)| \, \mathrm{d}t \right) \mathrm{d}y. \quad (9)\end{aligned}$$

The last equality follows from the Tonelli's theorem (with summation treated as integration with respect to the discrete measure).

Estimate (8) justifies the application of the Lebesgue dominated convergence theorem:

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{\Omega_{2n}\alpha^{2n}} \int_{B_{\mathbb{R}^n}(\delta,\alpha)} \sum_{|m|\geq N} \left(\int_{\prod_{j=1}^n [m_j-3/2, m_j+3/2]} |f(t+iy)| \, \mathrm{d}t \right) \mathrm{d}y \\ = \frac{1}{\Omega_{2n}\alpha^{2n}} \int_{B_{\mathbb{R}^n}(\delta,\alpha)} \lim_{N\to\infty} \sum_{|m|\geq N} \left(\int_{\prod_{j=1}^n [m_j-3/2, m_j+3/2]} |f(t+iy)| \, \mathrm{d}t \right) \mathrm{d}y \\ = 0. \quad (10)\end{aligned}$$

Taking $\limsup_{N\to\infty}$ in (9) and considering (10), we conclude that

$$\lim_{N\to\infty} \sup_{x\in Q^n} \sum_{|m|\geq N} |f(x+m+i\delta)| = 0.$$

Thus, the series $\sum_{m\in\Lambda} |f(x+m+i\delta)|$ converges uniformly on $x \in Q^n$, whence the series $\sum_{m\in\Lambda} f(x+m+i\delta)$ also converges uniformly on $x \in Q^n$. □

Proof of Theorem 1 In our standard notation, $z = x + iy$, $x \in \mathbb{R}^n$, $y \in \Gamma$, and $\widehat{f}(t) = \widehat{f_y}(t)\, e^{2\pi(y,t)}$. Therefore, (4) can be rewritten in the following form:

$$\sum_{m\in\Lambda} f_y(x+m) = \sum_{m\in\Lambda} \widehat{f_y}(m)\, e^{2\pi i(x,m)}, \qquad x \in \mathbb{R}^n,\ y \in \Gamma. \quad (11)$$

Considering the periodicity of both the sides of (11), it is sufficient to prove this formula for $x \in Q^n$.

Let us fix an arbitrary $y \in \Gamma$. As we already know, $f_y \in L^1(\mathbb{R}^n)$. Hence, [19, Chapter VII, Section 2, Theorem 2.4] implies that the series $\sum_{m\in\Lambda} f_y(x+m)$ converges in $L^1(Q_n) = L^1(\mathbb{T}^n)$. Its sum has the Fourier series:

$$\sum_{m\in\Lambda} \widehat{f_y}(m)\, e^{2\pi i(x,m)}. \tag{12}$$

Since $\widehat{f_y}(m) = e^{-2\pi(y,m)}\widehat{f}(m)$, supp $\widehat{f} \subset \Gamma^*$, and

$$\left|\widehat{f}(t)\right| \le A(n, p, \Gamma)\, \|f\|_{H^p}\, |t|^{n\left(\frac{1}{p}-1\right)}, \quad t \in \Gamma^* \tag{13}$$

(see [22, Section 1] and [5, Lemma 4]), we obtain that for any $x \in \mathbb{R}^n$, $m \in \Gamma^*$,

$$\left|\widehat{f_y}(m)\, e^{2\pi i(x,m)}\right| = \left|\widehat{f}(m)\, e^{-2\pi(y,m)}\right| \le A(n, p, \Gamma)\, \|f\|_{H^p}\, |m|^{n\left(\frac{1}{p}-1\right)}\, e^{-2\pi(y,m)}.$$

Since clearly

$$\sum_{m\in\Lambda\cap\Gamma^*} |m|^{n\left(\frac{1}{p}-1\right)}\, e^{-2\pi(y,m)} < \infty,$$

we conclude that the series (12) converges uniformly and absolutely on $x \in Q^n$. Thus, the function $\sum_{m\in\Lambda} \widehat{f_y}(m)\, e^{2\pi i(x,m)}$ is continuous on $x \in Q^n$ (or $\mathbb{T}^n$).

Applying [19, Chapter VII, Section 1, Corollary 1.8] to $F(x) = \sum_{m\in\Lambda} f_y(x+m)$, we obtain that

$$\sum_{m\in\Lambda} f_y(x+m) = \sum_{m\in\Lambda} \widehat{f_y}(m)\, e^{2\pi i(x,m)}, \quad \text{for a.e. } x \in Q^n. \tag{14}$$

As we already noticed, the right-hand side of (14) is continuous on Q^n. The left-hand side is continuous in view of Lemma 2. Hence, (14) holds for every $x \in Q^n$, which completes the proof of (4).

Finally, using estimate (13) and the fact that supp $\widehat{f} \subset \Gamma^*$, it is easy to see that the series in the right-hand side of (4) converges locally uniformly in T_Γ. This implies that both the sides of (4) are holomorphic in T_Γ functions. □

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