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One helpful property of functions generating Pólya frequency sequences

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Abstract In this work we study solutions of the equation $z^p R(z^k) = \alpha$ with non-zero complex α, integer p, k and $R(z)$ generating a (possibly doubly infinite) totally positive sequence. It is shown that the zeros of $z^p R(z^k) - \alpha$ are simple (or at most double in the case $\operatorname{Im} \alpha^k = 0$) and split evenly among the sectors $\{\frac{j}{k}\pi \leqslant \operatorname{Arg} z \leqslant \frac{j+1}{k}\pi\}$, $j = 0, \ldots, 2k-1$. Our approach rests on the fact that $z(\ln z^{p/k} R(z))'$ is an $\mathcal{R}$-function (i.e. maps the upper half of the complex plane into itself). This result guarantees the same localization to zeros of entire functions

$$f(z^k) + z^p g(z^k) \quad \text{and} \quad g(z^k) + z^p f(z^k)$$

provided that $f(z)$ and $g(-z)$ have genus 0 and only negative zeros. As an application, we deduce that functions of the form $\sum_{n=0}^{\infty} (\pm i)^{n(n-1)/2} a_n z^n$ have simple zeros distinct in absolute value under a certain condition on the coefficients $a_n \geqslant 0$. This includes the "disturbed exponential" function corresponding to $a_n = q^{n(n-1)/2}/n!$ when $0 < q \leqslant 1$, as well as the partial theta function corresponding to $a_n = q^{n(n-1)/2}$ when $0 < q \leqslant q_* \approx 0.7457224107$.

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1 Introduction

The present paper studies quite a general equation of the form $z^p R(z^k) = \alpha$; however the simple case $k = 2$ considered in Sects. 8 and 9 has the most interesting applications. In particular, Corollary 34 introduces sufficient conditions on a function of the form $\sum_{n=0}^{\infty} i^{\pm\frac{n(n-1)}{2}} f_n z^n$, where $f_0 \neq 0$ and $f_n \geqslant 0$ for all n, which assure the simplicity of its zeros. It turns out that zeros of such functions as

$$\mathcal{F}(z; \pm iq) = \sum_{n=0}^{\infty} \frac{1}{n!} (\pm iq)^{\frac{n(n-1)}{2}} z^n \quad \text{when} \quad 0 < q \leqslant 1 \quad \text{and}$$
$$\Theta_0(z; \pm iq) = \sum_{n=0}^{\infty} (\pm iq)^{\frac{n(n-1)}{2}} z^n \quad \text{when} \quad 0 < q \leqslant q_* \approx 0.7457224107$$

are simple and distinct in absolute value. What is more, the inequality $0 < q \leqslant q_*$ does not seem to be necessary: computer simulations show that the constant q_* can be replaced with a noticeably greater number so that all zeros of $\Theta_0(z; \pm iq)$ still keep their simplicity and distinctness in absolute value. The former function $\mathcal{F}(z; q)$ gives a solution to the functional-differential problem

$$\mathcal{F}'(z) = \mathcal{F}(qz), \quad \mathcal{F}(0) = 1,$$

while the latter is the partial theta function satisfying

$$\Theta_0(z; q) = 1 + z\Theta_0(zq; q).$$

The partial theta function participates in a number of beautiful Ramanujan-type relations ([3, Chapter 6], [29])and is related to q-series and some types of modular forms. Both $\mathcal{F}$ and Θ_0 appear in problems of statistics and combinatorics (see e.g. [25,26]) and their zeros are the subjects of conjectures by Alan Sokal. The details can be found in Sect. 9.

Nevertheless, general statements offer a better insight into the problem, given an opportunity to determine factors on which the result depends and to find possible generalizations. Their main drawback is an excessive amount of specific cases in Sects. 4–6. To give a survey of our results, we briefly introduce two special classes of functions and definitions of α-sets and α-points.

Definitions. A doubly infinite sequence $\left(\rho_n\right)_{n=-\infty}^{\infty}$ is called totally positive if all of the minors of the (four-way infinite) Toeplitz matrix $\left(\rho_{n-j}\right)_{n,j=-\infty}^{\infty}$ are non-negative (i.e. the matrix is totally non-negative). Unless we have $\rho_n = \rho_0^{1-n}\rho_1^n$ for every n, the correspondent power series $\sum_{n=-\infty}^{\infty}\rho_n z^n$ converges in some annulus to a function of the following form:

$$Cz^p e^{Az+\frac{A_0}{z}} \cdot \frac{\prod_{\nu>0}\left(1+\frac{z}{a_\nu}\right)}{\prod_{\mu>0}\left(1-\frac{z}{b_\mu}\right)} \cdot \frac{\prod_{\nu>0}\left(1+\frac{z^{-1}}{c_\nu}\right)}{\prod_{\mu>0}\left(1-\frac{z^{-1}}{d_\mu}\right)} \tag{1}$$

with absolutely convergent products, integer p and coefficients satisfying $A, A_0, C \geqslant 0$, $a_\nu, b_\mu, c_\nu, d_\mu > 0$ for all ν, μ. The converse is also true: every function with the representation (1) generates (i.e. its Laurent coefficients form) a doubly infinite totally positive sequence. Both these facts are first proved in the paper [10].

In the case of $\cdots = \rho_{j-2} = \rho_{j-1} = 0 \neq \rho_j$, we assume the sequence to be terminating on the left of ρ_j and call it totally positive. A totally positive sequence can be infinite when it contains no zeros to the right of ρ_j or finite otherwise. These sequences were studied earlier than doubly infinite ones in [1]. They are generated by functions of the form (1), where the products in the last fraction are empty and $A_0 = 0$. Note that the term Pólya frequency sequence is often used as a synonym for totally positive sequence.

Herein, it is convenient to use the notion of α-point. Given a complex number α, the *α-set* of a function $f(z)$ is the set $\{z \in \mathbb{C} : f(z) = \alpha\}$ and points of this set are called *α-points*. A non-constant meromorphic function can clearly have only isolated α-points. We say that an α-point z_* of a function f has multiplicity $n \in \mathbb{Z}_{>0}$ whenever $f'(z_*) = \cdots = f^{(n-1)}(z_*) = 0 \neq f^{(n)}(z_*)$. The α-point is simple if its multiplicity equals one.

The present work aims at describing the behaviour of α-points of functions which can be represented[1] as $z^p R(z^k)$, where p is an integer, k is a positive integer and $R(z)$ is not constant and generates a (possibly doubly infinite) totally positive sequence. We confine ourselves to the case when $\gcd(|p|, k) = 1$: other cases can be treated by introducing the variable $\eta := z^{\gcd(|p|,k)}$. As a main tool, we use a relation of such functions to the so-called $\mathcal{R}$-functions (also known as the Pick or Nevanlinna functions).

By definition, $\mathcal{R}$-functions are real (i.e. real at every real point of continuity) functions mapping the upper half of the complex plane into itself. They are characterized by the integral representation (see e.g. [16, Expression (S1.1.1)], [7, p. 201] or [19, p.311]; it is usually attributed to Rolf Nevanlinna)

$$C_1 + C_2 z + \int_{-\infty}^{\infty} \frac{1+\lambda z}{\lambda - z}\, d\varsigma(\lambda), \tag{2}$$

[1] Functions of this form are the kth root transforms of $z^p R^k(z)$. In the particular case when $R(z)$ and $R'(z)$ are holomorphic and non-zero at $z = 0$, the function $zR^k(z)$ is univalent in some disk centred at the origin. Then, $zR(z^k)$ will be a univalent function with k-fold symmetry in this disk in the sense that the image of $zR(z^k)$ will be k-fold rotationally symmetric (see e.g. [8, Sect. 2.1] for the details). The term "functions with k-fold symmetry" could be good under the narrower conditions imposed; however, we study a more general case assuming no such regularity at the origin and allowing any integer p satisfying $\gcd(|p|, k) = 1$.

where $C_2 \geqslant 0$ and C_1 are real constants and $\varsigma(\lambda)$ is a non-decreasing function of bounded variation. It is quite common to extend $\mathcal{R}$-functions into the lower half of the complex plane by complex conjugation, thus keeping the expression (2). Basic properties of $\mathcal{R}$-functions are summarized in [16].

Results. Our first goal is to describe the α-set of the expression $z^B R(z)$ in the closed upper half of the complex plane $\overline{\mathbb{C}_+} := \{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0\}$, where $R(z)$ is as above, B is real and $\alpha \in \mathbb{C}\backslash\{0\}$. This is done in Theorem 11: if the equation $z^B R(z) = \alpha$ has solutions in $\mathbb{C}_+$, then the α-points are simple and distinct in absolute value. The α-points on the real line (excepting the origin) may be either simple or double. For real constants a and $b_1 \neq b_2$ Theorem 13 shows that solutions to $z^B R(z) = ae^{ib_1}$ and to $z^B R(z) = ae^{ib_2}$ alternate when ordered in absolute value (under the additional condition that none of them fall onto the real line). The corresponding properties of α-points in the whole complex plane are described in Theorem 15 and Remark 16. Our approach is based on Lemma 1: a function $\psi(z)$ is univalent in the upper half of the complex plane provided that $z\psi'(z)$ is an $\mathcal{R}$-function. In fact, this lemma is an "appropriate" reformulation of classical results; however, we need a construction from its proof. Section 3 then considers the properties of $\psi(z)$ on the real line under the additional assumption that $\psi(z)$ is meromorphic in $\overline{\mathbb{C}_+}\backslash\{0\}$. It is interesting to note that Theorem 11 can be interpreted as a wide generalisation of the main theorem in [5].

The second goal of the present work—to study α-points of $z^p R(z^k)$—is presented in Theorems 20, 22–25. To derive these theorems we track the solutions to

$$z^{p/k} R(z) = \alpha \cdot \exp\left(i\frac{2\pi n}{k}\right) \quad \text{and} \quad z^{p/k} R(z) = \overline{\alpha} \cdot \exp\left(i\frac{2\pi n}{k}\right), \quad n \in \mathbb{Z},$$

under the change of variable $z \mapsto z^k$. If we split the complex plane into $2k$ sectors

$$Q_j = \left\{\frac{n}{k}\pi < \operatorname{Arg} z < \frac{n+1}{k}\pi\right\}, \quad j = 0, \dots, 2k-1,$$

then Theorem 20 states that for $\operatorname{Im}\alpha^k \neq 0$ all α-points are inner points of the sectors, simple, and those in distinct sectors strictly interlace with respect to their absolute value. In other words, if α-points of $z^p R(z^k)$ are denoted by z_i so that $\cdots \leqslant |z_{-1}| \leqslant |z_0| \leqslant |z_1| \leqslant \cdots$, then $\cdots < |z_{-1}| < |z_0| < |z_1| < \cdots$ and $z_i \in Q_n$ implies that $z_{i+1}, \dots, z_{i+2k-1} \notin Q_n$ and[2] that $z_{i+2k} \in Q_n$. In fact, there is a formula for m such that $z_{i+1} \in Q_m$, which is trivial for $p = \pm 1$ or $k = 2$. Theorem 22 provides analogous properties in the case $\operatorname{Im}\alpha^k = 0$. In particular, it asserts that there are at most two α-points sharing the same absolute value, which are simple unless they occur at a sector boundary where they may collapse into a double α-point.

In turn, Theorems 23, 24 and 25 answer the question about which sector contains the α-point that is minimal in absolute value for a meromorphic function $R(z)$. This

[2] As soon as the α-set of the function $z^p R(z^k)$ actually contains the point z_{i+2k}: Theorem 20 asserts nothing about existence of α-points, nor does Theorem 22.

automatically extends to the α-point that is maximal in absolute value when $R\left(\frac{1}{z}\right)$ is meromorphic.

Theorems 20 and 22–25 describe zeros of entire functions of the form

$$f(z^k) + z^j g(z^k) \quad \text{or} \quad g(z^k) + z^j f(z^k), \quad j, k \in \mathbb{Z}_{>0}, \tag{3}$$

where (complex) entire functions $f(z)$ and $g(-z)$ are of genus[3] 0 and have only negative zeros. Since $f(z^k)/f(0)$ and $g(z^k)/g(0)$ become real functions, the correspondence is provided by

$$f(z^k) + z^{-p} g(z^k) = z^{-p}\left(g(z^k) + z^p f(z^k)\right) = 0 \iff z^p \frac{f(z^k)/f(0)}{g(z^k)/g(0)} = -\frac{g(0)}{f(0)}$$

on setting $p := \pm j$. We can allow $f(z)$ and $g(-z)$ to be any functions generating totally positive sequences up to constant complex factors. Then the functions of the form (3) can be identified by the condition on their Maclaurin or Laurent coefficients. See Sect. 7 for further details.

Our third goal is attained in the last two sections. There we apply the above results in the setting $k = 2$; the results are summarized in Theorems 29 and 30. For a (complex) entire function H of the complex variable z consider its decomposition into odd and even parts such that $H(z) = f(z^2) + zg(z^2)$. Theorem 29 from Sect. 8 answers the following question: how are the zeros of the function $H(z)$ distributed if the ratio $\frac{f(z)}{g(z)}$ has only negative zeros and positive poles? The case when the ratio $\frac{f(z)}{g(z)}$ has only negative poles and positive zeros is treated by Theorem 30. The question appears to be connected to the Hermite-Biehler theorem. This is a well-known fact asserting that if the function $H(z)$ is a real polynomial, then its stability[4] is equivalent to the fact that $f(z)$ and $g(z)$ only have simple negative interlacing[5] zeros and $f(0) \cdot g(0) > 0$. This correspondence expressed as conditions on the Hurwitz matrix is at the heart of the Routh-Hurwitz theory (see e.g. [4,7,12,24,27]). With a proper extension of the notion of stability, this criterion extends to entire (see [7,19,24]), rational (see [4]) and further towards meromorphic functions. Furthermore, if $H(z)$ is a polynomial and we additionally allow the ratio $\frac{f(z)}{g(z)}$ to have positive zeros and poles, then we will obtain the "generalized Hurwitz" polynomials as introduced in [27]. In the same paper [27, Sect. 4.6], its author describes "strange" polynomials (related to stable polynomials) with interesting behaviour. Item (2) of our Theorem 29 and Item (5) of our Theorem 30 explain the nature of their "strangeness".

There are related questions which are not considered in the current work and can become the subject of forthcoming studies. One of them is to obtain more precise

[3] The definition of genus can be found in e.g. [7, p. 92] or [19, p. 9]. These books also introduce further basic notions of the theory of entire functions.

[4] The polynomial is called (Hurwitz) stable if all of its roots have negative real parts.

[5] The zeros of two functions are called interlacing if between each two consecutive zeros of the first function there is exactly one zero of the second function and vice versa. The Hermite-Biehler theorem assumes the interlacing property to be strict, which means that the functions have no common zeros.

estimates on arguments and absolute values of solutions to $z^p R(z^k) = \alpha$ when $R(z)$ belongs to specific subclasses of functions generating totally positive sequences. In this way, we can find such estimates for α-points lying close to the origin, which are not covered by the standard theory of value distribution. Another question (possibly related to the first one) is to make further progress toward proving the conjectures stated in Sect. 9.

2 Connection between $\mathcal{R}$-functions and univalent functions

Let us use the notation "arg" for the multivalued argument function and "Arg" for the principal branch of argument, $-\pi < \operatorname{Arg} z \leqslant \pi$ for any z. We start from the following useful observation:[6]

Lemma 1 *Let ϕ be a function holomorphic in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ with values in $\mathbb{C}_+$ and let ψ be a fixed holomorphic branch of $\int \frac{\phi(z)}{z}\,\mathrm{d}z$. Then the function ψ is univalent in $\mathbb{C}_+$. Moreover, if for some $z_1, z_2 \in \mathbb{C}_+$ we have*

$$a := \operatorname{Re}\psi(z_1) = \operatorname{Re}\psi(z_2) \quad \textit{and} \quad \operatorname{Im}\psi(z_1) < \operatorname{Im}\psi(z_2), \tag{4}$$

then $|z_1| < |z_2|$.

Proof First let us approximate the upper half-plane $\mathbb{C}_+$ by the set

$$\mathbb{C}_\delta := \left\{z \in \mathbb{C} : \ \delta < \operatorname{Arg} z < \pi - \delta, \ |z| > \delta\right\}, \quad \delta > 0.$$

For $z = re^{i\theta}$ we have $\dfrac{\partial z}{\partial r} = \dfrac{z}{|z|}$ and $\dfrac{\partial z}{\partial \theta} = iz$, so

$$r\frac{\partial}{\partial r}\operatorname{Im}\psi(z) = \operatorname{Im}\left(\frac{zr}{r}\psi'(z)\right) = \operatorname{Im}\phi(z) = -\frac{\partial}{\partial \theta}\operatorname{Re}\psi(z),$$

which is the Cauchy-Riemann equation. By hypothesis $\operatorname{Im}\phi(z) > 0$ for $z \in \overline{\mathbb{C}_\delta}$ therefore,

$$\frac{\partial}{\partial r}\operatorname{Im}\psi(z) > 0 \quad \text{and} \tag{5}$$

$$\frac{\partial}{\partial \theta}\operatorname{Re}\psi(z) < 0. \tag{6}$$

The latter inequality implies that for each $r > 0$ there can be at most one value of $\theta \in [\delta, \pi - \delta]$ such that $\operatorname{Re}\psi(re^{i\theta}) = a$. Moreover, the set $\Gamma_\delta := \left\{z \in \mathbb{C}_\delta : \operatorname{Re}\psi(z) = a\right\}$ only consists of analytic arcs because $\operatorname{Re}\psi$ is a function harmonic in $\overline{\mathbb{C}_\delta}$. In other words, we obtain the following:

[6] Many similar facts are well known. For example, considering functions $\Phi(\zeta) := \phi(e^{-\zeta})$ gives the problem from [31] but in a strip. However, we place this lemma here since we need the relation between $|z_1|$ and $|z_2|$ satisfying (4) rather than the univalence itself.

(a) For each $r > 0$ there is at most one point $z \in \Gamma_\delta$ satisfying $|z| = r$. That is, every arc of Γ in polar coordinates (r, θ) can be set by a function $\theta(r)$.
Furthermore, for every $R > \delta$ the domain $D := \{z \in \mathbb{C}_\delta : |z| < R\}$ contains at most a finite number of the arcs. Suppose that it contains an infinite number of them, then the ray $\{re^{i\theta} : r > 0\}$ for an appropriate fixed $\theta \in [\delta, \pi - \delta]$ meets $\overline{\Gamma_\delta}$ at an infinite number of points of D (since each arc has two ends on the boundary of D). The function $\mathrm{Re}\psi(re^{i\theta})$ is analytic in $r > 0$ as a function of two variables θ and r with θ fixed. Consequently, $\mathrm{Re}\psi(re^{i\theta})$ must be constant on that ray, because it attains the same value in points of a sequence converging to an internal point of its domain of analyticity. So, we have a contradiction unless Γ_δ is equal to $\{re^{i\theta} : r > 0\}$. However, in the case $\Gamma_\delta = \{re^{i\theta} : r > 0\}$ the curve Γ_δ contains only one arc.
Denote by $\gamma_1, \gamma_2, \dots$ the connected components of Γ_δ according to their distance to the origin so that[7] $\mathrm{dist}(0, \gamma_1) \leqslant \mathrm{dist}(0, \gamma_2) \leqslant \dots$. To count all arcs in this manner is possible because D contains only a finite number of them for any $R > \delta$. It is enough to justify two additional statements, which together with (a) imply the lemma.

(b) On each arc $\gamma_i,\ i = 1, 2, \dots$, the value of $\mathrm{Im}\psi$ increases (strictly) for increasing $|z|$.

(c) If we pass from γ_i to γ_{i+1} (due to (a) it corresponds to the grow of $|z|$), then $\mathrm{Im}\psi$ cannot decrease (in fact, we will show that these arcs can be connected by a line segment of $\partial\mathbb{C}_\delta$ where $\mathrm{Im}\psi$ increases).

To wit, the assertions (a)–(c) provide that any distinct points of $\mathbb{C}_\delta$ giving the same $\mathrm{Re}\psi$ give distinct $\mathrm{Im}\psi$ such that the conditions (4) imply $|z_1| < |z_2|$. In particular, this yields the univalence of ψ in $\mathbb{C}_\delta$. Furthermore, since δ is an arbitrary positive number, the lemma will hold in the whole open half-plane $\mathbb{C}_+$.

For the arc $\gamma_i,\ i = 1, 2, \dots$, consider its natural parameter τ. Orienting the arc according to the growth of r, we obtain $\frac{\partial\tau}{\partial r} > 0$. In addition, let us consider a coordinate ν changing in a direction orthogonal to τ, i.e. such that (τ, ν) form an orthogonal coordinate system. Then, with the help of inequality (5) and one of the Cauchy-Riemann equations, we deduce that[8]

$$\begin{aligned} 0 < \frac{\partial \mathrm{Im}\psi(z)}{\partial r} &= \frac{\partial \mathrm{Im}\psi(z)}{\partial \tau}\frac{\partial \tau}{\partial r} + \frac{\partial \mathrm{Im}\psi(z)}{\partial \nu}\frac{\partial \nu}{\partial r} \\ &= \frac{\partial \mathrm{Im}\psi(z)}{\partial \tau}\frac{\partial \tau}{\partial r} \pm \frac{\partial \mathrm{Re}\psi(z)}{\partial \tau}\frac{\partial \nu}{\partial r} = \frac{\partial \mathrm{Im}\psi(z)}{\partial \tau}\frac{\partial \tau}{\partial r} \end{aligned}$$

for $z \in \Gamma_\delta$. Therefore, it is true that $z_1, z_2 \in \gamma_i$ and $|z_1| < |z_2|$ imply $\mathrm{Im}\psi(z_1) < \mathrm{Im}\psi(z_2)$, which is equivalent to (b).

[7] Here $\mathrm{dist}(0, \gamma_i) := \inf_{z\in\gamma_i} |z|$ is the distance between the origin and the component $\gamma_i,\ i = 1, 2, \dots$.

[8] In fact we have more: $\partial \mathrm{Im}\psi(z)/\partial\nu = 0$ implies that the gradient of $\mathrm{Im}\psi$ on γ_i is tangential to γ_i.

Now, given two consecutive arcs γ_i and γ_{i+1} consider the arguments θ_1 and θ_2 of their adjacent points, i.e.

$$\theta_1 := \lim_{\substack{|z|\to r_1\\ z\in\gamma_i}} \operatorname{Arg} z \quad \text{and} \quad \theta_2 := \lim_{\substack{|z|\to r_2\\ z\in\gamma_{i+1}}} \operatorname{Arg} z, \text{ where } r_1 = \sup_{z\in\gamma_i}|z|,\ r_2 = \inf_{z\in\gamma_{i+1}}|z|.$$

The arguments can be either $\pi-\delta$ or δ, because the arcs are regular and hence can only end at the boundary of $\mathbb{C}_\delta$. Observe that $\theta_1=\theta_2$. Indeed, let for example $\theta_1=\pi-\delta$. Then (6) yields $\operatorname{Re}\psi(z)>a$ as $|z|=r_1,\ z\in\mathbb{C}_\delta$. However, $\theta_2=\delta$ in its turn would imply $\operatorname{Re}\psi(z)<a$ when $|z|=r_2$. So, in the "semi-annulus" $\{z\in\mathbb{C}_\delta : r_1<|z|<r_2\}$ there would be such z that $\operatorname{Re}\psi(z)=a$, i.e. $z\in\Gamma_\delta$ which contradicts the fact that γ_i and γ_{i+1} are consecutive arcs of Γ_δ.

Since $\theta_1=\theta_2$, the ray $\Theta := \left\{re^{i\theta_1}, r>\delta\right\}$ meets both arcs γ_i and γ_{i+1} in the limiting points $r_1e^{i\theta_1}$ and $r_2e^{i\theta_1}$, respectively. As a consequence, we obtain that $\operatorname{Im}\psi(r_1e^{i\theta_1})<\operatorname{Im}\psi(r_2e^{i\theta_1})$ since $\operatorname{Im}\psi$ grows everywhere on Θ by the condition (5). Then (b) implies that $\sup_{z\in\gamma_i}\operatorname{Im}\psi(z)\leqslant\inf_{z\in\gamma_{i+1}}\operatorname{Im}\psi(z)$. Thus, the condition (c) is satisfied as well. ◻

3 Properties of α-points on the real line

Lemma 2 *Under the conditions of Lemma 1, let the function ϕ admit an analytic continuation through the interval $(x_1,x_2)\subset\mathbb{R}\backslash\{0\}$. Then the function ψ defined as in Lemma 1 has no α-points with multiplicity more than two in (x_1,x_2).*

Proof The assertion of this lemma is exactly that $\phi(z)=z\psi'(z)$ has no multiple zeros in (x_1,x_2). However, if ϕ could have a double zero x_0, then $\operatorname{Im}\phi(z)$ in the semi-disk $\{z\in\mathbb{C}_+ : |z-x_0|<\varepsilon\ll 1\}$ must have values of both signs (since $\phi(z)$ is close to $(z-x_0)^2$ for such z). In its turn, this contradicts $\phi(\mathbb{C}_+)\subset\mathbb{C}_+$. ◻

Further in this section, we restrict the $\mathcal{R}$-functions ϕ_1,ϕ_2 to be meromorphic in $\mathbb{C}$ and real on the real line (where finite), i.e. to have the (absolutely convergent) Mittag-Lefler representation

$$B+Az-\frac{A_0}{z}-\sum_{\nu\neq0}\frac{zA_\nu/a_\nu}{z-a_\nu}, \tag{7}$$

where $B,a_\nu\in\mathbb{R};\ a_\nu\neq0;\ A,A_0\geqslant0$ and $A_\nu>0$ for all $\nu\neq0$, such that $\phi_1,\phi_2\not\equiv$ const. Non-constant real meromorphic functions of this form (and only of this form) map $\mathbb{C}_+$ into $\mathbb{C}_+$, see [7, Theorem 1 of Chapter V, Sect. 1], [30, II.8] or [19, Theorem 2 of Chapter VII, Sect. 1].

A non-constant function $\phi(z)$ is supposed to have the more general representation $\phi_1(z)-\phi_2(1/z)$, where $\phi_1(z)$ and $\phi_2(z)$ are as given by (7). Note that both mappings $z\mapsto\frac{1}{z}$ and $z\mapsto -z$ are real and map the upper half of the complex plane $\mathbb{C}_+$ into the lower half-plane, so $\phi(z)$ is necessarily an $\mathcal{R}$-function.

Remark 3 If $z\psi'(z)$ has the form (7), then $\psi(z)$ can be represented as

$$\psi(z) = \int \frac{z\psi'(z)}{z}\mathrm{d}z = C + B\ln z + Az + \frac{A_0}{z} - \sum_\nu \frac{A_\nu}{a_\nu}\ln\left(1 - \frac{z}{a_\nu}\right)$$

for some complex constant C. This implies the equality

$$\mathrm{Re}\psi(z) = \mathrm{Re}C + B\ln|z| + \left(A + \frac{A_0}{|z|^2}\right)\mathrm{Re}\ z - \sum_\nu \frac{A_\nu}{a_\nu}\ln\left|1 - \frac{z}{a_\nu}\right|. \qquad (8)$$

Remark 4 If $z\psi'(z) = \phi(z) = \phi_1(z) - \phi_2(1/z)$, then we introduce two auxiliary functions ψ_1 and ψ_2 (single-valued in $\overline{\mathbb{C}_+}$ where regular) so that $z\psi_1'(z) =: \phi_1(z)$ and $\psi(z) - \psi_1(z) =: \psi_2(z)$. These settings then imply $z\psi_2'(z) = -\phi_2(\frac{1}{z}) = z^2\left(\frac{1}{z}\right)' \cdot \phi_2\left(\frac{1}{z}\right)$, that is $\psi_2\left(\frac{1}{z}\right) = \int \frac{\phi_2(z)}{z}\mathrm{d}z$. Both $\phi_1(z)$ and $\phi_2(z)$ satisfy (7); therefore

$$\mathrm{Re}\psi(z) = \mathrm{Re}\psi_1(z) + \mathrm{Re}\psi_2(z),$$

where both $\mathrm{Re}\psi_1(z)$ and $\mathrm{Re}\psi_2(1/z)$ have the form (8). In particular, the function ψ has a logarithmic singularity in each pole $x_* \neq 0$ of ϕ, and $\mathrm{Re}\psi(z) \to +\infty \cdot x_*$ when $z \to x_*$. The notation $+\infty \cdot x_*$ stands for $+\infty$ if $x_* > 0$, and for $-\infty$ if $x_* < 0$.

Lemma 5 *If $x\psi'(x) = \phi(x) = \phi_1(x) - \phi_2(1/x)$, where $x \in \mathbb{R}$ and $\phi_1(x)$, $\phi_2(x)$ have the form* (7)*, then the following assertions are true:*

- (a) *The function* $\mathrm{Im}\psi(x)$ *can change its value only at the origin and in poles of ϕ.*
- (b) *Between every two consecutive negative poles $x_2 < x_1$ of ϕ, there is exactly one local maximum of* $\mathrm{Re}\psi$.
- (c) *Between every two consecutive positive poles $x_1 < x_2$ of ϕ, there is exactly one local minimum of* $\mathrm{Re}\psi$.
- (d) *In (b) and (c), x_1 can be set to zero provided that ϕ is regular between* 0 *and x_2, and* $\lim_{t\to 0+}\left|\phi(tx_2)\right| = \infty$*. In this case we have* $\mathrm{Re}\psi(tx_2) \to +\infty \cdot x_2$ *as $t \to$* $0+$.

Proof Take a real $x \neq 0$ such that both functions $\phi_1(x)$ and $-\phi_2(1/x)$ are regular. Since their values are real on the real line, the condition

$$x\frac{\partial \mathrm{Im}\psi(x)}{\partial x} = r\frac{\partial \mathrm{Im}\psi(x)}{\partial r} = \mathrm{Im}\phi(x) = \mathrm{Im}\phi_1(x) - \mathrm{Im}\phi_2(1/x) = 0$$

is satisfied. So the assertion (a) is true.

The function $x\frac{\partial \mathrm{Re}\psi(x)}{\partial x} = \mathrm{Re}\phi(x) = \phi_1(x) - \phi_2(1/x)$ strictly increases from $-\infty$ to $+\infty$ between the points x_1 and x_2, and hence it changes its sign exactly once in the interval $(\min(x_1, x_2), \max(x_1, x_2))$. That is, $\mathrm{sign}x \cdot \mathrm{Re}\psi(x)$ changes from decreasing to increasing on this interval, which is giving us the assertions (b) and (c) for both zero and non-zero x_1.

Suppose that the function ϕ is regular between 0 and x_2 and $\lim_{t\to 0+}|\phi(tx_2)|$ is infinite. Then ϕ increases in this interval, so $\lim_{t\to 0+}\phi(tx_2) = -\infty\cdot x_2$. Therefore, $-\psi'(tx_2) = -\frac{\phi(tx_2)}{tx_2} > \frac{1}{t}$ for small enough $t > 0$ and

$$\begin{aligned}\operatorname{Re}\psi(tx_2) &= \operatorname{Re}\psi\left(\tfrac{1}{2}x_2\right) + \int_{\frac{1}{2}x_2}^{tx_2}\frac{\phi(x)}{x}\mathrm{d}x\\ &= \operatorname{Re}\psi\left(\tfrac{1}{2}x_2\right) + x_2\int_t^{\frac{1}{2}}\left(-\frac{\phi(sx_2)}{sx_2}\right)ds \to +\infty\cdot x_2 \quad \text{as} \quad t\to 0+,\end{aligned}$$

which is (d). □

Lemma 6 *In addition to the conditions of Lemma 5, suppose that ϕ is a regular function in the interval $\mathfrak{I} = (\min\{0, x_2\}, \max\{0, x_2\}) \subset \mathbb{R}$, x_2 is a pole of ϕ and the limit $\mathfrak{B} := \lim_{t\to 0+}\phi(tx_2)$ is finite.*[9]

(a) *If $\mathfrak{B}x_2 > 0$, then $\operatorname{Re}\psi(x)$ is an increasing function in $\mathfrak{I}$ such that $\operatorname{Re}\psi(\mathfrak{I}) = \mathbb{R}$, and furthermore, $\operatorname{Re}\psi(z) \neq \operatorname{Re}\psi(x)$ on condition that $|z| \leqslant |x|$ with $x \in \mathfrak{I}$ and $z \in \overline{\mathbb{C}_+}\setminus\{x\}$.*
(b) *If $\mathfrak{B}x_2 < 0$, then $\operatorname{Re}\psi(x)$ has exactly one local extremum in $\mathfrak{I}$ and tends to $+\infty\cdot x_2$ as x approaches 0 or x_2.*
(c) *If $\mathfrak{B} = 0$, then $\operatorname{Re}\psi(x)$ is an increasing function in $\mathfrak{I}$ and the inequality $\operatorname{Re}\psi(z) \neq \operatorname{Re}\psi(x)$ holds provided that $|z| \leqslant |x|$ with $z \in \overline{\mathbb{C}_+}\setminus\{x\}$, $x \in \mathfrak{I}$. Moreover, $\lim_{t\to 0+}\frac{\phi(tx_2)}{tx_2}$ is positive or $+\infty$. If additionally $\operatorname{Re}\psi(tx_2)$ is unbounded as $t\to 0+$, then $\operatorname{Re}\psi(\mathfrak{I}) = \mathbb{R}$.*

Proof In the interval $\mathfrak{I}$, the function $x\frac{\partial \operatorname{Re}\psi(x)}{\partial x} = \phi(x)$ strictly increases, and hence changes its sign at most once. Therefore, $\operatorname{Re}\psi(x)$ has at most one local extremum: maximum for $x_2 < 0$ and minimum for $x_2 > 0$. Suppose that $0 < |\mathfrak{B}| < \infty$. Then the equality $x\frac{\partial \operatorname{Re}\psi(x)}{\partial x} = \phi(x)$ yields the following relation:

$$\operatorname{Re}\psi(tx_2) = \operatorname{Re}\psi\left(\tfrac{1}{2}x_2\right) + \int_{\frac{1}{2}x_2}^{tx_2}\frac{\phi(x)-\mathfrak{B}}{x}\mathrm{d}x + \mathfrak{B}\ln\frac{tx_2}{\frac{1}{2}x_2} \sim \mathfrak{B}\ln t \xrightarrow{t\to 0+} -\infty\cdot\mathfrak{B}.$$

On account of $\operatorname{Re}\psi(x) \to +\infty\cdot x_2$ when $x \to x_2$ (see Remark 4) this relation implies the assertion (b) and that $\operatorname{Re}\psi$ increases in $\mathfrak{I}$ from $-\infty$ to $+\infty$ if $\mathfrak{B}x_2 > 0$. Therefore, to obtain (a) it is enough to use the inequality

$$\operatorname{Re}\psi(-|z|) < \operatorname{Re}\psi(z) < \operatorname{Re}\psi(|z|), \quad \text{where} \quad \operatorname{Im} z > 0, \tag{9}$$

which is a consequence of (6). Indeed, if for example $x_2 < 0$, then we have $\operatorname{Re}\psi(x) \leqslant \operatorname{Re}\psi(-|z|) < \operatorname{Re}\psi(z)$ for each $x \in \mathfrak{I}$ satisfying $|x| \geqslant |z|$.

Since $\phi(x)$ is increasing, the condition $\mathfrak{B} = 0$ implies $\frac{\phi(x)}{x} > 0$ in the interval $\mathfrak{I}$, i.e. that $\operatorname{Re}\psi$ is growing independently of the sign of x_2. The inequality

[9] This limit exists since the function ϕ increases in $\mathfrak{I}$.

$\lim_{t\to 0+} \frac{\phi(tx_2)}{tx_2} \neq 0$ follows from the fact that $\mathcal{R}$-functions cannot vanish faster then linearly.[10] Furthermore, $\mathrm{Re}\psi$ runs through the whole $\mathbb{R}$ on condition that it is unbounded near the origin, as asserted in (c). If $|z| \leqslant |x|$ with $z \in \overline{\mathbb{C}_+}\setminus\{x\}$ and $x \in \mathfrak{I}$, then the inequality (9) provides $\mathrm{Re}\psi(z) \neq \mathrm{Re}\psi(x)$. ◻

Remark 7 In Lemmas 5 and 6, the value of x_2 can be taken equal to $+\infty$ or $-\infty$ at the cost of some of the conclusions. With such a choice, the condition $\mathrm{Re}\psi(x) \to +\infty \cdot x_2$ as $x \to x_2$ may be violated. This, in turn, implies that the function $\mathrm{Re}\psi(x)$ in (b), (c) and (d) of Lemma 5 and (b) of Lemma 6 may lose the extremum and become monotonic. In cases (a) and (c) of Lemma 6, $\mathrm{Re}\psi(\mathfrak{I})$ becomes only a semi-infinite interval of the real line, instead of the equality $\mathrm{Im}\psi(\mathfrak{I}) = \mathbb{R}$.

4 Location of α-points in the closed upper half-plane

Lemma 8 *Let functions $\phi_1(z)$, $\phi_2(z)$ be of the form (7) and let $\psi(z)$ be a holomorphic branch of $\int \Big(\phi_1(z) - \phi_2\big(\frac{1}{z}\big)\Big)\frac{\mathrm{d}z}{z}$. If two points $z_1, z_2 \in \overline{\mathbb{C}_+}$ that are regular for ψ satisfy $|z_1| < |z_2|$ and $a := \mathrm{Re}\psi(z_1) = \mathrm{Re}\psi(z_2)$, then*

(a) $\mathrm{Im}\psi(z_1) \leqslant \mathrm{Im}\psi(z_2)$;
(b) *For each $\varrho \in (\mathrm{Im}\psi(z_1), \mathrm{Im}\psi(z_2))$ there exists $z \in \overline{\mathbb{C}_+}$ such that $|z_1| < |z| < |z_2|$ and $\psi(z) = a + i\varrho$;*
(c) *z_1 and z_2 can be connected by a piecewise analytic curve of a finite length, on which ψ is smooth and $\mathrm{Im}\psi(z)$ is a non-decreasing function of $|z|$; the curve is a subinterval of $\mathbb{R}$ if and only if equality holds in (a);*
(d) *Furthermore, equality holds in (a) if and only if $z_1, z_2 \in \mathbb{R}$, $z_1 \cdot z_2 > 0$ and $\psi(z) \neq a$ for all $z \in \overline{\mathbb{C}_+}$ such that $|z_1| < |z| < |z_2|$.*

Proof Recall that the function $\phi(z) = \phi_1(z) - \phi_2(1/z)$ maps $\mathbb{C}_+ \to \mathbb{C}_+$, i.e. satisfies Lemma 1. Thus if $z_3, z_4 \in \mathbb{C}_+$ and $\mathrm{Re}\, z_3 = \mathrm{Re}\, z_4$, then the condition $\mathrm{Im}\psi(z_3) > \mathrm{Im}\psi(z_4)$ induces $|z_3| > |z_4|$, and $\mathrm{Im}\psi(z_3) < \mathrm{Im}\psi(z_4)$ induces $|z_3| < |z_4|$. As a consequence, the assertion (a) holds provided that both z_1, z_2 are not real.

The real part of ψ goes to $\pm\infty$ on approaching a (non-zero) pole of ϕ, as stated in Remark 4. Consequently, it is impossible for a pole of ϕ to be a limiting point of the set

$$\Gamma := \big\{z \in \mathbb{C}_+ : \mathrm{Re}\psi(z) = a,\ |z_1| < |z| < |z_2|\big\},$$

so the function ψ is regular in a neighbourhood of Γ. (Recall that $z_1 = 0$ is allowed by the hypothesis of the lemma only if ψ is regular at the origin.)

Analogously to Γ_δ from the proof of Lemma 1, points of Γ form an analytic curve possibly containing multiple disconnected components—analytic arcs. Due to (6), for each $r > 0$ there exist at most one value of $\theta \in (0, \pi)$ such that $re^{i\theta} \in \Gamma$. That is, if

[10] This fact follows from the integral representation (2). It can be also deduced from the expression (7) when $\mathcal{R}$-functions have the form considered in this lemma.

some z_3, z_4 satisfy $|z_3| < |z_4|$, $z_3 \in \gamma_1$ and $z_4 \in \gamma_2$, where γ_1 and γ_2 are arbitrary distinct arcs of Γ, then necessarily $\sup_{z\in\gamma_1} |z| =: r_1 \leqslant r_2 := \inf_{z\in\gamma_2} |z|$.

Suppose that the arcs γ_1 and γ_2 are consecutive, i.e. that $\Gamma \cap \{z : r_1 < |z| < r_2\} = \varnothing$. Within this setting, the limits

$$\zeta_1 := \lim_{\substack{|z|\to r_1\\ z\in\gamma_1}} z \quad \text{and} \quad \zeta_2 := \lim_{\substack{|z|\to r_2\\ z\in\gamma_2}} z$$

exist and are real. Moreover, they have the same sign: for example, $\zeta_1 < 0 < \zeta_2$ implies that $\mathrm{Re}\psi(ir_2) < \mathrm{Re}\psi(\zeta_2) = a = \mathrm{Re}\psi(\zeta_1) < \mathrm{Re}\psi(ir_1)$ according to (6) and hence $\mathrm{Re}\psi(ir_*) = a$ for some $r_* \in (r_1, r_2)$ by continuity, which contradicts $\Gamma \cap \{z \in \mathbb{C}_+ : r_1 < |z| < r_2\} = \varnothing$. The supposition $\zeta_2 < 0 < \zeta_1$ implies a contradiction in a similar way. Consequently, one (and only one) of the inequalities $\mathrm{Re}\psi(z) < a$ and $\mathrm{Re}\psi(z) > a$ holds for all $z \in \mathbb{C}_+$ satisfying $r_1 < |z| < r_2$; the former inequality $\mathrm{Re}\psi(z) < a$ corresponds to the positive sign of ζ_1, ζ_2, while the latter corresponds to the negative sign.

Now, each non-zero singularity x_* of $\psi(z)$ is a pole of $\phi(z)$, and Lemma 5 states that $\mathrm{Re}\psi(x_*) \to +\infty \cdot x_*$. In other words, for any $z \in \mathbb{C}_+$ close enough to x_* we have $\mathrm{Re}\psi(z) > a$ when $x_* > 0$ and $\mathrm{Re}\psi(z) < a$ when $x_* < 0$. At the same time, we have seen that $\mathrm{Re}\psi(z) < a$ if $0 < \zeta_1 \leqslant |z| \leqslant \zeta_2$ and $z \in \mathbb{C}_+$, so the condition $0 < \zeta_1 < x_* < \zeta_2$ cannot be satisfied. Analogously, the inequality $\mathrm{Re}\psi(z) > a$ holds provided that $\zeta_2 \leqslant -|z| \leqslant \zeta_1 < 0$ and $z \in \mathbb{C}_+$, so the condition $\zeta_2 < x_* < \zeta_1 < 0$ cannot be satisfied. As a consequence, the $\phi(z)$ is regular in the interval $\mathfrak{I} := \big[\min\{\zeta_1, \zeta_2\}, \max\{\zeta_1, \zeta_2\}\big]$. The function $\mathrm{Re}\psi$ is non-constant and has at most one extremum inside $\mathfrak{I}$ by Lemma 5, satisfies $\mathrm{Re}\psi(\zeta_1) = \mathrm{Re}\psi(\zeta_2) = a$, so the equality $\mathrm{Re}\psi(z) = a$ is impossible in $\mathfrak{I}\setminus\{\zeta_1, \zeta_2\}$. As a summary, we obtain that one of the inequalities $\mathrm{Re}\psi(z) < a$ or $\mathrm{Re}\psi(z) > a$ hold for all $z \in \overline{\mathbb{C}_+}$ such that $r_1 < |z| < r_2$.

By Lemma 5, $\mathrm{Im}\psi(z)$ is constant in $\mathfrak{I}$ (this fact implies the equality in (a) for $z_1 = \zeta_1 \neq 0$ and $z_2 = \zeta_2$). We obtain that, on $\gamma_1 \cup \mathfrak{I} \cup \gamma_2$, the function $\psi(z)$ is regular and $\mathrm{Im}\psi(z)$ is continuous and non-decreasing as $|z|$ grows. In particular, $\mathrm{Im}\psi(z)$ attains all intermediate values. This reasoning is applicable for each pair of consecutive arcs constituting the set Γ. That is, any two points $z_1, z_2 \in \overline{\mathbb{C}_+}\setminus\{0\}$ with $\mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2)$ can be connected by a piecewise analytic curve containing intervals of the real line and all arcs of Γ. The case when $z_1 = 0$ remains to be checked. In this case, $\psi(z)$ is regular at the origin, and thus it is strictly increasing in some real interval enclosing z_1 (due to $\mathfrak{B} = \lim_{z\to 0} z(\psi(z))' = 0$, see the assertion (c) of Lemma 6). Then (9) shows that z_1 is the end of some arc from Γ. Choosing this arc as γ_1 allows us to apply the previous part of the proof. In particular $\mathrm{Im}\psi(z_1) < \mathrm{Im}\psi(z_2)$.

Note that, on the one hand, poles of $\phi(z) = \phi_1(z) - \phi_2(1/z)$ can concentrate only at the origin since both $\phi_1(z)$ and $\phi_2(z)$ are meromorphic. On the other hand, each interval between poles contains at most two ends of arcs from Γ. Therefore, the number of arcs in $\Gamma \cap \{z : |z_1| < |z| < |z_2|\}$ is finite. Each of the arcs has a finite length since ψ is smooth in a neighbourhood of Γ, so the length of the curve connecting z_1 with z_2 is finite. This implies the assertions (b) and (c) of the lemma.

Furthermore, we necessarily have $\operatorname{Im}\psi(z_1) < \operatorname{Im}\psi(z_2)$ unless this piecewise analytic curve is a segment of the real line. That is, the assertions (a) and (d) are proved. □

Lemma 9 *Suppose that $f(z)$ is holomorphic at z_0, $g(z)$ is holomorphic at $f_0 = f(z_0)$ such that $g'(f_0) \neq 0$ and that n is a positive integer number. Then $f'(z_0) = f''(z_0) = \cdots = f^{(n)}(z_0) = 0$ if and only if*

$$\frac{\mathrm{d}g(f(z))}{\mathrm{d}z}\Big|_{z=z_0} = \frac{\mathrm{d}^2 g(f(z))}{\mathrm{d}z^2}\Big|_{z=z_0} = \cdots = \frac{\mathrm{d}^n g(f(z))}{\mathrm{d}z^n}\Big|_{z=z_0} = 0. \tag{10}$$

Analogously, if a function $f(z)$ is holomorphic at z_0 such that $f'(z_0) \neq 0$ and $g(z)$ is holomorphic at $f_0 = f(z_0)$, then the condition (10) is equivalent to $g'(f_0) = g''(f_0) = \cdots = g^{(n)}(f_0) = 0$.

Proof Both facts follow from solving equations provided by the chain rule sequentially:

$$\begin{aligned}
\frac{\mathrm{d}g(f(z))}{\mathrm{d}z}\Big|_{z=z_0} &= g'(f_0)f'(z_0),\\
\frac{\mathrm{d}^2 g(f(z))}{\mathrm{d}z^2}\Big|_{z=z_0} &= g''(f_0)(f'(z_0))^2 + g'(f_0)f''(z_0),\\
&\cdots\\
\frac{\mathrm{d}^n g(f(z))}{\mathrm{d}z^n}\Big|_{z=z_0} &= g^{(n)}(f_0)(f'(z_0))^n + \cdots + g'(f_0)f^{(n)}(z_0).
\end{aligned}$$

□

The machinery presented in the previous sections is suitable for studying functions of the form

$$V(z) = e^{Az+C+\frac{A_0}{z}} z^B \frac{\prod_{\nu>0}\left(1+\frac{z}{a_\nu}\right)^{\kappa_\nu}}{\prod_{\mu>0}\left(1-\frac{z}{b_\mu}\right)^{\lambda_\mu}} \quad \text{(a branch regular in } \mathbb{C}_+), \tag{11}$$

where $C \in \mathbb{C}$, $B \in \mathbb{R}$, $A, A_0 \geqslant 0$, and $a_\nu, \kappa_\nu, b_\mu, \lambda_\mu$ are positive reals for all ν, μ. Along with functions as in (11), we study functions of the more general form

$$W(z) = e^{Az+C+\frac{A_0}{z}} z^B \frac{\prod_{\nu>0}\left(1+\frac{z}{a_\nu}\right)^{\kappa_\nu}}{\prod_{\mu>0}\left(1-\frac{z}{b_\mu}\right)^{\lambda_\mu}} \frac{\prod_{\nu>0}\left(1+\frac{1}{zc_\nu}\right)^{\widetilde{\kappa}_\nu}}{\prod_{\mu>0}\left(1-\frac{1}{zd_\mu}\right)^{\widetilde{\lambda}_\mu}} = V_1(z)\, V_2\left(\frac{1}{z}\right), \tag{12}$$

where both V_1 and V_2 admit the representation (11).

Remark 10 Let $V(z)$, $V_1(z)$ and $V_2(z)$ be as in (11) and $W(z) = V_1(z)\, V_2\left(\frac{1}{z}\right)$. Clearly both V and W are regular and non-zero outside the real line. Moreover, the expression $z(\ln V(z))'$ is a meromorphic function of the form (7); the function $z(\ln W(z))' = z(\ln V_1)'(z) - \frac{1}{z} \cdot (\ln V_2)'\left(\frac{1}{z}\right)$ is meromorphic for $z \neq 0$ and maps $\mathbb{C}_+ \to \mathbb{C}_+$. Consequently, they have an analytic continuation in a neighbourhood of each real α-point (excluding the origin) for $\alpha \neq 0$. This allows us to determine the multiplicity of such α-points.

Theorem 11 *If a function W defined in $\overline{\mathbb{C}_+}$ has the form (12) such that $W(z) \not\equiv e^C z^B$, then for any $\alpha \in \mathbb{C}\backslash\{0\}$ the α-points of $W(z)$ lying in $\overline{\mathbb{C}_+}$ (if they exist) are at most double and distinct in absolute value from other solutions to $\left|W(z)\right| = |\alpha|$. The α-points inside $\mathbb{C}_+$ must be simple.*

Remark 12 If $W(z)$ is regular and non-zero for $z = 0$, then it has the form (11) with $A_0 = B = 0$. Therefore, the equality $W(0) = \alpha \neq 0$ implies that $\ln W(z)$ is regular at the origin and

$$(\ln W)'(0) = A + \sum_{\nu>0} \frac{\kappa_\nu}{a_\nu} + \sum_{\mu>0} \frac{\lambda_\mu}{b_\mu} > 0,$$

so the α-point $z = 0$ can only be simple (since $\mathfrak{B} = \lim_{z\to 0} z(\ln W(z))' = 0$, applying Lemma 6 (c) also yields the simplicity of $z = 0$).

Proof of Theorem 11 Let $\psi(z)$ be a branch of $\ln W(z)$ continuous in $\mathbb{C}_+$; then $z\psi'(z) = \phi_1(z) - \phi_2(1/z)$ with $\phi_1(z)$ and $\phi_2(z)$ of the form (7) (cf. Remark 10). The function $z\psi'(z)$ is a non-constant $\mathcal{R}$-function, so $\operatorname{Im} z\,\psi'(z) > 0$ for every $z \in \mathbb{C}_+$. In particular, $\frac{W'(z)}{W(z)} = \psi'(z) \neq 0$, that is $W'(z) \neq 0$ and thus all non-real α-points of $W(z)$ are simple.

The inequality (6) for $r > 0$ implies $\operatorname{Re}\psi\left(re^{i\theta_1}\right) > \operatorname{Re}\psi\left(re^{i\theta_2}\right)$, that is

$$\left|W(re^{i\theta_1})\right| > \left|W(re^{i\theta_2})\right| \quad \text{on condition that} \quad 0 \leqslant \theta_1 < \theta_2 \leqslant \pi. \tag{13}$$

Consequently, if $W(z) = \alpha$, then $\left|W(|z|e^{i\theta})\right| \neq \left|W(z)\right| = |\alpha|$ for all $\theta \in [0, \pi]\backslash\{\operatorname{Arg} z\}$.

Each α-point of W is an $(\operatorname{Ln}\alpha + 2i\pi n)$-point of ψ, where $\operatorname{Ln}\alpha$ denotes the principal value of $\ln\alpha$ and n is some integer dependent on the α-point. Moreover, each α-point of W has the same multiplicity as the coinciding $(\operatorname{Ln}\alpha + 2i\pi n)$-point of ψ by Lemma 9. The multiplicities of real $(\operatorname{Ln}\alpha + 2i\pi n)$-points of ψ are at most 2 by Lemma 2. So all α-points of W on the real line are at most double. □

Theorem 13 *Under the assumptions of Theorem 11, if $|z_1| < |z_2|$, $W(z_1) = \alpha$ and $W(z_2) = \alpha e^{i\theta}$ with a real $\theta > 0$, then for every $\varrho \in (0, \theta)$ there exists $z_* \in C_{12} := \{z \in \overline{\mathbb{C}_+} : |z_1| < |z| < |z_2|\}$ such that $W(z_*) = \alpha e^{i\varrho}$, unless simultaneously $\theta = 0 (\operatorname{mod} 2\pi k)$, both z_1 and z_2 are real of the same sign, $W(z)$ is regular in $(\min\{z_1, z_2\}, \max\{z_1, z_2\})$ and $|W(z)| \neq |\alpha|$ in the semi-annulus C_{12}.*

Proof This is a straightforward corollary of Lemma 8 for $\psi(z)$ being a branch of $\ln W(z)$. Just as in the proof of Theorem 11, we only need to observe that the exponential function maps $\alpha + 2i\pi n$ for all integers n to the same point e^α. □

If the α-set of W is not empty, then α-points of W are assumed to be enumerated according to the growth of their absolute values, i.e. $\cdots \leqslant |z_0| \leqslant |z_1| \leqslant |z_2| \leqslant \cdots$ and $W(z) = \alpha \iff z \in \bigcup_k z_k$. Here, we count only once each multiple α-point.

Theorem 14 *For an α-point $z_i \in \mathbb{R}$ of the function W, only the following possibilities exist:*

(a) *The point z_i belongs to an interval between two consecutive positive poles or negative zeros of W. If z_i is double, then the interval contains no other α-points of W. If z_i is simple, then the interval contains exactly one another α-point: either z_{i-1} or z_{i+1}.*
(b) *The point z_i belongs to an interval between the origin and the maximal negative zero, or between the origin and the minimal positive pole. Then exactly one another α-point (if z_i is simple) or no other α-points (if z_i is double) lie on the same interval provided that $A_0 > 0$ or $Bz_i < 0$ in (12). If $A_0 = 0$ and $Bz_i \geqslant 0$, then z_i is the α-point minimal in absolute value (moreover, it is the minimal solution to $\big|W(z)\big| = |\alpha|$) and the same interval contains no other α-points.*
(c) *The point z_i lies on a ray of the real line, in which W has no poles or zeros. Then this ray contains at most one another α-point of W. If $A_0 = 0$, $Bz_i \geqslant 0$ and one end of this ray is the origin, then z_i is the only α-point on the ray and its absolute value is minimal among all solutions to $\big|W(z)\big| = |\alpha|$.*

In the cases (a)–(c), the number and multiplicities of α-points of W in the corresponding interval are equal to the number and multiplicities of $|\alpha|$-points of $|W|$.

Proof Let us denote by $\psi(z)$ some branch of the function $\ln W(z)$ which is continuous in $\mathbb{C}_+$; then $z\psi'(z) = \phi(z) := \phi_1(z) - \phi_2(1/z)$ with $\phi_1(z)$ and $\phi_2(z)$ of the form (7):

$$\phi(z) = B + Az - \sum_{\nu>0} \frac{-z\kappa_\nu}{z + a_\nu} - \sum_{\mu>0} \frac{z\lambda_\mu}{z - b_\mu} - \frac{A_0}{z} - \sum_{\nu>0} \frac{\widetilde{\kappa}_\nu}{zc_\nu + 1} - \sum_{\mu>0} \frac{\widetilde{\lambda}_\mu}{zd_\mu - 1}$$

(cf. Remark 10). Consequently, in each continuous interval $\mathfrak{I}$ of $\{z \in \mathbb{R} : z \neq 0,\ W(z) \neq 0,\ W(z) \neq \infty\}$, the function $\operatorname{Im}\psi(z)$ is constant [Lemma 5(a)]. Furthermore, $\operatorname{Re}\psi(z)$ has exactly one extremum between each pair of consecutive positive poles or negative zeros of $W(z)$ by Lemma 5(b, c), that is no, or one double, or two simple $(\ln|\alpha|)$-points. Each α-point of $W(z)$ is a $(\operatorname{Ln}\alpha + 2i\pi n)$-point of $\psi(z)$ with some integer n, and their multiplicities are the same by Lemma 9. The equality $(\operatorname{Im}\psi(z))' = 0$ for $z \in \mathfrak{I}$ then implies that all $(\operatorname{Ln}\alpha + 2i\pi n)$-points of $\psi(z)$ with the above-mentioned n and $(\ln|\alpha|)$-points of $\operatorname{Re}\psi(z)$ coincide with multiplicities in $\mathfrak{I}$. As a summary, we obtain (a). Moreover, the number of α-points of W and their multiplicities in $\mathfrak{I}$ is, therefore, equal to the number and multiplicities of $|\alpha|$-points of $|W|$.

The assertion (b) follows from Lemma 5(d) and from Lemma 6. Indeed, if x_2 denotes the maximal negative zero or the minimal positive pole, then $\operatorname{sign} z_i = \operatorname{sign} x_2$ and the limit determining the properties of z_i is

$$\mathfrak{B} = \lim_{t\to 0+} \phi(tx_2) = \begin{cases} -\lim_{t\to 0+} \frac{A_0}{tx_2} = -\infty \cdot x_2, & \text{if } A_0 > 0 \\ B, & \text{otherwise.} \end{cases}$$

Similarly, the assertion (c) is a corollary of Remark 7. □

In the following we consider only the case of $C = 0$; otherwise, the equality $W(z) = \alpha$ can be replaced with $W(z)e^{-C} = \alpha e^{-C}$.

Theorem 15 *Let $W(z)$ be a function of the form (12) distinct from z^B, such that κ_ν, $\widetilde{\kappa}_\nu$, λ_μ, $\widetilde{\lambda}_\mu$ are positive integers and $C = 0$. Choose the branch of z^B which is holomorphic in $\mathbb{C}\backslash(-\infty, 0]$ and positive for $z > 0$. Given a complex number $\alpha \notin \mathbb{R}$ such that $\alpha e^{\pm iB\pi} \notin \mathbb{R}$, each α-point of $W(z)$ in $\mathbb{C}\backslash\mathbb{R}$ is simple and distinct in absolute value from other α-points. If z_i, z_{i+1} are two consecutive points of the α-set, then $\operatorname{Im} z_i \cdot \operatorname{Im} z_{i+1} < 0$.*

Moreover, the equations $W(x) = \alpha$ and $W_\pm(-x) := \lim_{y\to\pm 0} W(-x + iy) = \alpha$ have no solution for $x > 0$.

Note that in the case of integer B, the conditions $\alpha e^{\pm iB\pi} \notin \mathbb{R}$ and $\alpha \notin \mathbb{R}$ of this theorem are equivalent; furthermore, the function $W(x)$ is defined for $x < 0$ and equal to $W_-(x) = W_+(x)$.

Proof On the one hand, for $x > 0$ the functions $W(x)$, $e^{-iB\pi} W_+(-x)$ and $e^{iB\pi} W_-(-x)$ are real. On the other hand, both α and $\alpha e^{\pm iB\pi}$ are non-real. Therefore, there is no solution to $W(x) = \alpha$ and to $W_\pm(-x) = \alpha$ when $x > 0$. Since $W(\overline{z}) = \overline{W(z)}$, we can find the solutions to $W(z) = \alpha$ in the rest of the complex plane $\mathbb{C}\backslash\mathbb{R}$ from the equations $W(z) = \alpha$ and $W(z) = \overline{\alpha}$ in the upper half-plane.

Now assume that z varies in $\mathbb{C}_+$. Theorem 11 implies that all α-points (as well as all $\overline{\alpha}$-points) of the function $W(z)$ are simple and distinct in absolute value. Furthermore, according to the remark following (13) absolute values of α-points and of $\overline{\alpha}$-points cannot coincide (due to $\alpha \neq \overline{\alpha}$). On account of $\overline{\alpha} = \alpha e^{-2i \arg \alpha}$, if we have two solutions z_i, z_{i+k} to $W(z) = \alpha$ where k is some positive integer, then there exists a solution z_* to $W(z) = \overline{\alpha} = \alpha e^{-2i \arg \alpha}$ such that $|z_i| < |z_*| < |z_{i+k}|$ by Theorem 13 with the setting $\theta = 2\pi$. Conversely, between each pair of $\overline{\alpha}$-points there is an α-point by the same theorem. That is, the absolute values of α- and $\overline{\alpha}$-points in $\mathbb{C}_+$ interlace with each other. This fact provides the theorem, because $W(z) = \overline{\alpha}$ is equivalent to $W(\overline{z}) = \alpha$. □

Remark 16 If in Theorem 15 we take the number $\alpha \neq 0$ real, then the equations $W(z) = \alpha$ and $W(\overline{z}) = \alpha$ are satisfied simultaneously. As a result, each α-point of $W(z)$ in $\mathbb{C}\backslash\mathbb{R}$ is simple and there is a unique α-point with the matching absolute value (which is the complex conjugate). For an α-point z_i on the real line (such points are positive unless z^B is real for $z < 0$, i.e. unless B is integer) there are only the possibilities (a)–(c) of Theorem 14. The α-set of W for $\alpha e^{\pm iB\pi} \in \mathbb{R}$ and $B \notin \mathbb{Z}$ can be studied similarly; the main distinction is that W is not continuous on the negative semi-axis, so the corresponding result will be concerned with the limiting values W_+ or W_-.

Remark 17 Functions of the form (11) generate totally positive sequences exactly when the exponents κ_ν, λ_μ are positive integers, $B \in \mathbb{Z}_{\geqslant 0}$, $C \in \mathbb{R}$ and $A_0 = 0$. The expression (12) determines a generating function of a doubly infinite totally positive sequence whenever $\kappa_\nu, \lambda_\mu, \widetilde{\kappa}_\nu, \widetilde{\lambda}_\mu \in \mathbb{Z}_{>0}$ and $B \in \mathbb{Z}$. See the subsection "Definitions" of Sect. 1 for further details.

Hereinafter we concentrate on the case $B = \frac{p}{k}$ of (12) with positive integers κ_ν, $\widetilde{\kappa}_\nu$, λ_μ, $\widetilde{\lambda}_\mu$, integer $k \geqslant 2$ and $p \neq 0$. We assume that $\gcd(|p|, k) = 1$, i.e.

the fraction $\frac{p}{k}$ is irreducible. The kth root is a k-valued holomorphic function in the punctured plane $\mathbb{C}\setminus\{0\}$. So, let $\sqrt[k]{\cdot}$ denote its branch that is holomorphic in $\overline{\mathbb{C}_+}\setminus\{0\}$ and maps the positive semi-axis into itself. Then

$$R(w) = \left(\sqrt[k]{w}\right)^p e^{Aw+A_0w^{-1}} \frac{\prod_{\nu>0}\left(1+\frac{w}{a_\nu}\right)}{\prod_{\mu>0}\left(1-\frac{w}{b_\mu}\right)} \frac{\prod_{\nu>0}\left(1+\frac{1}{wc_\nu}\right)}{\prod_{\mu>0}\left(1-\frac{1}{wd_\mu}\right)}, \tag{14}$$

where the coefficients satisfy $A, A_0 \geqslant 0$ and $a_\nu, b_\mu, c_\nu, d_\mu > 0$ for all ν, μ is a single-valued meromorphic function in $\overline{\mathbb{C}_+}\setminus\{0\}$ regular for $\operatorname{Im} w \neq 0$.

5 Composition with kth power function

In the current section we assume that a function $G \not\equiv z^p$ has the representation

$$G(z) := e^{Az^k+A_0z^{-k}} z^p \frac{\prod_{\nu>0}\left(1+\frac{z^k}{a_\nu}\right)}{\prod_{\mu>0}\left(1-\frac{z^k}{b_\mu}\right)} \frac{\prod_{\nu>0}\left(1+\frac{z^{-k}}{c_\nu}\right)}{\prod_{\mu>0}\left(1-\frac{z^{-k}}{d_\mu}\right)} \tag{15}$$

for some integers $k \geqslant 2$ and p, $\gcd(|p|, k) = 1$, in which the coefficients satisfy $A, A_0 \geqslant 0$ and $a_\nu, b_\mu, c_\nu, d_\mu > 0$ for all ν, μ. As we noted above, the case when $|p|$ and k are not coprime does not need any additional study: it can be treated by introducing the new variable $\eta := z^{1/\gcd(|p|,k)}$. Furthermore, the location of zeros and poles of $G(z)$ is clear from the expression (15), so we concentrate on the equation $G(z) = \alpha$ where $\alpha \in \mathbb{C}\setminus\{0\}$.

For the sake of brevity denote $e_m := \exp\left(i\frac{m}{k}\pi\right)$. The condition $\gcd(|p|, k) = 1$ implies that

- $(e_{mp})_{m=-k}^{k-1}$ is a cyclic group of order $2k$ generated by e_p when p is odd (thus $e_{mp} = e_n$ for $n \in \mathbb{Z}$ if and only if $mp \equiv n \pmod{2k}$);
- $(e_{mp})_{m=0}^{k-1}$ and $(e_{mp+1})_{m=0}^{k-1}$ are two disjoint cyclic groups of order k generated by e_p when p is even (the former group contains $e_0 = 1$ and the latter one contains $e_k = -1$).

Denote the sectors of the complex plane with the central angle $\frac{\pi}{k}$ by

$$Q_s := \left\{z \in \mathbb{C}\setminus\{0\} : 0 < \operatorname{Arg} ze_{-s} < \tfrac{\pi}{k}\right\} \quad \text{and}$$
$$\widetilde{Q}_s := \left\{z \in \mathbb{C}\setminus\{0\} : 0 \leqslant \operatorname{Arg} ze_{-s} < \tfrac{\pi}{k}\right\},$$

where $s \in \mathbb{Z}$, so that they are numbered in an anticlockwise direction and $Q_s = Q_{2k+s}$, $\widetilde{Q}_s = \widetilde{Q}_{2k+s}$.

The substitution $z \mapsto \widetilde{z}e_{2m}$ turns $G(z) = \alpha$ with a fixed α into the equivalent equation $G(\widetilde{z}e_{2m}) = G(\widetilde{z})e_{2pm} = \alpha$ where $m \in \mathbb{Z}$, which gives us the following remark (note that we suppress the trivial case $G(z)$ identically equal to z^p with no special attention):

Remark 18 Let $G(z)$ and $R(w)$ be as in (15) and (14), respectively, $\alpha \neq 0$ and $w \in \mathbb{C}_+ \cup (0, +\infty)$. Substituting $z = \sqrt[k]{w} e_{2m}$ into (15) shows that if

$$R(w) = \alpha e_{-2pm}, \quad \text{where} \quad m = 0, \ldots, k-1, \tag{16}$$

then $z = \sqrt[k]{w} e_{2m} \in \widetilde{Q}_{2m}$ solves the equation $G(z) = \alpha$. Analogously, if the equality

$$R(w) = \overline{\alpha} e_{2pm}, \quad \text{where} \quad m = 0, \ldots, k-1, \tag{17}$$

holds for some m, then $z = \overline{\sqrt[k]{w}} e_{2m} \in \widetilde{Q}_{2m-1}$ solves $G(z) = \alpha$. Conversely, for each solution of $G(z) = \alpha$ there exists an integer m (unique under the condition $0 \leqslant m < k$) such that $R(z^k) = \alpha e_{-2pm}$ provided that $z^k \in \mathbb{C}_+ \cup (0, +\infty)$, or such that $R(\overline{z}^k) = \overline{\alpha} e_{2pm}$ provided that $z^k \notin \mathbb{C}_+ \cup [0, +\infty)$. In this sense, the equation $G(z) = \alpha$ can be replaced with the relation

$$R(w) \in \Omega, \quad \text{where} \quad \Omega := \left\{\alpha e_{-2pm}\right\}_{m=0}^{k-1} \cup \left\{\overline{\alpha} e_{2pm}\right\}_{m=0}^{k-1} \tag{18}$$

for $w \in \overline{\mathbb{C}_+}$, and then all α-points of $G(z)$ can be determined from the solutions to (18).

Remark 19 The relation (18) shows that the equation $G(z) = \alpha$ has different properties depending whether $\operatorname{Im} \alpha^k$ is zero or not. The case of $\overline{\alpha} \in \{\alpha e_{-2pm}\}_{m=0}^{k-1}$ coincides[11] with $\operatorname{Im} \alpha e_{ps} = 0$ for some $s = 0, \ldots, k-1$, and thus to $\operatorname{Im} \alpha^k = 0$. If it occurs, then the equivalent equation $G(\zeta e_{-s}) e_{ps} = \alpha e_{ps}$ in $\zeta \in \mathbb{C}$ has real coefficients and, hence, solutions symmetric with respect to the real line. Consequently, each solution to $G(z) = \alpha$ has the reflected point $\overline{z} e_{-2s}$ with the same absolute value as a pair such that $G(\overline{z} e_{-2s}) = G(z) = \alpha$ (unless $\overline{z} e_{-2s} = z$). In the case of $\overline{\alpha} \notin \{\alpha e_{-2pm}\}_{m=0}^{k-1}$, which is equivalent to $\operatorname{Im} \alpha^k \neq 0$, the relation (18) has no real solutions, and solutions to (16) and (17) have distinct absolute values, as is shown in Theorem 11. Accordingly, then all solutions of $G(z) = \alpha$ are distinct in absolute value.

We examine these cases in detail in Theorem 22 and Theorem 20, respectively.

Definition Denote by Ξ the set of absolute values of all solutions to $G(z) = \alpha$ with G of the form (15), that is

$$\Xi := \left\{\xi > 0 : \exists \theta \in (-\pi, \pi] \text{ such that } G(\xi e^{i\theta}) = \alpha\right\}.$$

Let $\cdots < \xi_i < \xi_{i+1} < \cdots$ be the entries of Ξ, such that $\Xi = \left\{\xi_n\right\}_{n \in I}$, and let $\ldots, z_i, z_{i+1}, \ldots$ be the corresponding α-points or, more precisely, $|z_i| = \xi_i$ and $G(z_i) = \alpha$ for all $i \in I$ (that is, z_i stands for any of the α-points which correspond to the value of ξ_i). The corresponding index set $I = \{n \in \mathbb{Z} : \omega_1 < n < \omega_2\}$ is a finite or infinite interval of integers, $-\infty \leqslant \omega_1 < \omega_2 \leqslant +\infty$.

[11] On the one hand, the condition that $\overline{\alpha} = \alpha e_{-2p\widetilde{s}}$ for some integer $\widetilde{s} = 0, \ldots, k-1$ coincides with $\overline{\alpha e_{-p\widetilde{s}}} = \alpha e_{-p\widetilde{s}}$ and therefore to $\operatorname{Im} \alpha e_{-p\widetilde{s}} = 0$. On the other hand, changing the order gives $\overline{\alpha} \in \left\{\alpha e_{-2pm}\right\}_{m=0}^{k-1} = \left\{\alpha e_{2p(k-m)}\right\}_{m=0}^{k-1} = \left\{\alpha e_{2pm}\right\}_{m=0}^{k-1}$, which is $\overline{\alpha} = \alpha e_{2ps}$ for some integer $s = 0, \ldots, k-1$; the last expression can be written as $\overline{\alpha e_{ps}} = \alpha e_{ps}$, or equivalently $\operatorname{Im} \alpha e_{ps} = 0$.

For brevity's sake, we omit the index set I and write $|z_i| \in \Xi$ to specify that the integer $i \in I$ and thus z_i is an actual α-point of G. Accordingly, $|z_i| \notin \Xi$ means that $i \notin I$, which implies that $I \subsetneq \mathbb{Z}$ and $\Xi = \{|z_n|\}_{n\in I}$ is not a doubly infinite sequence. If $\omega_1 > -\infty$ then it is convenient to put $\omega_1 = -1$, so that z_0 becomes one of the α-points of G minimal in absolute value.

Theorem 20 *If* $\operatorname{Im}\alpha^k \neq 0$ *and* $G(z)$ *has the form* (15)*, then the* α*-set of* $G(z)$ *satisfies the following two properties:*

(a) *Each* α*-point* z_i *is simple, satisfies* $\operatorname{Im} z_i^k \neq 0$ *and is distinct in absolute value from other* α*-points of* G *(i.e.* $G(z) = \alpha$ *and* $|z| = |z_i| \implies z = z_i$*).*
(b) *For each two consecutive* α*-points* z_i, z_{i+1}*, the inclusions* $\alpha \in Q_{2q-\varkappa}$ *and* $z_i \in Q_{2m-\sigma}$ *with* $q, m \in \mathbb{Z}$ *and* $\varkappa, \sigma \in \{0, 1\}$ *imply that* $z_{i+1} \in Q_{2l-1+\sigma}$*, where* l *is an integer solution of* $p(l + m) \equiv 2q + 1 - \varkappa - \sigma \pmod{k}$*.*

Proof Note that each element of Ω raised to the kth power equals α^k or $\overline{\alpha}^k$. The expression (14) yields that $\operatorname{Im} R^k(w) = 0 \neq \operatorname{Im}\alpha^k$ and, hence, $R(w) \notin \Omega$ provided that $w \in \mathbb{R}$. Consequently, all solutions to (18) lie in the open upper half-plane $\mathbb{C}_+$. That is $G(z) \neq \alpha$ for $\operatorname{Im} z^k = 0$. The function $R(w)$ satisfies the conditions of Theorem 11; thus solutions to $R(w) \in \Omega$ in $\mathbb{C}_+$ are simple and (since the equality $|R(w)| = |\alpha|$ is necessary for $R(w) \in \Omega$) distinct in absolute value. Therefore, all α-points of G are simple by Lemma 9 and distinct in absolute value: if $G(z) = \alpha$ and $|z| = |z_i|$ for some integer i, then $z = z_i$.

Now let $|z_i|, |z_{i+1}| \in \Xi$. There exist integers q, m, l and $\varkappa, \sigma, \tau \in \{0, 1\}$ such that $\alpha \in Q_{2q-\varkappa}$, $z_i \in Q_{2m-\sigma}$ and $z_{i+1} \in Q_{2l-\tau}$. Without loss of generality we assume $0 \leqslant q, m, l \leqslant k - 1$. Note that z_i corresponds to a solution w_i of (16) or (17) when $\sigma = 0$ or $\sigma = 1$, respectively. Analogously, z_{i+1} corresponds to a solution w_{i+1} of (16) or (17) depending on whether τ is zero or not. Figure 1 illustrates the correspondence between α-points of $G(z)$ and solutions of (16)–(18).

First, suppose that $\operatorname{Im}\alpha^k > 0$, i.e. $\varkappa = 0$ and $\alpha \in Q_{2q}$. Then the points $\alpha e_{-2pm} \in Q_{2q-2pm}$ of the set Ω occur exactly once in each sector Q_j with the even indices $j = 0, 2, \ldots, 2k - 2$ when m runs over the integers $0, \ldots, k - 1$. Analogously, the points $\overline{\alpha} e_{2pm} \in Q_{-2q-1+2pm}$ of the set Ω occur exactly once in each sector Q_j with the odd indices $j = 1, 3, \ldots, 2k - 1$ when $m = 0, \ldots, k - 1$. Consequently, $\sigma = 0$ induces the equation $R(w_i) = \alpha e_{-2pm} \in Q_{2q-2pm}$, while $\sigma = 1$ induces $R(w_i) = \overline{\alpha} e_{-2pm} \in Q_{-2q-1+2pm}$. Combining these equalities together gives

$$R(w_i) \in \Omega \cap Q_{(-1)^\sigma((2q+\sigma)-2pm)}. \tag{19}$$

The same reasoning for w_{i+1} provides us with the condition

$$R(w_{i+1}) \in \Omega \cap Q_{(-1)^\tau((2q+\tau)-2pl)}. \tag{20}$$

Since $R(w_{i+1}) = R(w_i)e^{i\theta}$ with an appropriate real θ, for each $\rho \in (0, \theta)$ there exists w_* satisfying $|w_i| < |w_*| < |w_{i+1}|$ and $R(w_*) = R(w_i)e^{i\varrho}$ by Theorem 13. However, z_i and z_{i+1} are consecutive α-points, so $R(w_*)$ cannot belong to Ω for any $\rho \in (0, \theta)$. At the same time, Ω has exactly one point in each sector of the complex plane, and we necessarily have $R(w_{i+1}) \in \Omega \cap Q_{(-1)^\sigma((2q+\sigma)-2pm)+1}$ from (19). Thus, $(-1)^\tau\big((2q+\tau)-2pl\big) \equiv (-1)^\sigma\big((2q+\sigma)-2pm\big)+1 \pmod{2k}$ on account of the relation (20). Checking the parity immediately gives $\tau = 1-\sigma$. As a result,

$$\sigma = 0 \implies (2q+1)-2pl \equiv -(2q-2pm+1) = 2q+1-2(1+2q-pm) \quad \text{and}$$
$$\sigma = 1 \implies 2q-2pl \equiv -(2q+1-2pm)+1 = 2q-2(2q-pm)$$

modulo $2k$. These two relations imply that $2pl \equiv 2((1-\sigma)+2q-pm) \pmod{2k}$, or equivalently $p(l+m) \equiv 2q+1-\sigma \pmod{k}$.

Now let $\operatorname{Im}\alpha^k < 0$, that is to say $\varkappa = 1$ and $\alpha \in Q_{2q-1}$, so consequently $\alpha e_{-2pm} \in Q_{2q-1-2pm}$ and $\overline{\alpha} e_{2pm} \in Q_{-2q+2pm}$. It implies that

$$R(w_i) \in \Omega \cap Q_{(-1)^\sigma(2q-(1-\sigma)-2pm)} \quad \text{and}$$
$$R(w_{i+1}) \in \Omega \cap Q_{(-1)^\tau(2q-(1-\tau)-2pl)} = \Omega \cap Q_{(-1)^\sigma(2q-(1-\sigma)-2pm)+1}$$

analogously to the case of positive $\operatorname{Im}\alpha^k$. Due to the parity, we have $\tau = 1-\sigma$; thus

$$\sigma = 0 \implies -(2q-2pl) \equiv 2q-1-2pm+1 = 2q-2pm \pmod{2k}$$
$$\text{and} \quad \sigma = 1 \implies 2q-1-2pl \equiv -(2q-2pm)+1 = -2q+2pm+1 \pmod{2k}.$$

The last two equations are equivalent to $2pl \equiv 4q-2\sigma-2pm \pmod{2k}$, which coincides with $p(l+m) \equiv 2q-\sigma \pmod{k}$. □

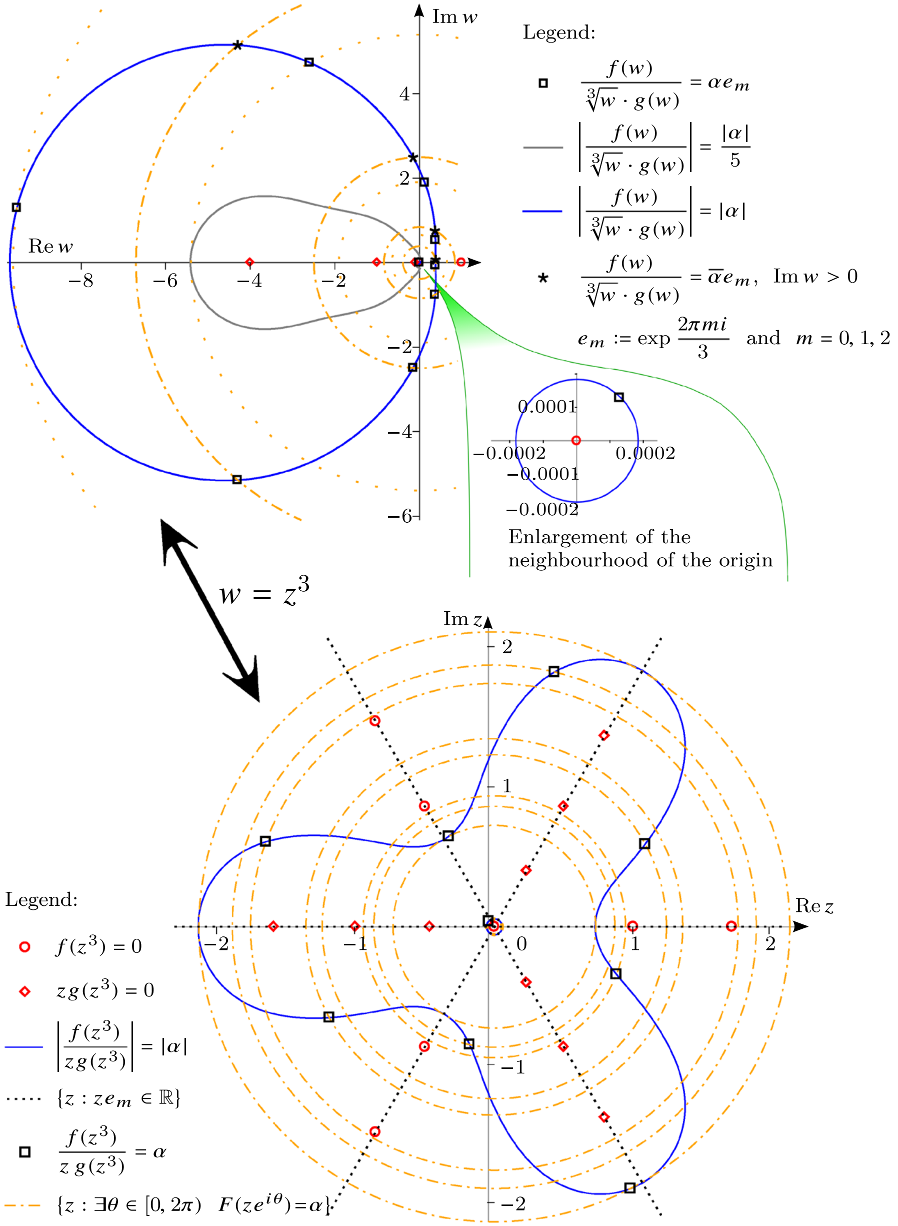
Remark 21 The rays of the line $\{z \in \mathbb{C} : \operatorname{Im} ze_s = 0\}$, which is given by $z = \overline{z}e_{-2s}$, can be expressed via the sectors Q_i of the complex plane by the following formula:

$$\overline{Q}_{2m} \cap \overline{Q}_{-2s-2m-1} \setminus \{0\} = \begin{cases} \{z \in \mathbb{C} : ze_s > 0\}, & \text{if } m \equiv -\left\lceil \frac{s}{2} \right\rceil \pmod{k}, \\ \{z \in \mathbb{C} : ze_s < 0\}, & \text{if } m \equiv -\left\lceil \frac{s-k}{2} \right\rceil \pmod{k}, \\ \varnothing & \text{otherwise}; \end{cases} \tag{21}$$

the notation $\lceil a \rceil$ stands for the minimal integer which is greater or equal to a real number a.

Theorem 22 *Let* $\operatorname{Im}\alpha^k = 0$, $\alpha \neq 0$ *and the integers* s, l, m *be such that* $\operatorname{Im}\alpha e_{ps} = 0$ *and* $p(m-l) \equiv 1 \pmod{k}$*; then*

(a) *point* z *satisfies the conditions* $G(z) = \alpha$ *and* $|z| = |z_i|$ *if and only if* $z \in \{z_i, z_i^*\}$, *where* $z_i^* := \overline{z}_i e_{-2s}$.
(b) *The inclusion* $z_i \in Q_{2m} \cup Q_{-2s-2m-1}$ *for some integer* m *implies that both* $z_i^* \neq z_i$ *are simple* α*-points and* $z_{i+1} \in \overline{Q}_{2l} \cup \overline{Q}_{-2s-2l-1}$ *(when* $|z_{i+1}| \in \Xi$*).*
(c) *The conditions* $z_i^* = z_i$ *and* $\arg z_i = \arg z_{i+1}$ *imply that both* z_i, z_{i+1} *are simple,* $\arg z_i \neq \arg z_{i-1}$ *provided that* $|z_{i-1}| \in \Xi$ *and* $\arg z_{i+1} \neq \arg z_{i+2}$ *provided that* $|z_{i+2}| \in \Xi$.
(d) *If* $z_i^* = z_i$ *and* $\arg z_i \neq \arg z_{i+1}$, *then* z_i *is simple or double (which corresponds to, respectively,* $\arg z_i = \arg z_{i-1}$ *or* $\arg z_i \neq \arg z_{i-1}$ *on condition that* $|z_{i-1}| \in \Xi$*). Furthermore,* $z_i \in \overline{Q}_{2m} \cap \overline{Q}_{-2s-2m-1}$ *with* m *given by* (21) *implies* $z_{i+1} \in \overline{Q}_{2l} \cup \overline{Q}_{-2s-2l-1}$.



The α-set of the function $F(z) = \frac{f(z^3)}{zg(z^3)} = \frac{(z^3+0.1)(z^3+1)(z^3+4)}{z(z^3-1)(z^3-5)}$ for $\alpha = -1 - i$ is presented in the bottom graph. The plot of the corresponding intermediary function $R(w)$ is in the top graph. The α-points of $F(z)$ coincide with zeros of the polynomial

$$z^9 + (1+i)z^7 + 5.1z^6 - 6(1+i)z^4 + 4.5z^3 + 5(1+i)z + 0.4.$$

Fig. 1 Illustration to Theorems 20, 22–25

(e) *If* $z_i^* = z_i$ *and* $|z_{i+1}| \notin \Xi$*, then the multiplicity of* z_i *is at most* 2.

In other words, if $\operatorname{Im} z_i e_s \neq 0$, then z_i is simple, $\operatorname{Im} z_i^k \neq 0$ and the reflected point $z_i^* = \overline{z}_i e_{-2s}$ also solves $G(z) = \alpha$; no other α-points share the same absolute value. Furthermore, $z_i \in Q_{2m}$ and $z_i^* \in Q_{-2s-2m-1}$ for some integer m (probably after exchanging $z_i \leftrightarrow z_i^*$).

If $\operatorname{Im} z_i e_s = 0$ i.e. $z_i \in \overline{Q}_{2m} \cap \overline{Q}_{-2s-2m-1}$ for some m satisfying (21), then Theorem 22 asserts that z_i is simple or double, and there are no other solutions of $G(z) = \alpha$ sharing the same absolute value. If z_i is not the first or the last α-point (with respect to the absolute value), then either z_i is double or exactly one other α-point adjacent to z_i has the same argument (in fact, it belongs to the same interval between two consecutive singularities of $\ln G$).

Proof The equality $G(z_i) = \alpha$ is equivalent to $G(\overline{z}_i e_{-2s}) = \alpha$ since

$$G(\overline{z}_i e_{-2s}) = \overline{G(z_i e_{2s})} = \overline{\alpha e_{2ps}} = \overline{\alpha e_{ps}} e_{-ps} = \alpha e_{ps} e_{-ps} = \alpha.$$

Consequently, $G(z_i) = \alpha$ if and only if $G(z_i^*) = \alpha$, where $z_i^* = \overline{z}_i e_{-2s}$. The points z_i and z_i^* coincide exactly when $z_i e_s$ is a real number (cf. Remark 19).

Choose the integer m satisfying $z_i \in \widetilde{Q}_{2m} \cup \widetilde{Q}_{-2m-2s-1}$, which implies the same inclusion for z_i^*. We constrain ourselves to the case $z_i \in \overline{Q}_{2m}$ and thus $z_i^* \in \overline{Q}_{-2m-2s-1}$: this causes no loss of generality since z_i and z_i^* are interchangeable with each other. The closed sector $\overline{Q}_{2m}$ replaces $\widetilde{Q}_{2m}$ due to the possibility $z_i = z_i^* \in \overline{Q}_{2m} \cap \overline{Q}_{-2m-2s-1}$ (*cf.* Remark 21). Note that the point $w_i := z_i^k = (z_i e_{-2m})^k \in \overline{\mathbb{C}_+}$ satisfies

$$R(w_i) = R\left((z_i e_{-2m})^k\right) = G(z_i e_{-2m}) = \alpha e_{-2pm}, \tag{22}$$

where the second equality is valid since $z_i e_{-2m} \in \overline{Q}_0$ and thus $\sqrt[k]{(z_i e_{-2m})^k} = z_i e_{-2m}$ (*cf.* Remark 18). Conversely, if $R(w_i) = \alpha e_{-2pm}$, then both $z_i = \sqrt[k]{w_i} e_{2m}$ and $z_i^* = \overline{\sqrt[k]{w_i}} e_{-2m-2s}$ are α-points of G.

The function $R(w)$ has the form (14) and hence satisfies the conditions of Theorem 11. Therefore, solutions of $R(w) \in \Omega$ in the closed upper half-plane $\overline{\mathbb{C}_+}$ are distinct in absolute value; those in $\mathbb{C}_+$ are additionally simple, and those on the real line are simple or double. In particular, if $R(w) \in \Omega$ and $|w| = |w_i|$, then $w = w_i$ which implies the assertion (a). Moreover, by Lemma 9 the multiplicities of z_i, z_i^* are equal to one in the assertion (b) and are at most two in the assertions (c)–(e). The assertion (e) is, therefore, proved because it only states that the multiplicity does not exceed two.

Now let $|z_{i+1}| \in \Xi$, which means that there is at least one α-point, z_{i+1}, with absolute value greater than $|z_i|$. Then, by analogy with z_i, the points z_{i+1} and $\overline{z}_{i+1} e_{-2s}$ are the only solutions of the equation $G(z) = \alpha$ which satisfy $|z| = |z_{i+1}|$. Furthermore, we can assume that $z_{i+1} \in \overline{Q}_{2l}$ for some integer l without loss of generality. Then $w_{i+1} := z_{i+1}^k \in \overline{\mathbb{C}_+}$ implies $z_{i+1} = \sqrt[k]{w_{i+1}} e_{2l}$ and, similarly to (22), the equality $R(w_{i+1}) = \alpha e_{-2pl}$.

Observe that the points $w_i, w_{i+1} \in \overline{\mathbb{C}_+}$ satisfy the conditions $|w_i| < |w_{i+1}|$, $R(w_i) = \alpha e_{-2pm}$ and $R(w_{i+1}) = \alpha e_{-2pl} = \alpha e_{-2pm+2\delta}$ for an appropriate integer δ.

Moreover, the quantity $\alpha e_{2pm}e^{i\varrho}$ cannot belong to Ω for all $\varrho \in \left(0, \frac{2\delta\pi}{k}\right)$; otherwise, there exists w_* satisfying $|w_i| < |w_*| < |w_{i+1}|$ and $R(w_*) \in \Omega$ by Theorem 13, which contradicts the fact that z_i and z_{i+1} are two consecutive α-points. As stated in Theorem 13, this is only possible in two cases: if $\delta = 1$ or if simultaneously: $\delta = 0$, $\operatorname{Arg} w_i = \operatorname{Arg} w_{i+1} \in \{0, \pi\}$ and $|R(w)| \neq |\alpha|$ provided that $|w_i| < |w| < |w_{i+1}|$. In the former case, we necessarily obtain the equation $-2pl \equiv -2pm + 2\delta \pmod{2k}$ with respect to the unknown l, that is $p(m - l) \equiv \delta = 1 \pmod{k}$. This proves the assertion (b) because in the corresponding case $z_i \in Q_{2m} \cup Q_{-2s-2m-1}$ we have that $\operatorname{Arg} w_i \notin \{0, \pi\}$, and the simplicity of z_i, z_i^* is shown above.

To obtain the remaining assertions (c)–(d), we assume that $z_i^* = z_i$ and thus $w_i \in \mathbb{R}$. Let $\mathfrak{I} \subset \{w \in \mathbb{R} : w \neq 0,\ R(w) \neq 0,\ R(w) \neq \infty\}$ be the maximal continuous subinterval containing w_i. Theorem 14 applied to $R(w)$ yields that

- The condition that w_i is double implies $\mathfrak{I} \not\ni w_{i-1}, w_{i+1}$;
- If w_i is simple and $\mathfrak{I} \ni w_{i+1}$, then $\mathfrak{I} \not\ni w_{i-1}$;
- If w_i is simple and $\mathfrak{I} \not\ni w_{i+1}$, then $\mathfrak{I} \ni w_{i-1}$ unless $\sqrt[k]{|w_{i-1}|} \notin \Xi$.

Let us show that $w_i \in \mathfrak{I} \not\ni w_{i+1}$ and $w_i \cdot w_{i+1} > 0$ together imply $\delta = 1$, and, therefore, $\arg z_i \neq \arg z_{i+1}$. Indeed, since $\mathfrak{I} \not\ni w_{i+1}$ the function $R(w)$ has a singularity in the interval between w_i and w_{i+1}, so Theorem 13 gives $\delta = 1$. Accordingly, $z_i \in \overline{Q}_{2m}$, $z_{i+1} \in \overline{Q}_{2l}$ with $l \not\equiv m \pmod{k}$ and hence $\arg z_i \neq \arg z_{i+1}$. In other words, we obtained that if $\arg z_i = \arg z_{i+1}$, then necessarily $w_i, w_{i+1} \in \mathfrak{I}$, and furthermore z_i and z_{i+1} are simple α-points by Lemma 9. The equality $\arg z_{i+1} = \arg z_{i+2}$ (or $\arg z_i = \arg z_{i-1}$) analogously yields that both z_{i+1}, z_{i+2} (or z_i, z_{i-1}) are simple and both w_{i+1}, w_{i+2} (or w_i, w_{i-1}, respectively) belong to the same subinterval of $\{w \in \mathbb{R} : w \neq 0,\ R(w) \neq 0,\ R(w) \neq \infty\}$. Consequently, the assertion (c) is true since at most two of the points $w_{i-1}, w_i, w_{i+1}, w_{i+2}$ can lie in $\mathfrak{I}$. Recall that if w_i is double, then $\mathfrak{I} \not\ni w_{i-1}, w_{i+1}$; this fact implies the assertion (d) using Lemma 9 with the above proof of (c). □

6 Location of the α-point that is minimal or maximal in absolute value

Let a function F have the form

$$F(z) := z^p e^{Az^k} \frac{\prod_{1 \leqslant \nu \leqslant \omega_1} \left(1 + \frac{z^k}{a_\nu}\right)}{\prod_{1 \leqslant \mu \leqslant \omega_2} \left(1 - \frac{z^k}{b_\mu}\right)}, \qquad F(z) \not\equiv z^p, \tag{23}$$

where k and p are integer such that $k \geqslant 2$ and $\gcd(|p|, k) = 1$, $0 \leqslant \omega_1, \omega_2 \leqslant +\infty$, $A \geqslant 0$ and $a_\nu, b_\mu > 0$ for all ν, μ. Such functions are the particular case of (15) and, therefore, satisfy conditions of Theorems 20 and 22. The next two theorems reveal another property of the α-set of F. Assuming that the α-set is non-empty, they determine which of the sectors contains the α-point (or α-points) of the function F that is minimal in absolute value.

Theorem 23 *Consider a complex number $\alpha \neq 0$ and a function F of the form (23) with $p > 0$. Let $q = 0, \dots, k-1$ and $\varkappa = 0, 1$ be chosen so that $\alpha \in \widetilde{Q}_{2q-\varkappa}$, and the integer m be such that $pm \equiv q \pmod{k}$.*

If $\alpha^k \not< 0$, then the α-point z_0 of $F(z)$ closest to the origin is simple and differs in argument and absolute value from the succeeding α-point (or points). Moreover, $\alpha \in Q_{2q-\varkappa}$ implies $z_0 \in Q_{2m-\varkappa}$. If $\alpha e_{-2q} > 0$, then $z_0 e_{-2m} > 0$.

If $\alpha^k < 0$, that is $\alpha e_{-2q+1} > 0$, then the two zeros of $F(z) - \alpha$ closest to the origin (counting double zeros as two) are equal in absolute value or in argument. In the latter case, both zeros belong to the ray $\{z e_{-2m+1} > 0\}$. In the former case, one of them belongs to Q_{2m-1} and another belongs to $Q_{2\widetilde{m}} = Q_{-2m-2s}$ where $\widetilde{m}$ satisfies $p\widetilde{m} \equiv q - 1 \pmod{k}$ and s is introduced in Remark 19.

Proof Let z_0 denote the solution of the equation $F(z) = \alpha$ that is minimal in absolute value. Consider the corresponding point $w_0 \in \overline{\mathbb{C}_+}$ determined by $w_0 = z_0^k$ if $\operatorname{Im} z_0^k \geqslant 0$ and by $w_0 = \overline{z}_0^k$ if $\operatorname{Im} z_0^k \leqslant 0$. Recall that (see Remark 18) the equality $F(z_0) = \alpha$ is equivalent to $R(w_0) \in \Omega$, where

$$\Omega = \left\{\alpha e_{-2pm}\right\}_{m=0}^{k-1} \cup \left\{\overline{\alpha} e_{2pm}\right\}_{m=0}^{k-1} = \left\{\alpha e_{2m}\right\}_{m=0}^{k-1} \cup \left\{\overline{\alpha} e_{2m}\right\}_{m=0}^{k-1}$$

and the function $R(w) = F\left(\sqrt[k]{w}\right)$ is defined in $\overline{\mathbb{C}_+}\backslash\{0\}$ by the equality (14), or more specifically,

$$R(w) = \left(\sqrt[k]{w}\right)^p e^{Aw} \frac{\prod_{1\leqslant\nu\leqslant\omega_1}\left(1 + \frac{w}{a_\nu}\right)}{\prod_{1\leqslant\mu\leqslant\omega_2}\left(1 - \frac{w}{b_\mu}\right)}. \tag{24}$$

Denote by w_* the point of the set $\left\{w \in \overline{\mathbb{C}_+} : |R(w)| = |\alpha|\right\}$ which is the closest to the origin. The assertions (b) and (c) of Theorem 14 imply the inequality $0 < w_* < b_1$ since $B = \frac{p}{k} > 0$ (moreover, the point w_* necessarily exists when $F(z)$ has poles). The function $R(w)$ has the form (24), that is $R(w_*) > 0$ and hence $R(w_*) = |\alpha|$. Putting $z_* := \sqrt[k]{w_*} e_{2m}$ we obtain $F(z_*) = |\alpha| e_{2pm}$. As suggested by the statement of the theorem, the integer m satisfies $pm \equiv q \pmod{k}$. Consequently, if $\alpha e_{-2q} = |\alpha| > 0$, then the point $z_0 := z_*$ satisfying the inequality $z_0 e_{-2m} > 0$ is the zero of $F(z) - \alpha$ that we are looking for; it is simple by Lemma 9 (the example is given in Fig. 3a, $\alpha = e^{i2\pi/3}$).

Recall that the α-point z_0 is minimal in absolute value; therefore $R(w) \notin \Omega$ on condition that $|w| < |w_0|$. Put

$$\theta := \begin{cases} \operatorname{Arg} R(w_0) & \text{if } \operatorname{Arg} R(w_0) \geqslant 0, \\ \operatorname{Arg} R(w_0) + 2\pi & \text{otherwise,} \end{cases}$$

so that $\operatorname{Arg} R(w_*) = \operatorname{Arg}|\alpha| = 0 \leqslant \theta < 2\pi$; then for each $\varrho \in (0, \theta)$ there exists $\widetilde{w} \in \overline{\mathbb{C}_+}$ by Theorem 13 such that $|w_*| < |\widetilde{w}| < |w_0|$ and $R(\widetilde{w}) = |\alpha| e^{i\varrho}$. Consequently, for each $\varrho \in (0, \theta)$ the condition $|\alpha| e^{i\varrho} \notin \Omega$ holds true when $\theta > 0$.

Suppose now that $\alpha \in Q_{2q}$, which is equivalent to $0 < \operatorname{Arg}(\alpha e_{-2q}) < \frac{\pi}{k}$. Since the set Ω contains no other points of Q_{2q}, the inequality in the expression $R(w_0) = |\alpha| e^{i\theta} \neq \alpha e_{-2q}$ implies $\frac{\pi}{k} < \theta < 2\pi$, leading us to the contradiction $|\alpha| e^{i\varrho} \in \Omega$ with $\varrho = \operatorname{Arg}(\alpha e_{-2q}) \in (0, \theta)$. Therefore, the equality $R(w_0) = \alpha e_{-2q}$ must be true.

In other words, we have $R(w_0) = \alpha e_{-2pm}$, and hence $z_0 = \sqrt[k]{w_0}e_{2m} \in Q_{2m}$ is the required α-point.

Analogously, suppose that $\alpha \in Q_{2q-1}$, that is $\overline{\alpha} \in Q_{-2q}$ and $0 < \operatorname{Arg}(\overline{\alpha}e_{2q}) < \frac{\pi}{k}$. Then the equality $R(w_0) = |\alpha|e^{i\theta} = \overline{\alpha}e_{2q}$ is satisfied, because the opposite condition $|\alpha|e^{i\theta} \neq \overline{\alpha}e_{2q}$ implies $\operatorname{Arg}(\overline{\alpha}e_{2q}) < \frac{\pi}{k} < \theta < 2\pi$, which is impossible by Theorem 13. Consequently, we obtain $R(w_0) = \overline{\alpha}e_{2q} = \overline{\alpha}e_{2pm}$ and, as stated in Remark 18, $z_0 = \overline{\sqrt[k]{w_0}}e_{2m} \in Q_{2m-1}$ (for the illustration see Fig. 3a with $\alpha = e^{i\pi/2}$). Combining the two last cases gives the implication $\alpha \in Q_{2q-\varkappa} \implies z_0 \in Q_{2m-\varkappa}$, while the simplicity of z_0 follows from Theorem 20.

The last case is $\alpha e_{-2q+1} > 0$, or equivalently[12] $\operatorname{Arg}(\overline{\alpha}e_{2q}) = \frac{\pi}{k} = \operatorname{Arg}(\alpha e_{2q+2ps})$. Just as in the previous case, we have $R(w_0) = \overline{\alpha}e_{2q} = \overline{\alpha}e_{2pm}$, and, therefore, the equality $z_0 = \overline{\sqrt[k]{w_0}}e_{2m} \in \widetilde{Q}_{2m-1}$ determines the α-point with the smallest absolute value. Unless $z_0e_{-2m+1} > 0$, Theorem 22 yields that there exists exactly one other α-point of F with the same absolute value, namely $z_0^* := \overline{z}_0e_{-2s} \in \widetilde{Q}_{-2m-2s}$ and that both z_0, z_0^* are simple. This situation appears in Fig. 2, $\alpha = \frac{i}{5}$, and Fig. 3a, $\alpha = e^{i\pi/3}$.

The case of $z_0e_{-2m+1} > 0$, that is $w_0 < 0$, needs a special attention. Let $-a_1$ be the maximal negative zero of $R(w)$. The interval $(-a_1, 0)$ contains one double (namely w_0) or two simple (w_0 and w_1) solutions to $R(w) \in \Omega$ as provided by (b) of Theorem 14. In the latter case, $R(w) \notin \Omega$ for all w satisfying $|w_0| < |w| < |w_1|$, which is given by Theorem 13. Lemma 9 then implies that these solutions determine the corresponding properties of the double α-point z_0 or, respectively, of the simple pair z_0, z_1 with $z_1e_{-2m+1} > 0$ (as it is shown in Fig. 2 for $\alpha = i$). When $R(w)$ has no zeros, the result is the same provided that F has at least two (or one double) α-points: see Theorem 14(c) and Remark 7. □

Theorem 24 *Let $\alpha^k < 0$ under the conditions of Theorem 23, and let the two zeros of $F(z) - \alpha$ closest to the origin (counting double zeros as two) be equal in argument. Then $p = 1$.*

Proof The case $\alpha = i$ in Fig. 2 illustrates that these conditions are consistent. In the proof we use the notation used in the proof of Theorem 23. The assertion of Theorem 24 can be stated as $z_0e_{-2m+1} > 0 \implies p = 1$ because all other situations are impossible (see the statement of Theorem 23).

Let $z_0e_{-2m+1} > 0$, which induces the inequality $w_0 < 0$. On the one hand, in this case $R(w_0) = |\alpha|e^{i\frac{\pi}{k}}$ and $R(w_*) = |\alpha|$ (see the proof of Theorem 23). Denote by $\psi(w)$ a branch of $\ln R(w)$ which is continuous in $\mathbb{C}_+$ and real at w_*; then $\operatorname{Im}\psi(w_*) = 0$ and $\operatorname{Im}\psi(w_0) = \frac{\pi}{k} + 2\pi n$ for some integer n. Item (b) of Lemma 8 yields that $n = 0$ since $R(w) \notin \Omega$ for all[13] $w_* < |w| < -w_0$. That is to say,

$$\frac{\pi}{k} = \operatorname{Im}\psi(w_0) - \operatorname{Im}\psi(w_*) = \operatorname{Im}\int_{w_*}^{w_0} \frac{R'(w)}{R(w)}\mathrm{d}w, \tag{25}$$

[12] The right-hand side follows from $\alpha e_{2q+2ps} = \alpha e_{ps} \cdot e_{2q+ps} = \overline{\overline{\alpha}e_{ps}} \cdot \overline{e_{-2q-ps}} = \overline{\overline{\alpha}e_{-2q}} = \overline{\alpha}e_{2q}$.

[13] If $R(w) \in \Omega$ for some $w \in \mathbb{C}_+$ satisfying $|w| < |w_0|$, then z_0 cannot be the α-point of $F(z)$ minimal in absolute value; see the proof of Theorem 23 for the details.

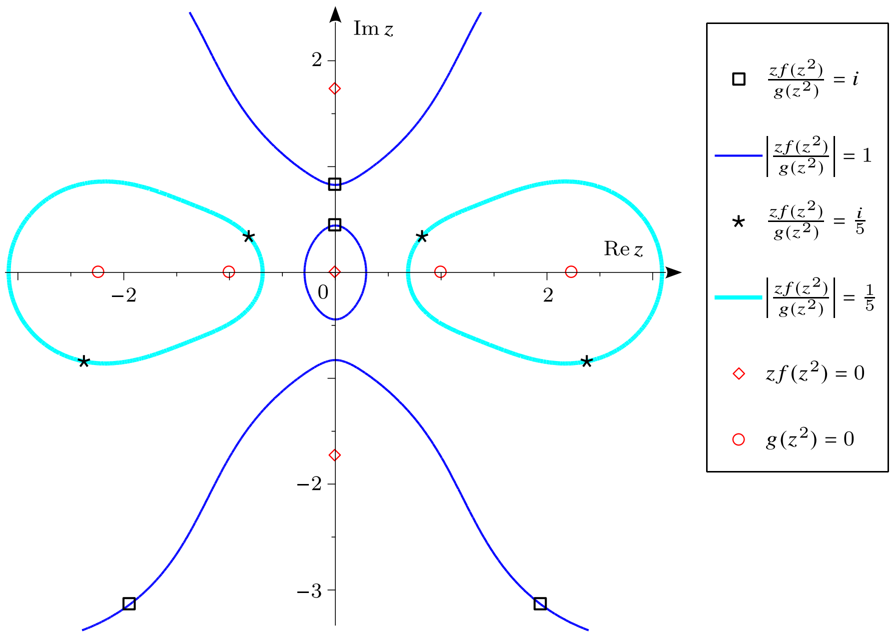


Fig. 2 The solutions to $\frac{zf(z^2)}{g(z^2)} = \alpha$, where $f(z) = z + 3$, $g(z) = (z - 1)(z - 5)$ and α is equal to i or $\frac{i}{5}$

where the integration is over any contour wholly lying in $\mathbb{C}_+$.

On the other hand, the function

$$R^k(w) = w^p e^{Akw} \frac{\prod_{\nu=1}^{\omega_1} \left(1 + \frac{w}{a_\nu}\right)^k}{\prod_{\mu=1}^{\omega_2} \left(1 - \frac{w}{b_\mu}\right)^k}$$

is meromorphic in $\mathbb{C}$. The domain $D = \left\{w \in \mathbb{C} : |w| < |w_0|,\ |R^k(w)| < |\alpha|^k\right\}$ is not empty since $p > 0$. Its boundary $\overline{D}\backslash D$ is the analytic curve $\left\{w \in \mathbb{C} : |w| \leqslant |w_0|,\ |R^k(w)| = |\alpha|^k\right\}$ because $\left|R\left(|w_0|e^{i\varrho}\right)\right| > |R(w_0)| = |\alpha|$ for any real $\varrho \in (-\pi, \pi)$ due to $\overline{R(w)} = R(\overline{w})$ and (13); this curve is closed but not necessarily connected. By definition, the closure of D cannot contain any pole of $R^k(w)$, so this function is holomorphic in $\overline{D}$. Cauchy's argument principle states that

$$\oint_{\overline{D}\backslash D} \frac{R'(w)}{R(w)} \mathrm{d}w = \frac{1}{k} \oint_{\overline{D}\backslash D} \frac{\left(R^k(w)\right)'}{R^k(w)} \mathrm{d}w = 2i\frac{Z}{k}\pi, \tag{26}$$

where Z is the number of zeros of $R^k(w)$ inside D counting multiplicities. Since $\overline{R(w)} = R(\overline{w})$, the contour $\overline{D}\backslash D$ is symmetric with respect to the real line. Consequently, the left-hand side of (26) can be modified in the following way:

$$\oint_{\overline{D}\backslash D} \frac{R'(w)}{R(w)} \mathrm{d}w = \int_\gamma \frac{R'(w)}{R(w)} \mathrm{d}w - \int_\gamma \frac{R'(\overline{w})}{R(\overline{w})} d\overline{w} = 2\int_\gamma \frac{R'(w)}{R(w)} \mathrm{d}w,$$

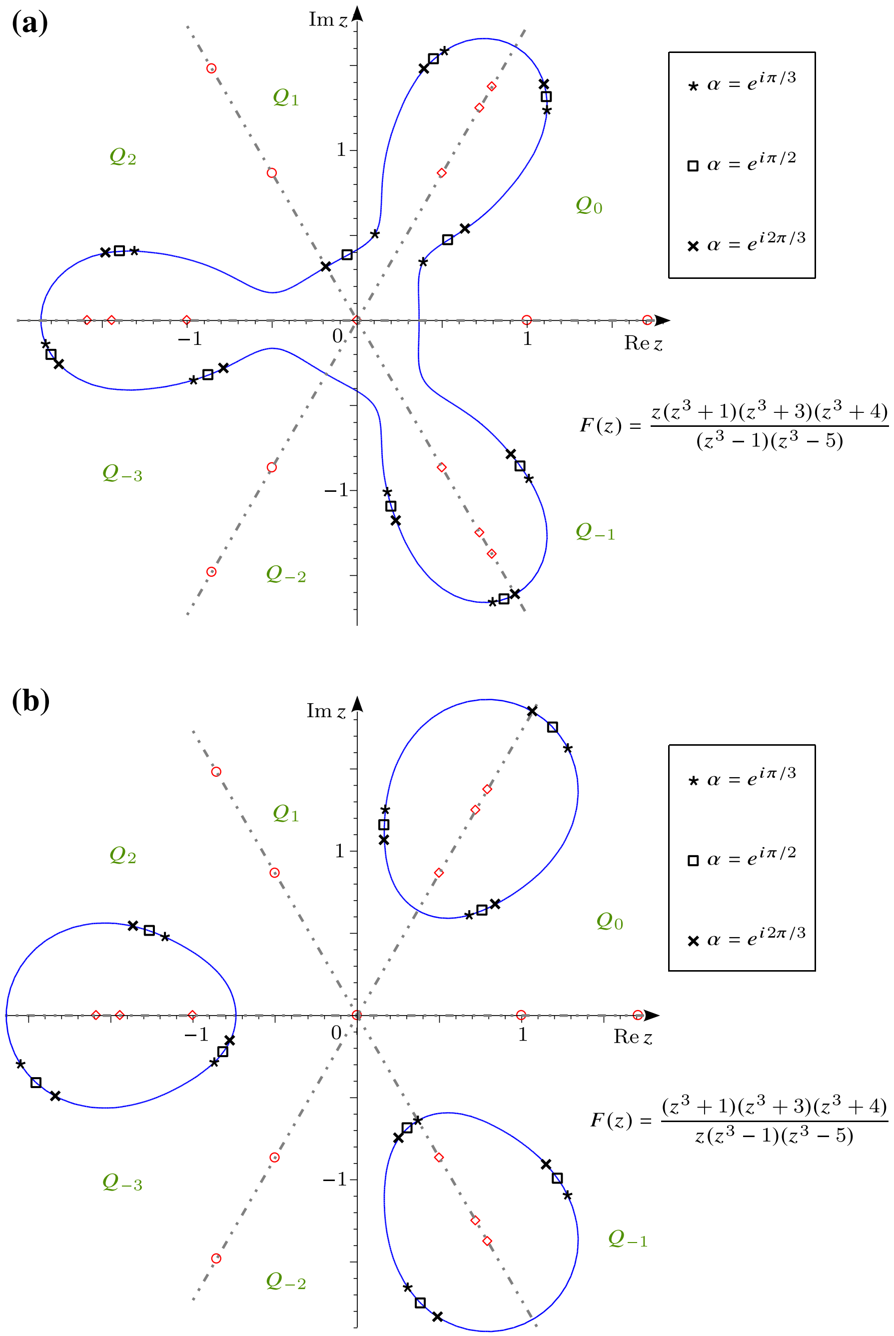


Fig. 3 The solutions to $F(z) = \alpha$ with $k = 3$, $p = \pm 1$ for different values of α (the isoline $|F(z)| = 1$ and zeros of the numerator and denominator of F have the same marks as in Fig. 2)

where the contour γ can be any contour lying wholly in $\mathbb{C}_+$, which starts at w_* and ends at w_0. On account of (26), we, therefore, have the following expression:

$$\operatorname{Im}\int_\gamma \frac{R'(w)}{R(w)}\mathrm{d}w = \frac{Z}{k}\pi$$

contradicting (25) unless $Z = 1$. However, p is the multiplicity of the zero of $R^k(w)$ at the origin, so $1 = Z \geqslant p \geqslant 1$. □

Observe that the change of variable $z \mapsto \zeta e_{-1}$ implies $z^k \mapsto -\zeta^k$. Hence, the function

$$\widetilde{F}(\zeta) := \frac{e_{-p}}{F(\zeta e_{-1})} = \zeta^{-p} e^{A\zeta^k} \frac{\prod_{\mu=1}^{\omega_2}\left(1 + \frac{\zeta^k}{b_\mu}\right)}{\prod_{\nu=1}^{\omega_1}\left(1 - \frac{\zeta^k}{a_\nu}\right)}$$

has the form (23) with a positive power of ζ as the first factor provided that $p < 0$. Moreover,

$$\begin{aligned} F(z) = \alpha \iff \widetilde{F}(\zeta) = \frac{e_{-p}}{\alpha} =: \widetilde{\alpha}, \\ \alpha \in Q_{2q-\varkappa} \iff \widetilde{\alpha} \in Q_{-2q+\varkappa-p-1} \quad \text{and} \\ \alpha e_{-2q+\varkappa} > 0 \iff \widetilde{\alpha} e_{2q-\varkappa+p} > 0. \end{aligned} \tag{27}$$

In this way the case of $p < 0$ can be reduced to the situation studied in the last two theorems. Unfortunately, the notation convenient in Theorem 23 does not suit this case well as it induces more complicated relations.

Theorem 25 *Suppose that all conditions of Theorem 23 hold except that* $p < 0$.

If $\alpha \in Q_{2q-\varkappa}$, *then the* α*-point* z_0 *of* $F(z)$ *closest to the origin is simple and differs in argument and absolute value from the succeeding* α*-point. Furthermore,* $z_0 \in Q_{2m-\sigma}$ *where* $\sigma := \varkappa$ *for even* p, $\sigma := 1 - \varkappa$ *for odd* p, *and the integer* m *satisfies*[14] $pm \equiv q - (-1)^\sigma \left\lceil \frac{p}{2} \right\rceil \pmod{k}$.

If $\alpha e_{-2q+\varkappa} > 0$, *where* p *and* $\varkappa$ *have the same parity, then the* α*-point* z_0 *of* $F(z)$ *closest to the origin is simple and differs in argument and absolute value from the succeeding* α*-point (or points). Moreover,* $z_0 e_{-2m+1} > 0$ *for* $pm \equiv q + \left\lceil \frac{p-1}{2} \right\rceil \pmod{k}$.

If $\alpha e_{-2q+\varkappa} > 0$, *where* p *and* $\varkappa$ *have distinct parity, then the two zeros of* $F(z) - \alpha$ *closest to the origin (counting double zeros as two) are equal in absolute value or in argument. In the latter case, which is only possible when* $p = -1$, *both the zeros belong to the ray* $\{z e_{-2m} > 0\}$. *In the former case, one of them belongs to* Q_{2m} *and another belongs to* $Q_{-2s-2m-1}$. *Here* m *solves* $pm \equiv q - \left\lceil \frac{p+1}{2} \right\rceil \pmod{k}$ *and* s *is as in Remark 19.*

[14] Recall that $\left\lceil \frac{p}{2} \right\rceil$ stands for the minimal integer greater than or equal to $\frac{p}{2}$. Here $\left|\left\lceil \frac{p}{2} \right\rceil\right| \leqslant \left|\frac{p}{2}\right|$ since $p < 0$.

Proof With the notation

$$\widetilde{q} := -q + \left\lceil \frac{\varkappa - p - 1}{2} \right\rceil \quad \text{and} \quad \widetilde{\varkappa} := 2\widetilde{q} + 2q - \varkappa + p + 1, \tag{28}$$

the relations (27) immediately yield

$$\alpha \in Q_{2q-\varkappa} \iff \widetilde{\alpha} \in Q_{2\widetilde{q}-\widetilde{\varkappa}} \xrightarrow{\text{Theorem 23}} \zeta_0 \in Q_{2\widetilde{m}-\widetilde{\varkappa}} \iff z_0 \in Q_{2\widetilde{m}-\widetilde{\varkappa}-1},$$

where $\widetilde{m}$ satisfies $(-p) \cdot \widetilde{m} \equiv \widetilde{q} \pmod{k}$ and ζ_0 is the solution to $\widetilde{F}(\zeta) = \widetilde{\alpha}$ minimal in absolute value. That is, modulo k we have

$$p\widetilde{m} \equiv q - \left\lceil \frac{\varkappa - p - 1}{2} \right\rceil = \begin{cases} q + \frac{p}{2}, & \text{if } p \text{ is even,} \\ q + \frac{p+1}{2} - \varkappa, & \text{if } p \text{ is odd.} \end{cases} \tag{29}$$

Let m denote an integer such that $z_0 \in Q_{2m-\sigma}$ for some $\sigma \in \{0, 1\}$. Then necessarily $2m - \sigma \equiv 2\widetilde{m} - \widetilde{\varkappa} - 1 \pmod{2k}$, which is satisfied by $m = \widetilde{m} - \widetilde{\varkappa}$ and $\sigma = 1 - \widetilde{\varkappa}$. Therefore, the second of the expressions (28) yields $\widetilde{\varkappa} = 1 - \varkappa$ if p is even and $\widetilde{\varkappa} = \varkappa$ if p is odd. The relation (29) within these settings becomes

$$pm \equiv \begin{cases} q + \frac{p}{2} - p\widetilde{\varkappa}, & \text{if } p \text{ is even;} \\ q + \frac{p+1}{2} - \widetilde{\varkappa}(p+1), & \text{if } p \text{ is odd} \end{cases} = \begin{cases} q + (-1)^{\widetilde{\varkappa}} \frac{p}{2}, & \text{if } p \text{ is even;} \\ q + (-1)^{\widetilde{\varkappa}} \frac{p+1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

modulo k. Since $p < 0$, the last equality implies

$$pm \equiv q + (-1)^{\widetilde{\varkappa}} \left\lceil \frac{p}{2} \right\rceil = q - (-1)^{\sigma} \left\lceil \frac{p}{2} \right\rceil \pmod{k}.$$

However, this relation coincides with the relation for m suggested by the statement of the theorem. For the corresponding illustration, see Fig. 3b, $\alpha = e^{i\pi/2}$.

When $\widetilde{\alpha}$ satisfies $\widetilde{\alpha} e_{-2\widetilde{q}} > 0$, from the relation (27) we have $-2\widetilde{q} \equiv 2q - \varkappa + p \pmod{2k}$, which determines the pair $q, \varkappa$ satisfying the inequality $\alpha e_{-2q+\varkappa} > 0$ instead of (28). In particular, p and $\varkappa$ have the same parity. So, $z_0 e_{-2m+1} = \zeta_0 e_{-2m} > 0$ for

$$pm = -(-p) \cdot m \equiv -\widetilde{q} \equiv q + \frac{p - \varkappa}{2} = q + \left\lceil \frac{p-1}{2} \right\rceil \pmod{k}.$$

The corresponding plot can be found in Fig. 3b, $\alpha = e^{i\pi/3}$.

When $\widetilde{\alpha} e_{-2\widetilde{q}+1} > 0$, the relation $-2\widetilde{q}+1 \equiv 2q - \varkappa + p \pmod{2k}$ provides another pair $q, \varkappa$ making the inequality $\alpha e_{-2q+\varkappa} > 0$ true. This gives us that $z_0 \in Q_{2\widetilde{m}-2}$ or $z_0 e_{-2\widetilde{m}+2} > 0$ (the latter is only possible for $p = -1$ by Theorem 24) whenever

$$p\widetilde{m} \equiv -\widetilde{q} \equiv q + \frac{p - 1 - \varkappa}{2} \pmod{k}.$$

The change $m := \widetilde{m} - 1$ gives $z_0 \in Q_{2m}$ or $z_0 e_{-2m} > 0$ whenever

$$pm \equiv q - p + \frac{p - 1 - \varkappa}{2} = q - \frac{p + 1 + \varkappa}{2} = q - \left\lceil \frac{p+1}{2} \right\rceil \pmod{k}.$$

For $z_0 \in Q_{2m}$, the integer s defined as in Remark 19 provides the expression $\overline{z}_0 e_{-2s}$ for the α-point of $F(z)$ which is equidistant with z_0 from the origin. See the relevant example in Fig. 3b, $\alpha = e^{i2\pi/3}$. □

Remark 26 Suppose that a function $H(z)$ has the form (23). Then the last three theorems give a straightforward conclusion concerning the solution of the equation $H(1/z) = \alpha$ with the maximal absolute value. It is of special interest when $H(z)$ is rational: then both $H(z)$ and $H(1/z)$ can be represented as in (23).

7 Zeros of entire functions

Let the positive integers j and k be coprime and $k \geqslant 2$. Theorems 20, 22–25 admit a transition to describing the zeros of functions of the forms

$$H_1(z) := f(z^k) + z^j g(z^k) \quad \text{and} \quad H_2(z) := g(z^k) + z^j f(z^k),$$

where the functions $f(z)$ and $g(-z)$ are entire, have genus 0 and only negative zeros. At least one of the functions f and g needs to be non-constant to exclude the trivial case. Note that both functions $f(z^k)/f(0)$ and $g(z^k)/g(0)$ must be real. They have no common zeros; therefore $f(z^k) \neq 0,\ \ g(z^k) \neq 0$ and $z \neq 0$ when $H_1(z) = 0$ or $H_2(z) = 0$.

To adapt the facts stated in Sects. 5 and 6 for studying zeros of the functions H_1 and H_2, put

$$F_1(z) := z^{-j}\frac{f(z^k)/f(0)}{g(z^k)/g(0)}, \quad F_2(z) := z^{j}\frac{f(z^k)/f(0)}{g(z^k)/g(0)} \quad \text{and} \quad \alpha := -\frac{g(0)}{f(0)}. \tag{30}$$

Then the following identities hold:

$$H_1(z) = \left(1 - \frac{F_1(z)}{\alpha}\right) z^j g(z^k) \quad \text{and} \quad H_2(z) = \left(1 - \frac{F_2(z)}{\alpha}\right) g(z^k). \tag{31}$$

Recall that $z^j g(z^k)$ and $H_i(z)$ have no common zeros for $i = 1, 2$. Therefore, the equalities (31) imply that

$$F_1(z) = \alpha \iff H_1(z) = 0 \quad \text{and} \quad F_2(z) = \alpha \iff H_2(z) = 0.$$

That is, the zero set of $H_i(z)$ coincides with the α-set of $F_i(z)$ for $i = 1, 2$. Moreover, the equalities (31) give that each α-point z_* of the function $F_i(z)$ is the zero of $H_i(z)$ with the same multiplicity.

Since the functions $F_i(z)$ have the form (23), the zeros of $H_i(z)$ for $i = 1, 2$ are located as is asserted about α-points of $F_i(z)$ by Theorems 20, 22–25 with $\alpha = -\frac{g(0)}{f(0)}$ and $p = (-1)^i \cdot j$.

Remark 27 Some extensions of the fact proposed in the current section are possible. Here we give two examples. However, it is unclear whether studying such functions is well-motivated.

1. Assume that $f(z)$ and $g(z)$ are functions regular and non-zero at the origin and that $\frac{f(z)}{g(-z)}$ does not coincide with z^p up to a constant (to suppress the trivial case). From the comparison of the formulae (30) with (15) and (23) it is seen that $f(z)/f(0)$ and $g(-z)/g(0)$ can be allowed to have the form

$$e^{Az} \cdot \frac{\prod_{\nu>0}\left(1+\frac{z}{a_\nu}\right)}{\prod_{\mu>0}\left(1-\frac{z}{b_\mu}\right)}, \quad \text{where } A \geqslant 0 \text{ and } a_\nu, b_\mu > 0 \quad \text{for all } \mu, \nu. \tag{32}$$

   In other words, if $f(0), g(0) \in \mathbb{C}\setminus\{0\}$ then $f(z)/f(0)$ and $g(-z)/g(0)$ can generate any totally positive sequences (see the subsection "Definitions" of Sect. 1) which start with 1. Indeed, after multiplying H_i by the denominators of $f(z^k)$ and $g(z^k)$ we obtain the entire function $\widetilde{H}_i$ with the same zeros as H_i. Then it is enough to note that the exponential factor originating from those of $f(z^k)$ and $g(z^k)$ is allowed in the representation (23). So, the result of the current section extends to such functions without any changes.
2. Let $f(z)$ and $g(-z)$ be non-trivial functions generating doubly infinite totally positive sequences up to complex constant factors, i.e. let them be of the form (1), where the constant C is an arbitrary number in $\mathbb{C}\setminus\{0\}$. In addition let $f(z) \not\equiv \text{const} \cdot z^p g(-z)$. With the help of analogous manipulations we still can obtain a transition of Theorems 20 and 22; it is enough additionally to factor some power of z out of H_i (this cannot change the zero set excepting the origin).

Remark 28 Allowing $f(z)$ and $g(-z)$ to be arbitrary functions of the forms (32) or (1) with $C \in \mathbb{C}\setminus\{0\}$ can be useful in the following sense. Consider the power series

$$f(z) = \sum_{n=-\infty}^{\infty} f_n z^n \quad \text{and} \quad g(z) = \sum_{n=-\infty}^{\infty} g_n z^n$$

such that $f_n \neq f_0^{1-n} f_1^n$ and $g_m \neq g_0^{1-m} g_1^m$ for some $n, m \in \mathbb{Z}$. Then (see the explanation on Page 2) the series converge and the functions $f(z)/f_0$ and $g(-z)/g_0$ generate totally positive sequences (possibly doubly infinite) if and only if the Toeplitz matrices $\left(f_{n-i}/f_0\right)_{i,n=-\infty}^{\infty}$ and $\left((-1)^{n-i} g_{n-i}/g_0\right)_{i,n=-\infty}^{\infty}$ have all their minors non-negative. However, then

$$H_1(z) = \sum_{n=-\infty}^{\infty} \left(f_n + z^j g_n\right) z^{kn} \quad \text{and} \quad H_2(z) = \sum_{n=-\infty}^{\infty} \left(g_n + z^j f_n\right) z^{kn}$$

are the Laurent series. This gives us the conditions in terms of the Laurent coefficients of $H_1(z)$ and $H_2(z)$ which show that the zeros of $H_1(z)$ and $H_2(z)$ are located according to Theorems 20 and 22 (and to Theorems 23–25 when the series $f(z)$, $g(z)$ are not doubly infinite, that is the limiting functions are meromorphic).

8 Conclusions for the case $k = 2$

Note that in the particular case of $p = \pm 1$ the relations modulo k from Theorems 20, 22–25 have obvious solutions. The setting $k = 2$ (implying that p is odd) also provides us with simple (and very useful) solutions. Let us restate the facts of Sects. 5 and 6 for this particular situation.

Denote $p = 2j + 1$. The congruence modulo k (a linear Diophantine equation) from Theorem 20 becomes $l \equiv 1 + \varkappa + \sigma + m \pmod 2$. If $\alpha \in \mathbb{R}$ (or $i\alpha \in \mathbb{R}$), then the constant s in Theorems 22–25 equals 0 (or 1, respectively). The congruence from Theorem 22 turns into $l = 1 + m \pmod 2$. The equation from Theorem 23 becomes $m = q \pmod 2$, and those from Theorem 25 become

$$m \equiv \begin{cases} q + j + 1 \pmod 2, & \text{if } \alpha \in Q_{2q-\varkappa} \text{ or } \alpha e_{-2q} > 0; \\ q + j \pmod 2, & \text{if } \alpha e_{-2q+1} > 0. \end{cases}$$

Let $N = (z_n)_{n=1}^{\omega}$ be the set of all α-points of $F(z)$, where $|z_{n-1}| \leqslant |z_n|$ for all n and each α-point counted according to its multiplicity. Then we have the following two theorems as a summary:

Theorem 29 *(cf. [9]) Let a function $F(z)$ have the form (23), $p = 2j + 1$, $j < 0$ and $k = 2$; then the α-points $N = (z_n)_{n=1}^{\omega}$ of $F(z)$ for $\alpha \neq 0$ are distributed as follows:*

(1) *If $\operatorname{Im} \alpha^2 \neq 0$, then all α-points are simple and satisfying $0 < |z_1| < |z_2| < \cdots$, $\operatorname{Im} z_n^2 \neq 0$ for every integer $n > 0$, and $z_n \in Q_l$ implies that $z_{n+1} \in Q_{l+\operatorname{sign}(\operatorname{Im}\alpha^2)}$. Moreover, $(-1)^j \operatorname{Im}\alpha \operatorname{Im} z_1 > 0$ and $\operatorname{Im} \alpha^2 \operatorname{Re} z_1 \operatorname{Im} z_1 < 0$.*

(2) *If $\operatorname{Im} \alpha = 0$, then α-points lie outside the imaginary axis and satisfy $0 < |z_1| \leqslant |z_2| < |z_3| \leqslant |z_4| < |z_5| \leqslant \ldots$; they are simple except real α-points which can be simple or double. Moreover, for each positive integer n the following five conditions hold:*

$$\begin{gathered} |z_{2n-1}| = |z_{2n}| \implies z_{2n-1} = \overline{z}_{2n}, \\ |z_{2n-1}| < |z_{2n}| \implies \operatorname{Arg} z_{2n-1} = \operatorname{Arg} z_{2n} \in \{0, \pi\}, \\ |z_1| < |z_2| \implies j = -1, \\ \operatorname{Re} z_{2n} \operatorname{Re} z_{2n+1} < 0 \quad \text{and} \quad (-1)^j \alpha \operatorname{Re} z_1 < 0. \end{gathered}$$

(3) *If $\operatorname{Re} \alpha = 0$, then α-points lie outside the real axis and satisfy $0 < |z_1| < |z_2| \leqslant |z_3| < |z_4| \leqslant |z_5| < \ldots$; they are simple except purely imaginary α-points which can be simple or double. Moreover, for each positive integer n the following five conditions hold:*

$$|z_{2n}| = |z_{2n+1}| \implies z_{2n} = -\overline{z}_{2n+1},$$

$$|z_{2n}| < |z_{2n+1}| \implies \operatorname{Arg} z_{2n} = \operatorname{Arg} z_{2n+1} \in \left\{-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right\},$$
$$\operatorname{Im} z_{2n-1}\operatorname{Im} z_{2n} < 0, \quad (-1)^j \operatorname{Im}\alpha \operatorname{Im} z_1 > 0 \quad \text{and} \quad \operatorname{Re} z_1 = 0.$$

Theorem 30 *Let a function $F(z)$ have the form (23), $p = 2j+1$, $j \geqslant 0$ and $k = 2$. Then the α-points $N = (z_n)_{n=1}^{\omega}$ of $F(z)$ for $\alpha \neq 0$ are distributed as follows:*

(4) *If* $\operatorname{Im}\alpha^2 \neq 0$, *then all α-points are simple and satisfying* $0 < |z_1| < |z_2| < \dots$, $\operatorname{Im} z_n^2 \neq 0$ *for every integer $n > 0$, and $z_n \in Q_l$ implies that $z_{n+1} \in Q_{l+\operatorname{sign}(\operatorname{Im}\alpha^2)}$. Moreover,* $\operatorname{Im}\alpha\operatorname{Im} z_1 > 0$ *and* $\operatorname{Re}\alpha\operatorname{Re} z_1 > 0$.

(5) *If* $\operatorname{Im}\alpha = 0$, *then α-points lie outside the imaginary axis and satisfy* $0 < |z_1| < |z_2| \leqslant |z_3| < |z_4| \leqslant |z_5| < \dots$*; they are simple except real α-points which can be simple or double. Moreover, for each positive integer n the following five conditions hold:*

$$|z_{2n}| = |z_{2n+1}| \implies z_{2n} = \overline{z}_{2n+1},$$
$$|z_{2n}| < |z_{2n+1}| \implies \operatorname{Arg} z_{2n} = \operatorname{Arg} z_{2n+1} \in \{0, \pi\},$$
$$\operatorname{Re} z_{2n-1}\operatorname{Re} z_{2n} < 0, \quad \operatorname{Re} z_1 = 0 \quad \text{and} \quad \alpha z_1 > 0.$$

(6) *If* $\operatorname{Re}\alpha = 0$, *then α-points lie outside the real axis and satisfy* $0 < |z_1| \leqslant |z_2| < |z_3| \leqslant |z_4| < |z_5| \leqslant \dots$*; they are simple except purely imaginary α-points which can be simple or double. Moreover, for each positive integer n the following five conditions hold:*

$$|z_{2n-1}| = |z_{2n}| \implies z_{2n-1} = -\overline{z}_{2n},$$
$$|z_{2n-1}| < |z_{2n}| \implies \operatorname{Arg} z_{2n-1} = \operatorname{Arg} z_{2n} \in \left\{-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right\},$$
$$|z_1| < |z_2| \implies j = 0,$$
$$\operatorname{Im} z_{2n}\operatorname{Im} z_{2n+1} < 0 \quad \text{and} \quad \operatorname{Im}\alpha\operatorname{Im} z_1 > 0.$$

Remark 31 The two last theorems have analogous statements if, instead of $F(z)$ satisfying (23), we take a function $G(z)$ of the more general form (15). Then, generally speaking, we cannot assert where the α-point of smallest absolute value occurs (it may not even exist).

Remark 32 Note that in all cases (1)–(6) the α-point split evenly among the quadrants of complex plane. That is, if the α-set of a function satisfies (1) or (4), then for any $r > 0$ the number of α-points in the finite sector $\{z \in Q_n : |z| < r\}$ can differ from the number of α-points in $\{z \in Q_j : |z| < r\}$ at most by 1 (here $n, j = 1, \dots, 4$). The cases appearing in (2), (3), (5) and (6) are the “degenerated” ones of (1) and (4) with possible ingress of some α-points into the real or imaginary axes.

Let us turn to zeros of entire functions by applying the idea of Sect. 7. An entire function $H(z) = \sum_{n=0}^{\infty} f_n z^n$, $f_0 \neq 0$, splits up into the even and odd parts according to

$$H(z) = f(z^2) + zg(-z^2), \quad \text{where} \tag{33}$$
$$f(z) = \sum_{n=0}^{\infty} f_{2n} z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} f_{2n+1} z^n. \tag{34}$$

Since

$$H(z)=0 \iff \frac{f(z^2)/f_0}{zg(-z^2)/f_1}=-\frac{f_1}{f_0},$$

zeros of $H(z)$ are distributed as stated in Theorem 29 if both $f(z)/f_0$ and $g(z)/f_1$ have only negative zeros and the genus equal to 0 up to factors of the form e^{cz}, $c\geqslant 0$. Similarly, zeros of $H(z)$ are distributed as stated in Theorem 30 provided that both $f(z)/f_0$ and $g(z)/f_1$ have only positive zeros and the genus equal to 0 up to factors of the form e^{-cz}, where $c\geqslant 0$.

Definition (*cf.* [24, *p.* 129]) A real entire function $\widetilde{H}(z)=f(z^2)+zg(z^2)$ is strongly stable if $f(z^2)+(1+\eta)zg(z^2)$ has no zeros in the closed right half of the complex plane for all complex η which are small enough. This is the "proper" extension to entire functions of the polynomial stability, and all stable polynomials are strongly stable. The strong stability can be understood as the "infinitesimal" robust stability; the robustness means a certain reserve against some kind of perturbations. Recall that $\widetilde{H}(z)$ is strongly stable whenever $\widetilde{H}(iz)$ belongs to the class $\mathcal{HB}$, which is introduced in [7, Chapter IV, Sect. 6] and in [19, p. 307].

Remark 33 If $\widetilde{H}(z)=f(z^2)+zg(z^2)$ is a strongly stable function of genus at most 1, then zeros of the function $H(z)$, which is defined by (33), are distributed as stated in Theorem 29. Indeed, the Hermite-Biehler theorem then implies that $f(g)$ and $g(z)$ have genus 0 and their zeros are negative, simple and interlacing. In that case, the interlacing property of $f(z)$ and $g(z)$ remains redundant.

Observe that if a complex number μ satisfies $\mu^4=-1$ (i.e. μ is a primitive 8th root of unity), then we have the identity

$$\begin{aligned}\sum_{n=0}^{\infty} f_n\mu^{n(n-1)}(\mu^{-1}z)^n &= \sum_{n=0}^{\infty} f_n\mu^{n(n-2)}z^n\\ &= \sum_{n=0}^{\infty} f_{2n}\mu^{4n(n-1)}z^{2n}+\sum_{l=0}^{\infty} f_{2l+1}\mu^{4l^2-1}z^{2l+1}\\ &= \sum_{n=0}^{\infty} f_{2n}1^{\frac{n(n-1)}{2}}z^{2n}+\sum_{l=0}^{\infty} f_{2l+1}(-1)^{l^2}\mu^{-1}z^{2l+1}\\ &= \sum_{n=0}^{\infty} f_{2n}z^{2n}+\overline{\mu}\sum_{l=0}^{\infty}(-1)^l f_{2l+1}z^{2l+1}.\end{aligned}$$

Consequently, the following fact is true:

Corollary 34 *Consider the functions*

$$h(z)=\sum_{n=0}^{\infty} i^{\frac{n(n-1)}{2}} f_n z^n \quad \textit{and} \quad \overline{h}(z)=\sum_{n=0}^{\infty} i^{-\frac{n(n-1)}{2}} f_n z^n,$$

where $f(z) = \sum_{n=0}^{\infty} f_{2n} z^n$ *and* $g(z) = \sum_{n=0}^{\infty} f_{2n+1} z^n$ *are entire functions of genus* 0 *and have only negative zeros. For* $\mu = \sqrt{i}$*, zeros of the function* $h(\overline{\mu} z)$ *are distributed as claimed in Theorem 29 for* $\alpha = \overline{\mu} f_1/f_0$ *and zeros of the function* $\overline{h}(\mu z)$ *are distributed as claimed in Theorem 29 for* $\alpha = \mu f_1/f_0$.

9 Two problems by A. Sokal

"Disturbed exponential" function. Sokal [25] put forward the hypothesis that

Conjecture 1 *The entire function*

$$\mathcal{F}(z; q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} z^n}{n!}, \tag{35}$$

where q *is a complex number,* $0 < |q| \leq 1$*, can have only simple zeros.*

The function $\mathcal{F}$ is the unique solution to the Cauchy problem

$$\mathcal{F}'(z) = \mathcal{F}(qz), \quad \mathcal{F}(0) = 1,$$

which can be checked by substitution. Moreover, when $|q| = 1$ this function has the exponential type 1, for q lower in absolute value the function $\mathcal{F}$ is of zero genus. The stronger version of the conjecture claims that

Conjecture 2 *The function* $\mathcal{F}(z; q)$ *for* $q \in \mathbb{C}, \quad 0 < |q| \leq 1$*, can have only simple zeros with distinct absolute values.*

The case of positive q was studied extensively. It is known that all zeros of $\mathcal{F}$ are negative (see [17, pp. 35, 177], [23, pp. 90, 104] and [21]), simple and satisfy Conjecture 2 as well as certain further conditions [18,20]. Conjecture 2 holds true for negative q as well, see e.g. [9, pp. 11–12, 17–18]. The properties of $\mathcal{F}(z; q)$ for complex q were studied in [2,11,28]. According to [25], Conjecture 2 is true if $|q| < 1$ and the zeros of $\mathcal{F}(z; q)$ are big enough in absolute value (Eremenko) as well as for small $|q|$.

Let us prove that Conjecture 2 also holds true for purely imaginary values of the parameter. As we pointed out, for positive $q \leqslant 1$ the function $\mathcal{F}(z; q) = f(z^2) + zg(z^2)$ has only negative zeros. In particular, it is stable. The Hermite-Biehler theorem (e.g. [7,19,24]) implies that the zeros of $f(z)$ and $g(z)$ are negative and interlacing. Therefore, by Corollary 34 the zeros of $\mathcal{F}(z; \pm iq)$ with $0 < q \leqslant 1$ are simple and their absolute values are distinct. □

The family of polynomials

$$P_N(z; q) = \sum_{n=0}^{N} \binom{N}{n} z^n q^{\frac{n(n-1)}{2}}$$

is relevant to the function $\mathcal{F}(z;q)$ and approximates this function in the sense that

$$P_N\left(zN^{-1};q\right)=\sum_{n=0}^{N}\frac{q^{\frac{n(n-1)}{2}}z^n}{n!}\left(1-\frac{1}{N}\right)\cdot\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{n-1}{N}\right)\xrightarrow{N\to\infty}\mathcal{F}(z;q).$$

The polynomial version of the conjecture has the following form:

Conjecture 3 *For all $N>0$ the polynomial $P_N(z;q)$ where $|q|<1$ can have only simple roots, separated in absolute value by at least the factor $|q|$.*

The original statement (which is equivalent to the one given here) is concerned with the family of polynomials $\left\{P_N\left(zw^{N-1};w^{-2}\right)\right\}_{N\in\mathbb{Z}_{>0}}$, where $w^{-2}=q$. Observe that Conjecture 3 $\implies$ Conjecture 2 $\implies$ Conjecture 1. The approach for $\mathcal{F}(z;q)$ extends to the polynomials $P_N(z;q)$ without changes. Their zeros are negative for positive q provided that the polynomials coincide with the action of the multiplier sequence[15] $\left(q^{n(n-1)/2}\right)_{n=0}^{\infty}$ on the polynomial $(z+1)^N$. This justifies the assertion of Conjecture 3 for purely imaginary q without bounds on the ratio of subsequent (by the absolute value) roots.

Partial theta function An analogous problem by Sokal appears in [15]. The partial theta function

$$\Theta_0(z;q)=\sum_{n=0}^{\infty}q^{\frac{n(n-1)}{2}}z^n$$

has only negative zeros whenever $0<q\leqslant\widetilde{q}\approx 0.3092493386$, which is shown in [15] (see also [14]). Splitting it into the even part $f(z^2)$ and the odd part $zg(z^2)$ gives

$$\Theta_0(z;q)=f(z^2)+zg(z^2),$$
$$f(z)=\sum_{n=0}^{\infty}q^{n(2n-1)}z^n=\sum_{n=0}^{\infty}q^{n(2n-2)}(qz)^n=\sum_{n=0}^{\infty}\left(q^4\right)^{\frac{n(n-1)}{2}}(qz)^n=\Theta_0(qz;q^4)$$

and

$$g(z)=\sum_{n=0}^{\infty}q^{n(2n+1)}z^n=\sum_{n=0}^{\infty}q^{n(2n-2)}\left(q^3z\right)^n=\Theta_0\left(q^3z;q^4\right).$$

Thus, both $f(z)$ and $g(z)$ have only negative zeros whenever $0<q^4\leqslant\widetilde{q}$. Therefore, by Corollary 34 all zeros of $\Theta_0(z;iq)$ are simple and distinct in absolute value if $0<q^4\leqslant\widetilde{q}$, that is if $0<q\leqslant q_*\approx 0.7457224107$. This is a partial positive answer to the following question:

[15] The definition and properties of multiplier sequences can be found in e.g. [23], [22, Chapter II] and [19, Chapter VIII, Sect. 3]. The fact that $\left(q^{n(n-1)/2}\right)_{n=0}^{\infty}$ is a multiplier sequence (of the first kind) for $0<q\leqslant 1$ was first shown by Laguerre [17, p. 35]. The more modern proof follows from Satz 10.1 of [22, p. 42] applied to the function $\Phi(z):=\exp\left(\frac{1}{2}z(z-1)\cdot\ln q\right)$.

Problem 4 (see [15, p. 832]) Is it true that all zeros of $\Theta_0(z;q)$ remain simple within the open disk $|q| < \widetilde{q}$?

With the help of exactly the same manipulations we could deduce that, for example, the (Jacobi) theta function

$$\Theta(z; iq) = \sum_{n=-\infty}^{\infty} (iq)^{\frac{n(n-1)}{2}} z^n, \quad 0 < q < 1,$$

also has its zeros simple, distinct in absolute value and residing in the quadrants of the complex plane rotated by $\pi/4$ (according to the Remark 31). However, this is redundant (although yet instructive) because the exact information is provided by the Jacobi triple product formula (see e.g. [13, Theorem 352]) which is valid for any complex $z \neq 0$ and $|q| < 1$:

$$\sum_{n=-\infty}^{\infty} q^{\frac{n(n-1)}{2}} z^n = \prod_{j=1}^{\infty} (1 - q^j) \left(1 + zq^{j-1}\right) \left(1 + \frac{q^j}{z}\right).$$

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