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Hyperbolic Distortion, Boundary Behaviour and Finite Blaschke Products

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Abstract We provide few results on the connections between the boundary regularity of an analytic self-map of the unit disk, and the boundary limits of its hyperbolic distortion. We give a characterization of finite Blaschke products via the boundary behaviour of a weighted local hyperbolic distortion of an analytic self-map of the unit disk.

Keywords Hyperbolic distortion · Boundary regularity · Finite Blaschke product · Angular derivative

Mathematics Subject Classification 30H05 · 30F45 · 30J10 · 30D40

1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane and let $H(\mathbb{D})$ be the space of all functions analytic on $\mathbb{D}$. For a non-constant function ϕ in $H(\mathbb{D})$ that maps the unit disk into itself and for $\alpha > 0$, let

$$\tau_{\phi,\alpha}(z) = \frac{(1-|z|^2)^\alpha |\phi'(z)|}{(1-|\phi(z)|^2)^\alpha}.$$

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We call $\tau_{\phi,\alpha}(z)$ the local α*-hyperbolic distortion* of ϕ at z. The motivation comes from the classical hyperbolic case with $\alpha = 1$.

Recall that for $z \in \mathbb{D}$, $\lambda(z) = \frac{1}{1-|z|^2}$ is the density of the hyperbolic metric on $\mathbb{D}$, and that for an analytic, non-constant map $\phi : \mathbb{D} \to \mathbb{D}$, the pull-back of the hyperbolic metric is defined by $\phi(\lambda)^*(z) = \frac{|\phi'(z)|}{1-|\phi(z)|^2}$. Thus,

$$\tau_\phi(z) = \tau_{\phi,1}(z) = \frac{(1-|z|^2)|\phi'(z)|}{(1-|\phi(z)|^2)} = \frac{\phi(\lambda)^*(z)}{\lambda(z)}$$

is the usual local hyperbolic distortion of ϕ at z (see, for example, [2] for further details on these and other related basic notions, results and references).

The local distortion, same as the absolute value of the usual derivative, provides a good measurement of the local geometric behaviour of an analytic function. One can also describe the boundary behaviour of an analytic self-map of the unit disc by taking limits of the local distortion, and that is precisely the direction that we will take in what follows.

Two other important notions which we will be using below are the notions of angular limit and angular derivative.

Note first that since an analytic self-map ϕ of the unit disk is in $H^\infty(\mathbb{D})$, it has radial (and angular) limits almost everywhere on the unit circle. We will denote the radial extension function with the same symbol ϕ.

For $\zeta \in \partial\mathbb{D}$ and $\gamma > 1$, the angular (or non-tangential) region $\Gamma_\gamma(\zeta)$ in $\mathbb{D}$ is defined by

$$\Gamma_\gamma(\zeta) = \{z \in \mathbb{D} : |\zeta - z| \le \gamma(1-|z|^2)\}.$$

If $z \to \zeta$ through the angular region $\Gamma_\gamma(\zeta)$, we say that the corresponding limit is an angular limit. We denote this type of limit by $\angle \lim_{z\to\zeta}$.

When ϕ maps the unit disk into itself, ϕ has an angular derivative at $\zeta \in \partial\mathbb{D}$ if there exists $\xi \in \partial\mathbb{D}$ such that the angular limit

$$\angle \lim_{z\to\zeta} \frac{\phi(z)-\xi}{z-\zeta}$$

exists. In this case we say that the value of the limit is the angular derivative of ϕ at ζ, and we denote it by $\phi'(\zeta)$.

By the Julia–Carathéodory theorem [14, p. 57], the existence of the angular derivative at ζ is equivalent to

$$\angle \lim_{z\to\zeta} \phi(z) = \xi \quad \text{and} \quad \angle \lim_{z\to\zeta} \phi'(z) = \bar{\zeta}\xi|\phi'(\zeta)|,$$

which is further more equivalent to

$$0 < \liminf_{z\to\zeta} \frac{1-|\phi(z)|}{1-|z|} = |\phi'(\zeta)| < \infty.$$

Moreover, the limit infimum is attained in an angular approach to ζ. Hence, note that if ϕ has an angular derivative at $\zeta \in \partial\mathbb{D}$, then

$$\angle \lim_{z\to\zeta} \tau_\phi(z) = 1.$$

The angular derivative is closely related to the geometry of the map close to the boundary and is a powerful tool that is used extensively in geometric function theory and elsewhere.

Let us also mention here, before we proceed any further, that there is a deep connection between the above-mentioned notions and ideas, and some other seemingly different mathematical areas. In particular, our motivation to look at the α- hyperbolic distortion comes originally from problems related to composition operators on the Bloch-type spaces. For more information on these topics and their relations, see for example [3,10,11,15].

2 Classical Hyperbolic Distortion

By the classical Schwarz–Pick lemma, $\tau_\phi(z) \leq 1$ for all $z \in \mathbb{D}$. If the equality holds for one $z \in \mathbb{D}$ then ϕ is a disk automorphism, and so the equality must hold for every $z \in \mathbb{D}$. These type of boundedness and maximality results give a nice geometric view of the hyperbolic distortion of self-maps of the unit disk: the distortion is always bounded by 1, i.e. every self-map of the unit disk is a hyperbolic contraction; and the maximal possible distortion is attained in the disk only when the map is a disk automorphism.

The following result, obtained by Heins in 1986, shows that the asymptotic maximality of the hyperbolic distortion τ_ϕ, as one approaches the boundary, also gives specific regularity restrictions on the map ϕ.

Theorem 2.1 [6] *Let ϕ be an analytic self-map of $\mathbb{D}$. Then ϕ is a finite Blaschke product if and only if* $\lim_{|z|\to 1} \tau_\phi(z) = 1$.

A more recent result of Kraus, Roth and Rucheweyh generalizes Heins's result to a characterization of local boundary behaviour of self-maps of $\mathbb{D}$ on subarcs of the unit circle. Thus, one can consider Heins's result as a special case of the following more general theorem.

Theorem 2.2 [8] *Let ϕ be an analytic self-map of $\mathbb{D}$ and let Γ be an open subarc of $\partial\mathbb{D}$. Then the following are equivalent:*

(a) *For every* $\zeta \in \Gamma$, $\liminf_{z\to\zeta} \tau_\phi(z) > 0$.
(b) *For every* $\zeta \in \Gamma$, $\lim_{z\to\zeta} \tau_\phi(z) = 1$.
(c) ϕ *has an analytic extension across* Γ *and* $\phi(\Gamma) \subset \partial\mathbb{D}$.

Using the previous subarc condition equivalences, one can push the argument further to the case when the arc Γ is reduced to a point. Thus, one gets the following connection between the boundary pointwise regularity of the map and the asymptotic boundary maximality of its hyperbolic distortion.

Proposition 2.1 *Let ϕ be an analytic self-map of $\mathbb{D}$, and let $\zeta \in \partial\mathbb{D}$. Then, the following are equivalent:*

(a) $\liminf_{z\to\zeta} \tau_\phi(z) > 0$.
(b) $\lim_{z\to\zeta} \tau_\phi(z) = 1$.
(c) *ϕ has an analytic extension at ζ and there exists an open arc Γ of $\partial\mathbb{D}$ containing ζ such that $\phi(\Gamma) \subset \partial\mathbb{D}$.*

Proof The inclusions (c)⇒(b)⇒(a) are trivial, and so we will be done with the proof if we show that (a) ⇒ (c).

Let $\liminf_{z\to\zeta} \tau_\phi(z) = c > 0$, and let $0 < \epsilon < c$. Then $\exists\delta > 0$ such that $\tau_\phi(z) > c - \epsilon$ whenever $z \in B(\zeta, \delta) \cap \mathbb{D}$. Let $\Gamma_\delta = B(\zeta, \delta) \cap \partial\mathbb{D}$. Then $\forall\xi \in \Gamma_\delta$ we have that

$$\liminf_{z\to\xi} \tau_\phi(z) \geq c - \epsilon > 0.$$

Using the equivalence of parts (a) and (c) from 2.2, we conclude that ϕ has an analytic extension across Γ_δ with $\phi(\Gamma_\delta) \subset \partial\mathbb{D}$. Hence, the claim (c) follows. □

Observe that if the value $\liminf_{z\to\zeta} \tau_\phi(z)$ is to be considered as the “boundary hyperbolic distortion” of ϕ at ζ when $\phi(\zeta) \in \partial\mathbb{D}$, the above result shows that its only possible values are: either 1, whenever ϕ has an analytic extension at ζ and maps an arc in $\partial\mathbb{D}$ containing ζ into $\partial\mathbb{D}$, or the “value of the boundary distortion” is 0 for all other cases. Thus, measuring the boundary distortion by $\liminf_{z\to\zeta} \tau_\phi(z)$ gives results which are very rigid.

A similar argument as in the proof of the previous proposition leads to yet another interesting observation, and gives further confirmation of the rigidity of the boundary limits of the hyperbolic distortion.

Proposition 2.2 *Let ϕ be an analytic self-map of $\mathbb{D}$, and let $\zeta \in \partial\mathbb{D}$ be such that $\lim_{z\to\zeta} \tau_\phi(z) = 0$. Then there exists Γ, an open subarc of $\partial\mathbb{D}$ containing the point ζ, such that the only possible subsets of Γ mapped by ϕ into $\partial\mathbb{D}$ are sets of measure zero.*

Proof Since $\lim_{z\to\zeta} \tau_\phi(z) = 0$, for any $0 < \epsilon < 1$, $\exists\delta > 0$ such that $\tau_\phi(z) < \epsilon$ whenever $z \in B(\zeta, \delta) \cap \mathbb{D}$. Let $\Gamma_\delta = B(\zeta, \delta) \cap \partial\mathbb{D}$. Then $\forall\xi \in \Gamma_\delta$ we have that

$$\angle \lim_{z\to\xi} \tau_\phi(z) \leq \epsilon < 1,$$

and so ϕ cannot have an angular derivative at any $\xi \in \Gamma_\delta$.

Let $E \subset \Gamma_\delta$ be such that $\phi(E) \subset \partial\mathbb{D}$.

If E is of positive measure, then the angular limit of ϕ' is infinite on a set of positive measure. But that is not possible, by the Privalov’s uniqueness theorem [12, p. 126] and so it must be that the measure of E is zero. □

The following simple example provides some motivation for a possible further direction of exploration.

Example 2.1 The map

$$\phi(z) = \frac{1+z}{2}$$

is an entire map which maps $\mathbb{D}$ onto the disk $|z - \frac{1}{2}| = \frac{1}{2}$. It follows from Proposition 2.1 that

$$\liminf_{z \to 1} \tau_\phi(z) = 0.$$

Note that here, it is also easy to calculate this directly by looking at the sequence $z_n = r_n e^{i\theta_n}$, with $r_n = 1 - \frac{1}{n}$ and $1 - r_n \cos\theta_n = \frac{1}{\sqrt{n}}$.

It is also easy to see that for $0 < r < 1$,

$$\tau_\phi(r) = \frac{1+r}{1+\phi(r)} \to 1$$

as $r \to 1$. Of course, since ϕ is analytic at 1 and $\phi(1) = 1$, by the Julia–Carathéodory Theorem, even more is true: ϕ has an angular derivative at 1, and so it must be that $\angle \lim_{z \to 1} \tau_\phi(z) = 1$.

Thus, a natural question to ask is if we can still guarantee some boundary regularity or rigidity of an analytic self-map of $\mathbb{D}$, if in the previous results we replace the boundary limits of the local hyperbolic distortion with angular boundary limits.

A result obtained by Kraus [7, Lem. 2.9, part (1)], says that if ϕ is a self-map of $\mathbb{D}$, I is a subset of $\partial\mathbb{D}$ and $\angle \lim_{z \to \zeta} \tau_\phi(z) = 1$ for every ζ in I, then ϕ has a finite angular derivative at almost every $\zeta \in I$. A special corollary of this results is stated as Theorem 2.3 below.

On the other hand, the existence of the angular limit of τ_ϕ at a single point does not imply that ϕ must have either an analytic extension, or even an angular derivative at that point. The following example from [13] shows that it is possible to have a map ϕ, self-map of $\mathbb{D}$, with $\phi(1) = 1$, $\angle \lim_{z \to 1} \tau_\phi(z) = 1$, and such that ϕ has no angular derivative at 1. We are grateful to Oliver Roth for his permission to include the example with few of its details.

Example 2.2 [13] Let $G = \{z \in \mathbb{D} : \Re z > 0\}$ and let h be the univalent map from G onto $\mathbb{D}$, such that $h(0) = 1$ and h is conformal at 0. Note that such a map can be obtained by a suitable composition of Cayley's map, its inverse, a branch of the root function and a rotation.

Let $c > 0$ be small enough so that $g(z) = -cz \log z$ maps G into G and let

$$\phi = h \circ g \circ h^{-1}.$$

Then $\phi : \mathbb{D} \to \mathbb{D}$, and if for $z \in \mathbb{D}$ we denote $w = h^{-1}(z) = re^{i\theta}$ then, as $z \to 1$, we have that $w \to 0$, i.e. $r \to 0$. Since then also

$$\frac{1 - |\phi(z)|}{1 - |z|} \approx c \log \frac{1}{r} + c\theta \tan\theta,$$

we get that

$$\liminf_{z\to 1}\frac{1-|\phi(z)|}{1-|z|}=\infty,$$

and so ϕ has no angular derivative at 1.

On the other hand, as $z\to 1$ in an angular region, w is also in an angular region, and so $|\theta|\le\gamma<\frac{\pi}{2}$, for some $\gamma>0$. A calculation shows that then (for such θ),

$$\angle\lim_{z\to 1}\tau_\phi(z)=\lim_{r\to 0}\frac{\sqrt{\log(r+1)^2+\theta^2}}{\theta\tan\theta-\log r}=1.$$

Note also that from Proposition 2.1, it must be that $\liminf_{z\to\zeta}\tau_\phi(z)=0$.

Observe also that if we request a slightly less restrictive condition, namely that $\angle\lim_{z\to 1}\tau_\phi(z)>0$, then there is a class of quite simple counterexamples of a similar type: the maps $\phi(z)=1-(1-z)^c, 0<c<1$ are such that $\phi(1)=1$, $\angle\lim_{z\to 1}\tau_\phi(z)=c>0$, and each ϕ does not have an angular derivative at 1. These maps where also used in [10] in order to provide examples of non-compact composition operators on the Bloch space, for which the inducing map touches the unit circle at one point, and does not have an angular derivative at that point.

Note that the examples above show that we cannot replace the limit conditions in Proposition 2.1 with angular limits.

Other types of boundary limits of the hyperbolic distortion, such as limits along certain simple curves Γ contained in a non-tangential region, were explored by Martin in [9]. The main result of the paper provides a concrete calculation of the value of the boundary limit of τ_ϕ for a univalent ϕ, such that the boundary of $\phi(\mathbb{D})$ has a corner at a point on the unit circle. The limit depends on the geometry of the corner and on the angle under which the curve Γ touches the unit circle.

Let us also mention here that there is no hope of characterizing the class of finite Blaschke products using angular derivatives in either Theorems 2.1 or 2.2, since it is known that there exist infinite Blaschke products with angular limits of modulus one everywhere on $\partial\mathbb{D}$, as was already observed in [8] (see [8] also for a concrete example of one such infinite Blaschke product).

The following result from [7] specifies the class of functions characterized by the existence of (non-zero) angular boundary limits of their hyperbolic distortion a.e. on the unit circle.

Note that the condition $\angle\lim_{z\to\zeta}\tau_\phi(z)=1$ in the theorem below can be replaced with a (seemingly) more general sufficient condition $\angle\lim_{z\to\zeta}\tau_\phi(z)>0$.

Theorem 2.3 [7] *Let ϕ be an analytic self-map of $\mathbb{D}$. Then*

$$\angle\lim_{z\to\zeta}\tau_\phi(z)=1$$

for almost every $\zeta\in\partial\mathbb{D}$ if and only if ϕ is an inner function with finite angular derivatives at almost every point in $\partial\mathbb{D}$.

As we will see in the next section, similar conditions on the angular boundary limits of a weighted hyperbolic distortion provide a characterization of a special subclass of inner functions, namely the class of finite Blaschke products.

3 Weighted Hyperbolic Distortion

In this section, we will expand our scope of investigation by considering the weighted hyperbolic local distortion $\tau_{\phi,\alpha}$ of an analytic self-map ϕ of the unit disk. Recall that for $\alpha > 0$ and $z \in \mathbb{D}$,

$$\tau_{\phi,\alpha}(z) = \frac{(1-|z|^2)^\alpha |\phi'(z)|}{(1-|\phi(z)|^2)^\alpha}.$$

It follows from the Schwarz lemma that

$$\frac{1-|\phi(0)|}{1+|\phi(0)|} \leq \frac{1-|\phi(z)|^2}{1-|z|^2},$$

for all $z \in \mathbb{D}$. Thus, using also the Schwarz–Pick lemma, one can easily see that for any analytic self-map ϕ of $\mathbb{D}$ and $\alpha \geq 1$, the local α-hyperbolic distortion $\tau_{\phi,\alpha}$ is bounded on $\mathbb{D}$, since

$$\tau_{\phi,\alpha}(z) \leq \left(\frac{1+|\phi(0)|}{1-|\phi(0)|}\right)^{\alpha-1}, \forall z \in \mathbb{D}.$$

This is not anymore the case for $0 < \alpha < 1$, as the following example shows.

Example 3.1 Let ϕ be the singular inner function

$$\phi(z) = \exp\left(-\frac{1+z}{1-z}\right),$$

and let $0 < \alpha < 1$. Then it is easy to see that as $z \to 1$ along the orocycle

$$C_l = \left\{z : \frac{1-|z|^2}{|1-z|^2} = l\right\},$$

with $l > 1$, the local α-hyperbolic distortion

$$\tau_{\phi,\alpha}(z) = \frac{2le^{-l}}{(1-e^{-2l})^{\alpha}} \frac{1}{(1-|z|^2)^{1-\alpha}},$$

goes to infinity as $z \to 1$.

Next, we provide a characterization of finite Blaschke products in the spirit of the previously mentioned results, while using only angular boundary limits of a distortion. Clearly, as Theorem 2.3 shows, one cannot use the classical hyperbolic distortion τ_{ϕ} in order to achieve this. As we show below, using instead the local α-hyperbolic distortion $\tau_{\phi,\alpha}$ with $\alpha > 1$ will do.

Theorem 3.1 *Let ϕ be a non-constant self-map of $\mathbb{D}$. Then ϕ is a finite Blaschke product if and only if there exist $\alpha > 1, c > 0$ such that for almost every $\zeta \in \partial\mathbb{D}$*

$$\angle \lim_{z\to\zeta} \tau_{\phi,\alpha}(z) \geq c.$$

Proof If ϕ is a finite Blaschke product, then ϕ has an analytic extension across the unit circle. Since also $|\phi(\zeta)| = 1$ for all $\zeta \in \partial\mathbb{D}$, ϕ has an angular derivative equal to the regular derivative of ϕ at ζ. Hence, using the Julia–Caratheodory theorem characterization of the angular derivative and using that ϕ' is continuos on $\overline{\mathbb{D}}$, one gets that for any $\alpha > 1$

$$\angle \lim_{z\to\zeta} \tau_{\phi,\alpha}(z) = |\phi'(\zeta)|^{1-\alpha} \geq ||\phi'||_{\infty}^{1-\alpha}.$$

For the other direction, let $\alpha > 1, c > 0$ be such that for almost every $\zeta \in \partial\mathbb{D}$

$$\angle \lim_{z\to\zeta} \tau_{\phi,\alpha}(z) = \angle \lim_{z\to\zeta} \left(\frac{1-|z|^2}{1-|\phi(z)|^2}\right)^{\alpha-1} \tau_{\phi}(z) \geq c.$$

Since by the Schwarz–Pick Lemma $\tau_{\phi}(z) \leq 1$ for every $z \in \mathbb{D}$, and since $\alpha - 1 > 0$, we have that

$$\angle \liminf_{z\to\zeta} \frac{1-|z|^2}{1-|\phi(z)|^2} \geq c^{\frac{1}{\alpha-1}}.$$

Thus, for almost every ζ in $\partial\mathbb{D}$

$$\angle \limsup_{z\to\zeta} \frac{1-|\phi(z)|^2}{1-|z|^2} \leq \left(\frac{1}{c}\right)^{\frac{1}{\alpha-1}},$$

and so, for almost every ζ in $\partial\mathbb{D}$ it must be that

$$|\phi'(\zeta)| = \liminf_{z\to\zeta} \frac{1-|\phi(z)|}{1-|z|} \leq 2\left(\frac{1}{c}\right)^{\frac{1}{\alpha-1}} < \infty.$$

But then by the Julia–Carathéodory theorem ϕ has an angular limit of modulus 1 almost everywhere on $\partial\mathbb{D}$, and so ϕ is inner. By a result of Ahern and Clark (see [1]), for an inner function ϕ we have that $|\phi'|$ is in $L^p(\partial\mathbb{D})$, $0 < p \le \infty$ if and only if ϕ' is in $H^p(\mathbb{D})$. Furthermore, if $\phi' \in H^{\frac{1}{2}}(\mathbb{D})$, then ϕ must be a Blaschke product, and any Blaschke product with a derivative in $H^1(\mathbb{D})$ must be a finite Blaschke product. Since in our case $|\phi'|$ is almost everywhere bounded by $2\left(\frac{1}{c}\right)^{\frac{1}{\alpha-1}}$ on $\partial\mathbb{D}$, by [1] we have that ϕ' belongs to $H^\infty(\mathbb{D})$, and so ϕ must be a finite Blaschke product. □

Note that when $0 < \alpha < 1$ the distortion $\tau_{\phi,\alpha}$ behaves very differently and might not even be bounded, as shown in Example 3.1. There is a more general reason why this example works: if $0 < \alpha < 1$ and ϕ is an inner function with bounded local α-hyperbolic distortion, then ϕ must be a finite Blaschke product. This can be seen, for example, by using the fact that the small α-Bloch-type spaces are Lipschitz spaces of order $1-\alpha$ see [4, p. 74]. We provide yet another proof of this result, that is more in line with the ideas already used above.

Proposition 3.1 *Let ϕ be an inner function and let $0 < \alpha < 1$. Then $\tau_{\phi,\alpha}$ is bounded if and only if ϕ is a finite Blaschke product.*

Proof If ϕ is a finite Blaschke product, then ϕ has an analytic extension across the unit circle, and so ϕ' is bounded on $\overline{\mathbb{D}}$. Using the Schwarz Lemma, we also have that $\frac{1-|z|^2}{1-|\phi(z)|^2} \le \frac{1+|\phi(0)|}{1-|\phi(0)}$, and so $\tau_{\phi,\alpha}(z)$ is bounded on $\mathbb{D}$.

For the converse, we will show first that if ϕ is inner and $\tau_{\phi,\alpha}$ is bounded, then the angular derivative of ϕ must exist and be bounded almost everywhere on $\partial\mathbb{D}$. Thus, by the results from [1], similar as in the proof of Theorem 3.1, ϕ must be a finite Blaschke product.

Let $M > 0$ be such that $\tau_{\phi,\alpha}(z) \le M$, $\forall z \in \mathbb{D}$ and suppose that there is $\zeta \in \partial\mathbb{D}$ such that the radial limit value $\phi(\zeta)$ exists, is of modulus one, but ϕ has no angular derivative at ζ, i.e.

$$\lim_{z\to\zeta} \frac{1-|\phi(z)|^2}{1-|z|^2} = \infty.$$

But then, since

$$\tau_{\phi,\alpha}(z) = \left(\frac{1-|\phi(z)|^2}{1-|(z)|^2}\right)^{1-\alpha} \tau_\phi(z) \le M,$$

it must be that $\lim_{z\to\zeta} \tau_\phi(z) = 0$.

By Proposition 2.2, there exists an arc Γ in $\partial\mathbb{D}$, containing the point ζ and such that the only possible subsets of Γ mapped by ϕ into $\partial\mathbb{D}$ are sets of measure zero. But then $\phi(\Gamma) \subset \mathbb{D}$ a.e., which contradicts the fact that ϕ is inner. Hence, ϕ has an angular derivative at every point at which the radial limit of ϕ exists (and is of modulus one), which is almost everywhere on $\partial\mathbb{D}$. Also, since

$$\frac{1-|\phi(z)|^2}{1-|z|^2} \le \left(\frac{M}{\tau_\phi(z)}\right)^{1/1-\alpha},$$

and $\angle \lim_{z\to\zeta} \tau_\phi(z) = 1$ whenever ϕ has an angular derivative at ζ, we get that $|\phi'(\zeta)| \le M^{1/1-\alpha}$ almost everywhere on $\partial\mathbb{D}$. As mentioned before, this implies that ϕ must be a finite Blaschke product. □

4 Further Remarks

As we have mentioned in the introduction, the α-hyperbolic local distortion plays an important role in determining the properties of composition operators on the Bloch-type spaces $\mathcal{B}^\alpha$. Without going into too many details, let us mention only that the types of results that we have dealt with here are closely related to the boundedness from below, which is further equivalent to semi-Fredholmness, of the composition operators on these spaces. This is why it is in some way natural to view all of the given boundary limit restrictions of $\tau_{\phi,\alpha}$ as "boundedness from below" close to the boundary or, as far as geometric function theory goes, as reverse Schwarz–Pick type inequalities.

There is a recent paper of Dyakonov [5], with exactly this title, and with some related results. For example, there is an interesting characterization of disk automorphisms by using "boundedness from below" of, on one hand, a slightly more general function, but on the other, with the inequality satisfied everywhere on $\mathbb{D}$.

Theorem 4.1 [5] *Let ϕ be an inner function with ϕ' in the Nevanlinna class $\mathcal{N}$. Then ϕ is a disk automorphism if and only if there exists a non-decreasing function $\eta : (0,\infty) \to (0,\infty)$ such that for all $z \in \mathbb{D}$*

$$\eta\left(\frac{1-|\phi(z)|^2}{1-|z|^2}\right) \le |\phi'(z)|.$$

Note that for $\alpha > 0$, $c > 0$, the function $\eta(t) = ct^\alpha$ is non-decreasing on $(0,\infty)$, and that in this case the condition $\eta\left(\frac{1-|\phi(z)|^2}{1-|z|^2}\right) \le |\phi'(z)|$ is nothing else than $\tau_{\phi,\alpha}(z) \ge c$. Thus, as a corollary to Theorem 4.1, for ϕ inner with $\phi' \in \mathcal{N}$, ϕ is a disk automorphism if and only if there exist $\alpha > 0$, $c > 0$ such that $\tau_{\phi,\alpha}(z) \ge c$, $\forall z \in \mathbb{D}$.

Also, recall that Theorem 2.1 says that ϕ is a finite Blaschke product if and only if there exist $c > 0$, $0 \le r < 1$ such that $\tau_\phi(z) \ge c$ for all $|z| > r$, i.e.

$$c\,\frac{1-|\phi(z)|^2}{1-|z|^2} \le |\phi'(z)|$$

on the open annulus centred at 0 and with radii r and 1. Thus, one direction of Theorem 4.1 can be viewed as a special case of Theorem 2.1 with $\eta(t) = ct$ and $r = 0$, when furthermore one gets that ϕ must be a Blaschke product of degree one, even without the assumption that ϕ is inner.

The angular boundary limits characterization of finite Blaschke products from Theorem 3.1 can be viewed as a similar type of a result. In this case though, one requires that $\tau_{\phi,\alpha}$ be bounded from below not on an annulus, but on a certain cone domain.

One interesting question thus is what is the smallest set on which the boundedness from below of (a bounded) $\tau_{\phi,\alpha}$, $0 < \alpha < 1$, determines that ϕ must be a finite

Blaschke product. More concretely, here is a question motivated by some operator theoretic results from [15].

Question 1 Let $0 < \alpha < 1$ and let ϕ be an analytic self-map of $\mathbb{D}$ such that $\tau_{\phi,\alpha}$ is bounded. Is it true that if for some $c > 0$ the set $\Omega_c = \{z \in \mathbb{D} : \tau_{\phi,\alpha} \geq c\}$ is such that $\partial\mathbb{D} \subset \overline{\phi(\Omega_c)}$, then ϕ must be a finite Blaschke product?

Another slightly more vague question motivated by the Dyakonov's results is:

Question 2 What are the conditions (if any) on the function η as in Theorem 4.1, such that if there exists a specific (smallest) proper subset $\Omega(\eta)$ of $\mathbb{D}$ with $\eta\left(\frac{1-|\phi(z)|^2}{1-|z|^2}\right) \leq |\phi'(z)|$, for all $z \in \Omega(\eta)$, then ϕ must be a finite Blaschke product?

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