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Fourier Multipliers in Hardy Spaces in Tubes over Open Cones

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Abstract We obtain effective sufficient conditions for multipliers of Fourier integrals acting from $H^p(T_\Gamma)$ to $H^q(T_\Gamma)$, $0 < p \le q \le 1$. We also show that they are sharp in some cases. Special attention is paid to the means of Fourier integrals with compactly supported radial kernels. As an application, the critical index for the Bochner–Riesz means to define a bounded linear operator from H^p to H^q is found. Surprisingly, it does not depend on p.

Keywords Fourier multiplier · Hardy spaces in tubes over open cones · Fourier integral · Multiplier defined by a radial function · Bochner–Riesz means · Nikol'skij type inequality · Non-increasing rearrangement

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Multipliers of Fourier series and integrals have been investigated and widely used since 1923, when they were introduced by Fekete [4]. It is well-known that the Fourier series of a 2π-periodic function may not converge, or it may converge not to its generating function. However, it is possible to introduce some multiplicative factors λ_n into the Fourier series, i.e., to consider a modified Fourier series

$$\Lambda f \sim \sum_{n\in\mathbb{Z}} c_n \lambda_n \mathrm{e}^{2\pi inx}$$

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that has better properties. This tool has been successfully applied to problems of approximation theory, differential equations, numerical analysis, etc., in particular if Λ defines a bounded linear operator on the corresponding function space. The first effective sufficient condition for boundedness of Λ in $L^p(\mathbb{T})$, $p \in (1, \infty)$, and its applications were found by Marcinkiewicz [14]. Later, for the non-periodic case of multipliers of Fourier integrals, these conditions were obtained by Michlin [15,16] and Hörmander [9] (see also [22, Ch. IV]). The cases most often investigated are $p = 1, 2, \infty$, which is not a surprise. Employing the Riesz–Thorin Theorem, it is easy to transfer such results to the case $p \in (1, \infty)$. These results and techniques became classical and are well described, e.g., in [23].

For $p \in (0, 1)$, L^p-spaces are pre-normed, and there are no linear continuous functionals, and no Fourier series in these spaces. This is the reason for considering the $H^p(\mathbb{D})$ spaces of functions analytic in the unit disk $\mathbb{D}$ and having their boundary values in $L^p(\mathbb{T})$.

For any $f \in H^p(\mathbb{D})$, $p > 0$, one can consider the Taylor series of f that coincides with the Fourier series of the limit values of f on the unit circle when $p \geq 1$. Since the functions under consideration are holomorphic, such investigation for $H^p(\mathbb{D})$, $p \in (0, 1)$, yields many interesting results. Several efficient conditions for multipliers in H^p spaces in polydisk $\mathbb{D}^m$, and their applications to various problems of approximation theory, were obtained by Trigub in [27]. Later, the results of [27] were extended to the case of H^p spaces in the Reinhardt domains by Volchkov [30].

The same type of problems naturally appear for Fourier integrals. Now, the multiplicative factor is some Lebesgue measurable function. For a function f with Fourier transform $\widehat{f}$, we can consider the linear operator defined in the following way

$$F_\varphi[f](x) = \int\limits_{\mathbb{R}^n} \varphi(t)\, \widehat{f}(t)\, \mathrm{e}^{2\pi i(x,t)}\, \mathrm{d}t,$$

where (x, t) denotes the usual dot product of two vectors in $\mathbb{R}^n$.

Owing to the de Leeuw Theorem [12], the case of multipliers for Fourier integrals in $L^p(\mathbb{R}^n)$, $p \in [1, \infty]$, may be reduced to the case of multipliers of Fourier series in $L^p(\mathbb{T}^n)$. A detailed explanation of this fact and related results could be found in [23, Ch. VII, Sect. 3].

For $p \in (0, 1)$, the situation is quite different. We need to investigate the multipliers for Fourier integrals separately. Moreover, as in the case of series' multipliers, one needs to study the Hardy spaces H^p instead of L^p. For the univariate case, it is H^p in the upper half-plane. Several useful sufficient conditions for such multipliers are obtained by Soljanik in [21]. They are also successfully applied for obtaining several two-sided estimates of approximation of a function by some means of its Fourier integrals.

Recent work of Heo, Nazarov and Seeger [7,8] is devoted to Fourier multipliers in $L^p(\mathbb{R}^n)$, $p \geq 1$, and Lorentz spaces. The main results of their articles are efficient estimates for the norms of Fourier multipliers from L^p to L^p and to Lorentz spaces $L^{p,\nu}$. The authors deal with general radial kernels. One of the most popular applications of these results is the Bochner–Riesz means.

Our aim is to find sufficient conditions for multipliers of Fourier integrals from $H^p(T_\Gamma)$ to $H^q(T_\Gamma)$, $0 < p \le q \le 1$, where T_Γ is the tube (in $\mathbb{C}^n$) with the base of an open cone $\Gamma \subset \mathbb{R}^n$. We also apply these conditions to find the critical index for Bochner–Riesz means. Despite the fact that we consider the most general case in $\mathbb{C}^n$, the results are new even for the univariate case of Hardy spaces in the upper half-plane.

Estimates for norms of multiplicative operators in L^p with $p \ge 1$ are usually obtained using Minkovskii's integral inequality or some of its variations. Since, in our case, $p < 1$, this inequality is not available, and we need another approach. The idea is to first use a version of the Three-Lines Theorem. It allows us to change the norm to $p > 1$, and then apply Minkowskii's integral inequality. This trick is used to prove Proposition 3.2 that is the basis for our main results. Once this proposition is proven, we use the method developed by Trigub [27] for multipliers of Fourier series, to obtain our main result—Theorem 1.3. Despite its limitation to the case of compactly supported functions, it could be easily extended to a more general case, see Theorem 1.6.

To show that the sufficient conditions obtained are relatively sharp, we carefully investigate the local behavior of multipliers. Special attention is paid to compactly supported radial functions. One of the tools here is non-increasing rearrangements. In particular, we prove Lemma 2.7 that generalizes a very well- known equality $\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x = \int_0^\infty (f^*(t))^p \, \mathrm{d}t$, where f^* denotes the non-increasing rearrangement of f. Another auxiliary result—Nikol'skij's type inequality given by Proposition 2.10—is of an independent interest.

1 Definitions and Main Results

1.1 General Notation

Let B be an open set in $\mathbb{R}^n$, $n \in \mathbb{N}$. Following [23, Ch. III], the tube with base B is

$$T_B = \left\{ z \in \mathbb{C}^n, z = x + iy : x \in \mathbb{R}^n, y \in B \right\}.$$

Despite the fact that this definition is related to an open set B, we will also use the same notation for not necessarily open B when proving some technical results in the next chapter. We will also use the notation B^o for the interior of the set B.

A non-empty open set $\Gamma \subset \mathbb{R}^n$ is called an open cone in $\mathbb{R}^n$ if $0 \notin \Gamma$ and whenever $x, y \in \Gamma$ and $\alpha, \beta > 0$, the linear combination $\alpha x + \beta y \in \Gamma$. The closure of an open cone is called a closed cone.

For any open cone Γ, the set

$$\Gamma^* = \left\{ x \in \mathbb{R}^n : (x, t) \ge 0, \ \forall t \in \Gamma \right\}$$

is closed. If Γ^* has non-empty interior, then it is a closed cone, and Γ is called a regular cone. The closed cone Γ^* is called the cone dual to Γ.

A holomorphic in T_B function belongs to the Hardy space $H^p(T_B)$, $p \in (0, \infty]$, if

$$\|f\|_{H^p} := \|f\|_{H^p(T_B)} := \begin{cases} \sup_{y \in B} \left(\int_{\mathbb{R}^n} |f(x+iy)|^p \, \mathrm{d}x \right)^{1/p}, & p \in (0, \infty), \\ \sup_{z \in T_B} |f(z)|, & p = \infty, \end{cases}$$

is finite. It is clear that the latter expression defines a norm in $H^p(T_B)$ for $p \in [1, \infty]$, and a pre-norm for $p \in (0, 1)$.

We will also use the following notation $f_y(\cdot) := f(\cdot + iy)$, $y \in B$. Thus, $\|g\|_{H^p(T_B)} = \sup_{y \in B} \left\| f_y \right\|_p$, where $\|\cdot\|_p$ is a standard norm (or pre-norm) in $L^p(\mathbb{R}^n)$.

Since the general case of an arbitrary open set B is too cumbersome and heavily dependent on the geometry of B even for $H^2(T_B)$ (see, e.g., [23, Ch. III, Sect. 2]), it is reasonable to restrict the investigation to the case of an open cone Γ.

If $f \in L^1(\mathbb{R}^n)$, its Fourier transform is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(t) \, \mathrm{e}^{-2\pi i (\xi, t)} \, \mathrm{d}t, \quad \xi \in \mathbb{R}^n.$$

We will also use the following notation $\widetilde{f}(\xi) = \widehat{f}(-\xi)$.

For a function from $H^p(T_\Gamma)$, $p \in [1, \infty)$, its Fourier transform may be defined as an L^p Fourier transform of a limit function, the existence of which is guaranteed by Theorem 5.6 in [23, Ch. III, Sect. 5]. For $p < 1$ and a general cone, it does not work, and we need to consider the limit function using tempered distributions. Having obtained a modification of Lemma 4 from [3] in [28, Thm. 1], the following definition of Fourier transform is justified.

Definition 1.1 The Fourier transform of a function $f \in H^p(T_\Gamma)$, $p \in (0, 1]$, is defined by

$$\widehat{f}(\xi) = \mathrm{e}^{2\pi (\xi, \delta)} \widehat{f_\delta}(\xi), \quad \xi \in \mathbb{R}^n \quad (\delta \in \Gamma\text{--arbitrary}). \tag{1.1}$$

It is easy to see that the right-hand side of (1.1) is independent of δ. Moreover, $f_\delta \in L^1(\mathbb{R}^n)$ (see, e.g., [23, Ch. III, Sect. 2, Lem. 2.12]). Hence, the Fourier transform is well-defined. Let us also note that for $p = 1$, our $\widehat{f}$ coincides with the classical Fourier transform of the limit function $f(x) = \lim_{\zeta \to 0, \, \zeta \in \Gamma} f_\zeta(x)$.

Furthermore, if $f \in H^p(T_\Gamma)$ for some $p \in (0, 1]$, then the following inversion formula holds true (see [28])

$$f(z) = \int_{\Gamma^*} \widehat{f}(t) \, \mathrm{e}^{2\pi i (z, t)} \, \mathrm{d}t, \quad z \in T_\Gamma. \tag{1.2}$$

Therefore, for any $p \in (0, 1]$, the space $f \in H^p(T_\Gamma)$ contains non-zero functions if and only if the cone Γ is regular (in fact, this is true for $p \in (0, \infty)$ since $f \in H^p$ implies $(f)^{[p]+1} \in H^s$ with $s = p/([p]+1) \in (0, 1]$). So, we will investigate only the case of a regular cone.

1.2 Fourier Multipliers

Since there are no non-trivial translation-invariant linear bounded operators from $H^p(T_\Gamma)$ to $H^q(T_\Gamma)$, $p > q$ (see [29, Thm. 2]), we assume $p \le q$.

Definition 1.2 Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$. A Lebesgue measurable function $\varphi : \Gamma^* \to \mathbb{C}$ is called a multiplier from $H^p(T_\Gamma)$ to $H^q(T_\Gamma)$ (notation: $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$), $0 < p \le q \le 1$, if for any function $f \in H^p(T_\Gamma)$, the function $\varphi\widehat{f}$ coincides almost everywhere on Γ^* with the Fourier transform of a function $F_\varphi[f] \in H^q(T_\Gamma)$, and

$$\|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} := \sup_{\|f\|_{H^p} \neq 0} \frac{\|F_\varphi[f]\|_{H^q}}{\|f\|_{H^p}} < \infty.$$

It follows immediately from (1.2) that the function $F_\varphi[f]$ is defined uniquely as

$$F_\varphi[f](z) = \int_{\Gamma^*} \varphi(t)\,\widehat{f}(t)\,e^{2\pi i (z,t)}\,dt, \quad z \in T_\Gamma.$$

Let us mention the basic properties of multipliers.

(1) $\|\varphi + \psi\|^q_{\mathcal{M}_{p,q}(T_\Gamma)} \le \|\varphi\|^q_{\mathcal{M}_{p,q}(T_\Gamma)} + \|\psi\|^q_{\mathcal{M}_{p,q}(T_\Gamma)}$.
(2) If $p \le q \le r$, then $\|\varphi\psi\|_{\mathcal{M}_{p,r}(T_\Gamma)} \le \|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} \|\psi\|_{\mathcal{M}_{q,r}(T_\Gamma)}$.
(3) For any real number $\alpha > 0$, $\|\varphi(\alpha\cdot)\|_{\mathcal{M}_{p,q}(T_\Gamma)} = \alpha^{n(1/q-1/p)} \|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)}$.
(4) *Local Property.* If for any point of Γ^*, including the point at infinity, there exists a neighborhood in which $\varphi : \Gamma^* \to \mathbb{C}$ coincides with a function from $\mathcal{M}_{p,q}(T_\Gamma)$, then $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$.

Properties (1)–(3) easily follow from Definition 1.2, while the Local Property will be proven later in Lemma 3.4. Moreover, Property (1) can also be extended to the case of an infinite sum. The precise statement is given in Proposition 3.1.

1.3 Conditions for Multipliers

Our first theorem deals with the case of a compactly supported multiplier only. However, the most popular kernels are, in fact, radial and compactly supported. Besides, our theorem is especially sharp in this case (see Theorems 1.4, 1.5 below).

Theorem 1.3 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$. Assume that a function $\varphi \in C(\mathbb{R}^n)$ satisfies* $\operatorname{supp}\varphi \subset [-\sigma, \sigma]^n$ *for some $\sigma > 0$. If $\widehat{\varphi} \in L^q(\mathbb{R}^n)$ for some $q \in (0, 1]$, then $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$ for any $p \in (0, q]$, and*

$$\|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} \le \frac{\gamma(n, p, q)}{(V_n(\Gamma))^{1/p-1}} \sigma^{n(1/p-1)} \|\widehat{\varphi}\|_q, \tag{1.3}$$

where

$$\gamma(n,p,q) = 2^{n\left(\frac{2}{p}+\frac{1}{q}-2\right)} \left(\frac{\pi^{\frac{n}{2}} n^{n\left(\frac{1}{2}+\frac{1}{q}\right)}}{\Gamma\left(\frac{n}{2}+1\right)} \right)^{\frac{1}{p}-1} \left(\sum_{m=0}^{\infty} \frac{1}{(m!)^q} \right)^{\frac{1}{q}}.$$

Here and in what follows, by γ, we denote some positive constants depending only on the parameters in parentheses. The following geometric characteristic of the cone Γ is also used throughout the article:

$$V_n(\Gamma) = \frac{1}{n!} \max \left\{ \left| \det \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{nn} \end{pmatrix} \right| : a_1, \dots, a_n \in \overline{\Gamma}, |a_1| = \dots = |a_n| = 1 \right\} \tag{1.4}$$

(here a_{kl} denotes the lth component of the vector a_k). Geometrically, $V_n(\Gamma)$ is a maximum possible volume of a simplex that could be built on n unit vectors contained in $\overline{\Gamma}$.

It is worth noting that the requirement $\widehat{\varphi} \in L^q$ is essential. The following theorem shows that it is also a necessary condition in a local sense.

Theorem 1.4 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, and let $\varphi \in C(\Gamma^*)$. If $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$ for some $0 < p \le q \le 1$, then for any point $x \in (\Gamma^*)^o$, and its every bounded neighborhood V_x such that $\overline{V_x} \subset (\Gamma^*)^o$, the function φ coincides in $\overline{V_x}$ with a compactly supported continuous function whose Fourier transform belongs to $L^q(\mathbb{R}^n)$.*

If our kernel is radial and compactly supported, then the requirement $\widehat{\varphi} \in L^q$ is crucial. Moreover, using the Local Property (Lemma 3.4), it is often easier to show that a radial function is a multiplier, and then conclude that its Fourier transform is in L^q (see, e.g., Corollary 3.9). Such an approach is justified by the following theorem.

Theorem 1.5 *Let $\varphi : \mathbb{R}^n \to \mathbb{C}$, $n \in \mathbb{N}$, be a continuous compactly supported radial function. Assume that in some neighborhood of the origin, φ coincides with a continuous compactly supported function whose Fourier transform belongs to $L^q(\mathbb{R}^n)$, for some $q \in (0, 1]$. If $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, for some regular cone Γ and $p \in (0, q]$, then $\widehat{\varphi} \in L^q(\mathbb{R}^n)$.*

It is also possible to get rid of the restriction that φ is compactly supported. We can require some smoothness instead. Using the method from [27], we can decompose our function into a sum of compactly supported functions whose Fourier transforms are in L^q. Owing to the Local Property of a multiplier, this approach seems very natural. This gives the following result.

Theorem 1.6 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, and $q \in (0, 1]$.*

(a) *If* $\varphi \in C^r(\mathbb{R}^n)$ *for some natural* $r > n\left(\frac{1}{q} - \frac{1}{2}\right)$, *and for some* $p \in (0, q]$, $\alpha, \beta \geq 0$, *the inequalities*

$$|\varphi(x)| \leq \frac{A}{1 + |x|^\alpha}; \quad \sum_{j=1}^{n} \left| \frac{\partial^r \varphi}{\partial x_j^r}(x) \right| \leq \frac{B}{1 + |x|^\beta}, \quad \forall x \in \mathbb{R}^n,$$

$$\min\{\beta - \alpha - r, 0\} + \frac{2qr\alpha}{n(2-q)} - \frac{2rq}{2-q}\left(\frac{1}{p} - \frac{1}{q}\right) > 0,$$

hold true, then $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, *and*

$$\|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} \leq \frac{\gamma(n, p, q, r, \alpha, \beta)}{(V_n(\Gamma))^{1/p-1}}(A + B).$$

In particular, if $\alpha = \beta > n\left(\frac{1}{p} - \frac{1}{2}\right)$, *then* $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$.

(b) *Suppose that* $\varphi \in C^s(\mathbb{R}^n)$ *for* $s = \left[\frac{n}{q} - \frac{n+1}{2}\right]$, *and* supp$\varphi \subset [-1, 1]^n$. *If*

$$\max_{j=1,\ldots,n} \sup_{t_j \neq 0} \sup_{x \in \mathbb{R}^n} \frac{\left| \frac{\partial^s \varphi}{\partial x_j^s}(x_1, \ldots, x_n) - \frac{\partial^s \varphi}{\partial x_j^s}(x_1, \ldots, x_{j-1}, x_j + t_j, x_{j+1}, \ldots, x_n) \right|}{|t_j|^\alpha} < \infty,$$

for some $\alpha > \frac{n}{q} - \frac{n+1}{2} - s$, *and for any* $j = 1, \ldots, n$, *the segment* $[-1, 1]$ *could be split into finite number of segments (bounded with regard to the rest of variables) such that, on any of these segments, the real and imaginary parts of* $\frac{\partial^s \varphi}{\partial x_j^s}$ *(as functions of* x_j*) are convex or concave, then* $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$ *for any* $p \in (0, q]$.

Employing the above results, we answer the question: When do the Bochner–Riesz means of the Fourier integral

$$R_h^{r,\alpha}(f; z) = \int\limits_{|x| \leq 1/h} \widehat{f}(x)\left(1 - h^{2r}|x|^{2r}\right)^\alpha e^{2\pi i(z,x)}\, dx, \quad z \in T_\Gamma,$$

define a bounded linear operator from $H^p(T_\Gamma)$ to $H^q(T_\Gamma)$?

Let us note that in L^p, with $1 \leq p \leq \infty$, the Bochner–Riesz means are well-investigated (see, e.g., [1,13], [2, Ch. 5], or [23, Ch. IV, Sect. 4; Ch. VII, Sect. 5]). For approximation of functions in H^p spaces, $0 < p \leq 1$, by their Bochner–Riesz means see, e.g., [21, Sect. 3], [27, Sect. 2], [28, Sect. 4]. In our case, the following statement holds true.

Proposition 1.7 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$. Assume $\alpha > 0$, $r \in \mathbb{N}$, and $0 < p \le q \le 1$. The function*

$$\varphi_{r,\alpha}(x) := \begin{cases} \left(1 - |x|^{2r}\right)^{\alpha}, & |x| \le 1, \\ 0, & |x| > 1, \end{cases}$$

belongs to $\mathcal{M}_{p,q}(T_\Gamma)$ if and only if

$$\alpha > \frac{n}{q} - \frac{n+1}{2}.$$

It may seem surprising, that the critical index does not depend on p. However, this is easily justified by Theorem 1.5.

It is interesting to find the critical index for Bochner–Riesz means for the case of fractional powers r. Unfortunately, the proof of Proposition 1.7 does not work since $\varphi_{r,\alpha}$ looses its smoothness at the origin.

2 Some Auxiliary Results

For two vectors, $a, b \in \mathbb{R}^n$ such that $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$, and $-\infty < a_j < b_j < \infty$, we will consider the open and closed rectangles in $\mathbb{R}^n$:

$$(a,b)_n := \prod_{j=1}^{n} \left(a_j, b_j\right), \quad [a,b]_n := \prod_{j=1}^{n} \left[a_j, b_j\right].$$

We will also use the following notation

$$\mathcal{V}(a,b) := \left\{\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n : \nu_j = a_j \text{ or } b_j,\ j = 1, \dots, n\right\}.$$

For $p \in (0, \infty]$, let us consider the pth means of a function $f : T_B \to \mathbb{C}$:

$$m_p(f, y) := \|f(\cdot + iy)\|_p = \begin{cases} \left(\int_{\mathbb{R}^n} |f(x+iy)|^p \, \mathrm{d}x\right)^{1/p}, & p \in (0, \infty), \\ \sup_{x \in \mathbb{R}^n} |f(x+iy)|, & p = \infty, \end{cases} \quad y \in B.$$

For an arbitrary set $E \subset \mathbb{R}^n$, let us denote $A(E)$ as the set of all functions continuous and bounded in E, and holomorphic in its interior, E^o.

We will need several statements of Hadamard Three-Lines Theorem type.

Lemma 2.1 *Suppose that B is a convex set in $\mathbb{R}^n$ with non-empty interior. For $y_0, y_1 \in B$ and $t \in [0,1]$ set $y_t := (1-t)y_0 + ty_1$. If $f \in A(T_B)$, then*

$$m_p(f, y_t) \le \max\left(m_p(f, y_0), m_p(f, y_1)\right), \quad p \in (0, \infty].$$

Note. Lemma 2.1 is already known. It was mentioned in [23, Ch. III, Sect. 6.1] that if $f \in H^p(T_B)$, then $\log \|f(\cdot + iy)\|_p$ is a convex function of $y \in B$. However, this source contains no references on the proof of this fact. To prove the lemma, we may use the Three-Lines Theorem for subharmonic functions (see, e.g., [20, Ch. 2, Sect. 2.3, Cor. 2.3.6]), and employ subharmonicity of $|f(z)|^p$. Now, we easily obtain the following result.

Corollary 2.2 *Suppose that B is a convex set in $\mathbb{R}^n$ with non-empty interior, and $f \in A(T_B)$. If K is a convex hull of a set $E \subset B$, then*

$$\sup_{y \in K} m_p(f, y) = \sup_{y \in E} m_p(f, y), \quad p \in (0, \infty].$$

Lemma 2.3 *Suppose that B is a convex set in $\mathbb{R}^n$ with non-empty interior, and $f \in A(T_B)$. If K is a convex hull of a set $E \subset B$, then for any $y_0 \in K^o$ and any p and q such that $0 < p \le q \le \infty$,*

$$m_q(f, y_0) \le \left(\frac{n!}{\pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + 1\right) (\operatorname{dist}(y_0, \partial K))^n} \right)^{\frac{1}{p} - \frac{1}{q}} \sup_{y \in E} m_p(f, y). \tag{2.1}$$

Proof For $q = p$, the statement is just Corollary 2.2. Since $p = \infty$ implies $q = \infty$, whence $p = q$ again, we will consider the case $0 < p < q \le \infty$. We will also suppose that the supremum in the right-hand side of (2.1) is finite, and that $K^o \ne \emptyset$. Otherwise, the statement is trivial.

We can use the approach of [23, Ch. III, Sect. 2, Lem. 2.12]. Let us fix an arbitrary x_0 in $\mathbb{R}^n$ and let $\varepsilon := \operatorname{dist}(y_0, \partial K) > 0$. Then, $B_n(y_0, \varepsilon) \subset K^o$ (here $B_n(y_0, \varepsilon)$ is the ball in $\mathbb{R}^n$ with the center at y_0 and of radius ε). If Ω_m denotes the volume of the unit ball in $\mathbb{R}^m$, then using the subharmonicity of $|f|^p$, we get

$$\begin{aligned} |f(x_0 + iy_0)|^p &\le \frac{1}{\varepsilon^{2n} \Omega_{2n}} \int\limits_{B_{2n}(x_0, y_0, \varepsilon)} |f(x + it)|^p \, dx \, dt \\ &\le \frac{1}{\varepsilon^{2n} \Omega_{2n}} \int\limits_{T_{B_n(y_0, \varepsilon)}} |f(x + it)|^p \, dx \, dt. \end{aligned} \tag{2.2}$$

Corollary 2.2 justifies changing the order of integration in (2.2), and we obtain the next result.

$$|f(x_0 + iy_0)|^p \le \frac{\left(\max_{y \in E} m_p(f, y)\right)^p}{\varepsilon^{2n} \Omega_{2n}} \int\limits_{B_n(y_0, \varepsilon)} dt = \frac{\left(\max_{y \in E} m_p(f, y)\right)^p \Omega_n}{\varepsilon^n \Omega_{2n}}.$$

Since $x_0 \in \mathbb{R}^n$ was taken arbitrarily, we get

$$m_\infty(f, y_0) \le \left(\frac{\Omega_n}{\varepsilon^n \Omega_{2n}} \right)^{\frac{1}{p}} \sup_{y \in E} m_p(f, y).$$

Now, for $q > p$, using the last inequality, we have

$$m_q(f, y_0) \le (m_\infty(f, y_0))^{\frac{q-p}{q}} \left(\int_{\mathbb{R}^n} |f(x + iy_0)|^p \, dx \right)^{\frac{1}{q}}$$
$$\le \left(\frac{\Omega_n}{\varepsilon^n \Omega_{2n}} \right)^{\frac{1}{p} - \frac{1}{q}} \sup_{y \in E} m_p(f, y).$$

Since $\varepsilon = \text{dist}(y_0, \partial K)$, and $\Omega_m = \frac{\pi^{m/2}}{\Gamma(m/2+1)}$, inequality (2.1) follows immediately. □

Applying Lemma 2.3 to $B = [a, b]_n$ and $E = \mathcal{V}(a, b)$, we obtain the following result.

Corollary 2.4 *Assume f is holomorphic in $T_{(a,b)_n}$ as well as bounded and continuous in $T_{[a,b]_n}$. Then, for any $0 < p \le q \le \infty$, the following inequality holds*

$$\sup_{y \in (a,b)_n} \|f(\cdot + iy)\|_q$$
$$\le \left(\frac{n!}{\pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + 1\right) \left(\min_{j=1,\dots,n} \left(\min\left(y_j - a_j, b_j - a_j \right) \right) \right)^n} \right)^{\frac{1}{p} - \frac{1}{q}}$$
$$\times \max_{\nu \in \mathcal{V}(a,b)} \|f(\cdot + i\nu)\|_p.$$

Let us return to $V_n(\Gamma)$ introduced in the previous section (see 1.4). As soon as the set $\left\{(x_1, \dots, x_n) \in \mathbb{R}^{n^2} : x_j \in \overline{\Gamma}, \left|x_j\right| = 1, \forall j = 1, \dots, n\right\}$ is compact in $\mathbb{R}^{n^2}$, then the maximum in (1.4) is attained on some set of vectors $e_1, \dots, e_n$. Since also Γ is open and non-empty, then $V_n(\Gamma) > 0$. Although the set of vectors $e_1, \dots, e_n$ may be not unique, let us fix some: $e := \{e_1, \dots, e_n\}$. We will consider only this fixed set in the following argument. Consider the linear map

$$\Psi_e := \begin{pmatrix} e_{11} & \dots & e_{n1} \\ \vdots & \ddots & \vdots \\ e_{1n} & \dots & e_{nn} \end{pmatrix},$$

and denote $\Gamma^e := \Psi_e\left(\left(\mathbb{R}^n_+\right)^o\right)$ (here $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_j \ge 0, \forall j = 1, \dots, n\}$, as usual). Since $|\det \Psi_e| = n! V_n(\Gamma) > 0$, this map is a bijection of $\mathbb{R}^n$ onto $\mathbb{R}^n$, $\left(\mathbb{R}^n_+\right)^o$ onto Γ^e, and $\mathbb{R}^n_+$ onto $\overline{\Gamma^e}$. It is also clear that $\Gamma^e \subset \Gamma$, and it is also an open cone.

Let us denote a translation of a cone Γ on a vector ζ by $\Gamma_\zeta := \{x + \zeta : x \in \Gamma\}$.

Lemma 2.5 *Let Γ be an open cone in $\mathbb{R}^n$, $n \in \mathbb{N}$. Assume that r and R are some points in $\left(\mathbb{R}^n_+\right)^o$ such that $r_j < R_j$, $\forall j = 1, \dots, n$. If a function F is holomorphic in $T_{\Psi_e((r,R)_n)}$ as well as bounded and continuous in $T_{\Psi_e([r,R]_n)}$, then for any $y \in$*

$\Psi_e\left((r,R)_n\right)$*, and for any* p *and* q *such that* $0 < p \le q \le \infty$*, the following inequality holds true*

$$\|F(\cdot + iy))\|_q \le \left(\frac{1}{\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)V_n(\Gamma)\left(\min_{j=1,\dots,n}\left(\min\left(\left(\left(\Psi_e^{-1}y\right)_j - r_j, R_j - \left(\Psi_e^{-1}y\right)_j\right)\right)\right)\right)^n} \right)^{\frac{1}{p}-\frac{1}{q}} \times \max_{\nu\in\mathcal{V}(r,R)} \|F(\cdot + i\Psi_e\nu)\|_p . \tag{2.3}$$

To prove the lemma, we only need to apply Corollary 2.4 to the function $G(z) = F(\Psi_e z)$, with $a = r$, $b = R$, $y = \Psi_e^{-1}y$, and get back to F.

Lemma 2.6 *Let* Γ *be a regular cone in* $\mathbb{R}^n$*,* $n \in \mathbb{N}$*, and* $\varphi : \Gamma^* \to \mathbb{C}$ *be a Lebesgue measurable function. Assume that there exists a Lebesgue measurable function* $\varphi^* : \mathbb{R}^n \to \mathbb{C}$ *such that*

(i) $\varphi^*(x) = \varphi(x)$ *almost everywhere on* Γ^**;*
(ii) $\varphi^*(\cdot)\,\mathrm{e}^{2\pi(\delta,\cdot)} \in L^1(\mathbb{R}^n)$*, for some* $\delta \in \Gamma$*.*

Then, for any function f*, belonging to* $H^p(T_\Gamma)$ *with some* $p \in (0,1]$*, the following equality holds true*

$$M_\varphi(f;x) := \int_{\Gamma^*} \varphi(t)\,\widehat{f}(t)\,\mathrm{e}^{2\pi i(x,t)}\,\mathrm{d}t = \int_{\mathbb{R}^n} f(x+t+i\delta)\,\widehat{\varphi^*}(t+i\delta)\,\mathrm{d}t, \quad x \in \mathbb{R}^n. \tag{2.4}$$

Proof Let us fix an arbitrary $x \in \mathbb{R}^n$. Since $\varphi^* = \varphi$ a.e. on Γ^*, and $\operatorname{supp}\widehat{f} \subset \Gamma^*$, then

$$M_\varphi(f;x) = \int_{\mathbb{R}^n} \varphi^*(t)\,\widehat{f}(t)\,\mathrm{e}^{2\pi i(x,t)}\,\mathrm{d}t.$$

As soon as $\varphi^*(\cdot)\,\mathrm{e}^{2\pi(\delta,\cdot)} \in L^1(\mathbb{R}^n)$, and $f_\delta \in L^1(\mathbb{R}^n)$ (as we already noticed), using Tonelli's Theorem, it is easy to see that the function $G(t,u) := \varphi^*(t)\,\mathrm{e}^{2\pi(\delta,t)} f_\delta(u)$ belongs to $L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Therefore, the function $\mathcal{G}(t,u) := G(t,u)\,\mathrm{e}^{-2\pi i(u-x,t)}$ is also there. Furthermore, let us write the Fourier transform (see Definition 1.1) of f with our δ:

$$\widehat{f}(t) = \mathrm{e}^{2\pi(\delta,t)}\widehat{f_\delta}(t) = \mathrm{e}^{2\pi(\delta,t)} \int_{\mathbb{R}^n} f_\delta(u)\,\mathrm{e}^{-2\pi i(u,t)}\,\mathrm{d}u, \quad t \in \mathbb{R}^n.$$

An application of Fubini's Theorem to $\mathcal{G}$ shows that $\varphi^*\widehat{f} \in L^1(\mathbb{R}^n)$, and allows us to change the order of integration in the equation below:

$$\begin{aligned} M_{\varphi}(f;x) &= \int\limits_{\mathbb{R}^n} \left(\varphi^*(t)\, \mathrm{e}^{-2\pi i(-x+i\delta,t)} \int\limits_{\mathbb{R}^n} f_\delta(u)\, \mathrm{e}^{-2\pi i(u,t)}\, \mathrm{d}u \right) \mathrm{d}t \\ &= \int\limits_{\mathbb{R}^n} \left(f_\delta(u) \int\limits_{\mathbb{R}^n} \varphi^*(t)\, \mathrm{e}^{-2\pi i(u-x+i\delta,t)}\, \mathrm{d}t \right) \mathrm{d}u \\ &= \int\limits_{\mathbb{R}^n} f_\delta(u)\, \widehat{\varphi^*}(u-x+i\delta)\, \mathrm{d}u \\ &= \int\limits_{\mathbb{R}^n} f_\delta(t+x)\, \widehat{\varphi^*}(t+i\delta)\, \mathrm{d}t \\ &= \int\limits_{\mathbb{R}^n} f(t+x+i\delta)\, \widehat{\varphi^*}(t+i\delta)\, \mathrm{d}t. \end{aligned}$$

Since $x \in \mathbb{R}^n$ was chosen arbitrarily, (2.4) holds. □

In the univariate case, (2.4) was proven in [21, Proof of Prop. 1].

Following [22, App. B.2], for a Lebesgue measurable function h on $\mathbb{R}^n$, we will consider its distribution function

$$\lambda_h(\alpha) := m\left\{x \in \mathbb{R}^n : |h(x)| > \alpha\right\}, \quad \alpha \geq 0,$$

with m—the Lebesgue measure on $\mathbb{R}^n$, as well as the non-increasing rearrangement of h given by

$$h^*(t) := \inf\{\alpha : \lambda_h(\alpha) \leq t\}, \quad t \geq 0.$$

As shown in [22, App. B.2], both functions λ_h and h^* are non-negative, non-increasing and right continuous. Moreover, h and h^* have the same distribution function, and

$$\int\limits_{\mathbb{R}^n} |h(x)|^p\, \mathrm{d}x = \int\limits_0^\infty \left(h^*(t)\right)^p \mathrm{d}t, \quad p \in (0,\infty). \tag{2.5}$$

For a function $\varphi \in L^2(\mathbb{R}^n)$, let us denote

$$a_\sigma(\varphi)_2 := \inf\left\{\|\varphi-\psi\|_2 : \psi \in L^2\left(\mathbb{R}^n\right), m\left(\operatorname{supp}\widehat{\psi}\right) \leq \sigma\right\}.$$

Since the Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$, then $\|\varphi-\psi\|_2 = \left\|\widehat{\varphi}-\widehat{\psi}\right\|_2$, whence

$$a_\sigma(\varphi)_2 = \inf\left\{\left(\int_{\mathbb{R}^n\setminus E} |\widehat{\varphi}(x)|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} : m(E) \leq \sigma\right\}$$
$$\leq \left(\int_{\mathbb{R}^n\setminus\left[\frac{-\sigma^{1/n}}{2},\frac{\sigma^{1/n}}{2}\right]_n} |\widehat{\varphi}(x)|^2 \, \mathrm{d}x\right)^{\frac{1}{2}}. \quad (2.6)$$

We also need a refined version of (2.5), given by the following statement. To the best of my knowledge, this result is new.

Lemma 2.7 *Let $f \in L^p(\mathbb{R}^n)$ for some $p \in (0,\infty)$, and f^* be its non-increasing rearrangement. Then, for any $\sigma > 0$,*

$$\sup_{E:\, m(E)\leq\sigma} \int_E |f(x)|^p \, \mathrm{d}x = \int_0^\sigma \left(f^*(t)\right)^p \, \mathrm{d}t. \quad (2.7)$$

Proof Let us take an arbitrary measurable set E so that $m(E) \leq \sigma$, and consider $h(x) := f(x)\,\chi_E(x)$, where χ_E is the indicator of E. Obviously, $h^*(t) \leq f^*(t)$, $t \geq 0$. It is also clear that $\lambda_h(\alpha) \leq \sigma$, for any $\alpha \geq 0$. Hence, $h^*(t) = 0$, $t \geq \sigma$. Now, from (2.5) we obtain

$$\int_E |f(x)|^p \, \mathrm{d}x = \int_{\mathbb{R}^n} |h(x)|^p \, \mathrm{d}x = \int_0^\infty \left(h^*(t)\right)^p \, \mathrm{d}t = \int_0^\sigma \left(h^*(t)\right)^p \, \mathrm{d}t$$
$$\leq \int_0^\sigma \left(f^*(t)\right)^p \, \mathrm{d}t. \quad (2.8)$$

Since E was chosen arbitrarily with the only requirement $m(E) \leq \sigma$, then

$$\sup_{E:\, m(E)\leq\sigma} \int_E |f(x)|^p \, \mathrm{d}x \leq \int_0^\sigma \left(f^*(t)\right)^p \, \mathrm{d}t. \quad (2.9)$$

Let us construct a set on which the supremum is attained. First, assume that f is bounded. Define

$$A := \sup\left\{\alpha : m\left(\left\{x \in \mathbb{R}^n : |f(x)| \geq \alpha\right\}\right) \geq \sigma\right\}.$$

If $A = 0$, then $m(\mathcal{B}_n) < \sigma$, for each $\mathcal{B}_m := \{x \in \mathbb{R}^n : |f(x)| \geq 1/m\}$, $m \in \mathbb{N}$. Hence,

$$m(\operatorname{supp} f) = m\left(\bigcup_{m=1}^\infty \mathcal{B}_m\right) = \lim_{m\to\infty} m(\mathcal{B}_m) \leq \sigma.$$

Thus, we could take $E = \operatorname{supp} f$, so that (2.8) becomes an equality, and (2.7) follows immediately.

Now, we will consider the case $A > 0$. Let us denote

$$M_f := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| = \inf \left\{ a : m\left(\left\{ x \in \mathbb{R}^n : |f(x)| > a \right\} \right) = 0 \right\}.$$

It is clear that $A \le M_f$, and if

$$m\left(\left\{ x \in \mathbb{R}^n : |f(x)| = M_f \right\} \right) < \sigma, \tag{2.10}$$

then, $A < M_f$.

Let us denote

$$\mathcal{U}_\alpha := \left\{ x \in \mathbb{R}^n : |f(x)| \ge \alpha \right\}, \ \alpha > 0,$$
$$\widetilde{\mathcal{U}}_A := \bigcup_{\alpha \in (A, M_f]} \mathcal{U}_\alpha = \left\{ x \in \mathbb{R}^n : |f(x)| > A \right\}.$$

Then, $\alpha_1 > \alpha_2$ implies $\mathcal{U}_{\alpha_1} \subset \mathcal{U}_{\alpha_2}$, and all $\mathcal{U}_\alpha$'s are Lebesgue measurable. Since $A = \sup \{\alpha : m(\mathcal{U}_\alpha) \ge \sigma\}$, then $\alpha > A$ implies $m(\mathcal{U}_\alpha) < \sigma$. Therefore, using the continuity of Lebesgue measure from below, we get $m\left(\widetilde{\mathcal{U}}_A\right) \le \sigma$.

Since $m\left(\widetilde{\mathcal{U}}_A\right) \le \sigma$ and $m(\mathcal{U}_A) \ge \sigma$, we can use the continuity of the Lebesgue measure to choose a Lebesgue measurable set E so that $\widetilde{\mathcal{U}}_A \subset E \subset \mathcal{U}_A$, and $m(E) = \sigma$.

If the requirement (2.10) is not satisfied, then take E as any subset of $\mathcal{U}_{M_f}$ with $m(E) = \sigma$.

Let us consider $g := f\chi_E$, and take an arbitrary $\alpha \ge A$ (remember, $A > 0$). Then,

$$\lambda_f(\alpha) = \lambda_g(\alpha), \quad \alpha \ge A. \tag{2.11}$$

Also note that $\lambda_f(\alpha) = \lambda_g(\alpha) = 0$, $\alpha > M_f$. Moreover, from the definition of A, we get

$$\lambda_f(\alpha) = m\left(\left\{ x \in \mathbb{R}^n : |f(x)| > \alpha \right\} \right) \ge m\left(\left\{ x \in \mathbb{R}^n : |f(x)| \ge A \right\} \right) \ge \sigma, \quad \alpha < A. \tag{2.12}$$

Since $\widetilde{\mathcal{U}}_A \subset E \subset \mathcal{U}_A$, then for any $\alpha \in (0, A)$, $x \in E$ implies $|g(x)| = |f(x)| \ge A > \alpha$. From another side, if $|g(x)| > \alpha$, then $|f(x)| > \alpha$. Hence, $x \in \widetilde{\mathcal{U}}_A \subset E$. Thus,

$$\left\{ x \in \mathbb{R}^n : |g(x)| > \alpha \right\} = E, \quad 0 < \alpha < A.$$

Therefore,

$$\lambda_g(\alpha) = m(E) = \sigma, \quad 0 < \alpha < A. \tag{2.13}$$

Considering (2.11), (2.12) and (2.13), for $t \in [0, \sigma)$, we obtain

$$g^* (t) = \inf \left\{ \alpha : \lambda_g (\alpha) \le t \right\} = (2.13) = \inf \left\{ \alpha : \alpha \ge A, \lambda_g (\alpha) \le t \right\} = (2.11) =$$

$$\inf \left\{ \alpha : \alpha \ge A, \lambda_f (\alpha) \le t \right\} = (2.12) = \inf \left\{ \alpha : \lambda_f (\alpha) \le t \right\} = f^* (t) . \quad (2.14)$$

Since also $\lambda_g (\alpha) \le m (\mathrm{supp} g) = \sigma$, for any $\alpha \ge 0$, then $g^* (t) = 0$ when $t \ge \sigma$. Now, using (2.14) and (2.5), we get

$$\int_E |f (x)|^p \, \mathrm{d}x = \int_{\mathbb{R}^n} |g (x)|^p \, \mathrm{d}x = \int_0^\infty \left(g^* (t)\right)^p \, \mathrm{d}t = \int_0^\sigma \left(g^* (t)\right)^p \, \mathrm{d}t$$
$$= \int_0^\sigma \left(f^* (t)\right)^p \, \mathrm{d}t.$$

Hence,

$$\sup_{E: m(E) \le \sigma} \int_E |f (x)|^p \, \mathrm{d}x \ge \int_0^\sigma \left(f^* (t)\right)^p \, \mathrm{d}t$$

that completes the proof for a bounded function f.

Let us get rid of this restriction. Since $f \in L^p (\mathbb{R}^n)$, then

$$m \left(\left\{ x \in \mathbb{R}^n : |f (x)| > f^* (\varepsilon) \right\} \right) \le \varepsilon, \quad \varepsilon > 0. \quad (2.15)$$

Let us consider functions

$$f_{(\varepsilon)} (x) := \min \left(|f (x)| , f^* (\varepsilon) \right\} , \quad x \in \mathbb{R}^n, \varepsilon > 0.$$

Clearly, they are in $L^p (\mathbb{R}^n)$ and are also bounded. Moreover, (2.15) implies that $f_{(\varepsilon)}$ coincides with $|f|$ everywhere except on some set of Lebesgue measure not more than ε.

Since for any $\alpha > 0$,

$$\left\{ x \in \mathbb{R}^n : |f (x)| > \alpha \right\} \subset \left\{ x \in \mathbb{R}^n : f_{(\varepsilon)} (x) > \alpha \right\} \cup \left\{ x \in \mathbb{R}^n : |f (x)| > f_{(\varepsilon)} (x) \right\} ,$$

then, $\lambda_f (\alpha) \le \lambda_{f_{(\varepsilon)}} (\alpha) + \varepsilon$. Hence,

$$f^* (t) = \inf \left\{ \alpha : \lambda_f (\alpha) \le t \right\} \le \inf \left\{ \alpha : \lambda_{f_{(\varepsilon)}} (\alpha) \le t - \varepsilon \right\} = f^*_{(\varepsilon)} (t - \varepsilon) , \quad t \ge \varepsilon.$$

Thus, applying (2.7) to the bounded function $f_{(\varepsilon)}$, and considering that $f^* \ge 0$, $\left| f_{(\varepsilon)} \right| \le |f|$, we get

$$\int_{\varepsilon}^{\sigma} \left(f^{*}(t)\right)^{p} \mathrm{d}t \le \int_{\varepsilon}^{\sigma+\varepsilon} \left(f^{*}(t)\right)^{p} \mathrm{d}t = \int_{\varepsilon}^{\sigma+\varepsilon} \left(f_{(\varepsilon)}^{*}(t-\varepsilon)\right)^{p} \mathrm{d}t = \int_{0}^{\sigma} \left(f_{(\varepsilon)}^{*}(t)\right)^{p} \mathrm{d}t$$
$$= \sup_{E:\, m(E)\le\sigma} \int_{E} \left|f_{(\varepsilon)}(x)\right|^{p} \mathrm{d}x \le \sup_{E:\, m(E)\le\sigma} \int_{E} |f(x)|^{p} \mathrm{d}x. \tag{2.16}$$

Applying Fatou's lemma, we can pass to the limit as $\varepsilon \to 0+$ to conclude

$$\int_{0}^{\sigma} \left(f^{*}(t)\right)^{p} \mathrm{d}t \le \sup_{E:\, m(E)\le\sigma} \int_{E} |f(x)|^{p} \mathrm{d}x.$$

Since the inverse inequality (2.9) was obtained without any assumption on boundedness of f, this completes the proof. □

From this lemma, (2.6) and (2.5), the next result follows immediately.

Corollary 2.8 *If $f \in L^{2}(\mathbb{R}^{n})$, then*

$$a_{\sigma}(f)_{2} = \left(\int_{\sigma}^{\infty} \left(\widehat{f}^{*}(t)\right)^{2} \mathrm{d}t\right)^{\frac{1}{2}}, \quad \sigma \ge 0.$$

Note that this statement is contained in [25, Proof of Thm. 2]. However, the source does not contain its detailed proof.

Corollary 2.9 *Let $f \in L^{2}(\mathbb{R}^{n})$. Then, for any $p \in (0, \infty)$, the following inequality holds true*

$$\int_{\mathbb{R}^{n}} \left|\widehat{f}(x)\right|^{p} \mathrm{d}x \le 2 \int_{0}^{\infty} \left(\frac{a_{\sigma}(f)_{2}}{\sqrt{\sigma}}\right)^{p} \mathrm{d}\sigma.$$

Proof The arguments used for this proof are the same as those used in the proof of [25, Thm. 2] just mentioned.

Since $f \in L^{2}(\mathbb{R}^{n})$, then $\widehat{f}$ also belongs there. Applying (2.6), Corollary 2.8, and considering that $\widehat{f}^{*}$ is non-increasing and non-negative, we get

$$\widehat{f}^{*}(2t) \le \frac{1}{\sqrt{t}} \left(\int_{t}^{2t} \left(\widehat{f}^{*}(u)\right)^{2} \mathrm{d}u\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{t}} \left(\int_{t}^{\infty} \left(\widehat{f}^{*}(u)\right)^{2} \mathrm{d}u\right)^{\frac{1}{2}} = \frac{a_{t}(f)_{2}}{\sqrt{t}}, \quad t > 0.$$

Therefore, (2.5) implies

$$\int_{\mathbb{R}^{n}} \left|\widehat{f}(x)\right|^{p} \mathrm{d}x = \int_{0}^{\infty} \left(\widehat{f}^{*}(t)\right)^{p} \mathrm{d}t = 2\int_{0}^{\infty} \left(\widehat{f}^{*}(2\sigma)\right)^{p} \mathrm{d}\sigma \le 2\int_{0}^{\infty} \left(\frac{a_{\sigma}(f)_{2}}{\sqrt{\sigma}}\right)^{p} \mathrm{d}\sigma$$

which completes the proof. □

We also need a Nikol'skij type inequality in a pointwise form. Unfortunately, it is not true without additional assumptions. The following statement is one of such 'constrained' forms.

Proposition 2.10 *Assume $\varphi \in C^r(\mathbb{R}^n)$, $r, n \in \mathbb{N}$, and there exist non-increasing functions $F, G : \mathbb{R}_+ \to \mathbb{R}_+$ such that for some $j = 1, \ldots, n$,*

$$|\varphi(x)| \leq F(|x|), \quad \left|\frac{\partial^r \varphi}{\partial x_j^r}(x)\right| \leq G(|x|), \quad x \in \mathbb{R}^n.$$

Then

$$\left|\frac{\partial^k \varphi}{\partial x_j^k}(x)\right| \leq \left(\frac{C_0 r}{k}\right)^k (F(|x|))^{1-\frac{k}{r}} (G(|x|))^{\frac{k}{r}}, \quad x \in \mathbb{R}^n,\ k = 1, \ldots, r-1,$$

where C_0 is an absolute constant.

Proof Suppose $g \in C^r(\mathbb{R})$, and for some $a, b \geq 0$,

$$|g(x)| \leq F\left(\sqrt{a+x^2}\right), \quad \left|g^{(r)}(x)\right| \leq G\left(\sqrt{b+x^2}\right), \quad x \in \mathbb{R}. \tag{2.17}$$

Then, fixing some $x \geq 0$ and applying the Nikol'skij type inequality on $\mathbb{R}_+$ (see, e.g., [24, Ch. 3, Sect. 3.10.2, (9)]) to the function $h(t) := g(t+x)$, we obtain

$$\begin{aligned}\sup_{t\geq x}\left|g^{(k)}(t)\right| = \sup_{t\geq 0}\left|h^{(k)}(t)\right| &\leq \left(\frac{C_0 r}{k}\right)^k \left(\sup_{t\geq 0}|h(t)|\right)^{1-\frac{k}{r}} \left(\sup_{t\geq 0}\left|h^{(r)}(t)\right|\right)^{\frac{k}{r}} \\ &= \left(\frac{C_0 r}{k}\right)^k \left(\sup_{t\geq x} F\left(\sqrt{a+t^2}\right)\right)^{1-\frac{k}{r}} \left(\sup_{t\geq x} G\left(\sqrt{b+t^2}\right)\right)^{\frac{k}{r}}.\end{aligned}$$

Since F and G are non-increasing, then $F\left(\sqrt{a+x^2}\right)$ and $G\left(\sqrt{b+x^2}\right)$ are non-increasing on $\mathbb{R}_+$, whence, for $x \geq 0$,

$$\left|g^{(k)}(x)\right| \leq \left(\frac{C_0 r}{k}\right)^k \left(F\left(\sqrt{a+x^2}\right)\right)^{1-\frac{k}{r}} \left(G\left(\sqrt{b+x^2}\right)\right)^{\frac{k}{r}}. \tag{2.18}$$

If $x < 0$, then considering $\mathcal{G}(t) := g(-t)$, we deduce that (2.18) holds for $x \in \mathbb{R}$.

Now, take any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and consider $g(t) := \varphi\left(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n\right)$. Applying (2.18) to this function with $a = b = \sum_{l=1,\ldots,n;\, l\neq j} x_l^2$, we get

$$\left|\frac{\partial^k \varphi}{\partial x_j^k}\left(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n\right)\right| = \left|g^{(k)}(t)\right|$$
$$\leq \left(\frac{C_0 r}{k}\right)^k \left(F\left(\sqrt{x_1^2 + \cdots + x_{j-1}^2 + t^2 + x_{j+1}^2 + \cdots + x_n^2}\right)\right)^{1-\frac{k}{r}}$$
$$\times \left(G\left(\sqrt{x_1^2 + \cdots + x_{j-1}^2 + t^2 + x_{j+1}^2 + \cdots + x_n^2}\right)\right)^{\frac{k}{r}}, \quad t \in \mathbb{R}.$$

Taking $t = x_j$ completes the proof. □

Corollary 2.11 *Assume $\varphi \in C^r(\mathbb{R}^n)$, $r, n \in \mathbb{N}$. If for some non-negative α, β, A and B, the following growth estimates*

$$|\varphi(x)| \leq \frac{A}{1+|x|^\alpha}; \quad \left|\frac{\partial^r \varphi}{\partial x_j^r}(x)\right| \leq \frac{B}{1+|x|^\beta}, \quad x \in \mathbb{R}^n, \ j = 1, \ldots, n,$$

are satisfied, then for $k = 1, \ldots, r-1$,

$$\left|\frac{\partial^k \varphi}{\partial x_j^k}(x)\right| \leq \left(\frac{C_0 r}{k}\right)^k \left(\frac{A}{1+|x|^\alpha}\right)^{1-\frac{k}{r}} \left(\frac{B}{1+|x|^\beta}\right)^{\frac{k}{r}}, \quad x \in \mathbb{R}^n, \tag{2.19}$$

where C_0 is an absolute constant.

Note that (2.19) is used in [27, Proof of Thm. 3b], but its justification is absent there.

Equipped with these statements, we can proceed to the proofs of our main results.

3 Conditions for Fourier Multipliers

3.1 Multipliers with Compactly Supported Kernel

The goal of this subsection is to prove Theorem 1.3. First, we need Proposition 3.2, which is rather technical, but it can be used for obtaining various conditions for Fourier multipliers. Let us start with a generalization of the basic Property (1) of a multiplier given by the following statement.

Proposition 3.1 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, $0 < p \leq q \leq 1$, and let $\{\varphi_m\}_{m=1}^\infty$ be a sequence of Fourier multipliers, $\varphi_m \in \mathcal{M}_{p,q}(T_\Gamma)$. Assume that $\sum_{m=1}^\infty |\varphi_m| \in L^\infty(\Gamma^*)$ and $\varphi(x) = \sum_{m=1}^\infty \varphi_m(x)$ almost everywhere on Γ^*.*

If $\sum_{m=1}^\infty \|\varphi_m\|_{\mathcal{M}_{p,q}(T_\Gamma)}^q < \infty$, then $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, and

$$\|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} \leq \left(\sum_{m=1}^\infty \|\varphi_m\|_{\mathcal{M}_{p,q}(T_\Gamma)}^q\right)^{\frac{1}{q}}.$$

Proof Since $\sum_{m=1}^{\infty}\varphi_m(x)$ converges to φ almost everywhere, and the Lebesgue measure in $\mathbb{R}^n$ is complete, then φ is measurable. Let us take an arbitrary $f\in H^p(T_\Gamma)$ and fix an arbitrary $y\in\Gamma$. Then, since the inversion formula (1.2) is true, we have $\widehat{f}(t)\,\mathrm{e}^{-2\pi(y,t)}\in L^1(\mathbb{R}^n)$. Using the assumption that $\sum_{m=1}^{\infty}|\varphi_m|\in L^\infty(\Gamma^*)$, we can apply the Lebesgue Dominated Convergence Theorem to derive

$$\sum_{m=1}^{\infty}\int_{\Gamma^*}\varphi_m(t)\,\widehat{f}(t)\,\mathrm{e}^{2\pi i(x+iy,t)}\,\mathrm{d}t=\int_{\Gamma^*}\varphi(t)\,\widehat{f}(t)\,\mathrm{e}^{2\pi i(x+iy,t)}\,\mathrm{d}t$$
$$=F_\varphi[f](x+iy),\quad x\in\mathbb{R}^n. \tag{3.1}$$

Again, since $\varphi\in L^\infty(\Gamma^*)$ and $\widehat{f}(t)\,\mathrm{e}^{-2\pi(y,t)}\in L^1(\mathbb{R}^n)$, then the Lebesgue Dominated Convergence Theorem implies $F_\varphi[f](\cdot+iy)$ is continuous on $\mathbb{R}^n$, and hence Lebesgue measurable.

Since all φ_m's belong to $\mathcal{M}_{p,q}(T_\Gamma)$, then $\left|F_{\varphi_m}[f](\cdot+iy)\right|^q\in L^1(\mathbb{R}^n)$, for any m. Since also

$$\sum_{m=1}^{\infty}\int_{\mathbb{R}^n}\left|F_{\varphi_m}[f](x+iy)\right|^q\,\mathrm{d}x\le\sum_{m=1}^{\infty}\|\varphi_m\|^q_{\mathcal{M}_{p,q}(T_\Gamma)}\|f\|^q_{H^p}<\infty,$$

the Dominated Convergence Theorem implies that the series $\sum_{m=1}^{\infty}\left|F_{\varphi_m}[f](x+iy)\right|^q$ converges almost everywhere on $\mathbb{R}^n$ to a function from $L^1(\mathbb{R}^n)$ (see, e.g., [5, Ch. 2, Sect. 2.3, Thm. 2.25]).

Using the triangle inequality in the power of q, (3.1) implies

$$\sum_{m=1}^{\infty}\left|F_{\varphi_m}[f](x+iy)\right|^q\ge\left|\sum_{m=1}^{\infty}F_{\varphi_m}[f](x+iy)\right|^q=\left|F_\varphi[f](x+iy)\right|^q,$$

and we immediately conclude that $\left|F_\varphi[f](\cdot+iy)\right|^q\in L^1(\mathbb{R}^n)$, and

$$\left\|F_\varphi[f](\cdot+iy)\right\|_q^q\le\sum_{m=1}^{\infty}\|\varphi_m\|^q_{\mathcal{M}_{p,q}(T_\Gamma)}\|f\|^q_{H^p}.$$

Passing to $\sup_{y\in\Gamma}$ in the last inequality, we get the desired result. □

Proposition 3.2 *Let Γ be a regular cone in $\mathbb{R}^n$, $n\in\mathbb{N}$. Assume $f\in H^p(T_\Gamma)$ for some $p\in(0,1]$, and $\varphi(\cdot)\,e^{2\pi\alpha\sqrt{n}|\cdot|}\in L^1(\mathbb{R}^n)$, for some $\alpha>0$. Then, for any $q\in[p,1]$, and $r,R\in\mathbb{R}^n_+$ such that $0<r_j<R_j$, $j=1,\dots,n$, and $|R|\le\alpha$, the following inequality holds*

$$\left\|M_\varphi(f)\right\|_{H^q}\le 2^{n\left(\frac{1}{p}+\frac{1}{q}-1\right)}\left(\frac{1}{\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)V_n(\Gamma)\left(\min_{j=1,\dots,n}\left(R_j-r_j\right)\right)^n}\right)^{\frac{1}{p}-1}$$
$$\times\max_{\nu\in\mathcal{V}(r,R)}\|\widehat{\varphi}(\cdot+i\Psi_e\nu)\|_q\,\|f\|_{H^p}. \tag{3.2}$$

Proof For any $y \in [r, R]_n$, $(\Psi_e y)_j = \sum_{k=1}^{n} e_{kj} y_k$. Since $e_1, \ldots, e_n$ are unit vectors, and $y \in [r, R]_n \subset \mathbb{R}_+^n$, applying the Cauchy–Schwartz inequality, we get

$$|\Psi_e y| \le \sqrt{n}\,|y| \le \sqrt{n}\,|R| \le \sqrt{n}\alpha. \tag{3.3}$$

Since $\varphi(\cdot)\,\mathrm{e}^{2\pi\alpha\sqrt{n}|\cdot|} \in L^1(\mathbb{R}^n)$, the function

$$\widehat{\varphi}(\Psi_e(x+iy)) = \int_{\mathbb{R}^n} \varphi(t)\,\mathrm{e}^{2\pi(\Psi_e y,t)}\mathrm{e}^{-2\pi i(\Psi_e x,t)}\,\mathrm{d}t$$

is holomorphic in $T_{(r,R)_n}$ as well as continuous and bounded in $T_{[r,R]_n}$. Since also Ψ_e is a non-singular linear transformation, $\widehat{\varphi}$ is holomorphic in $T_{\Psi_e((r,R)_n)}$, continuous and bounded in $T_{\Psi_e([r,R]_n)}$.

We will also use the fact that if $f \in H^p(T_\Gamma)$ for some p, then, for any $w \in \Gamma$, $f_w \in H^{p_0}(T_\Gamma)$ with any $p_0 \in [p, \infty]$ (see, e,g., [23, Ch. III, Sect. 2, Lem. 2.12]). Hence, $f_w \in H^p(T_\Gamma)$, and using the definition of the Fourier transform (1.1) with $\delta = w$, we have

$$\begin{aligned} M_\varphi(f; x+iw) &= \int_{\Gamma^*} \varphi(t)\,\widehat{f}(t)\,\mathrm{e}^{2\pi i(x+iw,t)}\,\mathrm{d}t = \int_{\Gamma^*} \varphi(t)\,\widehat{f}(t)\,\mathrm{e}^{-2\pi(w,t)}\mathrm{e}^{2\pi i(x,t)}\,\mathrm{d}t \\ &= \int_{\Gamma^*} \varphi(t)\,\widehat{f_w}(t)\,\mathrm{e}^{2\pi i(x,t)}\,\mathrm{d}t = M_\varphi(f_w; x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{3.4}$$

Let us choose an arbitrary $\rho \in (r, R)_n$. Then, (3.3) and the Cauchy-Schwarz inequality imply

$$|\varphi(t)|\,\mathrm{e}^{2\pi(\Psi_e\rho,t)} \le |\varphi(t)|\,\mathrm{e}^{2\pi|\Psi_e\rho||t|} \le |\varphi(t)|\,\mathrm{e}^{2\pi\sqrt{n}\alpha|t|} \in L^1\left(\mathbb{R}^n\right).$$

Using (3.4) and applying Lemma 2.6 with $\delta = \Psi_e\rho$ and $\varphi^* = \varphi$ to the function f_w, we conclude that

$$M_\varphi(f; x+iw) = \int_{\Gamma^*} f_w(x+u+i\Psi_e\rho)\,\widehat{\varphi}(u+i\Psi_e\rho)\,\mathrm{d}u, \quad x \in \mathbb{R}^n.$$

In the following, we will suppose that the maximum in the right-hand side of (3.2) is finite (otherwise, (3.2) is trivial). Under this assumption, we have that

$$\left\|M_\varphi(f; \cdot + iw)\right\|_q^q = \int_{\mathbb{R}^n} \left\|g_{\Psi_e\rho}(w, x; \cdot)\right\|_1^q\,\mathrm{d}x,$$

where $g(w, x; \cdot) := f_w(x+\cdot)\,\widehat{\varphi}(\cdot)$ (recall that $g_\beta(z) = g(z+i\beta)$). If we consider this function as a function of the last argument with fixed x and w, then it obviously satisfies the conditions of Lemma 2.5. Applying this statement with $q = 1$, $p = p$, $f(\cdot) = g(w, x; \cdot)$, we could continue our estimates:

$$\left\| M_\varphi (f; \cdot + iw) \right\|_q^q \le \Theta \int_{\mathbb{R}^n} \max_{\nu \in \mathcal{V}(r,R)} \| g(w, x; \cdot + i\Psi_e \nu) \|_p^q \, \mathrm{d}x,$$

where

$$\Theta := \left(\frac{1}{\pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}+1\right) V_n(\Gamma) \left(\min_{j=1,\ldots,n} \left(\min\left(\rho_j - r_j, R_j - \rho_j\right)\right)\right)^n} \right)^{\frac{q}{p}-q}.$$

Now, let us note that if $F_1, \ldots, F_N \in L^+(X, \mu)$, then

$$\int_X \max_{j=1,\ldots,N} F_j \, \mathrm{d}\mu \le \int_X (F_1 + \cdots + F_N) \, \mathrm{d}\mu \le N \max_{j=1,\ldots,N} \int_X F_j \, \mathrm{d}\mu.$$

Using this fact and changing variables ($x + u = t$), we get

$$\left\| M_\varphi (f; \cdot + iw) \right\|_q^q \le 2^n \Theta \max_{\nu \in \mathcal{V}(r,R)} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} | f_w (t + i\Psi_e \nu) \widehat{\varphi} (t - x + i\Psi_e \nu)|^p \, \mathrm{d}t \right)^{q/p} \mathrm{d}x. \quad (3.5)$$

Since $q/p > 1$, we can employ Minkovskii's integral inequality and obtain:

$$\begin{aligned} &\left\| M_\varphi (f; \cdot + iw) \right\|_q^q \\ &\le 2^n \Theta \max_{\nu \in \mathcal{V}(r,R)} \left(\int_{\mathbb{R}^n} | f_w (t + i\Psi_e \nu)|^p \left(\int_{\mathbb{R}^n} |\widehat{\varphi} (t - x + i\Psi_e \nu)|^q \, \mathrm{d}x \right)^{p/q} \mathrm{d}t \right)^{q/p} \\ &= 2^n \Theta \max_{\nu \in \mathcal{V}(r,R)} \|\widehat{\varphi} (\cdot + i\Psi_e \nu)\|_q^q \, \| f_w (\cdot + i\Psi_e \nu)\|_p^q \\ &\le 2^n \Theta \max_{\nu \in \mathcal{V}(r,R)} \|\widehat{\varphi} (\cdot + i\Psi_e \nu)\|_q^q \, \| f \|_{H^p}^q . \end{aligned}$$

Since the maximum in the right-hand side is assumed finite, taking $\sup_{w \in \Gamma}$, we have

$$\begin{aligned} \left\| M_\varphi (f) \right\|_{H^q} \le 2^{n/q} &\left(\frac{1}{\pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}+1\right) V_n(\Gamma) \left(\min_{j=1,\ldots,n} \left(\min\left(\rho_j - r_j, R_j - \rho_j\right)\right)\right)^n} \right)^{\frac{1}{p}-1} \\ &\times \max_{\nu \in \mathcal{V}(r,R)} \|\widehat{\varphi} (\cdot + i\Psi_e \nu)\|_q \, \| f \|_{H^p} . \end{aligned}$$

Since the left-hand side of this inequality does not depend on ρ, we could take $\rho = \frac{1}{2}(r + R)$, and the last inequality yields (3.2). □

Following [23, Ch. III, Sect. 4], a convex, compact and symmetric with respect to the origin set $K \subset \mathbb{R}^n$ with non-empty interior is called a symmetric body. Its polar set is defined by $K^* = \{t \in \mathbb{R}^n : (x, t) \leq 1, \ \forall x \in K\}$. Let us also set

$$\|z\| := \sup_{t \in K^*} |(z, t)| = \sup_{t \in K^*} |(z_1 t_1 + \cdots + z_n t_n)|.$$

Note that K^* is again a symmetric body, and $(K^*)^* = K$ (see, e.g., [23, Ch. III, Sect. 4, Lem. 4.7]).

It is said that an entire function f defined in $\mathbb{C}^n$ is of exponential type K, where K is a symmetric body, if for any $\varepsilon > 0$ there exists a constant $A_\varepsilon > 0$ such that

$$|f(z)| \leq A_\varepsilon \mathrm{e}^{2\pi(1+\varepsilon)\|z\|}, \quad \forall z \in \mathbb{C}^n.$$

The class of all entire functions of exponential type K is denoted by $\mathcal{E}(K)$.

Proof of Theorem 1.3 Since φ is compactly supported on convex body $K := [-\sigma, \sigma]^n = [-\sigma, \sigma] \times \cdots \times [-\sigma, \sigma]$, then, according to the multivariate Paley–Wiener Theorem [23, Ch. III, Sect. 4, Thm. 4.9],

$$\widehat{\varphi}(z) = \int\limits_{[-\sigma,\sigma]^n} \varphi(t)\, \mathrm{e}^{-2\pi(z,t)}\, \mathrm{d}t$$

is a function of $\mathcal{E}(K^*)$ class. Therefore,

$$|\widehat{\varphi}(z)| \leq A_\varepsilon \mathrm{e}^{2\pi(1+\varepsilon)\|z\|}, \tag{3.6}$$

where $\|z\| = \sup_{y \in K} |z_1 y_1 + \ldots z_n y_n|$. If we fix all the other variables except the jth, then, clearly, the function

$$\Phi_j(\xi) := \widehat{\varphi}\left(z_1, \ldots, z_{j-1}, \xi, z_{j+1}, \ldots, z_n\right)$$

is a univariate entire function of exponential type $2\pi\sigma$.

Applying the Bernstein inequality in L^p-metric (for $p \in (0, 1)$, the result is due to Rahman and Schmeisser [19, Cor. 1]), we get

$$\left\| \frac{\partial \widehat{\varphi}}{\partial x_j} \left(z_1, \ldots, z_{j-1}, \cdot, z_{j+1}, \ldots, z_n\right) \right\|_{L^q(\mathbb{R})} = \left\| \Phi_j' \right\|_{L^q(\mathbb{R})} \leq 2\pi\sigma \left\| \Phi_j \right\|_{L^q(\mathbb{R})}.$$

Thus,

$$\int\limits_{\mathbb{R}^{n-1}} \left(\int\limits_{\mathbb{R}} \left| \frac{\partial \widehat{\varphi}}{\partial x_j} \right|^q \mathrm{d}x_j \right) \mathrm{d}x_1 \ldots \mathrm{d}x_{j-1} \mathrm{d}x_{j+1} \ldots \mathrm{d}x_n \leq (2\pi\sigma)^q \|\widehat{\varphi}\|_q^q < \infty.$$

Applying Tonelli's Theorem, we obtain that $\frac{\partial\widehat{\varphi}}{\partial x_j} \in L^q(\mathbb{R}^n)$, and

$$\left\| \frac{\partial\widehat{\varphi}}{\partial x_j} \right\|_q \le 2\pi\sigma \, \|\widehat{\varphi}\|_q \,, \quad j = 1, \ldots, n. \tag{3.7}$$

Expanding exponential to the Taylor series, we have

$$\mathrm{e}^{2\pi(y,t)} = \sum_{m=0}^{\infty} \frac{(2\pi)^m}{m!} \left(\sum_{j=1}^{n} y_j t_j \right)^m, \quad y, t \in \mathbb{R}^n.$$

The following equality can be easily checked by induction

$$\left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \left\{ \mathrm{e}^{-2\pi i(x,t)} \right\} i^m = \left(\sum_{j=1}^{n} y_j t_j \right)^m \mathrm{e}^{-2\pi i(x,t)} \, (2\pi)^m \,. \tag{3.8}$$

Now,

$$\widehat{\varphi}(x+iy) = \int\limits_{[-\sigma,\sigma]^n} \varphi(t) \sum_{m=0}^{\infty} \frac{(2\pi)^m}{m!} \left(\sum_{j=1}^{n} y_j t_j \right)^m \mathrm{e}^{-2\pi i(x,t)} \, \mathrm{d}t. \tag{3.9}$$

Since

$$\left| \frac{(2\pi)^m}{m!} \left(\sum_{j=1}^{n} y_j t_j \right)^m \right| \le \frac{(2\pi)^m \sigma^m |y|^m n^{m/2}}{m!}, \quad t \in [-\sigma, \sigma]^n \,,$$

the series in the right-hand side of (3.9) converges uniformly (with respect to t) and absolutely on $[-\sigma, \sigma]^n$. Since $\varphi \in L^1([-\sigma, \sigma]^n)$, applying the Lebesgue Dominated Convergence Theorem, we can put the integral sign inside the series. Thus, using (3.8), we get

$$\widehat{\varphi}(x+iy) = \sum_{m=0}^{\infty} \frac{i^m}{m!} \int\limits_{[-\sigma,\sigma]^n} \varphi(t) \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \left\{ \mathrm{e}^{-2\pi i(x,t)} \right\} \mathrm{d}t.$$

Since $\varphi \in C(\mathbb{R}^n)$ and is compactly supported, then $|t|^k \varphi(t) \in L^1(\mathbb{R}^n)$, for any $k \in \mathbb{N}$, and we can take the differentiation operators outside of the integral. Hence,

$$\widehat{\varphi}(x+iy) = \sum_{m=0}^{\infty} \frac{i^m}{m!} \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \widehat{\varphi}(x) \,. \tag{3.10}$$

Now, (3.7) implies

$$\int_{\mathbb{R}^n} \left| \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right) \widehat{\varphi}(x) \right|^q \mathrm{d}x \le \sum_{j=1}^{n} |y_j|^q \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_j} \widehat{\varphi}(x) \right|^q \mathrm{d}x$$
$$\le (2\pi\sigma)^q \sum_{j=1}^{n} |y_j|^q \, \|\widehat{\varphi}\|_q^q .$$

Hence, by induction,

$$\int_{\mathbb{R}^n} \left| \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \widehat{\varphi}(x) \right|^q \mathrm{d}x \le (2\pi\sigma)^{mq} \left(\sum_{j=1}^{n} |y_j|^q \right)^m \|\widehat{\varphi}\|_q^q . \tag{3.11}$$

From (3.10), we obtain

$$|\widehat{\varphi}(x+iy)|^q \le \sum_{m=0}^{\infty} \frac{1}{(m!)^q} \left| \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \widehat{\varphi}(x) \right|^q . \tag{3.12}$$

Considering (3.11),

$$\sum_{m=0}^{\infty} \int_{\mathbb{R}^n} \frac{1}{(m!)^q} \left| \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \widehat{\varphi}(x) \right|^q \mathrm{d}x$$
$$\le \sum_{m=0}^{\infty} \frac{(2\pi\sigma)^{mq}}{(m!)^q} \left(\sum_{j=1}^{n} |y_j|^q \right)^m \|\widehat{\varphi}\|_q^q < \infty.$$

Therefore, the series in the right-hand side of (3.12) converges to a function from $L^1(\mathbb{R}^n)$, and its L^1-norm is (see [5, Ch. 2, Sect. 2.3, Thm. 2.25])

$$\sum_{m=0}^{\infty} \int_{\mathbb{R}^n} \frac{1}{(m!)^q} \left| \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \widehat{\varphi}(x) \right|^q \mathrm{d}x.$$

Now, (3.12) implies that $\widehat{\varphi}(\cdot + iy) \in L^q(\mathbb{R}^n)$, and

$$\|\widehat{\varphi}(\cdot + iy)\|_q \le \left(\sum_{m=0}^{\infty} \frac{(2\pi\sigma)^{mq}}{(m!)^q} \left(\sum_{j=1}^{n} |y_j|^q \right)^m \right)^{\frac{1}{q}} \|\widehat{\varphi}\|_q , \quad y \in \mathbb{R}^n. \tag{3.13}$$

Take

$$\tau := 2\pi\sigma n^{\frac{1}{2}+\frac{1}{q}}, \quad R := \left(\frac{1}{\tau}, \dots, \frac{1}{\tau} \right), \quad r := \left(\frac{\varepsilon}{\tau}, \dots, \frac{\varepsilon}{\tau} \right), \tag{3.14}$$

where ε (0, 1). If $\nu \in \mathcal{V}(r, R)$, then

$$\left|(\Psi_e \nu)_j\right| \leq \frac{\sqrt{n}}{\tau} = \frac{1}{2\pi \sigma n^{\frac{1}{q}}}.$$

Using (3.13) with $y = \Psi_e \nu$, we get

$$\|\widehat{\varphi}(\cdot + i\Psi_e \nu)\|_q \leq \left(\sum_{m=0}^{\infty} \frac{1}{(m!)^q}\right)^{1/q} \|\widehat{\varphi}\|_q .$$

Having applied Proposition 3.2 with r and R as in (3.14), we obtain

$$\left\|M_\varphi(f)\right\|_{H^q} \leq 2^{n\left(\frac{1}{p}+\frac{1}{q}-1\right)} \left(\frac{\left(2\pi \sigma n^{\frac{1}{2}+\frac{1}{q}}\right)^n}{\pi^{n/2}\Gamma\left(\frac{n}{2}+1\right) V_n(\Gamma)(1-\varepsilon)^n}\right)^{\frac{1}{p}-1} \times \left(\sum_{m=0}^{\infty} \frac{1}{(m!)^q}\right)^{\frac{1}{q}} \|f\|_{H^p} .$$

Passing to the limit as $\varepsilon \to 0+$ completes the proof. □

It is clear that if $\varphi \in C^\infty(\mathbb{R}^n)$ and is compactly supported, then it belongs to the Schwartz space $\mathcal{J}$. Applying Theorem 3.2 from [23, Ch. 1, Sect. 3], we get $\widehat{\varphi} \in \mathcal{J}$. Integrating in polar coordinates, we conclude $\widehat{\varphi} \in L^q(\mathbb{R}^n)$, for any $p \in (0, \infty]$. Applying Theorem 1.3, we easily deduce the following result.

Corollary 3.3 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$. If $\varphi \in C^\infty(\mathbb{R}^n)$ and is compactly supported, then $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, for any $0 < p \leq q \leq 1$.*

3.2 Local Property

The following lemma has already been identified as basic. Now, we are ready to present its proof.

Lemma 3.4 (Local Property) *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, and let $0 < p \leq q \leq 1$. Assume a function $\varphi : \Gamma^* \to \mathbb{C}$ has the following property: for any point $t \in \Gamma^*$, including the point at infinity, there exists a neighborhood V_t such that, in $V_t \cap \Gamma^*$, φ coincides with some function $\varphi_t \in \mathcal{M}_{p,q}(T_\Gamma)$. Then $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$.*

Proof Without loss of generality, we will assume that V_t's are open balls $V_t = B_n(t, r_t) = \{x \in \mathbb{R}^n : |x - t| < r_t\}$ of positive radius r_t, and $V_\infty := \{x \in \mathbb{R}^n : |x| > r_\infty\}$.

Since $\Gamma^* \setminus V_\infty$ is a compact in $\mathbb{R}^n$, there exists a finite subcovering of $\Gamma^* \setminus V_\infty$ by V_t's, i.e., $\Gamma^* \setminus V_\infty \subset \bigcup_{k=1}^{m} V_{t_k}$. For simplicity, let us denote $V_{t_{m+1}} := V_\infty$. Then, $\Gamma^* \subset \bigcup_{k=1}^{m+1} V_{t_k}$.

Using, e.g., [17, Ch. 1, Sect. 1.2, Thm. 1.2.3], it is clear that there exists a partition of unity subordinate to the open covering $\left\{V_{t_k}\right\}_{k=1}^{m+1}$ that is a family of C^{∞}-functions $\left\{\zeta_{(t_k)}\right\}_{k=1}^{m+1}$ such that

$$0 \le \zeta_{(t_k)} \le 1, \quad \operatorname{supp}\zeta_{(t_k)} \subset V_{t_k}, \quad k = 1, \dots, m+1,$$

the family $\left\{\operatorname{supp}\zeta_{(t_k)}\right\}$ is locally finite, and

$$\sum_{k=1}^{m+1} \zeta_{(t_k)}(x) = 1, \quad \forall x \in \Gamma^*. \tag{3.15}$$

It is clear that $\zeta_{(\infty)} = \zeta_{(t_{m+1})}$ is equal to 1 on $\Gamma^* \setminus \bigcup_{k=1}^{m} V_{t_k}$. Hence, $\eta_{(\infty)} := 1 - \zeta_{(\infty)}$ is also in $C^{\infty}(\mathbb{R}^n)$ satisfying

$$\eta_{(\infty)}(x) = 0, \quad \forall x \in \Gamma^* \setminus \bigcup_{k=1}^{m} V_{t_k}.$$

Hence, $\eta_{(\infty)}$ is also compactly supported. Corollary 3.3 now implies $\eta_{(\infty)} \in \mathcal{M}_{p,p}(T_\Gamma)$.

Since $\operatorname{supp}\zeta_{(t_k)} \subset V_{t_k}$ and $\varphi = \varphi_{(t_k)}$ on V_{t_k}, for $k = 1, \dots, m+1$, we have

$$\zeta_{(t_k)}(x)\,\varphi(x) = \zeta_{(t_k)}(x)\,\varphi_{(t_k)}(x), \quad x \in \Gamma^*,\ k = 1, \dots, m+1.$$

Multiplying (3.15) by $\varphi(x)$, we get

$$\varphi(x) = \sum_{k=1}^{m+1} \zeta_{(t_k)}(x)\,\varphi_{(t_k)}(x), \quad x \in \Gamma^*. \tag{3.16}$$

This implies that φ is Lebesgue measurable, since all $\varphi_{(t_k)}$ are multipliers, whence measurable, and $\zeta_{(t_k)}$ are continuous.

Since for any $k = 1, \dots, m$, the functions $\zeta_{(t_k)}$ are infinitely differentiable on $\mathbb{R}^n$ and compactly supported, Corollary 3.3 implies that $\zeta_{(t_k)} \in \mathcal{M}_{p,p}(T_\Gamma)$. Hence, using Property (2) of a multiplier, $\zeta_{(t_k)}\varphi_{(t_k)} \in \mathcal{M}_{p,q}(T_\Gamma)$.

Now, $\varphi_{(\infty)}\zeta_{(\infty)} = \varphi_{(\infty)} - \varphi_{(\infty)}\eta_{(\infty)} \in \mathcal{M}_{p,q}(T_\Gamma)$, because $\eta_{(\infty)} \in \mathcal{M}_{p,p}(T_\Gamma)$, and $\varphi_{(\infty)} \in \mathcal{M}_{p,q}(T_\Gamma)$.

Thus, all the summands in (3.16) belong to $\mathcal{M}_{p,q}(T_\Gamma)$, whence $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$. □

3.3 Necessary Conditions

The local property of a multiplier and Theorem 1.3 allow us to get efficient necessary conditions, and even criteria. This is especially useful for radial functions. In particular, we can easily obtain the critical index for Bochner–Riesz means (Proposition 1.7).

To prove Theorem 1.4, we need a couple of lemmas that may be of independent interest.

Lemma 3.5 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, $\varphi \in L^1_{loc}(\Gamma^*)$, and $0 < p \leq q \leq 1$. If $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, and ψ is a compactly supported function such that $\widetilde{\psi}(\cdot) = \widehat{\psi}(-\cdot) \in H^p(T_\Gamma)$, then $\widehat{\varphi\psi} \in L^q(\mathbb{R}^n)$.*

Proof Let us consider the function

$$g(z) := \int_{\Gamma^*} \varphi(t)\,\psi(t)\,e^{2\pi i(z,t)}\,dt, \quad z \in T_\Gamma. \tag{3.17}$$

Since $\widetilde{\psi}(\cdot) \in H^p(T_\Gamma)$, the inversion formula implies $\operatorname{supp}\psi \subset \Gamma^*$. As soon as $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, we also deduce

$$\|g\|_{H^q} \leq \|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} \left\|\widetilde{\psi}\right\|_{H^p}. \tag{3.18}$$

Since $\varphi \in L^1_{\text{loc}}(\Gamma^*)$, and ψ is continuous and compactly supported, then $\varphi\psi \in L^1(\Gamma^*)$. Moreover, $\left|e^{2\pi i(z,t)}\right| \leq 1$, for $z \in T_\Gamma$, $t \in \mathbb{R}^n$. Hence, applying the Lebesgue Dominated Convergence Theorem to (3.17), we obtain

$$\widehat{\varphi\psi}(-x) = g(x) := \lim_{y\to 0,\, y\in\Gamma} g(x+iy) = \int_{\Gamma^*} \varphi(t)\,\psi(t)\,e^{2\pi i(x,t)}\,dt, \quad x \in \mathbb{R}^n.$$

Note that $|g(x)|^q$ is also Lebesgue measurable on $\mathbb{R}^n$ as a limit of Lebesgue measurable functions $|g(x+iy)|^q$. Hence, using Fatou's Lemma and (3.18), we get

$$\left\|\widehat{\varphi\psi}\right\|_q \leq \liminf_{y\to 0,\, y\in\Gamma} \|g(\cdot+iy)\|_q \leq \|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} \left\|\widetilde{\psi}\right\|_{H^p} < \infty.$$

□

Lemma 3.6 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, $\varphi \in L^1_{loc}(\Gamma^*)$, and $0 < p \leq q \leq 1$. If $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, and $\psi \in C^\infty(\mathbb{R}^n)$ is compactly supported with $\operatorname{supp}\psi \subset (\Gamma^*)^o$, then $\widehat{\varphi\psi} \in L^q(\mathbb{R}^n)$.*

Proof Let us consider

$$\widetilde{\psi}(z) = \widetilde{\psi}(x+iy) = \int_{\mathbb{R}^n} \psi(t)\,e^{-2\pi(y,t)}e^{2\pi i(x,t)}\,dt, \quad z = x+iy \in T_\Gamma. \tag{3.19}$$

We need to prove that $\widetilde{\psi} \in H^p(T_\Gamma)$. Since $\psi \in L^2(\mathbb{R}^n)$ and is compactly supported, the Paley–Wiener Theorem implies that $\widetilde{\psi}$ is an entire function of exponential type.

Since ψ is compactly supported, then it is clear that for any $y \in \Gamma$, $x \in \mathbb{R}^n$, we have $\psi(\cdot)\,e^{-2\pi(y,\cdot)}e^{2\pi i(x,\cdot)} \in L^1(\mathbb{R}^n)$. According to Fubini's Theorem, we can choose the order of integration in (3.19) as we need.

If $g \in C^{\infty}(\mathbb{R}^n)$ and is compactly supported, then the Lebesgue integral is, in fact, a Riemann integral, and using integration by parts in the iterated integrals, and applying Leibnitz differentiation formula, we arrive at

$$\int_{\mathbb{R}^n} g(t)\, \mathrm{e}^{-2\pi(y,t)} \mathrm{e}^{2\pi i(x,t)}\, \mathrm{d}t$$

$$= \frac{i^k}{(2\pi)^k x_j^k} \sum_{l=0}^{k} \binom{k}{l} \left(-2\pi y_j\right)^{k-l} \int_{\mathrm{supp}g} \left(\frac{\partial^l}{\partial t_j^l} g(t)\right) \mathrm{e}^{-2\pi(y,t)} \mathrm{e}^{2\pi i(x,t)}\, \mathrm{d}t. \quad (3.20)$$

Since $\mathrm{supp}\psi \subset (\Gamma^*)^o$, then for any $t \in \mathrm{supp}\psi$ and $y \in \overline{\Gamma}$, we have $(y,t) > 0$. Since $\mathrm{supp}\psi$ is compact, then $\inf \left\{(y,t) \mid y \in \overline{\Gamma}, |y| = 1, t \in \mathrm{supp}\psi\right\}$ is attained at some couple, y_0 and t_0. Therefore,

$$a := \min\{(y,t) \mid y \in \Gamma, |y| = 1, t \in \mathrm{supp}\psi\} = (y_0, t_0) > 0,$$

whence

$$(y,t) \geq a\,|y|, \quad y \in \Gamma,\ t \in \mathrm{supp}\psi.$$

Investigating the function $h(\xi) := \xi^m \mathrm{e}^{-2\pi a \xi}$, $m \in \mathbb{Z}_+$ for its supremum on $(0, \infty)$, we find that $h(\xi) \leq \frac{m^m}{(2\pi a)^m} e^{-m}$. Thus, for $y \in \Gamma$, $t \in \mathrm{supp}\psi$, we have

$$\left|y_j\right|^m \mathrm{e}^{-2\pi(y,t)} \leq |y|^m \mathrm{e}^{-2\pi a|y|} \leq \gamma_1(m,a) := \begin{cases} \frac{m^m}{(2\pi a)^m} \mathrm{e}^{-m}, & m \in \mathbb{N}, \\ 1, & m = 0. \end{cases}$$

Now, applying (3.20) to $\psi(\cdot + iy)$, and using the last estimate, we obtain

$$\left|\widetilde{\psi}(x+iy)\right| \leq \frac{\gamma_2(n,k,\psi)}{\left|x_j\right|^k}, \quad x_j \neq 0,\ y \in \Gamma, \quad (3.21)$$

where

$$\gamma_2(n,k,\psi) := \frac{1}{(2\pi)^k} \sum_{l=0}^{k} \binom{k}{l} (2\pi)^{k-l} \gamma_1(k-l,a) \int_{\mathrm{supp}\psi} \left|\frac{\partial^l}{\partial t_j^l} \psi(t)\right| \mathrm{d}t < \infty$$

does not depend on x and y.

Using Hölder's inequality, we also have

$$|x|^{2m} = \left(\sum_{j=1}^{n} x_j^2\right)^m \leq \sum_{j=1}^{n} x_j^{2m} \left(\sum_{j=1}^{n} 1\right)^{1-\frac{1}{m}} = n^{1-\frac{1}{m}} \sum_{j=1}^{n} x_j^{2m}, \quad m \in \mathbb{N}.$$

Hence, from (3.21), we get

$$|x|^{2m}\left|\widetilde{\psi}(x+iy)\right| \leq n^{2-\frac{1}{m}}\gamma_2(n,2m,\psi), \quad x\in\mathbb{R}^n,\ y\in\Gamma,\ m\in\mathbb{N}. \tag{3.22}$$

It is also obvious that

$$\left|\widetilde{\psi}(x+iy)\right| \leq \|\psi\|_1 < \infty, \quad x\in\mathbb{R}^n,\ y\in\Gamma. \tag{3.23}$$

Integrating using polar coordinates and considering (3.22) and (3.23), we easily deduce that $\widetilde{\psi}\in H^p(T_\Gamma)$. Finally, Lemma 3.5 implies that $\widehat{\varphi\psi}\in L^q(\mathbb{R}^n)$. □

Proof of Theorem 1.4 Let us take an arbitrary $x\in(\Gamma^*)^o$ and its bounded neighborhood V_x such that $\overline{V_x}\subset(\Gamma^*)^o$. Let $\psi_{(x)}$ be a function with the following properties:

1. $\psi_{(x)}\in C^\infty(\mathbb{R}^n)$;
2. $\psi_{(x)}$ is compactly supported and $\mathrm{supp}\psi_{(x)}\subset(\Gamma^*)^o$;
3. $\psi_{(x)}\equiv 1$ on $\overline{V_x}$.

To prove that such a function exists, let us first note that since $\mathbb{R}^n$ is a normal topological space, there exists an open set U such that $\overline{V_x}\subset U\subset\overline{U}\subset(\Gamma^*)^o$. Then, [17, Ch. 1, Sect. 1.2, Cor. 1.2.6] guarantees the existence of a function $\psi_{(x)}$ with the desired properties.

Now, the function

$$G(t) := \varphi(t)\,\psi_{(x)}(t)$$

is continuous, compactly supported and coincides with φ on $\overline{V_x}$. Moreover, according to Lemma 3.6, $\widehat{G}\in L^q(\mathbb{R}^n)$, which completes the proof. □

As we can see, the requirement on the Fourier transform of a multiplier to be in $L^q(\mathbb{R}^n)$ is essential. This becomes even more important when we switch to compactly supported radial functions as in Theorem 1.5. To prove this theorem, we need the following statement.

Lemma 3.7 *Let* $\psi\in C^\infty(\mathbb{R}^n)$*,* $n\in\mathbb{N}$*, and be compactly supported. Assume that* $\varphi\in C(\mathbb{R}^n)$*, is also compactly supported and* $\widehat{\varphi}\in L^q(\mathbb{R}^n)$*, for some* $q\in(0,1]$*. Then,* $\widehat{\psi\varphi}\in L^q(\mathbb{R}^n)$.

Proof Since ψ is compactly supported, then there exists $R>0$ such that $\mathrm{supp}\psi\subset B(0,R)$. Take $a:=(R,\ldots,R)\in\left(\mathbb{R}^n_+\right)^o$. Then the function

$$\tau_a\psi(x)=\psi(x-a), \quad x\in\mathbb{R}^n,$$

also belongs to $C^\infty(\mathbb{R}^n)$, and $\mathrm{supp}\tau_a\psi\subset\left(\mathbb{R}^n_+\right)^o$. Obviously, $\tau_a\varphi$ is also continuous and compactly supported.

Since $\varphi\in L^1(\mathbb{R}^n)$, using the property of the Fourier transform of a translation, we get $\widehat{\tau_a\varphi}(x)=\mathrm{e}^{-2\pi i(a,x)}\widehat{\varphi}(x)$, and hence $\|\widehat{\tau_a\varphi}\|_q=\|\widehat{\varphi}\|_q<\infty$.

According to Theorem 1.3, $\tau_h\varphi \in \mathcal{M}_{p,q}\left(T_{(\mathbb{R}^n_+)^o}\right)$, for any $p \in (0, q]$. Now, Lemma 3.6 applied to $\tau_a\varphi$, $\tau_a\psi$ and the cone $\left(\mathbb{R}^n_+\right)^o$ implies $\widehat{\tau_a(\psi\varphi)} \in L^q(\mathbb{R}^n)$. Hence, $\widehat{\psi\varphi} \in L^q(\mathbb{R}^n)$. □

Proof of Theorem 1.5 Let us take an arbitrary $x \in \mathbb{R}^n$, $x \neq 0$. Since $(\Gamma^*)^o \neq \emptyset$, there exists a rotation T such that $Tx \in (\Gamma^*)^o$.

Since $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, according to Theorem 1.4, in any closed ball $\overline{B(Tx, r)} \subset (\Gamma^*)^o$, the function φ coincides with some continuous compactly supported $\varphi_{(Tx)}$ such that $\widehat{\varphi_{(Tx)}} \in L^q(\mathbb{R}^n)$.

Since T is a rotation, T maps $B(x, r)$ onto $B(Tx, r)$, and considering that φ is radial and T preserves the norm in $\mathbb{R}^n$, we have

$$\varphi(\xi) = \varphi(T\xi) = \varphi_{(Tx)}(T\xi), \quad \xi \in B(x, r).$$

Since the Fourier transform commutes with rotation, $\left(\varphi_{(Tx)} \circ T\right)^{\widehat{}} \in L^q(\mathbb{R}^n)$. Thus, in some open ball $B(t, r)$ of any point $t \in \mathbb{R}^n$ (the condition on the origin is given explicitly), φ coincides with some function $\varphi_{(t)}$ that is continuous, compactly supported and $\widehat{\varphi_{(t)}} \in L^q(\mathbb{R}^n)$.

Since suppφ is a compact set in $\mathbb{R}^n$, we can choose a finite number of the balls under consideration so that

$$\operatorname{supp}\varphi \subset \bigcup_{k=0}^{m} B(t_k, r_k).$$

Let us denote $B_k := B(t_k, r_k)$, $k = 0, \dots, m$, and let $B_{m+1} := \mathbb{R}^n \backslash \operatorname{supp}\varphi$. Thus, $\bigcup_{k=0}^{m+1} B_k$ is an open covering of $\mathbb{R}^n$.

According to [17, Ch. 1, Sect. 1.2, Thm. 1.2.3], for the open set $\bigcup_{k=0}^{m} B_k$, there exists a partition of unity subordinate to $\{B_k\}_{k=0}^{m+1}$ that is a family of C^∞-functions $\left\{\zeta_{(k)}\right\}_{k=0}^{m+1}$ such that

$$0 \le \zeta_{(k)} \le 1, \quad \operatorname{supp}\zeta_{(k)} \subset B_k, \quad k = 0, \dots, m+1,$$

the family $\left\{\operatorname{supp}\zeta_{(k)}\right\}$ is locally finite, and

$$\sum_{k=0}^{m+1} \zeta_{(k)}(x) = 1, \quad x \in \bigcup_{k=0}^{m} B_k.$$

Multiplying both sides by $\varphi(x)$ and considering that $\operatorname{supp}\zeta_{(m+1)} \subset B_{m+1}$, and $\varphi \equiv 0$ in B_{m+1}, we obtain

$$\varphi(x) = \sum_{k=0}^{m} \zeta_{(k)}(x)\,\varphi(x) = \sum_{k=0}^{m} \zeta_{(k)}(x)\,\varphi_{(t_k)}(x), \quad x \in \mathbb{R}^n. \tag{3.24}$$

Lemma 3.7 implies $\widehat{\zeta_{(k)}\varphi_{(t_k)}} \in L^q(\mathbb{R}^n)$, $k = 0, \ldots, m$. Hence, (3.24) yields $\widehat{\varphi} \in L^q(\mathbb{R}^n)$. □

From Theorems 1.3 and 1.5, we easily obtain

Corollary 3.8 *Let* $\varphi : \mathbb{R}^n \to \mathbb{C}$, $n \in \mathbb{N}$, *be a continuous compactly supported radial function. Assume that in some neighborhood of the origin,* φ *belongs to* $C^\infty(\mathbb{R}^n)$*-class. Then, for any* $0 < p \le q \le 1$ *and any regular cone* $\Gamma \subset \mathbb{R}^n$, $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$ *if and only if* $\widehat{\varphi} \in L^q(\mathbb{R}^n)$.

3.4 Sufficient Conditions Involving Growth of Partial Derivatives

Proof of Theorem 1.6 Our proof is very similar to [27, Proof of Thm. 3].

Proof of (b). It is clear that $\varphi \in L^2(\mathbb{R}^n)$. If $K := \left[-\frac{\sigma^{1/n}}{2}, \frac{\sigma^{1/n}}{2}\right]_n$, then the estimate (2.6), Paley–Wiener's and Plancherel's Theorems imply that

$$\inf\left\{\|\varphi - \psi\|_2 : \psi \in \mathcal{E}\left(K^*\right)\right\} = \inf\left\{\left\|\widehat{\varphi} - \widehat{\psi}\right\|_2 : \psi \in \mathcal{E}\left(K^*\right)\right\} \ge a_\sigma(\varphi)_2 .$$

Applying the direct theorem on approximation by entire functions of exponential type [18, Ch. 5, Sect. 5.2, Thm. 5.2.4 (see 5)], we obtain

$$a_\sigma(\varphi)_2 \le \frac{\gamma_0(s,n)}{\sigma^{s/n}} \max_{j=1,\ldots,n} \omega_2\left(\frac{\partial^s\varphi}{\partial x_j^s}; \frac{1}{\sigma^{1/n}}\right)_{2,j}, \tag{3.25}$$

where $\omega_2(g,h)_{2,j}$ denotes the partial (on jth variable) modulus of smoothness of g with the step h in $L^2(\mathbb{R}^n)$-norm.

Lemma 6 from [26] asserts that if g is bounded and piecewise convex function on $\mathbb{R}^n$, then for any $h > 0$ and $p \ge 1$, $\left\|\Delta_h^2 g\right\|_p \le M h^{1/p}\omega(g;h)_\infty$, where $\Delta_h^2 g$ is the forward difference of second order and step h (i.e., $\Delta_h^2 g(x) = g(x+2h) - 2g(x+h) + g(x)$), M depends only on the number of points dividing the intervals on which g is convex. In fact, the proof of this lemma only requires g to be convex or concave on each of the intervals, i.e., it may be convex on some of them and concave on the others.

Under our assumptions, we can apply the lemma with $p = 2$, and obtain

$$\omega_2\left(\frac{\partial^s\varphi}{\partial x_j^s}; h\right)_{2,j} \le M h^{\frac{1}{2}} \omega\left(\frac{\partial^s\varphi}{\partial x_j^s}; h\right)_\infty \le M C h^{\frac{1}{2}+\alpha},$$

where

$$C := \max_{j=1,\ldots,n} \sup_{t_j \ne 0} \sup_{x \in \mathbb{R}^n} \frac{\left|\frac{\partial^s\varphi}{\partial x_j^s}(x_1,\ldots,x_n) - \frac{\partial^s\varphi}{\partial x_j^s}\left(x_1,\ldots,x_{j-1},x_j+t_j,x_{j+1},\ldots,x_n\right)\right|}{\left|t_j\right|^\alpha},$$

which is finite according to the assumption of our theorem. Therefore, (3.25) implies

$$\int_1^\infty \left(\frac{a_\sigma(\varphi)_2}{\sqrt{\sigma}}\right)^q d\sigma \le (\gamma_0(s,n)MC)^q \int_1^\infty \left(\frac{1}{\sigma^{s/n}\sigma^{(\alpha+1/2)/n}\sigma^{1/2}}\right)^q d\sigma.$$

Since $\alpha > \frac{n}{q} - \frac{n+1}{2} - s$, then $s/n + (\alpha+1/2)/n + 1/2 > 1/q$ and the last integral is finite. Also considering that $a_\sigma(\varphi)_2 \le \|\varphi\|_2$, and applying Corollary 2.9, we obtain

$$\|\widehat{\varphi}\|_q^q \le 2\int_0^1 \frac{\|\varphi\|_2^q}{\sigma^{q/2}} d\sigma + 2\int_1^\infty \left(\frac{a_\sigma(\varphi)_2}{\sqrt{\sigma}}\right)^q d\sigma$$
$$\le 2\|\varphi\|_2^q \frac{1}{1-q/2} + 2(\gamma_0(s,n)MC)^q \int_1^\infty \left(\frac{1}{\sigma^{s/n}\sigma^{(\alpha+1/2)/n}\sigma^{1/2}}\right)^q d\sigma < \infty.$$

Now, an application of Theorem 1.3 completes the proof.

Proof of (a). Let us show that if φ and all $\frac{\partial^r\varphi}{\partial x_j^r}$, $j = 1, \dots, n$, belong to $L^2(\mathbb{R}^n)$ with some $r > n\left(\frac{1}{q} - \frac{1}{2}\right)$, then there exists some constant $\gamma_1(r,q,n)$ such that

$$\|\widehat{\varphi}\|_q^q \le \gamma_1(r,q,n)\|\varphi\|_2^{q-\frac{n}{r}(1-\frac{q}{2})} \max_{j=1,\dots,n} \left\|\frac{\partial^r\varphi}{\partial x_j^r}\right\|_2^{\frac{n}{r}(1-\frac{q}{2})}. \tag{3.26}$$

Indeed, (3.25) implies $a_\sigma(\varphi)_2 \le \frac{4\gamma_0(r,n)}{\sigma^{r/n}} \max_{j=1,\dots,n}\left\|\frac{\partial^r\varphi}{\partial x_j^r}\right\|_2$. Choosing σ_0 so that $\|\varphi\|_2\sigma_0^{r/n} = \max_{j=1,\dots,n}\left\|\frac{\partial^r\varphi}{\partial x_j^r}\right\|_2$, applying the last inequality, employing the condition $r > n\left(\frac{1}{q} - \frac{1}{2}\right)$, i.e., $q\left(\frac{1}{2} + \frac{r}{n}\right) > 1$, and considering that $a_\sigma(\varphi)_2 \le \|\varphi\|_2$, we get

$$\int_0^\infty \left(\frac{a_\sigma(\varphi)_2}{\sqrt{\sigma}}\right)^q d\sigma \le \|\varphi\|_2^q \int_0^{\sigma_0} \frac{d\sigma}{\sigma^{q/2}} + (4\gamma_0(r,n))^q \max_{j=1,\dots,n}\left\|\frac{\partial^r\varphi}{\partial x_j^r}\right\|_2^q \int_{\sigma_0}^\infty \frac{d\sigma}{\sigma^{q/2+rq/n}} =$$

$$= \left(\frac{1}{1-\frac{q}{2}} + \frac{(4\gamma_0(r,n))^q}{\frac{q}{2}+\frac{rq}{n}-1}\right)\|\varphi\|_2^{q-\frac{n}{r}(1-\frac{q}{2})} \max_{j=1,\dots,n}\left\|\frac{\partial^r\varphi}{\partial x_j^r}\right\|_2^{\frac{n}{r}(1-\frac{q}{2})}.$$

Now, Corollary 2.9 implies (3.26) immediately with

$$\gamma_1(r,q,n) := 2\left(\frac{1}{1-\frac{q}{2}} + \frac{(4\gamma_0(r,n))^q}{\frac{q}{2}+\frac{rq}{n}-1}\right).$$

Let us consider the following partition of unity. Take an arbitrary function $h_{(0)} \in C^{\infty}(\mathbb{R})$ satisfying the following three conditions: (i) $h_{(0)}(t) = 0$ for $t \le -1/2$; (ii) $\left\| h_{(0)} \right\|_{\infty} = 1$; (iii) $h_{(0)}(t) + h_{(0)}(-t) \equiv 1$, i.e., $h_{(0)} - 1/2$ is odd. For $\nu \in \mathbb{N}$, we also set

$$h_{(\nu)}(t) := h_{(0)}\left(\frac{t+1}{2^{\nu-1}} - \frac{3}{2}\right) h_{(0)}\left(\frac{3}{2} - \frac{t+1}{2^{\nu}}\right).$$

It is clear that $\operatorname{supp} h_{(\nu)} \subset \left[2^{\nu-1} - 1, 2^{\nu+1} - 1\right]$. Using the Leibniz differentiation formula, we get

$$\left| h_{(\nu)}^{(s)}(t) \right| \le \frac{3^s}{2^{\nu s}} \max_{k=0,\ldots,s} \left\| h_{(0)}^{(k)} \right\|_{\infty}^2, \quad \nu \in \mathbb{N},\ s \in \mathbb{Z}_+,\ t \in \mathbb{R}. \tag{3.27}$$

Let us also observe that

$$h_{(0)}\left(\frac{1}{2} - t\right) + \sum_{\nu=1}^{\infty} h_{(\nu)}(t) = 1, \quad t \ge 0. \tag{3.28}$$

Therefore, considering $\varphi_{(0)}(x) := \varphi(x) h_{(0)}\left(\frac{1}{2} - |x|^2\right)$, $\varphi_{(\nu)}(x) := \varphi(x) h_{(\nu)}\left(|x|^2\right)$ $x \in \mathbb{R}^n$, $\nu \in \mathbb{N}$, we obtain the following decomposition

$$\varphi(x) = \sum_{\nu=0}^{\infty} \varphi_{(\nu)}(x), \quad x \in \mathbb{R}^n. \tag{3.29}$$

Obviously,

$$\begin{aligned} &\operatorname{supp} \varphi_{(\nu)} \subset \left\{ x \in \mathbb{R}^n : 2^{\nu-1} - 1 \le |x|^2 \le 2^{\nu+1} - 1 \right\}, \quad \nu \in \mathbb{N}, \\ &\operatorname{supp} \varphi_{(0)} \subset \overline{B(0,1)}. \end{aligned} \tag{3.30}$$

It is also clear that the series in (3.29) converges absolutely (for any x, it is a finite sum) to $|\varphi(x)|$ that is bounded on $\mathbb{R}^n$ since φ is continuous and compactly supported.

If all the $\varphi_{(\nu)}$ belong to $\mathcal{M}_{p,q}(T_\Gamma)$, then Proposition 3.1 implies

$$\|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)}^q \le \sum_{\nu=0}^{\infty} \left\| \varphi_{(\nu)} \right\|_{\mathcal{M}_{p,q}(T_\Gamma)}^q, \tag{3.31}$$

whereas the series in the right-hand side of this inequality converges. To prove that, we need to estimate the norms $\left\| \widehat{\varphi}_{(\nu)} \right\|_q$.

Since $\alpha \ge 0$, the condition on the growth of φ yields $|\varphi(x)| \le \frac{A}{1+|x|^\alpha} \le \frac{A}{2^{-\alpha}\left(\sqrt{2}\right)^{\nu\alpha}}$. Using (3.30), we also get

$$m\left(\operatorname{supp}\varphi_{(\nu)}\right)=\begin{cases}\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}\left(\left(2^{\nu+1}-1\right)^{\frac{n}{2}}-\left(2^{\nu-1}-1\right)^{\frac{n}{2}}\right), & \nu\in\mathbb{N}, \leq 2^{\frac{\nu n}{2}}\frac{2^{\frac{n}{2}}\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.\\ \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}, & \nu=0\end{cases} \tag{3.32}$$

Hence, for $\nu\in\mathbb{Z}_+$,

$$\begin{aligned}\left\|\varphi_{(\nu)}\right\|_2 &\leq \max_{x\in\operatorname{supp}\varphi_{(\nu)}}|\varphi(x)|\left(m\left(\operatorname{supp}\varphi_{(\nu)}\right)\right)^{\frac{1}{2}}\left\|h_{(\nu)}\right\|_\infty\\ &\leq \frac{\left(\sqrt{2}\right)^{\frac{n}{2}}\pi^{\frac{n}{4}}}{2^{-\alpha}\left(\Gamma\left(\frac{n}{2}+1\right)\right)^{\frac{1}{2}}}\frac{A}{\left(\sqrt{2}\right)^{\nu\alpha}}\left(\sqrt{2}\right)^{\frac{\nu n}{2}}.\end{aligned} \tag{3.33}$$

Applying Faá di Bruno's formula for derivatives of a composition (see, e.g., [10, 11]), we have

$$\frac{\partial^s}{\partial x_j^s}\left(h_{(\nu)}\left(|x|^2\right)\right)=\sum_{k_1+2k_2=s;\, k_1,k_2\in\mathbb{Z}_+}\frac{s!}{k_1!k_2!}h_{(\nu)}^{(k_1+k_2)}\left(|x|^2\right)\left(2x_j\right)^{k_1}.$$

Since $h_{(\nu)}\left(|x|^2\right)\equiv 0$ when $|x|\geq\left(\sqrt{2}\right)^{\nu+1}\geq\sqrt{2^{\nu+1}-1}$, considering also (3.27), we get

$$\left|\frac{\partial^s}{\partial x_j^s}\left(h_{(\nu)}\left(|x|^2\right)\right)\right|\leq\frac{\gamma_2\left(s,h_{(0)}\right)}{\left(\sqrt{2}\right)^{\nu s}},\quad x\in\mathbb{R}^n,\ \nu\in\mathbb{Z}_+, \tag{3.34}$$

where

$$\gamma_2\left(s,h_{(0)}\right):=\max_{l=0,\dots,s}\left\|h_{(0)}^{(l)}\right\|_\infty^2\sum_{k_1+2k_2=s;\, k_1,k_2\in\mathbb{Z}_+}\frac{s!}{k_1!k_2!}3^{(k_1+k_2)}\left(2\sqrt{2}\right)^{k_1}.$$

Applying the Leibniz rule for differentiation of a product, from (3.34), we derive

$$\left|\frac{\partial^r}{\partial x_j^r}\varphi_{(\nu)}(x)\right|\leq 2^r\max_{k=0,\dots,r}\left(\frac{\left|\frac{\partial^k}{\partial x_j^k}\varphi(x)\right|\gamma_2\left(r-k,h_{(0)}\right)}{\left(\sqrt{2}\right)^{\nu(r-k)}}\right),$$
$$x\in\mathbb{R}^n,\ r,\nu\in\mathbb{Z}_+,\ j=1,\dots,n.$$

Now, Corollary 2.11 implies

$$\left|\frac{\partial^r}{\partial x_j^r}\varphi_{(\nu)}(x)\right| \leq 2^r \max_{k=0,\dots,r}\left(\left(\frac{A}{1+|x|^\alpha}\right)^{1-\frac{k}{r}}\left(\frac{B}{1+|x|^\beta}\right)^{\frac{k}{r}}\frac{\gamma_2\left(r-k,h_{(0)}\right)}{\left(\sqrt{2}\right)^{\nu(r-k)}}\right)$$
$$\times\max\left(\max_{k=1,\dots,r-1}\left(\frac{C_0 r}{k}\right)^k,1\right),\quad x\in\mathbb{R}^n,\ r,\nu\in\mathbb{Z}_+,\ j=1,\dots,n.$$

Since (3.30), if $a \geq 0$, then $x \in \operatorname{supp}\varphi_{(\nu)}$ implies $1+|x|^a \geq 2^{-a}\left(\sqrt{2}\right)^{\nu a}$. Using the last inequality, and

$$\max_{k=0,\dots,r}\frac{1}{\left(\sqrt{2}\right)^{\nu\alpha(1-k/r)+\nu\beta k/r+\nu(r-k)}} \leq \frac{1}{\left(\sqrt{2}\right)^{\nu(\alpha+r)}}+\frac{1}{\left(\sqrt{2}\right)^{\nu\beta}},$$

it is easy to conclude:

$$\left|\frac{\partial^r}{\partial x_j^r}\varphi_{(\nu)}(x)\right| \leq \gamma_3\left(\alpha,\beta,r,h_{(0)}\right)(A+B)\left(\frac{1}{\left(\sqrt{2}\right)^{\nu(\alpha+r)}}+\frac{1}{\left(\sqrt{2}\right)^{\nu\beta}}\right),\quad x\in\mathbb{R}^n,$$

where $r,\nu\in\mathbb{Z}_+$, $j=1,\dots,n$, and

$$\gamma_3\left(\alpha,\beta,r,h_{(0)}\right) := 2^r\left(2^\alpha+2^\beta\right)\max\left(\max_{k=1,\dots,r-1}\left(\frac{C_0 r}{k}\right)^k,1\right)\max_{k=0,\dots,r}\gamma_2\left(k,h_{(0)}\right).$$

Similarly to (3.33), considering (3.32), for any $\nu\in\mathbb{Z}_+$ and $j=1,\dots,n$, we obtain

$$\left\|\frac{\partial^r}{\partial x_j^r}\varphi_{(\nu)}\right\|_2 \leq \gamma_3\left(\alpha,\beta,r,h_{(0)}\right)\frac{\left(\sqrt{2}\right)^{\frac{n}{2}}\pi^{\frac{n}{4}}}{\left(\Gamma\left(\frac{n}{2}+1\right)\right)^{\frac{1}{2}}}(A+B)\left(\frac{1}{\left(\sqrt{2}\right)^{\nu(\alpha+r)}}+\frac{1}{\left(\sqrt{2}\right)^{\nu\beta}}\right)$$
$$\times\left(\sqrt{2}\right)^{\frac{\nu n}{2}}. \tag{3.35}$$

From (3.26), (3.33) and (3.35), we get

$$\left\|\widehat{\varphi}_{(\nu)}\right\|_q^q \leq \gamma_4\left(\alpha,\beta,n,r,q,h_{(0)}\right)(A+B)^q\left(\sqrt{2}\right)^{\frac{\nu n q}{2}-\nu\alpha q}$$
$$\times\left(\frac{1}{\left(\sqrt{2}\right)^{\nu r}}+\frac{1}{\left(\sqrt{2}\right)^{\nu(\beta-\alpha)}}\right)^{\frac{n}{r}\left(1-\frac{q}{2}\right)},$$

where

$$\gamma_4\left(\alpha,\beta,n,r,q,h_{(0)}\right) := \gamma_1(r,q,n)\left(\gamma_3\left(\alpha,\beta,r,h_{(0)}\right)\right)^{\frac{n}{r}\left(1-\frac{q}{2}\right)} \times\left(\frac{\left(\sqrt{2}\right)^{\frac{n}{2}}\pi^{\frac{n}{4}}}{\sqrt{\Gamma\left(\frac{n}{2}+1\right)}}\right)^{q} 2^{\alpha\left(q-\frac{n}{r}\left(1-\frac{q}{2}\right)\right)}.$$

Applying Theorem 1.3 with $\sigma=\sqrt{2^{\nu+1}-1}$, we obtain that $\varphi_{(\nu)}\in\mathcal{M}_{p,q}(T_\Gamma)$, and

$$\begin{aligned}\left\|\varphi_{(\nu)}\right\|^q_{\mathcal{M}_{p,q}(T_\Gamma)} &\le \frac{\gamma_5(n,p,q)}{(V_n(\Gamma))^{q\left(\frac{1}{p}-1\right)}}\left(\sqrt{2^{\nu+1}-1}\right)^{nq\left(\frac{1}{p}-1\right)}\left\|\widehat{\varphi}_{(\nu)}\right\|^q_q\\ &\le \frac{\gamma_5(n,p,q)\,\gamma_4\left(\alpha,\beta,n,r,q,h_{(0)}\right)\left(\sqrt{2}\right)^{nq\left(\frac{1}{p}-1\right)}}{(V_n(\Gamma))^{q\left(\frac{1}{p}-1\right)}}(A+B)^q\left(\sqrt{2}\right)^{\frac{\nu nq}{p}-\nu\alpha q-\nu n}\\ &\quad\times\left(1+\frac{1}{\left(\sqrt{2}\right)^{\nu(\beta-\alpha-r)}}\right)^{\frac{n}{r}\left(1-\frac{q}{2}\right)},\end{aligned}$$

where γ_5 is the constant from the estimate in Theorem 1.3. Since the series

$$\sum_{\nu=0}^{\infty}\left(\sqrt{2}\right)^{\frac{\nu nq}{p}-\nu\alpha q-\nu n}\left(1+\frac{1}{\left(\sqrt{2}\right)^{\nu(\beta-\alpha-r)}}\right)^{\frac{n}{r}\left(1-\frac{q}{2}\right)} \tag{3.36}$$

converges if and only if

$$\min(\beta-\alpha-r,0) > \frac{2rq}{2-q}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{2qr\alpha}{n(2-q)},$$

considering (3.31) and fixing some $h_{(0)}$ satisfying aforementioned conditions, we conclude that $\varphi\in\mathcal{M}_{p,q}(T_\Gamma)$, and

$$\|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} \le \frac{\gamma(n,p,q,r,\alpha,\beta)}{(V_n(\Gamma))^{\frac{1}{p}-1}}(A+B),$$

with $\gamma(n,p,q,r,\alpha,\beta) := \left(\sqrt{2}\right)^{n\left(\frac{1}{p}-1\right)}\Bigg(\gamma_5(n,p,q)\,\gamma_4\left(\alpha,\beta,n,r,q,h_{(0)}\right)$

$$\times \sum_{\nu=0}^{\infty} \left(\sqrt{2}\right)^{\frac{\nu n q}{p} - \nu\alpha q - \nu n} \left(1 + \frac{1}{\left(\sqrt{2}\right)^{\nu(\beta-\alpha-r)}} \right)^{\frac{n}{r}\left(1-\frac{q}{2}\right)} \Bigg)^{\frac{1}{q}}.$$

□

3.5 Bochner–Riesz Means

Applying Theorem 1.6 (b), it is easy to show that the Bocher–Riesz means of the Fourier integral belongs to $\mathcal{M}_{p,q}(T_\Gamma)$ under the assumptions of Proposition 1.7. However, we will give a more elegant proof of this statement based only on Theorem 1.5 and some known estimates.

Proof of Proposition 1.7 Let us show that for any $r \in \mathbb{N}$, the function

$$\varphi_{r,\alpha}(x) := \left(1 - |x|^{2r}\right)_+^{\alpha} = \begin{cases} \left(1 - |x|^{2r}\right)^{\alpha}, & |x| \le 1, \\ 0, & |x| > 1, \end{cases}$$

belongs to $\mathcal{M}_{p,q}(T_\Gamma)$ if and only if $\varphi_{2,\alpha} \in \mathcal{M}_{p,q}(T_\Gamma)$. Indeed, it is easy to check by induction that

$$\left(1 - |x|^{2r}\right)_+ = \left(1 - |x|^{2}\right)_+ \sum_{j=0}^{r-1} |x|^{2j}. \tag{3.37}$$

A similar equality was used in [21, Proof of Thm. 2]. Having taken some $h \in C^\infty(\mathbb{R}^n)$ so that $h \equiv 1$ in $\overline{B(0,1)}$ and $h \equiv 0$ outside of $B(0,2)$ (such a function exists due to [17, Ch. 1, Sect. 1.2, Cor. 1.2.6]), the Eq. (3.37) implies that

$$\varphi_{r,\alpha}(x) = \varphi_{2,\alpha}(x)\,\zeta(x), \quad x \in \mathbb{R}^n,$$

where

$$\zeta(x) := \left(\sum_{j=0}^{r-1} |x|^{2j}\right)^{\alpha} h(x).$$

Obviously, $\zeta \in C^\infty(\mathbb{R}^n)$ and is compactly supported. According to Corollary 3.3, $\zeta \in \mathcal{M}_{p,p}(T_\Gamma)$, for any $0 < p \le 1$, and any regular cone $\Gamma \subset \mathbb{R}^n$. If $\varphi_{2,\alpha} \in \mathcal{M}_{p,q}(T_\Gamma)$, then Property (2) of a multiplier yields $\varphi_{r,\alpha} \in \mathcal{M}_{p,q}(T_\Gamma)$.

On the other hand, (3.37) also implies

$$\varphi_{2,\alpha}(x) = \varphi_{r,\alpha}(x)\,\eta(x), \quad x \in \mathbb{R}^n,$$

where

$$\eta(x) := \left(\sum_{j=0}^{r-1} |x|^{2j} \right)^{-\alpha} h(x).$$

Using the same reasonings, $\varphi_{r,\alpha} \in \mathcal{M}_{p,q}(T_\Gamma)$ implies $\varphi_{2,\alpha} \in \mathcal{M}_{p,q}(T_\Gamma)$.

Now, since $\varphi_{2,\alpha}$ is radial and belongs to $C^\infty(B(0,1))$, according to Corollary 3.8, $\varphi_{2,\alpha} \in \mathcal{M}_{p,q}(T_\Gamma)$ if and only if its Fourier transform belongs to $L^q(\mathbb{R}^n)$.

As shown in [6, App. B.5], for any $\alpha > 0$,

$$\widehat{\varphi_{2,\alpha}}(t) = \frac{\Gamma(\alpha+1)}{\pi^\alpha |t|^{n/2+\alpha}} J_{n/2+\alpha}(2\pi|t|),$$

where J_ν is the Bessel function. The asymptotic behavior of J_ν is also well-known. Lemma 3.11 from [23, Ch. IV, Sect. 3] asserts that

$$J_\nu(s) = \sqrt{\frac{2}{\pi s}} \cos\left(s - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{s^{3/2}}\right), \quad s \to \infty.$$

Thus,

$$\widehat{\varphi_{2,\alpha}}(t) = \frac{\Gamma(\alpha+1)}{\pi^{\alpha+1}|t|^{n/2+\alpha+1/2}} \cos\left(2\pi|t| - \frac{\pi n}{4} - \frac{\pi\alpha}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{|t|^{n/2+\alpha+3/2}}\right), \quad |t| \to \infty.$$

Therefore, it is clear that $\widehat{\varphi_{2,\alpha}} \in L^q(\mathbb{R}^n)$ if and only if $n/2 + \alpha + 1/2 > n/q$. □

From Proposition 1.7 and Corollary 3.8, the following statement follows immediately.

Corollary 3.9 *Let $\alpha > 0$, $r, n \in \mathbb{N}$, and $q \in (0, 1]$. The Fourier transform of the function $\varphi_{r,\alpha}(x) = \left(1 - |x|^{2r}\right)_+^\alpha$ belongs to $L^q(\mathbb{R}^n)$ if and only if $\alpha > \frac{n}{q} - \frac{n+1}{2}$.*

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