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Normal Families: a Geometric Perspective

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Abstract In this largely expository paper we present an alternative to the common practice of discussing normal families of analytic maps in terms of the Euclidean metric and equicontinuity. Indeed, in most cases the hyperbolic metric and the Schwarz–Pick Lemma are available, and then equicontinuity is redundant and is replaced by a much stronger Lipschitz condition that is expressed in terms of conformally invariant metrics. Here, we discuss normal families in terms of (not necessarily analytic) maps that satisfy types of uniform Lipschitz conditions with respect to various conformal metrics, especially the hyperbolic and spherical metrics. A number of classical results for normal families of analytic maps extend to these broader classes of (not necessarily analytic) functions that satisfy types of uniform Lipschitz conditions.

Keywords Normal families · Schwarz–Pick lemma · Lipschitz functions · Conformal metrics

Mathematics Subject Classification (2000) 30 · 30D45 · 30F45 · 30C80 · 30G30

To the memory of our friend, Fred W. Gehring, for his many important contributions to mathematics.

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1 Introduction

We begin with an overview of the early history of normal families. One of the earliest results on the uniform convergence of a sequence of holomorphic functions is attributed to Weierstrass: if $f_1, f_2, \ldots$ are holomorphic in a plane region Δ, and if $f_n \to f$ uniformly on Δ, then f is holomorphic in Δ. We refer the reader to [30] for a comprehensive survey of, and references to, Weierstrass' work. The next major advance towards a theory of normal families was perhaps that of Stieltjes who, in 1894 in a paper on continued fractions [36], proved that if a sequence of holomorphic functions is uniformly bounded in a plane region Δ, and converges uniformly on some non-empty, open subset of Δ, then it converges, uniformly on each compact subset of Δ, to a function holomorphic in Δ. This may have been the first result in which convergence is obtained on a larger region than that covered by the given hypotheses. Next, in 1901 Osgood showed that for uniformly bounded sequences it is sufficient to assume that convergence occurs on a dense subset of Δ [26] (see also [4]). In 1904 Porter [28] showed that it is sufficient for convergence to occur on a curve in Δ, and then later, Vitali [38,39] and, independently, Porter [29], proved, again for uniformly bounded functions, that it is sufficient for the functions to converge on a sequence of points that converges to a point in Δ (see also [16,18]). Note that these results imply that a uniformly bounded family of functions, each holomorphic in a plane region Δ, is (according to the terminology introduced later by Montel) a normal family. Before this, Arzelà and Ascoli had been studying the convergence of real functions. In 1883 [5] Ascoli introduced the concept of an equicontinuous family of functions (although not in the context of complex analysis), and in 1895 Arzelà [3] proved the prototype for what is now known as the Arzelà–Ascoli theorem.

All of these results were established before Montel published his fundamental paper [22] in 1907 in which he proves that a sequence of uniformly bounded holomorphic functions has a subsequence that is locally uniformly convergent. In 1912 Montel [23] introduced the term *normal family*. He defined a family $\mathscr{F}$ of functions, holomorphic in a plane region Δ, to be a *normal family* if any sequence chosen from $\mathscr{F}$ has a subsequence that converges, uniformly on each compact subset of Δ, to some function f which is either holomorphic in Δ, or the constant function ∞. (Warning: Some later authors do not permit the limit function to be ∞ in their definition of a normal family.) Montel's great contribution was to recognize the far reaching and profound impact that equicontinuity and normal families have on complex analysis, and in 1927 he published his influential text [25] on normal, and quasi-normal, families of holomorphic, meromorphic and harmonic functions. Montel refers to Arzelà in the footnote on [25, p. 27], and from then, essentially all accounts of normal families in complex analysis (see, for example, [10,15,19,24,31,34,35]) have followed this development from equicontinuity and convergence to normal families. A more modern view is that a family is normal if it is relatively compact (that is, has compact closure) in some larger space of (usually continuous) functions endowed with the compact-open topology (see, for example, [13,14,17,37]). From this perspective (for example, in several complex variables), the emphasis shifts from sequential compactness to compactness.

Now we turn to the content of this paper. In this mostly expository paper we shall discuss some mechanisms that underlie normality in the context of one complex variable. Despite the persistent use of equicontinuity (which was designed for a much broader class of functions), there is ample evidence that it is more appropriate to base the theory of normal families of holomorphic functions on a uniform Lipschitz property. In many instances authors derive equicontinuity for a family of functions by proving a local uniform Lipschitz property and then ignore this in favor of the weaker property of equicontinuity. In the reverse direction, normal families of holomorphic functions do satisfy an appropriate uniform Lipschitz condition on compact subsets; in fact, loosely speaking, *with the correct metrics*, normality for families of holomorphic functions is *equivalent* to the existence of a uniform Lipschitz condition on compact subsets of the region. Given these facts, we would expect the Schwarz–Pick Lemma (which guarantees a uniform Lipschitz condition) to play a pivotal role, yet it hardly appears at all in the theory. Here, we shall show how the Schwarz–Pick Lemma, *and this alone*, provides all of the complex analytic information that is needed to develop in a coherent way the basic theory of normal families of holomorphic maps between hyperbolic regions. For maps of hyperbolic regions into non-hyperbolic regions Lipschitz conditions play the role of the Schwarz–Pick Lemma.

Here is an overview of the paper. Background material on conformal metrics, the space of continuous functions, and relative compactness in this space are presented in Sects. 2, 3 and 4. In Sect. 5 we establish necessary and sufficient conditions for a family of Möbius maps to be uniformly Lipschitz relative to the chordal distance. Section 6 is devoted to an investigation of various types of Lipschitz conditions. Sections 7 and 8 concern the relative compactness of families of analytic functions that satisfy types of uniform Lipschitz conditions. In Sect. 9 we relate Montel's approach to normal families of analytic functions to our Lipschitz families methodology. The remainder of the paper deals with families of non-analytic functions. Most of our basic results remain valid for families of (possibly non-analytic) functions that satisfy a type of uniform Lipschitz condition with respect to hyperbolic or spherical distances. This is carried out in Sects. 10 and 11. Thus (in common with some other areas in geometric function theory, for example the Denjoy–Wolff theorem and related results) this topic is essentially geometric in character and may be regarded as a distant relative of the well-known Contraction Mapping Theorem. In Sect. 10 we introduce the *Escher property* of the hyperbolic metric. Loosely speaking, the Escher property implies that in a hyperbolic region, for each positive number R, all hyperbolic disks with radius at most R are uniformly small in the spherical sense provided the hyperbolic center of the disk is sufficiently near the boundary. The Escher property plays a significant role in dealing with families that satisfy uniform Lipschitz conditions.

The approach to normal families of analytic maps through various types of Lipschitz conditions presented in this paper provides a systematic treatment of the subject that also extends to some families of non-analytic maps. Our immediate goal is not to answer open questions or suggest new problems. One of the most fruitful developments of normal families in recent times is the well-known rescaling lemma of Zalcman [40]. For surveys of the numerous applications of this lemma, see [6,41]. The ideas and techniques employed in this paper can be used to good effect in the context of rescaling results and will be discussed at length in a forthcoming paper by the same authors.

We close Sect. 1 with the statement of our extension of Montel's Fundamental Normality Criterion to Lipschitz families (a detailed explanation of the terms will follow later). We emphasize that the functions in $\mathscr{F}$ in Theorem 1.1 are not required to be analytic.

Theorem 1.1 *Let $\mathscr{F}$ be a family of maps from a hyperbolic plane region Δ to $\mathbb{C}_{0,1} = \mathbb{C}\backslash\{0, 1\}$. If $\mathscr{F}$ satisfies, on each compact subset of Δ, a uniform Lipschitz inequality with respect to the hyperbolic metrics of both Δ and $\mathbb{C}_{0,1}$, then $\mathscr{F}$ is a normal family in Δ.*

2 Conformal Semi-Metrics

Conformal semi-metrics play an important role in this paper, so we recall basic facts about them. We adopt the simpler terminology 'semi-metric' in place of 'conformal semi-metric' because all semi-metrics considered will be conformal. Suppose that Δ is a region in $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. A semi-metric on Δ is a continuous non-negative form $\tau(z)|\mathrm{d}z|$ which is positive except possibly for isolated zeros. If there are no zeros, then $\tau(z)|\mathrm{d}z|$ is called a conformal metric. We often write τ in place of the expression $\tau(z)|\mathrm{d}z|$. The latter expression is unnecessary when $\Delta \subseteq \mathbb{C}$; however, when $\infty \in \Delta$ the full expression $\tau(z)|\mathrm{d}z|$ reminds us that we must work in terms of a local coordinate at ∞. We avoid discussing a local coordinate when $\infty \in \Delta$ and leave any necessary modifications to the reader.

The distance function on Δ induced by a semi-metric τ is

$$d_\tau(z, w) = \inf \int_\gamma \tau(\zeta)|\mathrm{d}\zeta|,$$

where the infimum is taken over all paths γ in Δ joining z and w. We reserve the term 'metric' or 'semi-metric' for the infinitesimal form τ and use the term 'distance' for the integrated form. The distance function d_τ is compatible with the topology on Δ as a subset of $\mathbb{C}_\infty$. It is elementary that

$$\lim_{w\to z} \frac{d_\tau(z, w)}{|z - w|} = \tau(z).$$

A semi-metric τ on Δ is *complete* if the associated distance function d_τ is complete in the sense that all Cauchy sequences in (Δ, d_τ) are convergent. Recall that all closed disks are compact for complete metrics.

In general, if (X, d_X) is a distance space, $D_X(a, r)$ denotes the open disk in X with center a and radius r, and $\overline{D}_X(a, r)$ denotes the corresponding closed disk. Also, if $E \subset X$, then $\overline{E}$, E° and ∂E denote the closure, interior and boundary, respectively, of E. A subset E of X is *relatively compact* if its closure $\overline{E}$ is compact. Distance functions d_X and d'_X on X are *topologically equivalent* if the identity map of (X, d_X) to (X, d'_X) is a homeomorphism; in other words, d_X and d'_X determine the same topology on X. Distance functions d_X and d'_X on X are *bi-Lipschitz equivalent* if there exists $L > 0$ such that for all $u, v \in X$,

$$\frac{1}{L} d_X(u,v) \leqslant d'_X(u,v) \leqslant L d_X(u,v).$$

Bi-Lipschitz equivalent distance functions are topologically equivalent and not conversely.

Given regions Δ and Ω in $\mathbb{C}_\infty$, let $\mathscr{A}[\Delta,\Omega]$ designate the family of all analytic functions $f:\Delta\to\Omega$. We use the term ‘analytic’ to mean that a function is either holomorphic or meromorphic, as appropriate. If $\Omega\subseteq\mathbb{C}$ the functions in $\mathscr{A}[\Delta,\Omega]$ are holomorphic. The term meromorphic is appropriate when $\infty\in\Omega$. Observe that we regard the constant function $f=\infty$ to a meromorphic function, just as any other constant function is meromorphic.

If $f:\Delta\to\Omega$ is analytic and ρ is a semi-metric on Ω, then the *pull-back* of ρ by f is $f^*(\rho)=\rho(f(z))|f'(z)|$. Unless f is constant, the pull-back is a semi-metric in Δ. The quantity

$$D_{\tau,\rho} f(z) = \lim_{w\to z} \frac{d_\rho(f(z),f(w))}{d_\tau(z,w)} = \frac{\rho(f(z))|f'(z)|}{\tau(z)}$$

is the τ-to-ρ derivative of f; it measures the infinitesimal distortion of f from Δ with the distance d_τ to Ω with the distance d_ρ.

The three fundamental complete metrics with constant curvature are

(a) the Euclidean metric $1|\mathrm{d}z|$ on $\mathbb{C}$ with curvature zero and Euclidean distance function $e(z,w)=|z-w|$,
(b) the hyperbolic metric $\lambda_{\mathbb{D}}(z)| = \frac{2}{1-|z|^2}$ on $\mathbb{D}$ with curvature -1 and hyperbolic distance function $h_{\mathbb{D}}(z,w) = 2\tanh^{-1}\frac{|z-w|}{|1-\overline{w}z|}$, and
(c) the spherical metric $\sigma(z)=\frac{2}{1+|z|^2}$ on $\mathbb{C}_\infty$ with curvature $+1$ and spherical distance function $s(z,w) = 2\tan^{-1}\frac{|z-w|}{|1+\overline{w}z|}$.

It is sometimes convenient to utilize the *chordal distance* on $\mathbb{C}_\infty$ which is given by

$$\chi(z,w) = 2\sin\left(\tfrac{1}{2}s(z,w)\right) = \begin{cases} \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, & \text{if} \quad z,w\in\mathbb{C};\\ \frac{2}{\sqrt{1+|z|^2}}, & \text{if} \quad z\in\mathbb{C}, w=\infty. \end{cases}$$

Note that

$$\lim_{w\to z}\frac{\chi(z,w)}{|z-w|} = \frac{2}{1+|z|^2} = \sigma(z).$$

The spherical and chordal distances on $\mathbb{C}_\infty$ are bi-Lipschitz equivalent:

$$\tfrac{2}{\pi}s(z,w) \leqslant \chi(z,w) \leqslant s(z,w) \quad \text{for all } z,w\in\mathbb{C}_\infty.$$

For the record, and as far as we can tell, the chordal distance χ was introduced into complex analysis by Carathéodory in [8], and not (despite some remarks to the contrary in the literature) by Marty or Ostrowski, both of whom used the spherical distance in [20,27], respectively.

If Δ is a region in $\mathbb{C}$ and f is meromorphic on Ω, then the *spherical derivative* of f is

$$f^{\#}(z) = \lim_{w\to z} \frac{\chi(f(z), f(w))}{|z-w|} = \lim_{w\to z} \frac{s(f(z), f(w))}{|z-w|} = \frac{2|f'(z)|}{1+|f(z)|^2}.$$

This is the Euclidean-to-spherical derivative $D_{e,s} f$. If Δ is regarded as a subset of $\mathbb{C}_\infty$, then the spherical-to-spherical derivative of a meromorphic function f is

$$\mathscr{D} f(z) = \frac{f^{\#}(z)}{\sigma(z)} = \frac{(1+|z|^2) f^{\#}(z)}{2} = \lim_{w\to z} \frac{\chi(f(z), f(w))}{\chi(z, w)} = D_{s,s} f(s).$$

This derivative was introduced by Marty [20].

A region Δ in $\mathbb{C}_\infty$ is *hyperbolic* if $\mathbb{C}_\infty \backslash \Delta$ contains at least three points. A region Δ is hyperbolic if and only if there is an analytic covering $f : \mathbb{D} \to \Delta$. In this situation there is a unique complete metric λ_Δ on Δ with curvature -1 such that $f^*(\lambda_\Delta) = \lambda_{\mathbb{D}}$ for any analytic covering $f : \mathbb{D} \to \Delta$. The associated hyperbolic distance function on Δ is denoted by h_Δ.

The Schwarz–Pick Lemma plays a fundamental role in complex analysis. The general form of the Schwarz–Pick Lemma asserts that if $f : \Delta \to \Omega$ is analytic, where Δ and Ω are hyperbolic regions, then $f^*(\lambda_\Omega(z)) \leqslant \lambda_\Delta(z)$ and $h_\Omega(f(z), f(w)) \leqslant h_\Delta(z, w)$ for all $z, w \in \Delta$. In brief, analytic maps of hyperbolic regions are non-expansive relative to the hyperbolic metric.

Every region Δ in $\mathbb{C}_\infty$ has a complete metric. If Δ is a hyperbolic region in $\mathbb{C}$, then the hyperbolic metric is complete. The spherical metric σ is complete on $\mathbb{C}_\infty$. The remaining possibilities are that Δ is a once-punctured or twice-punctured sphere. If $\Delta = \mathbb{C}_\infty \backslash \{a\}$, $a \neq \infty$, then $f(z) = 1/(z-a)$ maps Δ conformally onto $\mathbb{C}$ and the pull-back of the Euclidean metric by f is complete. The quasi-hyperbolic metric $|dz|/|z|$ on $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ has curvature 0 and is complete; note that the pull-back of the quasi-hyperbolic metric by the exponential function is the Euclidean metric. If $\Delta = \mathbb{C}\backslash\{a, b\}$, then $g(z) = (z-a)/(z-b)$ is a conformal map of Δ onto $\mathbb{C}^*$ and the pull-back of the quasi-hyperbolic metric by g is a complete metric on Δ.

The following lemma will enable us to substitute one distance for another when discussing Lipschitz conditions on compact sets:

Lemma 2.1 *Suppose that τ_j is a conformal metric on Δ_j with associated distance function d_j, $j = 1, 2$, and $\Delta_1 \subseteq \Delta_2$. Then d_1 and d_2 are bi-Lipschitz equivalent on each compact subset of Δ_1.*

Proof The function $\psi : \Delta_1 \times \Delta_1 \to \mathbb{R}$ given by

$$\psi(z, w) = \begin{cases} d_2(z, w)/d_1(z, w), & \text{if} \quad z \neq w; \\ \tau_1(z)/\tau_2(z), & \text{if} \quad z = w; \end{cases}$$

is continuous and positive. Given a compact set E in Δ_1, ψ attains a positive minimum m and a finite maximum M on the compact set $E \times E$. Hence, $md_1(z, w) \leqslant d_2(z, w) \leqslant Md_1(z, w)$ for all $z, w \in E$. $\square$

3 The Space of Continuous Functions

Throughout this section, Δ and Ω denote regions in $\mathbb{C}_\infty$. Let $\mathscr{C}[\Delta, \Omega]$ be the space of continuous maps $f : \Delta \to \Omega$. Here continuity is relative to the restriction of the chordal distance on the domain Δ and co-domain Ω. Of course, we may replace the chordal distance on the domain or co-domain with any topologically equivalent distance function and still have the same class of continuous functions. For instance, if τ, ρ is a semi-metric on Δ, Ω, respectively, we may use d_τ in the domain and d_ρ in the co-domain. This ability to switch between equivalent metrics is convenient. Sometimes, we may even assume that τ or ρ is complete if this simplifies an argument.

We recall the standard construction of a metric on $\mathscr{C}[\Delta, \Omega]$, see [2, pp. 220–221] or [11, pp. 142–146]. Fix distances d_Δ and d_Ω that are topologically equivalent to the chordal distance on Δ and Ω, respectively. A compact exhaustion of Δ is a sequence K_n of compact subsets such that $K_n \subset K^\circ_{n+1}$ for all n and $\cup_{n=1}^\infty K_n = \Delta$. For f and g in $\mathscr{C}[\Delta, \Omega]$ set

$$d_n(f, g) = \sup\{d_\Omega(f(z), g(z)) : z \in K_n\}, \tag{3.1}$$

and

$$d_{\Delta,\Omega}(f, g) = \sum_{n=1}^\infty \frac{1}{2^n} \left(\frac{d_n(f, g)}{1 + d_n(f, g)} \right). \tag{3.2}$$

Then, as is well known, $d_{\Delta,\Omega}$ is a distance function on $\mathscr{C}[\Delta, \Omega]$, and $d_{\Delta,\Omega}(f_n, f) \to 0$ as $n \to \infty$ if and only if $f_n \to f$ uniformly on each compact subset of Δ. The metric $d_{\Delta,\Omega}$ on $\mathscr{C}[\Delta, \Omega]$ is based on a particular compact exhaustion of Δ, but since a metric topology is defined by its convergent sequences, we see that a metric constructed in the same way, but with another compact exhaustion, produces the same topology on $\mathscr{C}[\Delta, \Omega]$ as $d_{\Delta,\Omega}$ does. We call this (metrizable) topology the *topology of uniform convergence on compact sets* or the *topology of local uniform convergence* and denote it by $\mathscr{T}^{luc}$. Henceforth, when $f_n, f \in \mathscr{C}[\Delta, \Omega]$, the phrase, $f_n \to f$ in $\mathscr{C}[\Delta, \Omega]$, will mean local uniform convergence relative to d_Δ and d_Ω, or equivalently, convergence relative to the metric $d_{\Delta,\Omega}$ on $\mathscr{C}[\Delta, \Omega]$. As sequential compactness coincides with compactness in metric spaces, there is no need to distinguish between the two when discussing compact sets in $\mathscr{C}[\Delta, \Omega]$.

It is important to observe that the topology $\mathscr{T}^{luc}$ on $\mathscr{C}[\Delta, \Omega]$ is unchanged when the distances d_Δ and d_Ω are replaced by topologically equivalent distances. Perhaps the simplest way to see this is to note that $\mathscr{T}^{luc}$ is the compact-open topology $\mathscr{T}^{co}$ on $\mathscr{C}[\Delta, \Omega]$. We remind the reader of the definition of the compact-open topology. For each compact set K in Δ and each open set V in Ω, let $[K, V] = \{f \in \mathscr{C}[\Delta, \Omega] : f(K) \subset V\}$. The compact-open topology $\mathscr{T}^{co}$ on $\mathscr{C}[\Delta, \Omega]$ is the topology generated by the sets $[K, V]$, where K is any compact subset of Δ and V is any open set in Ω. This makes it clear that the topology $\mathscr{T}^{luc}$ on $\mathscr{C}[\Delta, \Omega]$ is unchanged when the distances d_Δ and d_Ω are replaced by topologically equivalent distances.

In general, the distance space $(\mathscr{C}[\Delta, \Omega], d_{\Delta,\Omega})$ need not be complete because functions can converge to a point in $\partial\Omega$. However, if (Ω, d_Ω) is complete, then $(\mathscr{C}[\Delta, \Omega], d_{\Delta,\Omega})$ is also complete. We omit the standard proof; see [11, p. 145] for a proof. Henceforth, we assume that d_Ω is complete, unless the contrary is explicitly stated.

Theorem 3.1 (Weierstrass) *$\mathscr{A}[\Delta, \Omega]$ is a complete subset of $\mathscr{C}[\Delta, \Omega]$.*

The reader should be careful about the assertion in Theorem 3.1. The classical form of Weierstrass' Theorem asserts that the local uniform limit f of a sequence f_n of holomorphic functions on a region Δ is holomorphic on Δ. This is a statement about the family $\mathscr{A}[\Delta, \mathbb{C}]$. If, in fact, $f_n(\Delta) \subseteq \Omega$ for all n, then the classical form of Weierstrass' Theorem in conjunction with Hurwitz' Theorem implies that either $f(\Delta) \subseteq \Omega$ or f is a constant map into $\partial\Omega$. The latter case does not occur when we regard $\mathscr{A}[\Delta, \Omega]$ as a subset of $\mathscr{C}[\Delta, \Omega]$ because the limit function f belongs to the complete space $\mathscr{C}[\Delta, \Omega]$. Similarly, the classical version of Weierstrass' Theorem for meromorphic functions asserts that if the sequence f_n of meromorphic functions converges locally uniformly relative to the chordal distance on $\mathbb{C}_\infty$, then the limit function f is meromorphic, including possibly the constant ∞. This deals with $\mathscr{A}[\Delta, \mathbb{C}_\infty]$, not $\mathscr{A}[\Delta, \Omega]$.

Lemma 3.1 *For $f \in \mathscr{A}[\Delta, \Omega]$ and a compact set $K \subseteq \Delta$, set $|f|_K = \max_{z\in K} D_{\tau,\rho} f(z)$. Then $f \mapsto |f|_K$ is a continuous functional on $\mathscr{A}[\Delta, \Omega]$ when τ is a metric.*

Proof We show that if $f_n \to f$ in $\mathscr{A}[\Delta, \Omega]$, then $|f_n|_K \to |f|_K$. We assume that $\Delta, \Omega \subseteq \mathbb{C}$. It suffices to show that $\rho(f_n(z))|f_n'(z)|/\tau(z) \to \rho(f(z))|f'(z)|/\tau(z)$ uniformly on K. The classical form of Weierstrass' Theorem implies that $f_n' \to f'$ locally uniformly. Therefore, $\rho(f_n(z))|f_n'(z)| \to \rho(f(z))|f'(z)|$ uniformly on K. Because τ is a positive continuous function, it follows that $\rho(f_n(z))|f_n'(z)|/\tau(z) \to \rho(f(z))|f'(z)|/\tau(z)$ uniformly on K. □

4 Relative Compactness in $\mathscr{C}[\Delta, \Omega]$

For completeness, we include a brief discussion of the Arzelà–Ascoli theorem for $\mathscr{C}[\Delta, \Omega]$. Let d_Δ and d_Ω (complete) be distance functions on Δ and Ω, respectively, that are topologically equivalent to the chordal distance.

Definition 4.1 Let $\mathscr{F} \subset \mathscr{C}[\Delta, \Omega]$.

(a) $\mathscr{F}$ is *equicontinuous at* $z_0 \in \Delta$ if, for every positive ε, there is a positive δ such that such that $d_\Omega(f(z), f(z_0)) < \varepsilon$ whenever $d_\Delta(z, z_0) < \delta$ and $f \in \mathscr{F}$.
(b) $\mathscr{F}$ is equicontinuous in Δ if it is equicontinuous at each $z_0 \in \Delta$.

By using the triangle inequality, the condition (a) can be written in a more convenient form, namely $\mathscr{F}$ is *equicontinuous at* z_0 if, for every positive ε, there is a positive δ such that $d_\Omega(f(z), f(w)) < \varepsilon$ whenever $z, w \in D_\Delta(z_0, \delta)$ and $f \in \mathscr{F}$. Equicontinuity was introduced by Ascoli in 1883 [5], and then developed by Arzelà [3] who proved a result which in modern form reads as follows (and whose proof is well documented in the modern literature; see, for example, [2,12,17,33]):

Theorem 4.1 (Arzelà–Ascoli) *Let Δ and Ω be regions in $\mathbb{C}_\infty$ and $\mathscr{F} \subseteq \mathscr{C}[\Delta, \Omega]$. $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \Omega]$ if and only if*

(a) *$\mathscr{F}$ is equicontinuous on Δ, and*
(b) *for each z in Δ, the set $\mathscr{F}(z) = \{f(z) : f \in \mathscr{F}\}$ is relatively compact in Ω.*

Let $\overline{\mathscr{F}}$ denote the closure of $\mathscr{F}$ in $\mathscr{C}[\Delta, \Omega]$. It is straightforward to verify that $\mathscr{F}$ is equicontinuous at z_0 if and only if $\overline{\mathscr{F}}$ is. Also, $\overline{\mathscr{F}}(z_0)$ is the closure of $\mathscr{F}(z)$. The Arzelà–Ascoli Theorem can be viewed as a characterization of the compact subsets of $\mathscr{C}[\Delta, \Omega]$. In a distance space, a set is compact if and only if it is complete and totally bounded. Condition (b) insures that $\overline{\mathscr{F}}$ is complete and together with (a) gives total boundedness.

This modern version of the Arzelà–Ascoli Theorem was not available to Montel in 1907 or 1912. Rather, Montel independently established an analog of the Arzelà–Ascoli Theorem in the context of holomorphic functions. Today most books use the modern version of the Arzelà–Ascoli Theorem to establish Montel's Theorem that a family of locally uniformly bounded holomorphic functions is normal.

Corollary 4.1 *Suppose that Δ and Ω are hyperbolic regions and $\mathscr{F} \subseteq \mathscr{A}[\Delta, \Omega]$. Then $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \Omega]$ if and only if there exists $z_0 \in \Delta$ such that $\mathscr{F}(z_0) = \{f(z_0) : f \in \mathscr{F}\}$ is relatively compact in Ω.*

Proof It is convenient to use the hyperbolic metrics λ_Δ and λ_Ω on the domain and codomain. Then the general form of the Schwarz–Pick Lemma implies that $\mathscr{A}[\Delta, \Omega]$, and so $\mathscr{F}$, is uniformly Lipschitz relative to the hyperbolic metrics h_Δ and h_Ω with Lipschitz constant 1. Hence, condition (a) of the Arzelà–Ascoli Theorem holds. It remains to verify that (b) holds. Because $\mathscr{F}(z_0)$ is relatively compact, there exists $w_0 \in \Omega$ and $R > 0$ such that $\mathscr{F}(z_0) \subset \overline{D}_\Omega(w_0, R)$. Consider any $z \in \Delta$ and set $r = h_\Delta(z_0, z)$. Then for all $f \in \mathscr{F}$, the Schwarz–Pick Lemma gives $h_\Omega(f(z_0), f(z)) \leqslant h_\Delta(z_0, z) = r$. Consequently,

$$h_\Omega(w_0, f(z)) \leqslant h_\Omega(w_0, f(z_0)) + h_\Omega(f(z_0), f(z)) \leqslant r + R,$$

so $\mathscr{F}(z) \subset \overline{D}_\Omega(w_0, R + r)$. Because the hyperbolic metric is complete, closed hyperbolic disks are compact. Hence, $\mathscr{F}(z)$ is relatively compact in Ω. □

Corollary 4.2 *Suppose that Ω and Σ are hyperbolic regions with $\overline{\Omega} \subset \Sigma$. Then for any hyperbolic region Δ, $\mathscr{A}[\Delta, \Omega]$ is relatively compact in $\mathscr{C}[\Delta, \Sigma]$.*

Proof The family $\mathscr{A}[\Delta, \Omega] \subset \mathscr{A}[\Delta, \Sigma]$, so by the Schwarz–Pick Lemma, $\mathscr{A}[\Delta, \Omega]$ satisfies an h_Δ-to-h_Σ Lipschitz condition. Because $\overline{\Omega} \subset \Sigma$, every orbit $\mathscr{A}[\Delta, \Omega](z_0)$ is relatively compact in Σ. Hence, $\mathscr{A}[\Delta, \Omega]$ is relatively compact in $\mathscr{C}[\Delta, \Sigma]$. □

Example 4.1 The family $\mathscr{A}[\mathbb{D}, \mathbb{D}]$ is not relatively compact in $\mathscr{C}[\mathbb{D}, \mathbb{D}]$ because the sequence $f_n(z) = (n-1)/n$ does not have a locally uniformly convergent subsequence in $\mathscr{C}[\mathbb{D}, \mathbb{D}]$. If we let $\Sigma = \{z : |z| < 2\}$, then $\overline{\mathbb{D}} \subset \Sigma$. Hence, $\mathscr{A}[\mathbb{D}, \mathbb{D}]$ is relatively compact in $\mathscr{C}[\mathbb{D}, \Sigma]$ by Corollary 4.2.

5 Relatively Compact Families of Möbius Maps

A Möbius map has the form $g : z \mapsto (az+b)/(cz+d)$, where $ad - bc = 1$. Möbius maps are the meromorphic homeomorphisms of $\mathbb{C}_\infty$. As the reader may verify, the family $\mathscr{M}$ of all Möbius maps is a closed (complete) subset of $\mathscr{C}[\mathbb{C}_\infty, \mathbb{C}_\infty]$. We will characterize relatively compact subsets $\mathscr{F}$ of $\mathscr{M}$. Because $\mathbb{C}_\infty$ is compact, the Arzelà–Ascoli Theorem implies that $\mathscr{F}$ is relatively compact in $\mathscr{M}$ if and only if it is equicontinuous on $\mathbb{C}_\infty$. We show that $\mathscr{F}$ is relatively compact in $\mathscr{M}$ if and only if it is uniformly Lipschitz with respect to the chordal distance.

Although a Möbius map g is not an isometry with respect to the chordal metric χ, it is a bi-Lipschitz map of $(\mathbb{C}_\infty, \chi)$ onto itself. First, as g determines the vector (a, b, c, d) up to the multiplicative factor -1, we can define the norm $\|g\|$ of g by

$$\|g\|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2.$$

As $ad - bc = 1$, we have $\|g\| = \|g^{-1}\|$. Although $\|g\|^2$ is defined algebraically, it is geometrically significant in several ways; first, it controls the distortion of g in its action on hyperbolic three-dimensional space (this will not concern us here), and it provides a Lipschitz constant for the action of g on $(\mathbb{C}_\infty, \chi)$.

Theorem 5.1 *Let g be a Möbius transformation. Then for all z and w in $\mathbb{C}_\infty$, we have*

$$\frac{\chi(z, w)}{\|g\|^2} \leqslant \chi(g(z), g(w)) \leqslant \|g\|^2 \chi(z, w). \tag{5.1}$$

Proof We begin with the elementary Euclidean identity for Möbius transformations

$$|g(z) - g(w)| = \sqrt{|g'(z)|\,|g'(w)|}\,|z - w|,$$

and express it in the equivalent spherical form

$$\chi\big(g(z), g(w)\big) = \sqrt{\mathscr{D}g(z)\mathscr{D}g(w)}\,\chi(z, w), \quad \text{for all } z, w \in \mathbb{C}_\infty, \tag{5.2}$$

where $\mathscr{D}g$ is the spherical-to-spherical, or Marty, derivative of g. If $g(z) = (az + b)/(cz + d)$, where $ad - bc = 1$, then (5.2) yields

$$\chi(g(z), g(w)) = \frac{2|z - w|}{(|az + b|^2 + |cz + d|^2)^{1/2}(|aw + b|^2 + |cw + d|^2)^{1/2}}.$$

The Cauchy–Schwarz inequality gives

$$\begin{aligned} |az+b|^2 + |cz+d|^2 &\leqslant (|a|^2+|b|^2)(1+|z|^2) + (|c|^2+|d|^2)(1+|z|^2) \\ &= (1+|z|^2)\|g\|^2, \end{aligned}$$

and so $\chi(g(z), g(w)) \geqslant \chi(z, w)\|g\|^2$. The other inequality in (5.1) follows by writing $h = g^{-1}$, $u = g(z)$ and $v = g(w)$, and noting that $\|g\| = \|h\|$. □

There is an elementary upper bound on the norm of a Möbius map in terms of the images of three points.

Theorem 5.2 *Let g be a Möbius transformation and suppose that*

$$\chi(g(0), g(1)) \geqslant m \quad \chi(g(1), g(\infty)) \geqslant m, \quad \chi(g(\infty), g(0)) \geqslant m,$$

where $m > 0$. Then $||g||^2 \leqslant 32/m^3$.

Proof Let $g(z) = (az+b)/(cz+d)$, where $ad - bc = 1$. Then

$$\begin{aligned} (|a|^2+|c|^2)(|b|^2+|d|^2) &= 4/\chi\big(g(0), g(\infty)\big)^2 \leqslant 4/m^2, \\ (|a|^2+|c|^2)(|a+b|^2+|c+d|^2) &= 4/\chi\big(g(\infty), g(1)\big)^2 \leqslant 4/m^2, \\ (|b|^2+|d|^2)(|a+b|^2+|c+d|^2) &= 4/\chi\big(g(1), g(0)\big)^2 \leqslant 4/m^2, \end{aligned}$$

so that

$$(|a|^2+|c|^2)(|b|^2+|d|^2)\big(|a+b|^2+|c+d|^2\big) \leqslant \frac{8}{m^3}.$$

The first of the four identities

$$\begin{aligned} a &= ad(a+b) - ab(c+d), \\ b &= -bc(a+b) + ab(c+d), \\ c &= cd(a+b) - cb(c+d), \\ d &= -cd(a+b) + ad(c+d), \end{aligned}$$

together with the Cauchy–Schwarz inequality yields

$$|a|^2 \leqslant \big(|ad|^2 + |ab|^2\big)\big(|a+b|^2 + |c+d|^2\big) \leqslant 8/m^3.$$

The other three identities produce similar bounds for $|b|^2$, $|c|^2$ and $|d|^2$, and together, these imply that $\|g\|^2 \leqslant 32/m^3$. □

Next, we give a geometric characterization of uniformly chordal-to-chordal Lipschitz families of Möbius maps.

Theorem 5.3 *For any family $\mathscr{G}$ of Möbius maps the following are equivalent:*

(a) *$\mathscr{G}$ is relatively compact in $\mathscr{M}$;*
(b) *$\mathscr{G}$ is uniformly chordal-to-chordal Lipschitz on $\mathbb{C}_\infty$;*
(c) $\sup_{g\in\mathscr{G}} \|g\| < +\infty$*;*
(d) *there is a positive number m such that for all g in $\mathscr{G}$,*

$$\chi(g(0), g(1)) \geqslant m, \quad \chi(g(1), g(\infty)) \geqslant m, \quad \chi(g(\infty), g(0)) \geqslant m.$$

Proof Theorem 5.2 shows (d) implies (c); Theorem 5.1 gives (c) implies (b); and the Arzelà–Ascoli theorem yields (b) implies (a). It remains to prove that (a) implies (d). Suppose that (a) holds. Then the closure $\overline{\mathscr{G}} \subset \mathscr{M}$ of $\mathscr{G}$ is compact. We establish the existence of a positive number m such that (d) holds for all $g \in \overline{\mathscr{G}}$. For distinct $u, v \in \mathbb{C}_\infty$, it is straightforward to verify that $g \mapsto \chi(g(u), g(v))$ is a positive continuous functional on $\mathscr{M}$. Because $\overline{\mathscr{G}}$ is compact, this functional attains a positive minimum value $m(u, v)$ on $\overline{\mathscr{G}}$. If we take $m = \min\{m(0, 1), m(1, \infty), m(\infty, 0)\}$, then (d) holds. □

6 Lipschitz Conditions

As Corollary 4.1 and Theorem 5.3 indicate, uniform Lipschitz conditions help to characterize certain relatively compact families of analytic functions. As we will see subsequently, these are two instances of a general phenomenon. Therefore, we investigate types of uniform Lipschitz conditions. Throughout this section Δ and Ω are regions in $\mathbb{C}_\infty$ with conformal semi-metrics τ and ρ and ρ is complete.

Definition 6.1 Suppose that $\mathscr{F} \subseteq \mathscr{C}[\Delta, \Omega]$.

(a) $\mathscr{F}$ is *locally uniformly d_τ-to-d_ρ Lipschitz* if for each $z_0 \in \Delta$ there exists $r > 0$ and a positive number L such that for all $z, w \in D_\tau(z_0, r)$ and all $f \in \mathscr{F}$, $d_\rho(f(z), f(w)) \leqslant L d_\tau(z, w)$.
(b) $\mathscr{F}$ is *uniformly d_τ-to-d_ρ Lipschitz on compact sets* if for each compact set $K \subset \Delta$ there exists a positive number L such that for all $z, w \in K$ and all $f \in \mathscr{F}$, $d_\rho(f(z), f(w)) \leqslant L d_\tau(z, w)$.
(c) $\mathscr{F}$ is *globally d_τ-to-d_ρ Lipschitz* if there exists a positive number L such that for all $z, w \in \Delta$ and all $f \in \mathscr{F}$, $d_\rho(f(z), f(w)) \leqslant L d_\tau(z, w)$.

Note that if $\mathscr{F} \subseteq \mathscr{C}[\Delta, \Omega]$ satisfies a type of uniform Lipschitz condition, then $\overline{\mathscr{F}}$ satisfies the same type of uniform Lipschitz condition. Of course, if Δ and Ω are hyperbolic regions endowed with their hyperbolic metrics, then the Schwarz–Pick Lemma implies that all $f \in \mathscr{A}[\Delta, \Omega]$ satisfy a global h_Δ-to-h_Ω Lipschitz condition. There is no analog of the Schwarz–Pick Lemma for maps of a hyperbolic region Δ either into $\mathbb{C}$ with the Euclidean metric or into $\mathbb{C}_\infty$ with the spherical metric. One may view the assumption of a local uniform h_Δ-to-e Lipschitz condition or a local uniform h_Δ-to-s Lipschitz condition as a replacement for the Schwarz–Pick Lemma.

Theorem 6.1 *A family $\mathscr{F} \subseteq \mathscr{C}[\Delta, \Omega]$ is locally uniformly d_τ-to-d_ρ Lipschitz if and only if it is uniformly d_τ-to-d_ρ Lipschitz on each compact subset of Δ.*

Proof If $\mathscr{F}$ is uniformly d_τ-to-d_ρ Lipschitz on compact sets, then it is locally uniformly d_τ-to-d_ρ Lipschitz because each point of Δ has a compact neighborhood.

On the other hand, suppose that $\mathscr{F}$ is locally uniformly d_τ-to-d_ρ Lipschitz. Consider any compact set K in Δ. Initially, we assume that τ is complete. Then we may assume that $K = \overline{D}_\tau(z_0, R)$ for some $z_0 \in \Delta$ and $R > 0$. We show that $\mathscr{F}$ is uniformly d_τ-to-d_ρ Lipschitz on K. Let $K^* = \overline{D}_\tau(z_0, 3R+1)$. By assumption, for each point $z \in K^*$ there is an open neighborhood $N(z)$ and a positive number $L(z)$ such that for all $u, v \in N(z)$ and all $f \in \mathscr{F}$, $d_\rho(f(u), f(v)) \leqslant L(z) d_\tau(u, v)$. As K^* is compact, a finite number of these neighborhoods, say $N(z_1), \dots, N(z_J)$ cover K^*. Let L be the maximum of the finite set of numbers $L(z_j)$, $1 \leqslant j \leqslant J$. By Lebesgue's Covering Theorem there is a positive number η such that if a subset Q of K^* has d_τ-diameter at most η, then $Q \subset N(z_j)$ for some j. We will show that if $u, v \in K$ and $f \in \mathscr{F}$, then $d_\rho(f(u), f(v)) \leqslant L d_\tau(u, v)$. Let $0 < \varepsilon \leqslant 1$ be given. Given any pair of points, u and v in K, $d_\tau(u, v) \leqslant 2R$. Let γ be a path in Δ from u to v such that

$$d_\tau(u, v) \leqslant \int_\gamma \tau(\zeta)|\mathrm{d}\zeta| < d_\tau(u, v) + \varepsilon.$$

If z is any point on γ, then

$$d_\tau(z_0, z) \leqslant d_\tau(z_0, u) + d_\tau(u, z) \leqslant R + \int_\gamma \tau(\zeta)|\mathrm{d}\zeta| < 3R + 1.$$

Therefore, γ lies in the compact set $K^* = \overline{D}_\tau(z_0, 3R+1)$. Suppose that the path γ is defined on the interval $[0, 1]$. Because γ is uniformly continuous on $[0, 1]$ we may determine $0 = t_1 < t_2 < \cdots < t_{q+1} = 1$ so that for each j, $1 \leqslant j < q$, the set $\gamma([t_j, t_{j+1}])$ has τ-diameter at most η. Then for any $f \in \mathscr{F}$,

$$\begin{aligned} d_\rho(f(u), f(v)) &\leqslant \sum_{j=1}^{q} d_\rho\big(f(\gamma(t_j)), f(\gamma(t_{j+1}))\big) \\ &\leqslant L \sum_{j=1}^{q} d_\tau\big(\gamma(t_j), \gamma(t_{j+1})\big) \\ &\leqslant L \sum_{j=1}^{q} \int_{\gamma_j} \tau(\zeta)|\mathrm{d}\zeta| \\ &= L \int_\gamma \tau(\zeta)|\mathrm{d}\zeta| \\ &\leqslant L\big(d_\tau(u, v) + \varepsilon\big), \end{aligned}$$

where γ_j denotes the restriction of γ to the interval $[t_j, t_{j+1}]$. Because $0 < \varepsilon \leqslant 1$ is arbitrary, this implies that if $u, v \in K$ and $f \in \mathscr{F}$, then $d_\rho(f(u), f(v)) \leqslant L d_\tau(u, v)$. Hence, $\mathscr{F}$ satisfies a uniform d_τ-to-d_ρ Lipschitz condition on K.

It τ is not complete, use a complete metric $\tilde{\tau}$ and then cite Lemma 3.1 to obtain the result for τ in place of $\tilde{\tau}$. □

Theorem 6.2 *Suppose that* $\mathscr{F} \subseteq \mathscr{A}[\Delta, \Omega]$.

(a) $\mathscr{F}$ *is uniformly* d_τ*-to-*d_ρ *Lipschitz on compact sets if and only if for each compact set* $K \subset \Delta$ *there exists a positive number* M *such that for all* $z \in K$ *and all* $f \in \mathscr{F}$, $D_{\tau,\rho} f(z) \leqslant M$.

(b) $\mathscr{F}$ *is globally* d_τ*-to-*d_ρ *Lipschitz if and only if there exists a positive number* M *such that for all* $z \in \Delta$ *and all* $f \in \mathscr{F}$, $D_{\tau,\rho} f(z) \leqslant M$.

Proof (a) There is no harm in assuming that τ is complete. Suppose that $\mathscr{F}$ is uniformly d_τ-to-d_ρ Lipschitz on compact sets. Fix a compact set K. We may assume that $K = \overline{D}_\tau(z_0, R)$ for some $z_0 \in \Delta$ and $R > 0$. There exists L such that for all $z, w \in \overline{D}(z_0, 2R)$ and all $f \in \mathscr{F}$, $d_\rho(f(z), f(w)) \leqslant L d_\tau(z, w)$. Then for $z \in K$ and $w \in \overline{D}(z_0, 2R)$,

$$\frac{d_\rho(f(z), f(w))}{|f(z) - f(w)|}\,\frac{|f(z) - f(w)|}{|z - w|} \leqslant L\,\frac{d_\tau(z, w)}{|z - w|}.$$

When $w \to z$, we obtain $\rho(f(z))|f'(z)| \leqslant L\tau(z)$, or $D_{\tau,\rho} f(z) \leqslant L$, for all $z \in K$ and $f \in \mathscr{F}$.

Conversely, suppose that for each compact set K there exists a positive number M such that for all $z \in K$ and all $f \in \mathscr{F}$, $D_{\tau,\rho} f(z) \leqslant M$. Fix a compact set in Δ, say $\overline{D}_\tau(z_0, R)$. For the compact set $\overline{D}_\tau(z_0, 3R + 2)$ determine M such that $\rho(f(z))|f'(z)| \leqslant M\tau(z)$ for all $z \in \overline{D}_\tau(z_0, 2R + 1)$ and all $f \in \mathscr{F}$. Consider any $z, w \in \overline{D}(z_0, R)$. There is a path γ in Δ from z_0 to z with

$$d_\tau(z_0, z) \leqslant \int_\gamma \tau(\zeta)|d\zeta| < R + 1.$$

As $d_\tau(z, w) \leqslant 2R$, there is a path δ in Δ from z to w with

$$d_\tau(z, w) \leqslant \int_\delta \tau(\zeta)|d\zeta| < 2R + 1. \tag{6.1}$$

Then γ followed by δ is a path in Δ from z_0 to w with τ-length at most $3R + 2$, so this path lies in $\overline{D}_\tau(z_0, 3R + 2)$. In particular, δ lies in $\overline{D}_\tau(z_0, 3R + 2)$. Then

$$\begin{aligned} d_\rho(f(z), f(w)) &\leqslant \int_{f\circ\delta} \rho(\omega)|d\omega| \\ &= \int_\delta \rho(f(\zeta))|f'(\zeta)||d\zeta| \end{aligned}$$

$$= \int\limits_{f \circ \delta} D_{\tau,\rho} f(\zeta)\tau(\zeta)|\mathrm{d}\zeta|$$
$$\leqslant M \int\limits_{\delta} \tau(\zeta)|\mathrm{d}\zeta|.$$

When we take the infimum over all paths δ in Δ from z to w that satisfy (6.1), we obtain $d_\rho(f(z), f(w)) \leqslant M d_\tau(z, w)$ for all $z, w \in \overline{D}_\tau(z_0, R)$ and every $f \in \mathscr{F}$.

(b) The proof is similar to, but simpler than, the proof of (a). □

Stated differently, uniform-type d_τ-to-d_ρ Lipschitz conditions on a family $\mathscr{F} \subseteq \mathscr{A}[\Delta, \Omega]$ are equivalent to uniform-type infinitesimal d_τ-to-d_ρ Lipschitz conditions, that is, uniform bounds on $D_{\tau,\rho} f$

Example 6.1 A holomorphic function f on $\mathbb{D}$ is called a *Bloch function* if $|f'(z)|/\lambda_{\mathbb{D}}(z)$ is uniformly bounded. Thus, Bloch functions are precisely the holomorphic functions on $\mathbb{D}$ that are globally $h_{\mathbb{D}}$-to-e Lipschitz. Similarly, a meromorphic function on $\mathbb{D}$ is a *normal function* if $f^{\#}(z)/\lambda_{\mathbb{D}}(z)$ is uniformly bounded, so normal functions are meromorphic functions on $\mathbb{D}$ that are $h_{\mathbb{D}}$-to-s Lipschitz. Finally, a meromorphic function f on $\mathbb{C}$ is a *Yosida function* if $f^{\#}(z)$ is uniformly bounded on $\mathbb{C}$, so Yosida functions are e-to-s Lipschitz functions.

7 Relative Compactness for Lipschitz Families

Throughout this section Δ and Ω are regions in $\mathbb{C}_\infty$ with complete conformal semi-metrics τ and ρ unless other metrics are indicated. Often, these semi-metrics will be the hyperbolic, Euclidean or spherical depending upon the context. For convenience, we reformulate the Arzelà–Ascoli Theorem for Lipschitz families.

Theorem 7.1 (Arzelà–Ascoli Theorem for Lipschitz families) *Suppose that $\mathscr{F} \subset \mathscr{C}[\Delta, \Omega]$ is locally uniformly d_τ-to-d_ρ Lipschitz. Then $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \Omega]$ if and only if there exists $z_0 \in \Delta$ such that $\mathscr{F}(z_0) = \{f(z_0) : f \in \mathscr{F}\}$ is relatively compact in Ω.*

Proof The necessity follows immediately from the Arzelà–Ascoli Theorem. For the sufficiency we again use the Arzelà–Ascoli Theorem. The local uniform d_τ-to-d_ρ Lipschitz condition implies that $\mathscr{F}$ is equicontinuous on Δ. Fix $z \in \Delta$. The uniform d_τ-to-d_ρ Lipschitz condition for the compact set $\{z_0, z\}$ together with the fact that $\mathscr{F}(z_0)$ is relatively compact implies that $\mathscr{F}(z)$ is relatively compact because ρ is complete. Thus, condition (b) of the Arzelà–Ascoli Theorem holds. □

For subfamilies $\mathscr{F}$ of $\mathscr{A}[\Delta, \Omega]$ a stronger theorem holds.

Theorem 7.2 (Arzelà–Ascoli Theorem for Lipschitz families of analytic maps) *Suppose that $\mathscr{F} \subseteq \mathscr{A}[\Delta, \Omega]$. Then $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \Omega]$ if and only if $\mathscr{F}$ is locally uniformly τ-to-ρ Lipschitz and there exists $z_0 \in \Delta$ such that $\mathscr{F}(z_0) = \{f(z_0) : f \in \mathscr{F}\}$ is relatively compact in Ω.*

Proof The sufficiency follows immediately from Theorem 7.1. For the necessity it suffices to show that if $\overline{\mathscr{F}}$ is compact in $\mathscr{C}[\Delta, \Omega]$, then it is locally uniformly d_τ-to-d_ρ Lipschitz. Suppose that $\overline{\mathscr{F}}$ is compact. Fix a compact set $K \subset \Delta$. Lemma 3.1 implies that the continuous functional $f \mapsto |f|_K$ attains a maximum value, say M, on the compact set $\overline{\mathscr{F}}$. Then Theorem 6.2 implies that $\overline{\mathscr{F}}$ is uniformly d_τ-to-d_ρ Lipschitz on compact subsets of Δ. □

Note that if τ, ρ are replaced by topologically equivalent metrics $\tilde{\tau}$, $\tilde{\rho}$ with $\tilde{\rho}$ complete, then Theorem 7.2 also holds for these metrics. This is obvious when the new metrics are bi-Lipschitz equivalent to the original metrics. However, only the weaker assumption of topologically equivalent metrics is needed.

Several known normality tests are special instances of Theorem 7.2.

Corollary 7.1 *Suppose that Δ is a hyperbolic region. Then $\mathscr{F} \subset \mathscr{A}[\Delta, \mathbb{C}]$ is relatively compact in $\mathscr{C}[\Omega, \mathbb{C}]$ if and only if $\mathscr{F}$ is locally uniformly h_Δ-to-e Lipschitz and there exists $z_0 \in \Delta$ such that $\mathscr{F}(z_0) = \{f(z_0) : f \in \mathscr{F}\}$ is relatively compact in $\mathbb{C}$.*

A different characterization is given by a theorem of Montel [11, p. 153]: $\mathscr{F} \subset \mathscr{A}[\Delta, \mathbb{C}]$ is relatively compact in $\mathscr{C}[\Omega, \mathbb{C}]$ if and only if it is locally uniformly bounded. Corollary 7.1 implies that the family $\mathscr{B}$ of all Bloch functions f on $\mathbb{D}$ with the normalization $f(0) = 0$ is relatively compact in $\mathscr{C}[\mathbb{D}, \mathbb{C}]$

Corollary 7.2 (Royden) *Suppose that Δ is a hyperbolic region and ρ is a semi-metric on $\mathbb{C}_\infty$. Then $\mathscr{F} \subset \mathscr{A}[\Delta, \mathbb{C}_\infty]$ is relatively compact in $\mathscr{C}[\Delta, \mathbb{C}_\infty]$ if and only if $\mathscr{F}$ is locally uniformly h_Δ-to-d_ρ Lipschitz.*

Royden [32] established this in the special case in which the semi-metric has a single zero at ∞. For example, he verified the normality of the family $\mathscr{F}$ of all functions $f \in \mathscr{A}[\Delta, \mathbb{C}_\infty]$ that satisfy $|f'| \leqslant e^{|f|}$ by using the semi-metric $\rho(z) = 1/e^{|z|}$ on $\mathbb{C}_\infty$ which has a zero at ∞.

Corollary 7.3 (Marty) *Suppose that Δ is a hyperbolic region. Then $\mathscr{F} \subset \mathscr{A}[\Delta, \mathbb{C}_\infty]$ is relatively compact in $\mathscr{C}[\Delta, \mathbb{C}_\infty]$ if and only if $\mathscr{F}$ is locally uniformly h_Δ-to-s Lipschitz.*

Marty's Theorem is the special instance of Royden's Theorem when ρ is the spherical metric. As Royden [32] pointed out, although Marty's result is necessary and sufficient for the relative compactness of a family of holomorphic or meromorphic functions, it may not be easy to apply in certain instances. Therefore, a result like Corollary 7.2 can be useful. The conclusion of Corollary 7.3 is equivalent to the assertion that for each compact set $K \subset \Delta$ there exists a positive number M such that for all $z \in K$ and all $f \in \mathscr{F}$, $f^\#(z)/\lambda_\Delta(z) \leqslant M$. When $\Delta \subset \mathbb{C}$ this is equivalent to the existence of a positive constant M such that $f^\#(z) \leqslant M$ for all $z \in K$ and all $f \in \mathscr{F}$ because the hyperbolic and Euclidean metrics are bi-Lipschitz equivalent on compact subsets of Δ. If $\Delta \subset \mathbb{C}_\infty$, then the equivalent condition is that for each compact set $K \subset \Delta$ there is a positive constant M such that $\mathscr{D}f(z) \leqslant M$ for all $z \in K$ and all $f \in \mathscr{F}$. This is the form of Corollary 7.3 originally stated by Marty [20]. Marty's Theorem implies that the family of normal functions and the family of Yosida functions are normal families.

8 Lipschitz Conditions Relative to a Larger Co-domain

Throughout this section Δ and Ω are regions in $\mathbb{C}_\infty$ with complete conformal semi-metrics τ and ρ unless other metrics are indicated. Moreover, Σ is a region with $\Omega \subseteq \Sigma$ and there is a complete semi-metric μ on Σ. In practice, these semi-metrics will usually be the hyperbolic, Euclidean or spherical, depending upon the context.

Here is the motivation for this section. A locally uniformly d_τ-to-d_ρ Lipschitz family $\mathscr{F} \subset \mathscr{C}[\Delta, \Omega]$ need not be relatively compact in $\mathscr{C}[\Delta, \Omega]$. By Theorem 7.2, $\mathscr{F}$ is not relatively compact if and only if for each $z_0 \in \Delta$, the orbit $\mathscr{F}(z_0)$ has an accumulation point in $\partial\Omega$. In this situation it is possible that $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \Sigma]$, where Σ is a region containing Ω.

Example 8.1 We illustrate this idea in a simple situation. The family $\mathscr{F}$ of functions $f_n(z) = z + n$, $n \in \mathbb{N}$, is uniformly e-to-e Lipschitz. It is not relatively compact in $\mathscr{C}[\mathbb{C}, \mathbb{C}]$ because no point of $\mathbb{C}$ has a relatively compact orbit under $\mathscr{F}$. The situation changes when the co-domain is taken to be $\mathbb{C}_\infty$ with the spherical metric, rather than $\mathbb{C}$ with the Euclidean metric. Then every orbit is relatively compact. Because $2|dz|/(1+|z|^2) \leqslant 2|dz|$, the spherical distance s satisfies a Lipschitz condition relative to the Euclidean distance;explicitly, $s(z, w) \leqslant 2|z - w|$ for all $z, w \in \mathbb{C}$. Therefore, if $\mathscr{F} \subset \mathscr{C}[\mathbb{C}, \mathbb{C}]$ is locally uniformly e-to-e Lipschitz, then it is also locally uniformly e-to-s Lipschitz as a subset of $\mathscr{C}[\mathbb{C}, \mathbb{C}_\infty]$. Hence, $\mathscr{F}$ is relatively compact in $\mathscr{C}[\mathbb{C}, \mathbb{C}_\infty]$.

Definition 8.1 Suppose that Ω and Σ are regions with conformal metrics ρ and μ, respectively, and $\Omega \subset \Sigma$. The metric μ satisfies a Lipschitz condition relative to ρ if there is a positive number L such that $\mu(z) \leqslant L\rho(z)$ for all $z \in \Omega$.

It is easy to see that $\mu \leqslant L\rho$ on Ω if and only if $d_\mu(z, w) \leqslant L d_\rho(z, w)$ for all $z, w \in \Omega$. As noted in Example 8.1, the spherical distance s satisfies a Lipschitz condition relative to the Euclidean distance.

Theorem 8.1 *Suppose that $\overline{\Omega}$ is a compact subset of Σ and μ satisfies a Lipschitz condition relative to ρ. If $\mathscr{F} \subset \mathscr{C}[\Delta, \Omega]$ is locally uniformly d_τ-to-d_ρ Lipschitz, then $\mathscr{F}$ is locally uniformly d_τ-to-d_μ Lipschitz and relatively compact in $\mathscr{C}[\Delta, \Sigma]$.*

Proof Because $d_\mu(u, v) \leqslant L d_\rho(u, v)$ for all $u, v \in \Omega$, it is elementary that $\mathscr{F}$ is locally uniformly d_τ-to-d_μ Lipschitz. As $\overline{\Omega}$ is a compact subset of Σ, for each $z_0 \in \Delta$, the orbit $\mathscr{F}(z_0)$ is relatively compact in Σ. Hence, $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \Sigma]$ by Theorem 7.1. □

Corollary 8.1 *Suppose that Δ is a hyperbolic region. If $\mathscr{F} \subset \mathscr{C}[\Delta, \mathbb{C}]$ is locally uniformly h_Δ-to-e Lipschitz, then $\mathscr{F}$ is locally uniformly h_Δ-to-s Lipschitz and relatively compact in $\mathscr{C}[\Delta, \mathbb{C}_\infty]$.*

We establish a simple sufficient condition for the existence of a Lipschitz condition between metrics.

Lemma 8.1 *Suppose that $\Omega \subset \Sigma$, $\overline{\Omega}$ is a compact subset of Σ and $\lim_{z\to\zeta} \frac{\mu(z)}{\rho(z)} = 0$ for all $\zeta \in \partial\Omega$. Then μ satisfies a Lipschitz condition with respect to ρ.*

Proof The positive function μ/ρ becomes a continuous non-negative function on the compact set $\overline{\Omega}$ when we define its value on $\partial\Omega$ to be 0. Therefore, it attains a maximum value L on the compact set $\overline{\Omega}$, so $\mu(z) \leqslant L\rho(z)$ for all $z \in \Omega$. □

Theorem 8.2 *Suppose that Ω is a hyperbolic region in $\mathbb{C}_\infty$.*

(a) *If $\Omega \subset \mathbb{C}$, then $\lim_{z\to\zeta} \lambda_\Omega(z) = +\infty$ for each $\zeta \in \partial\Omega \cap \mathbb{C}$.*
(b) *In general, $\lim_{z\to\zeta} \lambda_\Omega(z)/\sigma(z) = +\infty$ for all $\zeta \in \partial\Omega$.*

Proof It is known that $\lambda_{0,1}(z) \to +\infty$ as $z \to 0, 1$; this is not true when $z \to \infty$ (see [1, p. 18]). This implies that if $a, b \in \mathbb{C}$ and $\mathbb{C}_{a,b} = \mathbb{C}\backslash\{a, b\}$, then $\lambda_{a,b}(z) \to +\infty$ as $z \to a, b$.

(a) If Ω is a hyperbolic in region, then $\mathbb{C}\backslash\Omega$ has at least two boundary points in $\mathbb{C}$. Fix $a, b \in \partial\Omega \cap \mathbb{C}$. Then $\Omega \subset \mathbb{C}_{a,b}$ and the monotonicity property of the hyperbolic metric gives $\lambda_{a,b} \leqslant \lambda_\Omega$. Hence, $\lambda_\Omega(z) \to +\infty$ as $z \to \zeta$.
(b) If $\zeta \in \partial\Omega \cap \mathbb{C}$, then (b) follows from (a). It remains to establish (b) when $\zeta = \infty \in \partial\Omega$.The Möbius transformation $j(z) = 1/z$ (a rotation of $\mathbb{C}_\infty$) maps Ω onto a region Ω' that has 0 as a boundary point. Because the hyperbolic metric is conformally invariant and $j(\Omega') = \Omega$, $j^*(\lambda_\Omega) = \lambda_{\Omega'}$. Also, $j^*(\sigma) = \sigma$ since the spherical metric is invariant under rotations of $\mathbb{C}_\infty$. Therefore,

$$\frac{\lambda_{\Omega'}(z)}{\sigma(z)} = \frac{\lambda_\Omega(j(z))}{\sigma(j(z))}.$$

As 0 is a boundary point of Ω' in $\mathbb{C}$,

$$+\infty = \lim_{z\to 0} \frac{\lambda_{\Omega'}(z)}{\sigma(z)} = \lim_{z\to 0} \frac{\lambda_\Omega(j(z))}{\sigma(j(z))} = \lim_{z\to\infty} \frac{\lambda_\Omega(z)}{\sigma(z)}.$$

□

Corollary 8.2 *Suppose that Ω is a hyperbolic region in $\mathbb{C}_\infty$.*

(a) *If $\overline{\Omega} \subset \mathbb{C}$, then the Euclidean metric satisfies a Lipschitz condition relative to λ_Ω in Ω.*
(b) *The spherical metric σ satisfies a Lipschitz condition relative to λ_Ω in Ω.*

Similarly, if $\overline{\Omega} \subset \Sigma$, where Σ is a hyperbolic region, then λ_Σ satisfies a Lipschitz condition relative to λ_Ω. The family of all analytic maps between hyperbolic regions becomes relatively compact when we replace the co-domain by $\mathbb{C}$ or $\mathbb{C}_\infty$, as appropriate.

Theorem 8.3 *Suppose that Δ and Ω are hyperbolic regions.*

(a) *If $\overline{\Omega} \subset \mathbb{C}$, then the family $\mathscr{A}[\Delta, \Omega]$ satisfies a uniform h_Δ-to-e Lipschitz condition and is relatively compact in $\mathscr{C}[\Delta, \mathbb{C}]$.*
(b) *The family $\mathscr{A}[\Delta, \Omega]$ satisfies a uniform h_Δ-to-s Lipschitz condition and is relatively compact in $\mathscr{C}[\Delta, \mathbb{C}_\infty]$.*

Proof (a) By Corollary 8.2(a), the Euclidean distance satisfies a Lipschitz condition relative to h_Ω, say $|u - v| \leqslant L h_\Omega(u, v)$ for all $u, v \in \Omega$. If $f \in \mathscr{A}[\Delta, \Omega]$, then the general version of the Schwarz–Pick Lemma implies that for all $z, w \in \Delta$, $h_\Omega(f(z), f(w)) \leqslant h_\Delta(z, w)$. Hence, $|f(z) - f(w)| \leqslant L h_\Delta(z, w)$ for all $z, w \in \Delta$.

(b) The proof of (b) is similar with the spherical metric in place of the Euclidean metric and Corollary 8.2(b) rather than Corollary 8.2(a).

Corollary 8.3 (Montel) *Suppose that Δ is a hyperbolic region. Then the family $\mathscr{A}[\Delta, \mathbb{C}_{0,1}]$ of holomorphic maps of Δ into $\mathbb{C}_{0,1}$ satisfies a uniform h_Δ-to-s Lipschitz condition, so $\mathscr{A}[\Delta, \mathbb{C}_{0,1}]$ is relatively compact in $\mathbb{C}[\Delta, \mathbb{C}_\infty]$.*

When we combine Corollary 8.3 with Theorem 5.3, we obtain a strengthening of Montel's Theorem due to Carathéodory and Landau [7]; see also [9].

Theorem 8.4 (Carathéodory and Landau) *Let Δ be a hyperbolic region. Suppose that $\mathscr{F}$ is a family of meromorphic functions in Δ, such that for each f in $\mathscr{F}$, the image region $f(\Omega)$ omits three points a_f, b_f and c_f in $\mathbb{C}_\infty$ and*

$$\inf_{f \in \mathscr{F}} \chi(a_f, b_f) \,.\, \chi(b_f, c_f) \,.\chi(c_f, a_f) > 0. \tag{8.1}$$

Then $\mathscr{F}$ satisfies a uniform h_Δ-to-s Lipschitz condition and is relatively compact in $\mathscr{C}[\Delta, \mathbb{C}_\infty]$.

Proof It is elementary to verify that (8.1) is equivalent to the assumption that there exists a positive constant m such that for every $f \in \mathscr{F}$,

$$\chi(a_f, b_f) \geqslant m \quad \chi(b_f, c_f) \geqslant m, \quad \chi(c_f, a_f) \geqslant m. \tag{8.2}$$

The idea of the proof is to write $\mathscr{F}$ as the composition of two Lipschitz families. For each $f \in \mathscr{F}$, let g_f be the unique Möbius map that takes a_f, b_f, c_f to 0, 1, ∞, respectively. Then $g_f \circ f$ is a holomorphic function in $\mathscr{A}[\Delta, \mathbb{C}_{0,1}]$ and $f = h_f \circ (g_f \circ f)$, where $h_f = g_f^{-1}$. The inequalities (8.2) and Theorem 5.3(d) imply that $\mathscr{H} = \{h_f : f \in \mathscr{F}\}$ is a χ-to-χ Lipschitz family of Möbius maps, so there exists a positive constant M such that for all $h_f \in \mathscr{H}$ and all $u, v \in \mathbb{C}_\infty$, $\chi(h_f(u), h_f(v)) \leqslant M\chi(u, v)$. By Corollary 8.3, there is a constant L such that for all $z, w \in \Delta$ and all $f \in \mathscr{F}$, $\chi(g_f \circ f(z), g_f \circ f(w)) \leqslant L h_\Delta(z, w)$ because $\chi \leqslant s$. Then for all $z, w \in \Omega$ and any $f \in \mathscr{F}$,

$$\begin{aligned} \chi(f(z), f(w)) &= \chi(h_f \circ g_f \circ f(z), h_f \circ g_f \circ f(w)) \\ &\leqslant M\chi(g_f \circ f(z), g_f \circ f(w)) \\ &\leqslant M L h_\Delta(z, w). \end{aligned}$$

Thus, $\mathscr{F}$ is uniformly h_Δ-to-s Lipschitz. □

9 A Retrospective Look at Normal Families

Some authors assert that a family $\mathscr{F}$ of holomorphic functions defined on a region Δ is a 'normal family' if each sequence from $\mathscr{F}$ has a subsequence that is locally uniformly convergent to a holomorphic function. Ahlfors [2] uses the expression "normal with respect to $\mathbb{C}$" for such a family. This is equivalent to asserting that $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \mathbb{C}]$. A characterization of such normal families in terms of Lipschitz conditions is given in Corollary 7.1.

Montel's original definition of a normal family of holomorphic functions requires that each sequence from $\mathscr{F}$ has a subsequence that converges locally uniformly and the limit function is either holomorphic or the constant ∞. This definition does not make use of a topology on the space of continuous functions; the study of topology was in its infancy at the time of Montel's early work. Montel's definition is equivalent to $\mathscr{F}$ being relatively compact in $\mathscr{C}[\Delta, \mathbb{C}_\infty]$ because a sequence of holomorphic functions that is locally uniformly convergent relative to the chordal, or equivalently, the spherical, distance on the co-domain is either holomorphic or the constant ∞, see [2, p. 226] or [11, p. 156]. Marty's Theorem and Royden's Theorem characterize such families in terms of Lipschitz conditions. This motivates the following definition. We want a definition that includes Montel's original definition and also applies to (non-analytic) families that satisfy a Lipschitz condition:

Definition 9.1 Suppose that $\mathscr{F} \subseteq \mathscr{C}[\Delta, \Omega]$, where Δ, Ω and $\Sigma \supseteq \Omega$ are regions. Then $\mathscr{F}$ is a *normal family relative to* Σ if $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \Sigma]$ and the closure of $\mathscr{F}$ in $\mathscr{C}[\Delta, \Sigma]$ is the closure of $\mathscr{F}$ in $\mathscr{C}[\Delta, \Omega]$ together with constant maps into $\partial\Omega$, the boundary of Ω as a subset of Σ.

For $\Omega = \mathbb{C}$ and $\Sigma = \mathbb{C}_\infty$ this is Montel's original definition of a normal family of holomorphic functions. If Δ and Ω are hyperbolic regions and $\Sigma = \mathbb{C}_\infty$, then by Theorem 8.3(b) $\mathscr{A}[\Delta, \Omega]$ is uniformly h_Δ-to-s Lipschitz and relatively compact in $\mathscr{C}[\Delta, \mathbb{C}_\infty]$. Hurwitz' Theorem implies that the limit in $\mathscr{C}[\Delta, \mathbb{C}_\infty]$ of any sequence from $\mathscr{A}[\Delta, \Omega]$ either belongs to $\mathscr{A}[\Delta, \Omega]$ or is a constant map into $\partial\Omega$. Hence, according to Definition 9.1, $\mathscr{A}[\Delta, \Omega]$ is normal relative to $\mathbb{C}_\infty$.

Let us consider the situation when Δ and Ω are hyperbolic regions and $\mathscr{F} \subset \mathscr{C}[\Delta, \Omega]$ is a locally uniformly h_Δ-to-h_Ω Lipschitz. Because the spherical distance s satisfies a Lipschitz condition relative to h_Ω, $\mathscr{F}$ is locally uniformly h_Δ-to-s Lipschitz and so is relatively compact in $\mathscr{C}[\Delta, \mathbb{C}_\infty]$. If f_n is a sequence from $\mathscr{F}$ that is convergent to $f : \Delta \to \mathbb{C}_\infty$, one may show that either $f(\Delta) \subseteq \Omega$, in which case $f_n \to f$ in $\mathscr{C}[\Delta, \Omega]$, or else for each $z \in \Delta$, $\{f_n(z) : n = 1, 2, \ldots\}$ has no limit points in Ω. This means that the limit function f maps Δ into $\partial\Omega$. In the latter case, for an analytic function the Open Mapping Theorem implies that f must be a constant map of Ω into $\partial\Omega$. Do locally uniformly Lipschitz families have the property that limiting maps of Δ into $\partial\Omega$ must be constant?

Example 9.1 Lipschitz maps need not be open or orientation preserving. A simple example is the *folding map* $F : \mathbb{D} \to \mathbb{D}$ defined by

$$F(z) = \begin{cases} z, & \text{if} \quad \mathrm{Im} z \geqslant 0; \\ \bar{z}, & \text{if} \quad \mathrm{Im} z < 0. \end{cases}$$

Then F is a self-map of $\mathbb{D}$ that is non-expansive relative to $h_{\mathbb{D}}$. Clearly, F is neither orientation preserving nor an open mapping. Thus, Lipschitz maps are unlike analytic functions in these two senses. Nevertheless, we are able to show that locally Lipschitz families are normal in the sense of Definition 9.1; see Theorem 11.1 and its corollaries.

10 The Escher Condition

Definition 10.1 Suppose that $\Omega \subseteq \Sigma$ and ρ, μ are metrics on Ω, Σ, respectively. Then ρ satisfies an *Escher condition* relative to μ if for all $R > 0$ and $\varepsilon > 0$ there is a compact set K in Ω such that if $z \in \Omega \backslash K$ and $d_\rho(z, w) < R$, then $d_\mu(z, w) \leqslant \varepsilon d_\rho(z, w)$.

We note that in Definition 10.1 the point w is not required to lie in $\Omega \backslash K$. An Escher condition is a type of restricted Lipschitz condition between d_ρ and d_μ near $\partial\Omega$ that is only valid when z and w are not too far apart relative to d_ρ.

Example 10.1 To help convey the meaning of an Escher condition, we indicate a geometric consequence. Suppose that ρ satisfies an Escher condition relative to μ and we choose ε/R in place of ε. It follows that if $a \in \Omega\backslash K$, then $d_\rho(a, z) < R$ implies $d_\mu(a, z) < \varepsilon$; hence, $D_\rho(a, R) \subset D_\mu(a, \varepsilon)$ for all $a \in \Omega\backslash K$. In words, all ρ-disks with a fixed radius R are contained in small μ-disks when the center of the ρ-disk is sufficiently close to $\partial\Omega$. That the hyperbolic metric on $\mathbb{D}$ satisfies this shrinking disk condition relative to the Euclidean metric is evident in the well-known 'limit-circle' prints of M. C. Escher. Also, it is clear that the Euclidean metric satisfies an Escher condition relative to the spherical metric.

Theorem 10.1 *Suppose that ρ and μ are metrics on a region Ω.*

(a) *If ρ satisfies an Escher condition relative to μ, then $\lim_{z\to\zeta} \frac{\mu(z)}{\rho(z)} = 0$ for all $\zeta \in \partial\Omega$.*
(b) *If ρ is a complete metric on Ω and $\lim_{z\to\zeta} \frac{\mu(z)}{\rho(z)} = 0$ for all $\zeta \in \partial\Omega$, then ρ satisfies an Escher condition relative to μ.*

Proof (a) Assume that ρ satisfies an Escher condition relative to μ. Suppose that $z \in \Omega\backslash K$ and $d_\rho(z, w) < R$ implies $d_\mu(z, w) \leqslant \varepsilon d_\rho(z, w)$. Fix $z \in \Omega\backslash K$. Then for all w sufficiently close to z,

$$\frac{d_\mu(z, w)}{|z - w|} \leqslant \varepsilon \frac{d_\rho(z, w)}{|z - w|}.$$

As $w \to z$, we obtain $\mu(z) \leqslant \varepsilon\rho(z)$ for all $z \in \Omega \backslash K$. This implies that $\mu(z)/\rho(z) \to 0$ as $z \to \zeta \in \partial\Omega$.

(b) The hypothesis implies that if we extend the definition of μ/ρ to $\partial\Omega$ by setting it equal to 0 on $\partial\Omega$, then μ/ρ becomes a continuous function on the compact set $\overline{\Omega}$ that vanishes on the boundary. In particular, given $\varepsilon > 0$, there is a compact set $K^* \subset \Omega$ such that $\mu(z)/\rho(z) < \varepsilon$ for all $z \in \Omega\backslash K^*$. Because ρ is complete, we may assume that $K^* = \overline{D}_\rho(z_0, S)$ for some $z_0 \in \Omega$ and positive number S. Set $K = \overline{D}_\rho(z_0, R + S)$. We will show that if $w \in \Omega\backslash K$, and $d_\rho(z, w) < R$, then $d_\mu(z, w) \leqslant \varepsilon d_\rho(z, w)$. Fix $w \in \Omega\backslash K$ and assume that $d_\rho(z, w) < R$. Given $\eta > 0$, there is a path γ in Ω from z to w such that

$$d_\rho(z, w) \leqslant \int_\gamma \rho(\zeta)|\mathrm{d}\zeta| < \eta + d_\rho(z, w) < R.$$

The path $\gamma : [0, 1] \to \Omega$ must lie in $\Omega\backslash K^*$. If not, then there exists $t \in [0, 1]$ such that $d_\rho(z_0, \gamma(t)) \leqslant S$. Because $d_\rho(w, \gamma(t)) < R$, we have

$$d_\rho(w, \gamma(t)) \geqslant d_\rho(w, z_0) - d_\rho(z_0, \gamma(t)) > (R + S) - S = R,$$

a contradiction. Thus, γ lies in $\Omega\backslash K^*$, and so

$$d_\mu(z, w) \leqslant \int_\gamma \mu(\zeta)|\mathrm{d}\zeta| < \varepsilon \int_\gamma \rho(\zeta)|\mathrm{d}\zeta| < \varepsilon\big(\eta + d_\rho(z, w)\big).$$

Because $\eta > 0$ is arbitrary, we deduce that $d_\mu(z, w) \leqslant \varepsilon d_\rho(z, w)$. □

Depending upon the hyperbolic region, the hyperbolic metric satisfies an Escher property relative to the Euclidean metric or the spherical metric.

Corollary 10.1 *Suppose that Ω is hyperbolic region in $\mathbb{C}_\infty$.*

(a) *If Ω is a bounded region in $\mathbb{C}$, then λ_Ω satisfies an Escher condition relative to the Euclidean metric in Ω.*
(b) *In general, λ_Ω satisfies an Escher condition relative to the spherical metric σ in Ω.*

Proof Both (a) and (b) follow from Theorems 8.2 and 10.1. □

Example 10.2 The hyperbolic metric on an unbounded region $\Omega \subset \mathbb{C}$ need not satisfy an Escher condition relative to the Euclidean metric. For instance, in the upper half plane $\mathbb{H}$, a hyperbolic disk with center it and fixed hyperbolic radius R gets larger in the Euclidean sense when $t \to +\infty$.

11 Normal Families of Lipschitz Maps

For local uniform Lipschitz families, relative compactness together with an Escher condition on the co-domain is equivalent to normality.

Theorem 11.1 *Assume that Δ and $\Omega \subseteq \Sigma$ are regions with metrics τ, ρ and μ, respectively, with ρ and μ complete. Suppose that $\mathscr{F} \subseteq \mathscr{C}[\Delta, \Omega]$ is locally uniformly d_τ-to-d_ρ Lipschitz, Ω is a relatively compact subset of Σ, and ρ satisfies an Escher condition relative to μ. Then $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \Sigma]$ if and only if $\mathscr{F}$ is a normal family relative to Σ.*

Proof There is no harm in assuming that τ is complete. As already noted, the normality of $\mathscr{F}$ relative to Σ implies that $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \Sigma]$. Conversely, suppose that $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \Sigma]$. Let f_n be a sequence in $\mathscr{F}$. Because $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \Sigma]$, we may assume that f_n is locally uniformly convergent to a function $f \in \mathscr{C}[\Delta, \Sigma]$. We show that either f_n tends to a constant map of Δ to $\partial\Omega$, or f maps into Ω and $f_n \to f$ in $\mathscr{C}[\Delta, \Omega]$. Fix $z_0 \in \Delta$ and set $\zeta = \lim_{n\to+\infty} f_n(z_0) = f(z_0)$.

First, we assume that $\zeta \in \partial\Omega$. We use the Escher property to verify that $f_n \to \zeta$ locally uniformly in Δ. Fix $\varepsilon > 0$ and a compact subset K of Δ. We may assume that the compact set is $K = \overline{D}_\tau(z_0, R)$ for some $R > 0$. The local uniform d_τ-to-d_ρ Lipschitz condition implies that there exists a positive L such that for all $z, w \in K$ and all $f \in \mathscr{F}$, $d_\rho(f(z), f(w)) \leqslant L d_\tau(z, w)$. The Escher condition implies that there is a compact set E in Ω such that if $v \in \Omega \setminus E$ and $d_\rho(u, v) < LR$, then $d_\mu(u, v) \leqslant (\varepsilon/(LR)) d_\rho(u, v)$. Because E is disjoint from $\partial\Omega$, there is a positive $\delta < \varepsilon$ so that if $v \in \Omega$ and $d_\mu(\zeta, v) < \delta$, then $v \in \Omega \setminus E$. Because $f_n(z_0) \to \zeta$, we may assume that $d_\mu(f_n(z_0), \zeta) < \delta$ for all $n \geqslant N$. Suppose that $n \geqslant N$ and $d_\tau(z_0, z) \leqslant R$. Then $d_\rho(f_n(z), f_n(z)) \leqslant LR$ and $f_n(z_0) \in \Omega \setminus E$, so that

$$d_\mu(f_n(z_0), f_n(z)) \leqslant \frac{\varepsilon}{LR} d_\rho(f_n(z_0), f_n(z)) \leqslant \varepsilon.$$

Hence, for $n \geqslant N$ and $d_\tau(z_0, z) \leqslant R$, $d_\mu(f_n(z), \zeta) < 2\varepsilon$. Thus, $f_n \to \zeta$ locally uniformly.

Otherwise, $\zeta \in \Omega$. The local uniform d_τ-to-d_ρ Lipschitz condition can be used to prove that for any $z \in \Delta$, $\{f_n(z) : n = 1, 2, \ldots\}$ is relatively compact in Ω. Therefore, $f(z) \in \Omega$ for all $z \in \Delta$. Suppose that $d_\tau(z_n, z) \to 0$. Then continuous convergence implies that $d_\rho(f_n(z_n), f(z)) \to 0$. As $w_n = f_n(z_n)$, $w = f(z) \in \Omega$, this implies that $d_\rho(w_n, w) \to 0$. Therefore, $f_n \to f$ in $\mathscr{F}[\Delta, \Omega]$.

Finally, we verify that $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \Sigma]$. Fix a point $z_0 \in \Delta$. Because Ω is relatively compact in Σ, $\mathscr{F}(z_0)$ is relatively compact in Σ. Also, the Escher condition implies there is a positive constant m such that $m d_\rho \leqslant d_\mu$ on Ω. Therefore, $\mathscr{F}$ is locally uniformly d_τ-to-d_μ Lipschitz as a family of maps in $\mathscr{C}[\Delta, \Sigma]$. Theorem 7.1 implies that $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \Sigma]$. □

Corollary 11.1 *Let $\mathscr{F} \subset \mathscr{C}[\Delta, \Omega]$, where Δ and $\Omega \subset \mathbb{C}_\infty$ are a hyperbolic regions. If $\mathscr{F}$ is locally uniformly h_Δ-to-h_Ω Lipschitz, then $\mathscr{F}$ is normal relative to $\mathbb{C}_\infty$.*

Proof The hyperbolic metric λ_Ω satisfies an Escher condition relative to the spherical metric σ. □

Note that Theorem 1.1 is a special case of Corollary 11.1. Also, Corollary 11.1 strengthens Theorem 8.3(a).

Corollary 11.2 *Suppose that Δ is a hyperbolic region, $\mathscr{F} \subset \mathscr{C}[\Delta, \mathbb{C}]$ and $\mathscr{F}$ is locally uniformly hyperbolic-to-Euclidean Lipschitz. Then $\mathscr{F}$ is relatively compact in $\mathscr{C}[\Delta, \mathbb{C}_\infty]$ and a normal family relative to $\mathbb{C}_\infty$.*

Proof The Euclidean metric satisfies an Escher condition relative to the spherical metric. □

Corollary 11.2 improves Corollary 8.1.

12 Concluding Remarks

Although we have considered subfamilies of the family of continuous maps between regions in $\mathbb{C}_\infty$, our results remain valid for families of maps between Riemann surfaces. We state the results that extend. We first state the context explicitly. Assume that Δ and Ω are Riemann surfaces. Let $\mathscr{C}[\Delta, \Omega]$ be the family of continuous maps $f : \Delta \to \Omega$ and $\mathscr{A}[\Delta, \Omega]$ the subfamily of analytic maps. Theorem 7.2 holds in this Riemann surface setting. Also, Theorem 8.1 is valid when Σ is a Riemann surface containing Ω.

If Σ is a compact Riemann surface of genus $g \geqslant 2$ and $\Omega \subsetneq \Sigma$, then λ_Ω satisfies an Escher condition relative to λ_Σ. Therefore, $\mathscr{A}[\Delta, \Omega]$ is relatively compact in $\mathscr{C}[\Delta, \Sigma]$. In fact, Theorem 11.1 is valid in this general setting. Next, suppose that Σ is a compact Riemann surface of genus $g = 1$. Then Σ is not hyperbolic but it does have a complete conformal metric μ with constant zero curvature that is unique up to multiplication by a positive constant. If $\Omega \subsetneq \Sigma$, then Ω is hyperbolic and λ_Ω satisfies an Escher condition relative to μ. Again, Theorem 11.1 extends to this setting.

In fact, many results of this paper even extend to Lipschitz families of maps between appropriate types of metric spaces.

Miniowitz [21] established an analog of Zalcman's Rescaling Lemma for normal families of quasimeromorphic mappings. The main tool used by Miniowitz is an analog of Marty's Normality Criterion; a family of K-quasimeromorphic functions in a region $\Delta \subset \mathbb{R}^n$ is a normal family if and only if it satisfies an α-Hölder condition on compact subsets of Δ, where $\alpha = K^{1/(1-n)}$. This rescaling result of Miniowitz has proved very useful. There might be analogs of this paper for families of quasi-regular (and other) functions that satisfy various uniform α-Hölder conditions.

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