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A Reverse Schwarz–Pick Inequality

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Abstract We prove a kind of "reverse Schwarz–Pick lemma" for holomorphic self-maps of the disk. The result becomes especially clear-cut for inner functions and casts new light on their derivatives.

Keywords Schwarz–Pick lemma · Outer function · Inner function · Nevanlinna class

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1 Introduction and Main Result

We write $\mathbb{D}$ for the disk $\{z \in \mathbb{C}: |z| < 1\}$, $\mathbb{T}$ for its boundary, and m for the normalized arclength measure on $\mathbb{T}$; thus $dm(\zeta) = (2\pi)^{-1}|d\zeta|$. Further, H^∞ will denote the algebra of bounded holomorphic functions on $\mathbb{D}$, equipped with the usual supremum norm $\|\cdot\|_\infty$.

The classical Schwarz lemma and its invariant version, known as the Schwarz–Pick lemma (see [7, Ch. I]), lie at the heart of function theory on the disk. The Schwarz–Pick lemma tells us, in particular, that every function $\varphi \in H^\infty$ with $\|\varphi\|_\infty \le 1$ satisfies

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$$|\varphi'(z)| \le \frac{1-|\varphi(z)|^2}{1-|z|^2}, \qquad z \in \mathbb{D}. \tag{1.1}$$

Moreover, equality holds at some—or each—point of $\mathbb{D}$ if and only if φ is either a *Möbius transformation* (i.e., has the form

$$z \mapsto \lambda \frac{z-a}{1-\bar{a}z}$$

for some $\lambda \in \mathbb{T}$ and $a \in \mathbb{D}$) or a constant of modulus 1. It seems natural to ask how large the gap between the two quantities in (1.1) may be in general. To be more precise, we would like to know whether some sort of reverse estimate has a chance to hold, i.e., whether the quotient

$$\frac{1-|\varphi(z)|^2}{1-|z|^2} =: Q_\varphi(z)$$

admits an upper bound in terms of φ'. And if it does, what is the appropriate φ'-related majorant for that quantity?

Of course, there are limits to what can be expected. Assume, from now on, that φ is a *non-constant* H^∞-function of norm at most 1. First of all, while φ' may happen to vanish or become arbitrarily small at certain points of $\mathbb{D}$, it follows from Schwarz's lemma that Q_φ is always bounded away from zero; in fact,

$$Q_\varphi(z) \ge \frac{1-|\varphi(0)|}{1+|\varphi(0)|}, \qquad z \in \mathbb{D}. \tag{1.2}$$

As another example, consider the case where φ is analytic in a neighborhood of some closed arc $\gamma \subset \mathbb{T}$ and satisfies $\sup_{\zeta\in\gamma} |\varphi(\zeta)| < 1$. In this case, too, the two sides of (1.1) exhibit different types of behavior as z approaches γ. Indeed, $|\varphi'(z)|$ is then well-behaved, whereas $Q_\varphi(z)$ blows up like a constant times $(1-|z|)^{-1}$. Thus, the ratio $Q_\varphi(z)/|\varphi'(z)|$ is sometimes huge.

On the other hand, suppose that φ has an *angular derivative* (in the sense of Carathéodory) at a point $\zeta \in \mathbb{T}$. This means, by definition, that φ and φ' both have non-tangential limits at ζ and, once we agree to denote the two limits by $\varphi(\zeta)$ and $\varphi'(\zeta)$, the former of these satisfies $|\varphi(\zeta)| = 1$. The classical Julia–Carathéodory theorem (see [3, Ch. VI], [4, Ch. I] or [10, Ch. VI]) asserts that this happens if and only if

$$\liminf_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} < \infty,$$

a condition that can be further rewritten as

$$\liminf_{z\to\zeta} Q_\varphi(z) < \infty. \tag{1.3}$$

And if any of these holds, the theorem tells us also that $\varphi'(\zeta)$ coincides with the limit of the difference quotient

$$\frac{\varphi(z)-\varphi(\zeta)}{z-\zeta}$$

as $z \to \zeta$ non-tangentially, while the non-tangential limit of $Q_\varphi(z)$ equals $|\varphi'(\zeta)|$. In addition, this last number agrees with the value of the (unrestricted) lim inf in (1.3).

Consequently, if φ happens to possess an angular derivative $\varphi'(\zeta)$ at every point ζ of a set $\mathcal{E} \subset \mathbb{T}$, then the two sides of (1.1) have the same boundary values on $\mathcal{E}$. We may therefore expect the two quantities to be reasonably close near $\mathcal{E}$, so a certain "reverse Schwarz–Pick type estimate" is likely to hold on a suitable region of $\mathbb{D}$ adjacent to $\mathcal{E}$. And the more massive $\mathcal{E}$ is, the stronger should our reverse inequality become. Our main result, Theorem 1.1 below, provides such an estimate under the hypotheses that the "good" set $\mathcal{E}$ is (Lebesgue) measurable and the function $\log|\varphi'|$ is integrable on $\mathcal{E}$; of course, only the case $m(\mathcal{E}) > 0$ is of interest.

Before stating the result, let us recall that the *harmonic measure* ω_z associated with a point $z \in \mathbb{D}$ is given by

$$\mathrm{d}\omega_z(\zeta) = \frac{1-|z|^2}{|\zeta - z|^2}\,\mathrm{d}m(\zeta), \qquad \zeta \in \mathbb{T}.$$

For a measurable set $E \subset \mathbb{T}$, the quantity $\omega_z(E) = \int_E \mathrm{d}\omega_z$ is, thus, the value at z of the harmonic extension (into $\mathbb{D}$) of the characteristic function χ_E; this quantity can be roughly thought of as the normalized angle at which E is seen from z. We also recall that if h is a non-negative function on $\mathbb{T}$ with $\log h \in L^1(\mathbb{T})$, then

$$\mathcal{O}_h(z) := \exp\left(\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\log h(\zeta)\,dm(\zeta)\right), \qquad z \in \mathbb{D},$$

is a holomorphic function on $\mathbb{D}$ whose modulus has non-tangential boundary values h almost everywhere on $\mathbb{T}$. In fact,

$$|\mathcal{O}_h(z)| = \exp\left(\int_{\mathbb{T}} \log h\,\mathrm{d}\omega_z\right),$$

whence the preceding statement follows. This function $\mathcal{O}_h$ is known as the *outer function with modulus* h. Functions of the form $\lambda\mathcal{O}_h$, with $\lambda \in \mathbb{T}$ and h as above, will also be referred to as outer; see [7, Ch. II] for a further discussion of outer functions and their properties.

Theorem 1.1 *Let $\varphi \in H^\infty$ be a non-constant function with $\|\varphi\|_\infty = 1$, and let $\mathcal{E}$ be a measurable subset of $\mathbb{T}$ such that φ has an angular derivative almost everywhere* *on $\mathcal{E}$. Assume that* $\log|\varphi'| \in L^1(\mathcal{E})$ *and write* $\mathcal{O}$ *for the outer function with modulus* $|\varphi'|\chi_{\mathcal{E}} + \chi_{\mathbb{T}\setminus\mathcal{E}}$. *Then*

$$Q_\varphi(z) \le |\mathcal{O}(z)| \left(\frac{1}{1-\omega_z(\mathcal{E})} \cdot \frac{1+|z|}{1-|z|} \right)^{1-\omega_z(\mathcal{E})} \tag{1.4}$$

for all $z \in \mathbb{D}$.

The first factor on the right being

$$|\mathcal{O}(z)| = \exp\left(\int_{\mathcal{E}} \log|\varphi'|\, d\omega_z \right),$$

we may indeed view (1.4) as a reverse Schwarz–Pick inequality, since it provides us with an upper bound for Q_φ in terms of $|\varphi'|$. If, in addition, $|\varphi'|$ happens to be essentially bounded on $\mathcal{E}$, then we clearly have

$$|\mathcal{O}(z)| \le \|\varphi'\|_{\infty,\mathcal{E}}^{\omega_z(\mathcal{E})}, \tag{1.5}$$

where $\|\cdot\|_{\infty,\mathcal{E}} = \|\cdot\|_{L^\infty(\mathcal{E})}$. Combining (1.4) with (1.5) and with the elementary fact that

$$\sup\left\{t^{-t} : 0 < t < 1\right\} = e^{1/e}$$

leads to the weaker, but perhaps simpler, estimate

$$Q_\varphi(z) \le e^{1/e} \|\varphi'\|_{\infty,\mathcal{E}}^{\omega_z(\mathcal{E})} \left(\frac{1+|z|}{1-|z|} \right)^{1-\omega_z(\mathcal{E})}, \qquad z \in \mathbb{D}. \tag{1.6}$$

Both (1.4) and (1.6) reflect the influence of the "good" set $\mathcal{E}$ and of the "bad" set $\mathbb{T} \setminus \mathcal{E}$, depending on the location of z, in the spirit of Nevanlinna's *Zweikonstantensatz* (the two constants theorem). The latter supplies a sharp bound on $|f(z)|$ for an H^∞-function f whose modulus is bounded by two given constants on two mutually complementary subsets of the boundary; see, e.g., [6, Ch. VIII].

The next section contains some applications of Theorem 1.1 to inner functions, while the proof of the theorem is given in Sect. 3.

2 Inner Functions and Their Derivatives

We recall that a function $\theta \in H^\infty$ is said to be *inner* if $\lim_{r\to 1^-} |\theta(r\zeta)| = 1$ for almost all $\zeta \in \mathbb{T}$. Also involved in what follows is the *Nevanlinna class* $\mathcal{N}$, defined as the set of all holomorphic functions f on $\mathbb{D}$ that satisfy

$$\sup_{0<r<1} \int_{\mathbb{T}} \log^+ |f(r\zeta)|\, dm(\zeta) < \infty.$$

Equivalently, $\mathcal{N}$ is formed by the ratios u/v with $u, v \in H^\infty$ and with v zero-free on $\mathbb{D}$; see [7, Ch. II]. Imposing the additional restriction that v be outer, one arrives at the definition (or a characterization) of the *Smirnov class* $\mathcal{N}^+$.

Now, if θ is an inner function with $\theta' \in \mathcal{N}$, then θ has an angular derivative a. e. on $\mathbb{T}$ and $\log|\theta'| \in L^1(\mathbb{T})$. Therefore, when applying Theorem 1.1 to $\varphi = \theta$, we may take $\mathcal{E} = \mathbb{T}$. This yields the following result.

Corollary 2.1 *Let θ be a non-constant inner function with $\theta' \in \mathcal{N}$, and let $\mathcal{O} = \mathcal{O}_{|\theta'|}$ be the outer factor of θ' (i.e., the outer function with modulus $|\theta'|$ on $\mathbb{T}$). Then*

$$Q_\theta(z) \le |\mathcal{O}(z)|, \qquad z \in \mathbb{D}. \tag{2.1}$$

A similar estimate can be found in [5]. We also mention that there is an alternative route to Corollary 2.1 via subharmonicity, which hinges on Lemma 1.1 from [8]; this approach was kindly brought to my attention by Haakan Hedenmalm.

As a consequence of the preceding result, we now derive an amusing characterization of Möbius transformations.

Corollary 2.2 *Given a non-constant inner function θ with $\theta' \in \mathcal{N}$, the following are equivalent.*

(i) *θ is a Möbius transformation.*
(ii) *θ' is an outer function.*
(iii) *There is a non-decreasing function $\eta\colon (0,\infty) \to (0,\infty)$ such that*

$$\eta\left(Q_\theta(z)\right) \le |\theta'(z)|, \qquad z \in \mathbb{D}. \tag{2.2}$$

Before proving this, we recall that an inner function θ with $\theta' \in \mathcal{N}$ will automatically have θ' in $\mathcal{N}^+$, a fact established by Ahern and Clark in [1].

Proof of Corollary 2.2 The (i) $\Longrightarrow$ (ii) part is straightforward, while the converse follows from (2.1). Indeed, if θ' is outer, then $|\mathcal{O}(z)|$ on the right-hand side of (2.1) coincides with $|\theta'(z)|$. Combining this with the Schwarz–Pick inequality (1.1), where we put $\varphi = \theta$, gives

$$|\theta'(z)| = Q_\theta(z), \qquad z \in \mathbb{D}. \tag{2.3}$$

Thus, the current choice of φ ensures equality in (1.1), therefore θ must be a Möbius transformation.

Now that (i) and (ii) are known to be equivalent, it suffices to show that (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii). The first of these implications is obvious, since every Möbius transformation satisfies (2.3), so that (2.2) holds with $\eta(t) = t$. Finally, assuming (iii) and using (1.2) with $\varphi = \theta$, we deduce that $|\theta'(z)|$ is bounded away from zero on $\mathbb{D}$, whence $1/\theta' \in H^\infty$. Because $\theta' \in \mathcal{N}^+$, it follows that θ' is an outer function, and we arrive at (ii). □

The (ii) $\Longrightarrow$ (i) part of Corollary 2.2 can be rephrased by saying that the derivative θ' of a non-Möbius inner function θ is never outer, as long as it is in $\mathcal{N}$. Some special cases of this statement have been known. In particular, it was proved by Ahern and Clark (see [1, Corollary 4]) that, within the current class of θ's, the inner part of θ' will be non-trivial provided that θ has a singular factor (because such factors are actually inherited by θ'). On the other hand, if θ is a finite Blaschke product with at least two zeros, then θ' is known to have zeros in $\mathbb{D}$ (see [11] for more information on the location of these), so it is clear, once again, that θ' is non-outer. The same conclusion is obviously true for those inner functions θ which have multiple zeros in $\mathbb{D}$.

The interesting case is, therefore, that of an infinite Blaschke product with simple zeros. Note that if B is such a Blaschke product, then, unlike in the rational case, B' may well be zero-free on $\mathbb{D}$. For instance, this happens for

$$B_\alpha(z) := \frac{S(z) - \alpha}{1 - \overline{\alpha} S(z)},$$

where S is the "atomic" singular inner function given by

$$S(z) := \exp\left(\frac{z+1}{z-1}\right)$$

and α is a point in $\mathbb{D} \setminus \{0\}$. One easily verifies that B_α is indeed a Blaschke product, while the inner factor of B'_α is S.

We conclude this section with a question. Let $\mathfrak{J}$ stand for the set of non-constant inner functions. *Which inner functions occur as inner factors (and/or divisors of such factors) for functions in* $\mathcal{N} \cap \{\theta' : \theta \in \mathfrak{J}\}$ *?* One immediate observation is that if I is inner and $I' \in \mathcal{N}$, then I divides the inner part of $(I^2)'$, and this last function is in $\mathcal{N}$.

3 Proof of Theorem 1.1

For all $z \in \mathbb{D}$ and almost all $\zeta \in \mathcal{E}$, Julia's lemma (see [7, p. 41]) yields

$$\frac{|\varphi(\zeta) - \varphi(z)|^2}{1 - |\varphi(z)|^2} \le |\varphi'(\zeta)| \cdot \frac{|\zeta - z|^2}{1 - |z|^2}, \tag{3.1}$$

or equivalently,

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \cdot \left| \frac{1 - \overline{\varphi(z)}\varphi(\zeta)}{1 - \overline{z}\zeta} \right|^2 \le |\varphi'(\zeta)| \tag{3.2}$$

(recall that $|\varphi(\zeta)| = 1$ whenever φ has an angular derivative at ζ). Keeping $z \in \mathbb{D}$ fixed for the rest of the proof, we now introduce the H^∞-function

$$F_z(w) := \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \cdot \left(\frac{1 - \overline{\varphi(z)}\varphi(w)}{1 - \overline{z}w} \right)^2 \tag{3.3}$$

and go on to rewrite (3.2) in the form

$$|F_z(\zeta)| \le |\varphi'(\zeta)|, \qquad \zeta \in \mathcal{E}. \tag{3.4}$$

Next, we define G_z to be the outer function with modulus

$$|G_z(\zeta)| = |\varphi'(\zeta)| \cdot \chi_{\mathcal{E}}(\zeta) + |F_z(\zeta)| \cdot \chi_{\widetilde{\mathcal{E}}}(\zeta), \qquad \zeta \in \mathbb{T},$$

where $\widetilde{\mathcal{E}} := \mathbb{T} \setminus \mathcal{E}$, and note that

$$|F_z(\zeta)| \le |G_z(\zeta)|, \qquad \zeta \in \mathbb{T}. \tag{3.5}$$

Indeed, for $\zeta \in \mathcal{E}$ this last inequality coincides with (3.4), while for $\zeta \in \widetilde{\mathcal{E}}$ it reduces to an obvious equality.

Since G_z is outer, (3.5) implies a similar estimate on $\mathbb{D}$, that is,

$$|F_z(w)| \le |G_z(w)|, \qquad w \in \mathbb{D}.$$

In particular, setting $w = z$, we obtain

$$|F_z(z)| \le |G_z(z)|. \tag{3.6}$$

It is clear from (3.3) that

$$|F_z(z)| = F_z(z) = \frac{1 - |\varphi(z)|^2}{1 - |z|^2} = Q_\varphi(z), \tag{3.7}$$

and we take further efforts to estimate $|G_z(z)|$.

We have

$$\log|G_z(z)| = \int_{\mathbb{T}} \log|G_z(\zeta)|\, \mathrm{d}\omega_z(\zeta) = I_1(z) + I_2(z), \tag{3.8}$$

where

$$I_1(z) := \int_{\mathcal{E}} \log|\varphi'(\zeta)|\, \mathrm{d}\omega_z(\zeta) = \log|\mathcal{O}(z)| \tag{3.9}$$

and

$$I_2(z) := \int_{\widetilde{\mathcal{E}}} \log|F_z(\zeta)|\, \mathrm{d}\omega_z(\zeta). \tag{3.10}$$

The function $t \mapsto \log t$ being concave for $t > 0$, we find that

$$
\begin{aligned}
I_2(z) &= \omega_z(\widetilde{\mathcal{E}}) \cdot \int_{\widetilde{\mathcal{E}}} \log|F_z(\zeta)| \frac{\mathrm{d}\omega_z(\zeta)}{\omega_z(\widetilde{\mathcal{E}})} \\
&\le \omega_z(\widetilde{\mathcal{E}}) \cdot \log\left(\frac{1}{\omega_z(\widetilde{\mathcal{E}})} \int_{\widetilde{\mathcal{E}}} |F_z(\zeta)|\,\mathrm{d}\omega_z(\zeta) \right). \qquad (3.11)
\end{aligned}
$$

We proceed by observing that

$$
|F_z(\zeta)| \le \frac{1+|z|}{1-|z|} \cdot \frac{|1-\overline{\varphi(z)}\varphi(\zeta)|^2}{1-|\varphi(z)|^2}, \qquad \zeta \in \mathbb{T}. \qquad (3.12)
$$

Furthermore,

$$
\begin{aligned}
\int_{\mathbb{T}} |1-\overline{\varphi(z)}\varphi(\zeta)|^2 \,\mathrm{d}\omega_z(\zeta) &= \int_{\mathbb{T}} \left[1 - 2\,\mathrm{Re}\left(\overline{\varphi(z)}\varphi(\zeta)\right) + |\varphi(z)|^2|\varphi(\zeta)|^2\right] \mathrm{d}\omega_z(\zeta) \\
&\le \int_{\mathbb{T}} \left[1 - 2\,\mathrm{Re}\left(\overline{\varphi(z)}\varphi(\zeta)\right) + |\varphi(z)|^2\right] \mathrm{d}\omega_z(\zeta) \\
&= 1 - |\varphi(z)|^2.
\end{aligned}
$$

(Here, the last step consists in integrating a harmonic function against $d\omega_z$, so that the output is the function's value at z). In conjunction with (3.12), this gives

$$
\int_{\mathbb{T}} |F_z(\zeta)|\,\mathrm{d}\omega_z(\zeta) \le \frac{1+|z|}{1-|z|},
$$

whence *a fortiori*

$$
\int_{\widetilde{\mathcal{E}}} |F_z(\zeta)|\,\mathrm{d}\omega_z(\zeta) \le \frac{1+|z|}{1-|z|}. \qquad (3.13)
$$

Plugging (3.13) into (3.11), we now get

$$
I_2(z) \le \omega_z(\widetilde{\mathcal{E}}) \log\left(\frac{1}{\omega_z(\widetilde{\mathcal{E}})} \cdot \frac{1+|z|}{1-|z|} \right). \qquad (3.14)
$$

Finally, we combine (3.8) with (3.9) and (3.14) to infer that

$$
\log|G_z(z)| \le \log|\mathcal{O}(z)| + \omega_z(\widetilde{\mathcal{E}}) \log\left(\frac{1}{\omega_z(\widetilde{\mathcal{E}})} \cdot \frac{1+|z|}{1-|z|} \right)
$$

and hence

$$|G_z(z)| \le |\mathcal{O}(z)| \left(\frac{1}{\omega_z(\widetilde{\mathcal{E}})} \cdot \frac{1+|z|}{1-|z|} \right)^{\omega_z(\widetilde{\mathcal{E}})}. \tag{3.15}$$

This done, a juxtaposition of (3.6), (3.7) and (3.15) yields the required estimate (1.4) and completes the proof. □

Remark The function F_z, as defined by (3.3), can be written in the form

$$F_z(w) = \frac{k^2_{\varphi,z}(w)}{k_{\varphi,z}(z)},$$

where

$$k_{\varphi,z}(w) := \frac{1-\overline{\varphi(z)}\varphi(w)}{1-\bar{z}w}$$

is the reproducing kernel for the de Branges–Rovnyak space $\mathcal{H}(\varphi)$; see [10]. Even though this space does not show up in our proof, the appearance of its kernel functions may not be incidental. It should be mentioned that Hilbert space methods have been previously employed, in the $\mathcal{H}(\varphi)$ setting, in connection with generalized Schwarz–Pick inequalities [2] and with the Julia–Carathéodory theorem on angular derivatives [9,10].

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