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# Pseudo almost periodic solutions for a class of nonlinear Duffing equations on time scales

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**Abstract**
In this paper, we study the pseudo almost periodic solutions for a class of nonlinear Duffing equations with $S^p$-pseudo almost periodic coefficients and delays on time scales. For this purpose, we establish a result of the existence and uniqueness of pseudo almost periodic solution for an abstract linear equation with $S^p$-almost periodic coefficients and $S^p$-pseudo almost periodic forcing term. Meanwhile, to deal with the delay, we extend some concepts of functions from $\mathbb{T} \to \mathbb{R}$ to $\mathbb{T} \to \Pi$, where $\mathbb{T}$ is a time scale with translation set $\Pi$, and give some basic properties for these concepts. Then, applying these results, we obtain some results on the existence and uniqueness of pseudo almost periodic solutions for the Duffing equation. Moreover, some examples are given to illustrate our main results.

**Keywords** $S^p$-pseudo almost periodic · Duffing equation · Pseudo almost periodic solution · Time scales

**Mathematics Subject Classification** 34N05 · 34C27

## 1 Introduction

In recent years, the dynamic behaviors of nonlinear Duffing equations have been widely investigated in Burton (1986); Hale (1977); Kuang (2012); Yoshizawa (1975) due to the potential use in the areas of physics, mechanics and other engineering technique fields. Among them, the existence of almost periodic solutions and pseudo almost periodic solutions have attracted many authors. Some results on the existence of almost periodic solutions were obtained in the literature (see e.g., Zhou and Liu (2009); Peng and Wang (2010); Xu (2012); Liu and Tunç (2015)).

Recently, Zhou and Liu (2009) considered the following model for a nonlinear Duffing equation with a deviating argument:

$$x''(t) - ax(t) + bx^m(t - \tau(t)) = p(t), \tag{1}$$

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where $\tau(t)$ and $p(t)$ are almost periodic functions on $\mathbb{R}$, $m > 1$ is an integer, $a > 0$ and $b \neq 0$ are constants. By setting

$$y = x'(t) + \delta x(t),$$

where $\delta > 1$ is a constant, (1) transforms into the following system:

$$\begin{cases} x'(t) = -\delta x(t) + y(t), \\ y'(t) = \delta y(t) + (a - \delta^2)x(t) - bx^m(t - \tau(t)) + p(t). \end{cases}$$

The authors gave some criteria for the existence of almost periodic solutions for (1).

Then, Peng and Wang (2010) considered the following model for a nonlinear Duffing equation with a deviating argument:

$$x''(t) + cx'(t) - ax(t) + bx^m(t - \tau(t)) = p(t), \tag{2}$$

where $\tau(t)$ and $p(t)$ are almost periodic functions on $\mathbb{R}$, $m > 1$ is an integer and $a, b, c$ are constants. By the transformation

$$y(t) = x'(t) + \xi x(t) - Q_1(t), \quad Q_2(t) = p(t) + (\xi - c)Q_1(t) - Q_1'(t),$$

where $\xi > 1$ is a constant and $Q_1(t)$ is continuous and differentiable, (2) transforms into the following system:

$$\begin{cases} x'(t) = -\xi x(t) + y(t) + Q_1(t), \\ y'(t) = -(c - \xi)y(t) + (a - \xi(\xi - c))x(t) - bx^m(t - \tau(t)) + Q_2(t), \end{cases} \tag{3}$$

and then proved the existence of positive almost periodic solutions of (2) and (3).

After that, system (3) has been naturally extended by Xu (2012) to the following system with time-varying coefficients and delays:

$$\begin{cases} \dfrac{dx(t)}{dt} = -\delta_1(t)x(t) + y(t) + Q_1(t), \\ \dfrac{dy(t)}{dt} = \delta_2(t)y(t) + (a(t) - \delta_2^2(t))x(t) - b(t)x^m(t - \tau(t)) + Q_2(t), \end{cases} \tag{4}$$

where $a(t)$, $b(t)$, $\tau(t)$, $\delta_1(t)$, $\delta_2(t)$, $Q_1(t)$, $Q_2(t)$ are almost periodic functions on $\mathbb{R}$, $m > 1$ is an integer and $a(t) > 0$, $b(t) \neq 0$, and gave some sufficient conditions for the existence of almost periodic solutions of (4).

Based on the work of Xu (2012), Liu and Tunç (2015) considered the system (4) with $\delta_1, \delta_2 \in AP(\mathbb{R}; \mathbb{R})$, $a, b, \tau, Q_1, Q_2 \in PAP(\mathbb{R}; \mathbb{R})$, and $a > 0, b \neq 0$ for $t \in \mathbb{R}$. They gave some sufficient conditions for the existence and uniqueness of pseudo almost periodic solutions of (4). Their results improved the results in the literature (Peng and Wang 2010; Xu 2012).

Moreover, Yang and Li (2014) considered the Duffing equation on time scales:

$$(x^{\Delta})^{\Delta}(t) + c(t)x^{\Delta}(t) - a(t)x(t) + b(t)x^m(t - \tau(t)) = p(t), \tag{5}$$

where $\mathbb{T}$ is an invariant time scale, $t \in \mathbb{T}$, $t - \tau(t) \in \mathbb{T}$ and $m > 1$ is a constant, and presented the existence and global exponential stability of almost periodic solutions for (5).

To combine continuous and discrete issues, Hilger proposed the idea of time scales in his Ph.D. thesis (Hilger 1988) in 1988. Several mathematicians have been interested in this theory since it provides an efficient mathematical technique for studying economics, biomathematics, and quantum physics, among other subjects.

Motivated by the above works, in this paper, we study the pseudo almost periodic solutions for the nonlinear Duffing Eq. (5) with $S^p$-pseudo almost periodic coefficients and delays on time scales. For this purpose, we establish a result of the existence and uniqueness of pseudo almost periodic solution for an abstract linear equation with $S^p$-almost periodic coefficients and $S^p$-pseudo almost periodic forcing term (see Theorem 3.1). Meanwhile, to deal with the delay $\tau(t)$, we extend some concepts of functions from $\mathbb{T} \to \mathbb{R}$ to $\mathbb{T} \to \Pi$, where $\mathbb{T}$ is a time scale with translation set $\Pi$ (see Definition 2.9, 2.10, 2.13), and give some basic properties for these concepts including the composition result (see Lemma 2.4, 2.6, 2.7). Then applying these results and Banach fixed point theorem, we get the existence and uniqueness of the pseudo almost periodic solution for the Duffing Eq. (5) (see Theorem 3.2, Theorem 3.3). Moreover, some examples are given to illustrate our main results at the end of this work.

## 2 Preliminaries

We refer to the sets of positive integers, integers, real numbers and non-negative real numbers, respectively, as $\mathbb{N},\ \mathbb{Z},\ \mathbb{R}$ and $\mathbb{R}^+$ throughout this work. The space of all $n \times n$ real-valued matrices with matrix norm $\|\cdot\|$ is denoted by $\mathbb{R}^{n\times n}$, while the Euclidian space $\mathbb{R}^n$ or $\mathbb{C}^n$ with Euclidian norm $|\cdot|$ is denoted by $\mathbb{E}^n$.

### 2.1 Time scale

Let $\mathbb{T} \subset \mathbb{R}$ be a time scale, that is, $\mathbb{T} \neq \emptyset$ is closed. The forward and backward jump operators $\sigma,\ \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

If $\sigma(t) > t$, we say $t$ is right-scattered; otherwise, $t$ is right-dense. Similarly, if $\rho(t) < t$, we say $t$ is left-scattered; otherwise $t$ is left-dense.

If $\mathbb{T}$ has a left-scattered maximum $m$, then $\mathbb{T}^\kappa = \mathbb{T} \setminus m$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

**Definition 2.1** A time scale $\mathbb{T}$ is called invariant under translations if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T},\ t \in \mathbb{T}\} \neq \{0\},$$

and define

$$\mathcal{K} = \begin{cases} \inf\{|\tau| : \tau \in \Pi, \tau \neq 0\}\ , & \text{if } \mathbb{T} \neq \mathbb{R}; \\ 1 & , \text{if } \mathbb{T} = \mathbb{R}. \end{cases}$$

In fact, if $\mathbb{T} \neq \mathbb{R}$, we have $\mathcal{K} > 0$ and one can show that $\Pi = \mathcal{K}\mathbb{Z}$. We say $\Pi$ the translation set of $\mathbb{T}$ (see e.g, Tang and Li (2017)).

In this paper, we always assume that $\mathbb{T}$ is invariant under translations.

**Definition 2.2** (Bohner and Peterson (2001))

(i) A function $f : \mathbb{T} \to \mathbb{E}^n$ is continuous on $\mathbb{T}$ if $f$ is continuous at every right-dense point and at every left-dense point.
(ii) A function $f : \mathbb{T} \to \mathbb{E}^n$ is rd-continuous on $\mathbb{T}$ if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limit exists at all left-dense points in $\mathbb{T}$.

For $t, s \in \mathbb{T},\ t < s$, denote $(t, s),\ [t, s],\ (t, s],\ [t, s)$ the standard intervals in $\mathbb{R}$, and use the following symbols:

$$(t, s)_{\mathbb{T}} = (t, s) \cap \mathbb{T},\ [t, s]_{\mathbb{T}} = [t, s] \cap \mathbb{T},\ (t, s]_{\mathbb{T}} = (t, s] \cap \mathbb{T},\ [t, s)_{\mathbb{T}} = [t, s) \cap \mathbb{T}.$$

Denote

$$\begin{aligned}
&C(\mathbb{T}; \mathbb{E}^n) = \left\{ f : \mathbb{T} \to \mathbb{E}^n : f \text{ is continuous} \right\},\\
&C(\mathbb{T} \times D; \mathbb{E}^n) = \left\{ f : \mathbb{T} \times D \to \mathbb{E}^n : f \text{ is continuous} \right\},\\
&BC(\mathbb{T}; \mathbb{E}^n) = \left\{ f : \mathbb{T} \to \mathbb{E}^n : f \text{ is bounded and continuous} \right\},\\
&BUC(\mathbb{T}; \mathbb{E}^n) = \left\{ f : \mathbb{T} \to \mathbb{E}^n : f \text{ is uniformly continuous and bounded} \right\},\\
&BC(\mathbb{T} \times D; \mathbb{E}^n) = \left\{ f : \mathbb{T} \times D \to \mathbb{E}^n : f \text{ is bounded and continuous} \right\},
\end{aligned}$$

where $D \subset \mathbb{E}^n$ is an open set.

**Definition 2.3** (Bohner and Peterson (2001)) For $f : \mathbb{T} \to \mathbb{E}^n$ and $t \in \mathbb{T}^\kappa$, $f^\Delta(t) \in \mathbb{E}^n$ is called the delta derivative of $f(t)$ if for a given $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

**Lemma 2.1** (Cabada and Vivero (2006)) *Fix a point $\omega \in \mathbb{T}$ and an interval $[\omega, \omega + \mathcal{K})_{\mathbb{T}}$, there are at most countably many right-scattered points $\{t_i\}_{i \in I},\ I \subseteq \mathbb{N}$ in this interval. If we denote $t_{ij} = t_i + j\mathcal{K}, i \in I, j \in \mathbb{Z}$, we get all the right-scattered points, and we have $\mu(t_{ij}) = \mu(t_i)$.*

Let $\mathcal{F}_1 = \{[t, s)_{\mathbb{T}} : t, s \in \mathbb{T}\ with\ t \leqslant s\}$. Define a countably additive measure $m_1$ on $\mathcal{F}_1$ by assigning to every $[t, s)_{\mathbb{T}} \in \mathcal{F}_1$ its lengths, i.e.

$$m_1([t, s)_{\mathbb{T}}) = s - t.$$

Using $m_1$, we can generate the outer measure $m_1^*$ on the power set $\mathcal{P}(\mathbb{T})$ of $\mathbb{T}$: for $E \in \mathcal{P}(\mathbb{T})$

$$m_1^*(E) = \begin{cases} \inf_{\mathcal{B}} \left\{ \sum\limits_{i \in I_{\mathcal{B}}} (s_i - t_i) \right\} \in \mathbb{R}^+, & \beta \notin E;\\ +\infty, & \beta \in E, \end{cases}$$

where $\beta = \sup \mathbb{T}$ and

$$\mathcal{B} = \left\{ \{[t_i, s_i) \in \mathcal{F}_1\}_{i \in I_{\mathcal{B}}} : I_{\mathcal{B}} \subset \mathbb{N}, E \subset \bigcup_{i \in I_{\mathcal{B}}} [t_i, s_i)_{\mathbb{T}} \right\}.$$

A set $A \subset \mathbb{T}$ is called $\Delta-$measurable if for $E \subset \mathbb{T}$, we have

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \setminus A)).$$

Let $\mathcal{M}(m_1^*) = \{A : A \text{ is a } \Delta - \text{measurable subset in } \mathbb{T}\}$. Restricting $m_1^*$ to $\mathcal{M}(m_1^*)$, we get the Lebesgue $\Delta-$measure, which is denoted by $\mu_\Delta$.

**Definition 2.4** (Cabada and Vivero (2006))

(i) A function $\mathcal{S} : \mathbb{T} \to \mathbb{E}^n$ is said to be simple if $\mathcal{S}$ takes a finite number of values $c_1, c_2, \cdots, c_N$. Let $E_j = \{s \in \mathbb{T} : \mathcal{S}(s) = c_j\}$, then

$$\mathcal{S} = \sum_{j=1}^{N} c_j \chi_{E_j},$$

where $\chi_{E_j}$ is the characteristic function of $E_j$, that is

$$\chi_{E_j}(s) = \begin{cases} 1, & \text{if } s \in E_j; \\ 0, & \text{if } s \in \mathbb{T} \setminus E_j. \end{cases}$$

(ii) Assume that $E$ is a $\Delta-$measurable subset of $\mathbb{T}$ and $\mathcal{S} : \mathbb{T} \to \mathbb{E}^n$ is a $\Delta-$measurable simple function, then the Lebesgue $\Delta-$integral of $\mathcal{S}$ on $E$ is defined as

$$\int_E \mathcal{S}(s)\Delta s = \sum_{j=1}^{N} c_j \mu_\Delta(E_j \cap E).$$

(iii) A function $g : \mathbb{T} \to \mathbb{E}^n$ is a $\Delta-$integrable function if there exists a simple function sequence $\{g_k : k \in \mathbb{N}\}$ such that $g_k(s) \to g(s)\ a.e.\ in\ \mathbb{T}$, then the integral of $g$ is defined as

$$\int_{\mathbb{T}} g(s) = \lim_{k\to\infty} \int_{\mathbb{T}} g_k(s)\Delta s.$$

(iv) For $p \geqslant 1$, $g : \mathbb{T} \to \mathbb{E}^n$ is called locally $L^p$ $\Delta-$integrable if $g$ is $\Delta-$measurable and for any compact $\Delta-$measurable set $E \subset \mathbb{T}$, the $\Delta-$integral

$$\int_E |g(s)|^p \Delta s < \infty.$$

The set of all $L^p$ $\Delta-$integrable functions is denoted by $L^p_{loc}(\mathbb{T}; \mathbb{X})$.

**Definition 2.5** (Tang and Li (2018)) Define $\|\cdot\|_{S^p} : L^p_{loc}(\mathbb{T}; \mathbb{E}^n) \to \mathbb{R}^+ \cup \{+\infty\}$ as

$$\|g\|_{S^p} := \sup_{s\in\mathbb{T}} \left( \frac{1}{\mathcal{K}} \int_s^{s+\mathcal{K}} |g(r)|^p \Delta r \right)^{\frac{1}{p}}.$$

where $\mathcal{K}$ is defined in Definition 2.1. A function $g \in L^p_{loc}(\mathbb{T}; \mathbb{E}^n)$ is called $S^p$-bounded if $\|g\|_{S^p} < \infty$. The space of all $S^p$-bounded functions is denoted by $BS^p(\mathbb{T}; \mathbb{E}^n)$; if $\mathbb{T} = \mathbb{R}$, denote it by $BS^p(\mathbb{E}^n)$.

## 2.2 Almost periodicity and pseudo almost periodicity on $\mathbb{T}$

**Definition 2.6** (Wang and Agarwal (2015)) A set $A \subset \mathbb{T}$ is called relatively dense in $\mathbb{T}$ if there exists $l > 0$ such that $[s, s+l]_{\mathbb{T}} \cap A \neq \emptyset$, $s \in \mathbb{T}$, we call $l$ the inclusion length.

**Definition 2.7** (Li and Wang (2011))

(i) A function $g \in C(\mathbb{T}; \mathbb{X})$ is almost periodic on $\mathbb{T}$ if for $\varepsilon > 0$,

$$T(g, \varepsilon) = \{\tau \in \Pi : \|g(s+\tau) - g(s)\| < \varepsilon \text{ for } s \in \mathbb{T}\}$$

is a relatively dense set in $\Pi$. We call $T(g, \varepsilon)$ the $\varepsilon$-translation set of $g$ and $\tau$ the $\varepsilon-$translation period of $g$, and the set of all almost periodic functions on $\mathbb{T}$ is denoted by $AP(\mathbb{T}; \mathbb{X})$.

(ii) Let $D \subset \mathbb{E}^n$ be open. The set $AP(\mathbb{T} \times D; \mathbb{E}^n)$ consists of all functions $f : \mathbb{T} \times D \to \mathbb{E}^n$ such that $f(\cdot, x) \in AP(\mathbb{T}; \mathbb{E}^n)$ uniformly for each $x \in K$ where $K$ is any compact subset of $D$.

**Definition 2.8** (Li and Wang (2011)) A continuous function $g : \mathbb{T} \to \mathbb{E}^n$ is said to be normal on $\Pi$ if for any sequence $\{\alpha'_n\} \subset \Pi$, there is a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that $\{g(t+\alpha_n)\}$ converges uniformly for $t \in \mathbb{T}$.

**Lemma 2.2** (Li and Wang (2011)) *A continuous function $g : \mathbb{T} \to \mathbb{E}^n$ is almost periodic on $\mathbb{T}$ if and only if it is normal on $\Pi$.*

To ensure $t - \tau(t) \in \mathbb{T}$, we have to give a restriction: $\tau(t) \in \Pi$. So we extend some concepts of functions from $\mathbb{T} \to \mathbb{R}$ to $\mathbb{T} \to \Pi$ below.

**Definition 2.9** A function $f : \mathbb{T} \to \Pi$ is continuous if $f$ is continuous at every right-dense point and at every left-dense point.

Denote

$$\begin{aligned} C(\mathbb{T}; \Pi) &= \{f : \mathbb{T} \to \Pi : f \text{ is continuous}\}, \\ BC(\mathbb{T}; \Pi) &= \{f : \mathbb{T} \to \Pi : f \text{ is bounded and continuous}\}. \end{aligned}$$

**Definition 2.10** A function $g \in C(\mathbb{T}; \Pi)$ is almost periodic on $\mathbb{T}$ if for $\varepsilon > 0$,

$$T(g, \varepsilon) = \{\tau \in \Pi : \|g(s+\tau) - g(s)\| < \varepsilon \text{ for } s \in \mathbb{T}\}$$

is a relatively dense set in $\Pi$. We call $T(g, \varepsilon)$ the $\varepsilon$-translation set of $g$ and $\tau$ the $\varepsilon-$translation period of $g$, and the set of all almost periodic functions on $\mathbb{T}$ is denoted by $AP_m(\mathbb{T}; \Pi)$.

***Remark 2.1*** For $\mathbb{T} = \mathbb{R}$, we have $\Pi = \mathbb{R}$, $AP_m(\mathbb{T}; \Pi) = AP(\mathbb{R}; \mathbb{R})$.

Denote the set

$$\begin{aligned} PAP_0(\mathbb{T}; \mathbb{E}^n) = \Big\{ & f \in BC(\mathbb{T}; \mathbb{E}^n) : \lim_{r\to\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} |f(s)| \Delta s = 0, \\ & \text{where } t_0 \in \mathbb{T}, r \in \Pi \Big\}, \\ PAP_0(\mathbb{T} \times D; \mathbb{E}^n) = \Big\{ & f \in BC(\mathbb{T} \times D; \mathbb{E}^n) : f(\cdot, x) \in PAP_0(\mathbb{T}; \mathbb{E}^n) \\ & \text{uniformly in } x \in D \Big\}, \\ PAP_0(\mathbb{T}; \Pi) = \Big\{ & f \in BC(\mathbb{T}; \Pi) : \lim_{r\to\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} |f(s)| \Delta s = 0, \\ & \text{where } t_0 \in \mathbb{T}, r \in \Pi \Big\}. \end{aligned}$$

**Definition 2.11** (Li and Wang (2012)) A closed subset $C$ of $\mathbb{T}$ is said to be an ergodic zero set in $\mathbb{T}$ if

$$\frac{\mu_\Delta(C \cap ([t_0 - r, t_0 + r]) \cap \mathbb{T})}{2r} \to 0 \text{ as } r \to \infty, \text{ for } t_0 \in \mathbb{T}.$$

**Definition 2.12** (Li and Wang (2012))

(i) A function $f \in BC(\mathbb{T}; \mathbb{E}^n)$ is called pseudo almost periodic if $f = g + \phi$, where $g \in AP(\mathbb{T}; \mathbb{E}^n)$ and $\phi \in PAP_0(\mathbb{T}; \mathbb{E}^n)$. We denote by $PAP(\mathbb{T}; \mathbb{E}^n)$ the set of all pseudo almost periodic functions.
(ii) A function $f \in BC(\mathbb{T} \times D; \mathbb{E}^n)$ is called pseudo almost periodic if $f = g + \phi$, where $g \in AP(\mathbb{T} \times D; \mathbb{E}^n)$ and $\phi \in PAP_0(\mathbb{T} \times D; \mathbb{E}^n)$. We denote by $PAP(\mathbb{T} \times D; \mathbb{E}^n)$ the set of all pseudo almost periodic functions.

**Definition 2.13** A function $f \in BC(\mathbb{T}; \Pi)$ is called pseudo almost periodic if $f = g + \phi$, where $g \in AP_m(\mathbb{T}; \Pi)$ and $\phi \in PAP_0(\mathbb{T}; \Pi)$. We denote by $PAP(\mathbb{T}; \Pi)$ the set of all pseudo almost periodic functions.

**Lemma 2.3** (Li and Wang (2012))

(i) *If $f \in PAP(\mathbb{T}; \mathbb{E}^n)$ and $\phi \in PAP_0(\mathbb{T}; \mathbb{E}^n)$, then for any $\tau \in \Pi$, $f(\cdot + \tau) \in PAP(\mathbb{T}; \mathbb{E}^n)$ and $\phi(\cdot + \tau) \in PAP_0(\mathbb{T}; \mathbb{E}^n)$.*
(ii) *$PAP(\mathbb{T}; \mathbb{E}^n)$ and $PAP_0(\mathbb{T}; \mathbb{E}^n)$ are Banach spaces under the sup norm.*

**Lemma 2.4** *Assume that $\mathbb{T} \neq \mathbb{R}$.*

(i) *Let $f \in AP_m(\mathbb{T}; \Pi)$, then $f$ is periodic.*
(ii) *$AP_m(\mathbb{T}; \Pi)$ is a $\mathbb{Z}$-module.*

***Proof*** (i) For $\varepsilon > 0$, $T(f, \varepsilon) = \{\tau \in \Pi : \|f(\cdot + \tau) - f(\cdot)\| < \varepsilon \text{ for } s \in \mathbb{T}\}$ is relatively dense in $\Pi$. Let $\tau \in \Pi$, $f(t + \tau) - f(t) \in \Pi = \mathcal{K}\mathbb{Z}$ for $t \in \mathbb{T}$. Let $\varepsilon < \mathcal{K}$, we can get that $\|f(\cdot + \tau) - f(\cdot)\| < \varepsilon$ if and only if $f(t + \tau) - f(t) = 0$ for $t \in \mathbb{T}$. Thus, $f$ is periodic.

(ii) Let $f_1, f_2 \in AP_m(\mathbb{T}; \Pi)$ with period $T_1 = n_1\mathcal{K}$, $T_2 = n_2\mathcal{K}$, respectively. Then we have $f_1 + f_2$ is of period $T = [n_1, n_2]\mathcal{K}$, where $[n_1, n_2]$ denotes the least common multiple of $n_1$ and $n_2$, we get that $f_1 + f_2 \in AP_m(\mathbb{T}; \Pi)$ and thus $AP_m(\mathbb{T}; \Pi)$ is an additive group. Then it is easy to check that $AP_m(\mathbb{T}; \Pi)$ is a $\mathbb{Z}$-module.

□

***Remark 2.2*** For $\mathbb{T} \neq \mathbb{R}$, obviously, $AP_m(\mathbb{T}; \Pi)$ is not a vector space on $\mathbb{R}$.

**Lemma 2.5** (Zhang (1995)) *A function $\phi_0 \in BC(\mathbb{R}; \mathbb{R})$ is in $PAP_0(\mathbb{R}; \mathbb{R})$ if and only if, for $\varepsilon > 0$, the set $C_\varepsilon = \{t \in \mathbb{R} : |\phi_0(t)| \geqslant \varepsilon\}$ is an ergodic zero subset of $\mathbb{R}$.*

**Lemma 2.6** *A bounded continuous function $\phi_0 \in PAP_0(\mathbb{T}; \Pi)$ if and only if for $\varepsilon > 0$, the set $C_\varepsilon = \{t \in \mathbb{T} : |\phi_0(t)| \geqslant \varepsilon\}$ is an ergodic zero subset of $\mathbb{T}$.*

***Proof*** If $\mathbb{T} = \mathbb{R}$, the conclusion follows from Lemma 2.5. Assume that $\mathbb{T} \neq \mathbb{R}$. Let $\phi_0 \in PAP_0(\mathbb{T}; \Pi)$, by contradiction, suppose that $C_\varepsilon$ is not an ergodic zero subset of $\mathbb{T}$. Then there exists a constant $\varepsilon_0 > 0$ such that

$$\limsup_{r\to\infty} \frac{\mu_\Delta(C_\varepsilon \cap ([t_0 - r, t_0 + r]) \cap \mathbb{T})}{2r} \geqslant \varepsilon_0, \text{ for some } t_0 \in \mathbb{T}.$$

We can derive that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} |\phi_0(s)|\Delta s \geqslant \limsup_{r\to\infty} \frac{1}{2r} \int_{C_\varepsilon\cap[t_0-r,t_0+r]} |\phi_0(s)|\Delta s \geqslant \varepsilon_0\varepsilon > 0,$$

which contradicts that $\phi_0 \in PAP_0(\mathbb{T}; \Pi)$ and then $C_\varepsilon$ is an ergodic zero subset of $\mathbb{T}$.

On the other hand, for $\varepsilon > 0$ and $C_\varepsilon$ is an ergodic zero set. Without loss of generality, we can choose $\varepsilon < \mathcal{K}$, then we have $\phi_0(t) = 0$ for $t \in \mathbb{T} \setminus C_\varepsilon$. Let $M = \sup_{t\in\mathbb{T}} |\phi_0(t)|$, for $t_0 \in \mathbb{T}$, we obtain that

$$\begin{aligned}\frac{1}{2r}\int_{t_0-r}^{t_0+r} |\phi_0(s)|\Delta s &= \frac{1}{2r}\int_{C_\varepsilon\cap[t_0-r,t_0+r]} |\phi_0(s)|\Delta s + \int_{([t_0-r,t_0+r]\cap\mathbb{T})\setminus C_\varepsilon} |\phi_0(s)|\Delta s \\ &= \frac{1}{2r}\int_{C_\varepsilon\cap[t_0-r,t_0+r]} |\phi_0(s)|\Delta s \\ &\leqslant M \cdot \frac{\mu_\Delta(C_\varepsilon \cap [t_0-r, t_0+r]\cap\mathbb{T})}{2r} \to 0, \text{ as } r \to \infty.\end{aligned}$$

Thus, we have $\phi_0 \in PAP_0(\mathbb{T}; \Pi)$. □

**Lemma 2.7** (Liu and Tunç (2015)) *Suppose that* $F \in PAP(\mathbb{R}; \mathbb{R}) \cap BUC(\mathbb{R}; \mathbb{R})$ *and* $\phi \in PAP(\mathbb{R}; \mathbb{R})$. *Then* $F(\cdot - \phi(\cdot)) \in PAP(\mathbb{R}; \mathbb{R})$.

**Lemma 2.8** *For* $\mathbb{T} \neq \mathbb{R}$, *suppose that* $F \in PAP(\mathbb{T}; \mathbb{E}^n)$ *and* $\phi \in PAP(\mathbb{T}; \Pi)$. *Then* $F(\cdot - \phi(\cdot)) \in PAP(\mathbb{T}; \mathbb{E}^n)$.

***Proof*** Let $F = F_1 + F_0$, $\phi = \phi_1 + \phi_0$ with $F_1 \in AP(\mathbb{T}; \mathbb{E}^n)$, $F_0 \in PAP_0(\mathbb{T}; \mathbb{E}^n)$ and $\phi_1 \in AP_m(\mathbb{T}; \Pi)$, $\phi_0 \in PAP_0(\mathbb{T}; \Pi)$. Note that, for $t \in \mathbb{T}$,

$$\begin{aligned}F(t-\phi(t)) &= F_1(t-\phi(t)) + F_0(t-\phi(t)) \\ &= F_1(t-\phi_1(t)) + (F_1(t-\phi(t)) - F_1(t-\phi_1(t))) + F_0(t-\phi(t)).\end{aligned}$$

We first prove the almost periodicity of $F_1(t - \phi_1(t))$. From Lemma 2.4, we know that $\phi_1(t)$ is periodic on $\mathbb{T}$, then for $\{\alpha''_n\} \subset \Pi$, there exists a subsequence $\{\alpha'_n\} \subset \{\alpha''_n\}$ such that $\phi_1(t + \alpha'_n) = \phi_1(t + \alpha'_m) = \tau_0$ for $n, m \in \mathbb{N}$. Since $F_1 \in AP(\mathbb{T}; \mathbb{E}^n)$, by Lemma 2.2, for $\{\alpha'_n\}$, we can extract a subsequence $\{\alpha_n\}$ such that $\{F_1(t + \alpha_n)\}$ converges uniformly for $t \in \mathbb{T}$. Thus, $F_1(t + \alpha_n - \phi_1(t + \alpha_n)) = F_1(t + \alpha_n - \tau_0)$ converges uniformly for $t \in \mathbb{T}$, and $F_1(\cdot - \phi_1(\cdot))$ is normal on $\Pi$. Hence, $F_1(\cdot - \phi_1(\cdot)) \in AP(\mathbb{T}; \mathbb{E}^n)$ by Lemma 2.2 again.

Then we only need to show that $h = (F_1(\cdot - \phi(\cdot)) - F_1(\cdot - \phi_1(\cdot))) + F_0(\cdot - \phi(\cdot)) \in PAP_0(\mathbb{T}; \mathbb{E}^n)$. First we show that $F_1(\cdot - \phi(\cdot)) - F_1(\cdot - \phi_1(\cdot)) \in PAP_0(\mathbb{T}; \mathbb{E}^n)$. For $0 < \delta < \mathcal{K}$, let $C_\delta = \{|\phi_0(\cdot)| \geqslant \delta\}$. By Lemma 2.6, we can get that $C_\delta$ is an ergodic zero set in $\mathbb{T}$. This means that for $\varepsilon > 0$, there exists $T > 0$ such that when $r > T$, $t_0 \in \mathbb{T}$,

$$\frac{\mu_\Delta([t_0-r, t_0+r]\cap\mathbb{T}\cap C_\delta)}{2r} < \frac{\varepsilon}{2\|F_1\|}.$$

It is obvious that if $t \in [t_0 - r, t_0 + r] \cap \mathbb{T}\setminus C_\delta$, $\phi_0(t) = \phi(t) - \phi_1(t) = 0$. So we have

$$\begin{aligned}&\frac{1}{2r}\int_{t_0-r}^{t_0+r} |F_1(s-\phi(s)) - F_1(s-\phi_1(s))|\Delta s \\ &= \frac{1}{2r}\Bigg(\int_{t\in[t_0-r,t_0+r]\cap\mathbb{T}\setminus C_\delta} |F_1(s-\phi(s)) - F_1(s-\phi_1(s))|\Delta s \\ &\quad + \int_{t\in[t_0-r,t_0+r]\cap\mathbb{T}\cap C_\delta} |F_1(s-\phi(s)) - F_1(s-\phi_1(s))|\Delta s\Bigg) \\ &\leqslant 0 + 2\|F_1\| \cdot \frac{\mu_\Delta([t_0-r, t_0+r]\cap\mathbb{T}\cap C_\delta)}{2r} < \varepsilon.\end{aligned}$$

Therefore, $F_1(\cdot - \phi(\cdot)) - F_1(\cdot - \phi_1(\cdot)) \in PAP_0(\mathbb{T}; \mathbb{E}^n)$.

Next we show that $F_0(\cdot - \phi(\cdot)) \in PAP_0(\mathbb{T}; \mathbb{E}^n)$. Since $\phi$ is bounded, $\phi(\mathbb{T}) \subset \Pi = \mathcal{K}\mathbb{Z}$ is of finite number of values, denote them by $\{k_1, k_2, \ldots, k_n\}$, where $k_i \in \Pi,\ i = 1, 2, \ldots, n$. By Lemma 2.3 (i), we have $F_0(\cdot - k_i) \in PAP_0(\mathbb{T}; \mathbb{E}^n),\ i = 1, 2, ..., n$. So for $\varepsilon > 0$, there exists $T_1 > 0$ such that for $r > T_1$,

$$\frac{1}{2r}\int_{t_0-r}^{t_0+r} |F_0(s - k_i)|\Delta s < \frac{\varepsilon}{n},\ i = 1, 2, ..., n.$$

Then we can get

$$\begin{aligned}\frac{1}{2r}\int_{t_0-r}^{t_0+r} |F_0(s - \phi(s))|\Delta s &\leqslant \frac{1}{2r}\sum_{i=1}^{n}\int_{t_0-r}^{t_0+r} |F_0(s - k_i)|\Delta s \\ &< n \cdot \frac{\varepsilon}{n} = \varepsilon.\end{aligned}$$

This implies that $F_0(\cdot - \phi(\cdot)) \in PAP_0(\mathbb{T}; \mathbb{E}^n)$. □

### 2.3 $S^p$-almost periodic functions and $S^p$-pseudo almost periodic functions

**Definition 2.14** (Tang and Li (2018)) A function $g \in L^p_{loc}(\mathbb{T}; \mathbb{E}^n)$ is $S^p$-almost periodic on $\mathbb{T}$ if given $\varepsilon > 0$, the $\varepsilon$-translation set of $g$

$$T(g, \varepsilon) = \{\tau \in \Pi : \|g(\cdot + \tau) - g(\cdot)\|_{S^p} < \varepsilon\}$$

is a relatively dense set in $\Pi$. The space of all these functions is denoted by $S^pAP(\mathbb{T}; \mathbb{E}^n)$ with norm $\|\cdot\|_{S^p}$.

Define the norm operator $\mathcal{N}_p$ on $BS^p(\mathbb{T}; \mathbb{E}^n)$ as follows:

$$\mathcal{N}_p(f)(t) := \left(\frac{1}{\mathcal{K}}\int_t^{t+\mathcal{K}} |f(s)|^p \Delta s\right)^{\frac{1}{p}} \text{ for } f \in BS^p(\mathbb{T}; \mathbb{E}^n), t \in \mathbb{T}.$$

**Lemma 2.9** (Tang and Li (2018)) *The norm operator $\mathcal{N}_p$ maps $BS^p(\mathbb{T}; \mathbb{E}^n)$ in to $BC(\mathbb{T}; \mathbb{R})$ and maps $S^pAP(\mathbb{T}; \mathbb{E}^n)$ into $AP(\mathbb{T}; \mathbb{R})$. Moreover, for $f, g \in BS^p(\mathbb{T}; \mathbb{E}^n),\ t \in \mathbb{T}$,*

$$\|\mathcal{N}_p(f)\|_\infty = \|f\|_{S^p},\ |\mathcal{N}_p(f)(t) - \mathcal{N}_p(g)(t)| \leqslant \mathcal{N}_p(f \pm g)(t) \leqslant \mathcal{N}_p(f)(t) + \mathcal{N}_p(g)(t).$$

**Lemma 2.10** *Let $f \in BS^p(\mathbb{T}; \mathbb{E}^n),\ g \in BS^q(\mathbb{T}; \mathbb{E}^n)$ with $p, q \geqslant 1, \frac{1}{p} + \frac{1}{q} = 1$. Then we have, for $t \in \mathbb{T}$,*

$$\mathcal{N}_1(f \cdot g)(t) \leqslant \mathcal{N}_p(f)(t) \cdot \mathcal{N}_q(g)(t) \leqslant \|f\|_{S^p} \cdot \mathcal{N}_q(g)(t). \tag{6}$$

*In addition, if $f$ is bounded and continuous, we have*

$$\mathcal{N}_1(f \cdot g)(t) \leqslant \mathcal{N}_q(f \cdot g)(t) \leqslant \|f\| \cdot \mathcal{N}_q(g)(t). \tag{7}$$

***Proof*** If $p = 1, q = +\infty$, it is obvious. Now suppose that $p, q > 1$. By Hölder inequality, for $t \in \mathbb{T}$, we have

$$\begin{aligned}\mathcal{N}_1(f \cdot g)(t) &= \frac{1}{\mathcal{K}} \int_t^{t+\mathcal{K}} |f(s)g(s)|\Delta s \\ &\leqslant \frac{1}{\mathcal{K}} \left( \int_t^{t+\mathcal{K}} |f(s)|^p \Delta s \right)^{\frac{1}{p}} \cdot \left( \int_t^{t+\mathcal{K}} |g(s)|^q \Delta s \right)^{\frac{1}{q}} \\ &= \left( \frac{1}{\mathcal{K}} \int_t^{t+\mathcal{K}} |f(s)|^p \Delta s \right)^{\frac{1}{p}} \cdot \left( \frac{1}{\mathcal{K}} \int_t^{t+\mathcal{K}} |g(s)|^q \Delta s \right)^{\frac{1}{q}} \\ &= \mathcal{N}_p(f)(t) \cdot \mathcal{N}_q(g)(t) \leqslant \|f\|_{S^p} \cdot \mathcal{N}_q(g)(t).\end{aligned}$$

If $f$ is bounded and continuous, then we have

$$\begin{aligned}\mathcal{N}_1(f \cdot g)(t) &= \frac{1}{\mathcal{K}} \int_t^{t+\mathcal{K}} |f(s)g(s)|\Delta s \leqslant \frac{1}{\mathcal{K}} \left( \int_t^{t+\mathcal{K}} |f(s)g(s)|^q \Delta s \right)^{\frac{1}{q}} \cdot \left( \int_t^{t+\mathcal{K}} 1^p \Delta s \right)^{\frac{1}{p}} \\ &= \left( \frac{1}{\mathcal{K}} \int_t^{t+\mathcal{K}} |f(s)g(s)|^q \Delta s \right)^{\frac{1}{q}} \leqslant \|f\| \cdot \left( \frac{1}{\mathcal{K}} \int_t^{t+\mathcal{K}} |g(s)|^q \Delta s \right)^{\frac{1}{q}} \\ &= \|f\| \cdot \mathcal{N}_q(g)(t)\end{aligned}$$

□

**Definition 2.15** (Tang and Li (2018)) A function $f \in BS^p(\mathbb{T}; \mathbb{E}^n)$ is said to be ergodic if $\mathcal{N}_p(f) \in PAP_0(\mathbb{T}; \mathbb{R})$. We denote by $S^p PAP_0(\mathbb{T}; \mathbb{E}^n)$ the set of all ergodic functions from $\mathbb{T}$ to $\mathbb{E}^n$.

**Definition 2.16** (Tang and Li (2018)) A function $f \in BS^p(\mathbb{T}; \mathbb{E}^n)$ is called $S^p$-pseudo almost periodic if $f = g + \phi$, where $g \in S^p AP(\mathbb{T}; \mathbb{E}^n)$ and $\phi \in S^p PAP_0(\mathbb{T}; \mathbb{E}^n)$. We denote by $S^p PAP(\mathbb{T}; \mathbb{E}^n)$ the set of all such functions $f$.

**Lemma 2.11** (Tang and Li (2018))

(i) $PAP(\mathbb{T}; \mathbb{E}^n) \subset S^p PAP(\mathbb{T}; \mathbb{E}^n)$.
(ii) $S^q PAP(\mathbb{T}; \mathbb{E}^n) \subset S^p(\mathbb{T}; \mathbb{E}^n)$ *for* $1 \leqslant p \leqslant q$.
(iii) *Assume that* $f \in BS^p(\mathbb{T}; \mathbb{E}^n)$. *For* $t_0 \in \mathbb{T}$, *we have* $\int_{t_0}^{t_0+\mathcal{K}} |f(s)|\Delta s \leqslant \mathcal{K}\|f\|_{S^p}$.

**Lemma 2.12** *For* $f = f_1 + f_2 \in PAP(\mathbb{T}; \mathbb{E}^n)$ *and* $g = g_1 + g_2 \in S^p PAP(\mathbb{T}; \mathbb{E}^n)$ *with* $f_1 \in AP(\mathbb{T}; \mathbb{E}^n)$, $f_2 \in PAP_0(\mathbb{T}; \mathbb{E}^n)$, $g_1 \in S^p AP(\mathbb{T}; \mathbb{E}^n)$ *and* $g_2 \in S^p PAP_0(\mathbb{T}; \mathbb{E}^n)$. *Then* $f \cdot g \in S^1 PAP(\mathbb{T}; \mathbb{E}^n)$.

***Proof*** For convenience, we denote $f^\tau(\cdot) = f(\cdot + \tau)$ in the proof. In fact, we have $f \cdot g = f_1 \cdot g_1 + f_2 \cdot g_1 + f \cdot g_2$. Now we prove that $f \cdot g \in S^1 PAP(\mathbb{T}; \mathbb{E}^n)$ by the following 3 steps.

Step 1: We prove that $f_1 \cdot g_1 \in S^1 AP(\mathbb{T}; \mathbb{E}^n)$. For $\varepsilon > 0$, choose $\tau \in T(f_1, \varepsilon) \cap T(g_1, \varepsilon)$, by Lemma 2.9 and (7), we can get that

$$\begin{aligned}\|f_1(\cdot+\tau)g_1(\cdot+\tau)-f_1(\cdot)g_1(\cdot)\|_{S^p} &= \sup_{t\in\mathbb{T}}\mathcal{N}_p((f_1\cdot g_1)^\tau - f_1\cdot g_1)(t)\\ &= \sup_{t\in\mathbb{T}}\mathcal{N}_p(f_1^\tau\cdot(g_1^\tau-g_1)+(f_1^\tau-f_1)\cdot g_1)(t)\\ &\leqslant \sup_{t\in\mathbb{T}}\mathcal{N}_p(f_1^\tau\cdot(g_1^\tau-g_1))(t)+\sup_{t\in\mathbb{T}}\mathcal{N}_p((f_1^\tau-f_1)\cdot g_1)(t)\\ &\leqslant \|f_1\|\cdot\sup_{t\in\mathbb{T}}\mathcal{N}_p(g_1^\tau-g_1)(t)+\|f_1^\tau-f_1\|\cdot\sup_{t\in\mathbb{T}}\mathcal{N}_p(g_1)(t)\\ &\leqslant (\|f_1\|+\|g_1\|_{S^p})\varepsilon,\end{aligned}$$

which means that $f_1\cdot g_1 \in S^pAP(\mathbb{T};\mathbb{E}^n)$ and by Lemma 2.11 (ii), we have $f_1\cdot g_1 \in S^1AP(\mathbb{T};\mathbb{E}^n)$.

Step 2: We prove that $f\cdot g_2\in S^1PAP_0(\mathbb{T};\mathbb{E}^n)$. By Lemma 2.11 (iii) and (6), we have

$$\frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}_1(f\cdot g_2)(t)\Delta t\leqslant\|f\|\cdot\frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}_p(g_2)(t)\Delta t, \tag{8}$$

for a fixed $t_0\in\mathbb{T}$ and $r\in\Pi$. Let $r\to\infty$ in (8) we derive that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}_p(f\cdot g_2)\Delta t=0,$$

since $g_2\in S^pPAP_0(\mathbb{T};\mathbb{E}^n)$. Thus, $f\cdot g_2\in S^1PAP_0(\mathbb{T};\mathbb{E}^n)$.

Step 3: We prove that $f_2\cdot g_1 \in S^1PAP_0(\mathbb{T};\mathbb{E}^n)$. By Lemma 2.11 (i) we can get that $f_2\in S^qPAP_0(\mathbb{T};\mathbb{E}^n)$ where $\frac{1}{q}+\frac{1}{p}=1$. For a fixed $t_0\in\mathbb{T}$, $r\in\Pi$, by (6), we have

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}_1(f_2\cdot g_1)(t)\Delta t &\leqslant \lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}_q(f_2)(t)\cdot\mathcal{N}_p(g_1)(t)\Delta t\\ &\leqslant \|g_1\|_{S^p}\cdot\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}_q(f_2)(t)\Delta t=0.\end{aligned}$$

Thus, we get $f_2\cdot g_1\in S^1PAP_0(\mathbb{T};\mathbb{E}^n)$. □

## 2.4 Exponential functions

For a function $p:\mathbb{T}\to\mathbb{R}$, if we have $1+\mu(t)p(t)\neq 0$, $t\in\mathbb{T}^\kappa$, we say that $p$ is regressive. Denote the set of all regressive and rd-continuous function $p:\mathbb{T}\to\mathbb{R}$ by $\mathcal{R}=\mathcal{R}(\mathbb{T})=\mathcal{R}(\mathbb{T};\mathbb{R})$ and define the set $\mathcal{R}^+=\mathcal{R}^+(\mathbb{T};\mathbb{R})=\{p\in\mathcal{R}:1+\mu(t)p(t)>0 \text{ for } t\in\mathbb{T}\}$. We can see that the set $\mathcal{R}(\mathbb{T};\mathbb{R})$ is an Abelian group with addition $\oplus$ defined by $p\oplus q=p+q+\mu(t)pq$, and the additive inverse in this Abelian group is defined by $\ominus p=-\frac{p}{1+\mu(t)p}$.

**Definition 2.17** (Bohner and Peterson (2001)) For $p\in\mathcal{R}$, the exponential function is defined by

$$e_p(t,s)=\exp\left(\int_s^t\xi_{\mu(\tau)}(p(\tau))\Delta\tau\right),$$

for $t,s\in\mathbb{T}$ with the cylinder transformation

$$\xi_h(z)=\begin{cases}\frac{1}{h}\mathrm{Log}(1+hz)\ , & \text{if } h\neq 0;\\ z\ , & \text{if } h=0,\end{cases}$$

where Log is the principal logarithm.

**Definition 2.18** (Bohner and Peterson (2001)) For a matrix-valued function $A : \mathbb{T} \to \mathbb{R}^{n\times n}$, we say that $A(\cdot)$ is regressive if $I + \mu(t)A(t)$ is invertible for $t \in \mathbb{T}^\kappa$, and denote the set of all such regressive and rd-continuous functions by $\mathcal{R}(\mathbb{T}; \mathbb{R}^{n\times n})$.

**Definition 2.19** (Bohner and Peterson (2001)) Let $A \in \mathcal{R}(\mathbb{T}; \mathbb{R}^{n\times n})$. The initial value problem

$$X(t)^\Delta = A(t)X(t), X(t_0) = I,\ t, t_0 \in \mathbb{T}$$

has a unique solution which is denoted by $e_A(\cdot, t_0)$. We say that $e_A(\cdot, t_0)$ is the matrix exponential function at $t_0$.

**Lemma 2.13** (Bohner and Peterson (2001)) *Let $t, s \in \mathbb{T}$.*

(i) $e_p(t, t) = 1,\ e_A(t, t) = I.$
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s).$
(iii) $e_p(t, s)e_p(s, r) = e_p(t, r), e_A(t, s)e_A(s, r) = e_A(t, r).$

**Lemma 2.14** (Tang and Li (2018)) *Let $\alpha > 0$ be a constant and $t, s \in \mathbb{T}$.*

(i) $e_{\ominus\alpha}(t, s) \leqslant 1$ *if* $t > s$.
(ii) $e_{\ominus\alpha}(t + \tau, s + \tau) = e_{\ominus\alpha}(t, s)$ *for* $\tau \in \Pi$.
(iii) *There exists $N_\alpha > 0$ depending on $\alpha$ such that $n_{ts}\mathcal{K}e_{\ominus\alpha}(t, s) \leqslant N_\alpha$ for $t \geqslant s$, where $(n_{ts} - 1)\mathcal{K} \leqslant t - s < n_{ts}\mathcal{K}$.*
(iv) *The series $\sum_{j=1}^{\infty} e_{\ominus\alpha}(t, \sigma(t) - (j - 1)\mathcal{K})$ converges uniformly for $t \in \mathbb{T}$. Moreover, for all $t \in T$,*

$$\sum_{j=1}^{\infty} e_{\ominus\alpha}(t, \sigma(t) - (j - 1)\mathcal{K}) \leqslant \lambda_\alpha = \begin{cases} \frac{1}{1-e^{-\alpha}} &, \text{ if } \mathbb{T} = \mathbb{R}; \\ 2 + \alpha\bar{\mu} + \frac{1}{\alpha\bar{\mu}} &, \text{ if } \mathbb{T} \neq \mathbb{R}, \end{cases}$$

*where* $\bar{\mu} = \sup_{t\in\mathbb{T}} \mu(t)$.

**Lemma 2.15** *Assume that $A \in \mathcal{R}(\mathbb{T}; \mathbb{R}^{n\times n})$ is $S^p$-almost periodic and*

$$\|e_A(t, s)\| \leqslant Ce_{\ominus\alpha}(t, s), t \geqslant s, \tag{9}$$

*where $C$ and $\alpha$ are positive real numbers. Let $M = \begin{cases} C^2(1 + \alpha\mathcal{K})N_\alpha, \text{ if } \mathbb{T} \neq \mathbb{R}, \\ C^2 N_\alpha, \text{ if } \mathbb{T} = \mathbb{R} \end{cases}$ with $N_\alpha$ the constant in Lemma 2.14* (iii)*, and for $\varepsilon > 0$,*

$$\Upsilon(\varepsilon) = \{r \in \Pi : \|e_A(t + r, \sigma(s) + r) - e_A(t, \sigma(s))\| < \varepsilon, t, s \in \mathbb{T}, t \geqslant \sigma(s)\}.$$

*Then $T(A, \varepsilon/M) \subset \Upsilon(\varepsilon)$.*

***Proof*** For $\varepsilon > 0$, let $r \in T(A, \varepsilon/M)$ and $U(t, \sigma(s)) = e_A(t + r, \sigma(s) + r) - e_A(t, \sigma(s))$. Differentiate $U$ with respect to $t$ and denote by $\frac{\partial_\Delta U}{\partial_\Delta t}$ the partial derivative, then

$$\begin{aligned} \frac{\partial_\Delta U}{\partial_\Delta t} &= A(t + r)e_A(t + r, \sigma(s) + r) - A(t)e_A(t, \sigma(s)) \\ &= A(t)U(t, \sigma(s)) + (A(t + r) - A(t))e_A(t + r, \sigma(s) + r). \end{aligned}$$

Note that $U(\sigma(s), \sigma(s)) = 0$, then by the variation of constants formula,

$$U(t, \sigma(s)) = \int_{\sigma(s)}^{t} e_A(t, \sigma(\tau))(A(\tau + r) - A(\tau))e_A(\tau + r, \sigma(s) + r)\Delta\tau.$$

For $\mathbb{T} = \mathbb{R}$,

$$\begin{aligned}
\|U(t, s)\| &\leqslant \int_{s}^{t} \|e_A(t, \tau)\| \cdot \|A(\tau + r) - A(\tau)\| \cdot \|e_A(\tau + r, s + r)\| d\tau \\
&\leqslant C^2 \int_{s}^{t} e^{-\alpha(t-\tau)} e^{-\alpha(\tau-s)} \|A(\tau + r) - A(\tau)\| d\tau \\
&= C^2 e^{-\alpha(t-s)} \int_{s}^{t} \|A(\tau + r) - A(\tau)\| d\tau \\
&\leqslant C^2 e^{-\alpha(t-s)} \int_{t-n_{ts}}^{t} \|A(\tau + r) - A(\tau)\| d\tau \\
&= C^2 e^{-\alpha(t-s)} \sum_{j=1}^{n_{ts}} \int_{t-j}^{t-(j-1)} \|A(\tau + r) - A(\tau)\| d\tau \\
&\leqslant C^2 n_{ts} e^{-\alpha(t-s)} \|A(\cdot + r) - A(\cdot)\|_{S^p} \\
&\leqslant C^2 N_\alpha \varepsilon / M = \varepsilon.
\end{aligned}$$

For $\mathbb{T} \neq \mathbb{R}$, by Lemma 2.11, 2.13, 2.14 and the fact that $\mu(\tau) \leqslant \mathcal{K}$, $\tau \in \mathbb{T}$, for $t, s \in \mathbb{T}$, $t \geqslant \sigma(s)$,

$$\begin{aligned}
\|U(t, \sigma(s))\| &\leqslant \int_{\sigma(s)}^{t} \|e_A(t, \sigma(\tau))\| \cdot \|A(\tau + r) - A(\tau)\| \cdot \|e_A(\tau + r, \sigma(s) + r)\| \Delta\tau \\
&\leqslant C^2 \int_{\sigma(s)}^{t} e_{\ominus\alpha}(t, \sigma(\tau)) e_{\ominus\alpha}(\tau + r, \sigma(s) + r) \|A(\tau + r) - A(\tau)\| \Delta\tau \\
&= C^2 e_{\ominus\alpha}(t, \sigma(s)) \int_{\sigma(s)}^{t} e_{\ominus\alpha}(\tau, \sigma(\tau)) \|A(\tau + r) - A(\tau)\| \Delta\tau \\
&\leqslant C^2 (1 + \alpha\bar{\mu}) e_{\ominus\alpha}(t, \sigma(s)) \int_{\sigma(s)}^{t} \|A(\tau + r) - A(\tau)\| \Delta\tau \\
&\leqslant C^2 (1 + \alpha\mathcal{K}) e_{\ominus\alpha}(t, \sigma(s)) \int_{t-n_{ts}\mathcal{K}}^{t} \|A(\tau + r) - A(\tau)\| \Delta\tau \\
&= C^2 (1 + \alpha\mathcal{K}) e_{\ominus\alpha}(t, \sigma(s)) \sum_{j=1}^{n_{ts}} \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} \|A(\tau + r) - A(\tau)\| \Delta\tau \\
&\leqslant C^2 (1 + \alpha\mathcal{K}) n_{ts} \mathcal{K} e_{\ominus\alpha}(t, \sigma(s)) \|A(\cdot + r) - A(\cdot)\|_{S^p} \\
&\leqslant C^2 (1 + \alpha\mathcal{K}) N_\alpha \varepsilon / M = \varepsilon.
\end{aligned}$$

This implies that $T(A, \varepsilon/M) \subset \Upsilon(\varepsilon)$, and $\Upsilon(\varepsilon)$ is relatively dense in $\Pi$. □

## 3 Main results

Let $y(t) = x^{\Delta}(t) + \delta_1(t)x(t)$, Eq. (5) transforms into the following system:

$$\begin{cases} x^{\Delta}(t) = -\delta_1(t)x(t) + y(t), \\ y^{\Delta}(t) = -\delta_2(t)y(t) + \beta(t)x(t) - b(t)x^m(t-\tau(t)) + p(t), \end{cases} \tag{10}$$

where $\delta_2(t) = c(t) - \delta_1(\sigma(t))$, $\beta(t) = a(t) + \delta_1^{\Delta}(t) + \delta_1(t)\delta_2(t)$. To study (10), we first consider the following abstract linear equation:

$$x^{\Delta}(t) = A(t)x(t) + f(t), t \in \mathbb{T}, \tag{11}$$

where $f = g + \phi \in S^p PAP(\mathbb{T}; \mathbb{E}^n) \cap C(\mathbb{T}; \mathbb{E}^n)$.

**Lemma 3.1** (Tang and Li (2018)) *Assume that $A \in \mathcal{R}(\mathbb{T}; \mathbb{R}^{n\times n})$ with (9) satisfied. Then (11) admits a unique bounded continuous solution $u(t)$ given by*

$$u(t) = \int_{-\infty}^{t} e_A(t, \sigma(s)) f(s)\Delta s, t \in \mathbb{T}, \tag{12}$$

*and $|u(t)| \leqslant C\lambda_\alpha \mathcal{K}\|f\|_{S^p}$, where $\lambda_\alpha$ is given in Lemma 2.14* (iv).

**Theorem 3.1** *Assume that all conditions in Lemma 2.15 are satisfied. Then (11) admits a unique pseudo almost periodic solution given by (12).*

***Proof*** By Lemma 3.1, it suffices to prove that $u \in PAP(\mathbb{T}; \mathbb{E}^n)$. For $t \in \mathbb{T}$, let

$$u(t) = \int_{-\infty}^{t} e_A(t, \sigma(s)) f(s)\Delta s = \sum_{j=1}^{\infty} u_j(t),$$

where

$$\begin{aligned} u_j(t) &= \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} e_A(t, \sigma(s)) f(s)\Delta s \\ &= \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} e_A(t, \sigma(s)) g(s)\Delta s + \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} e_A(t, \sigma(s)) \phi(s)\Delta s \\ &:= g_j(t) + \phi_j(t), \qquad j \in \mathbb{N}. \end{aligned}$$

Now we prove $u_j \in PAP(\mathbb{T}; \mathbb{E}^n)$. For $\varepsilon > 0$, it follows from Lemma 2.15 that $\Upsilon(\varepsilon) \cap T(g, \varepsilon)$ is relatively dense in $\Pi$. For $r \in \Upsilon(\varepsilon) \cap T(g, \varepsilon)$, by Lemma 2.1 and 2.11,

$$\begin{aligned} &|g_j(t+r) - g_j(t)| \\ &= \left| \int_{t+r-j\mathcal{K}}^{t+r-(j-1)\mathcal{K}} e_A(t+r, \sigma(s)) g(s)\Delta s - \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} e_A(t, \sigma(s)) g(s)\Delta s \right| \\ &= \left| \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} e_A(t+r, \sigma(s)+r) g(s+r) - e_A(t, \sigma(s)) g(s)\Delta s \right| \\ &\leqslant \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} \|e_A(t+r, \sigma(s)+r) - e_A(t, \sigma(s))\| \cdot |g(s)|\Delta s \\ &\quad + \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} \|e_A(t, \sigma(s))\| \cdot |g(s+r) - g(s)|\Delta s \end{aligned}$$

$$\leqslant \varepsilon \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} |g(s)|\Delta s + Ce_{\ominus\alpha}(t,\sigma(t)-(j-1)\mathcal{K}) \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} |g(s+r)-g(s)|\Delta s$$
$$\leqslant \mathcal{K}\|g\|_{S^p}\varepsilon + C(1+\alpha\bar{\mu})\mathcal{K}\|g(\cdot+r)-g(\cdot)\|_{S^p}$$
$$\leqslant \mathcal{K}\|g\|_{S^p}\varepsilon + C(1+\alpha\bar{\mu})\mathcal{K}\varepsilon = (\mathcal{K}\|g\|_{S^p} + C(1+\alpha\bar{\mu})\mathcal{K})\varepsilon,$$

which means that $g_j(t)$ is almost periodic for $j \in \mathbb{N}$.

Next, we prove that $\phi_j(t) \in PAP_0(\mathbb{T};\mathbb{E}^n)$.

$$|\phi_j(t)| \leqslant \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} \|e_A(t,\sigma(s))\| \cdot |\phi(s)|\Delta s \leqslant C \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} e_{\ominus\alpha}(t,\sigma(s)) \cdot |\phi(s)|\Delta s$$
$$\leqslant Ce_{\ominus\alpha}(t,\sigma(t)-(j-1)\mathcal{K}) \int_{t-j\mathcal{K}}^{t-(j-1)\mathcal{K}} |\phi(s)|\Delta s \leqslant C(1+\alpha\bar{\mu})\mathcal{K}\mathcal{N}_p(\phi)(t-j\mathcal{K}).$$

Notice that $\phi \in S^pPAP_0(\mathbb{T};\mathbb{E}^n)$. Thus, for a fixed $t_0 \in \mathbb{T}$,

$$\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}|\phi_j(t)|\Delta t \leqslant C(1+\alpha\bar{\mu})\mathcal{K}\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\mathcal{N}_p(\phi)(t-j\mathcal{K})\Delta t = 0.$$

This implies that $\phi_j \in PAP_0(\mathbb{T};\mathbb{E}^n)$, and then $u_j(t) \in PAP(\mathbb{T};\mathbb{E}^n)$. This together with the boundedness of $u(t)$ yields that $u(t) \in PAP(\mathbb{T};\mathbb{E}^n)$. □

The following conditions will be useful in the proof of our main results.

(H$_1$) $\delta_1,\delta_2 \in C(\mathbb{T};\mathbb{R}^+) \cap S^pAP(\mathbb{T};\mathbb{R}^+)$ and $-\delta_1,-\delta_2 \in \mathcal{R}^+$. We denote $\delta_i^- = \inf_{t\in\mathbb{T}}\delta_i(t)$, $i=1,2$, $\bar{\delta} = \min(\delta_1^-,\delta_2^-)$.

(H$_2$) $\beta,\ b,\ p \in S^pPAP(\mathbb{T};\mathbb{R}),\ \tau \in PAP(\mathbb{T};\Pi)$.

(H$_3$) $\theta_1 = \max\left\{\dfrac{1}{\delta_1^-}, \lambda_{\delta_2^-}\mathcal{K}(\|\beta\|_{S^p} + m\|b\|_{S^p})\right\} < 1.$

We note that, in this work, $z(t) = (z_1(t), z_2(t))$ is assumed to be a column vector function without any further comments. In the rest of this work, we will use the following norm for $PAP(\mathbb{T};\mathbb{R}^2)$, which is equivalent to the one mentioned in Lemma 2.3 (ii):

$$\|\varphi\|_* = \max\left\{\sup_{t\in\mathbb{T}}|\varphi_1(t)|,\ \sup_{t\in\mathbb{T}}|\varphi_2(t)|\right\},$$

for $\varphi = (\varphi_1,\varphi_2) \in PAP(\mathbb{T};\mathbb{R}^2)$. For $\mathbb{T} \neq \mathbb{R}$, let

$$E_1^* = \left\{\varphi \in PAP(\mathbb{T};\mathbb{R}^2) : \|\varphi - \varphi^0\|_* \leqslant \frac{\theta_1\lambda}{1-\theta_1}\right\},$$

where

$$\varphi^0(t) = \left(0, \varphi_2^0(t)\right),\ \varphi_2^0(t) = \int_{-\infty}^{t} e_{-\delta_2}(t,\sigma(s))p(s)\Delta s,\ \lambda = \lambda_{\delta_2^-}\mathcal{K}\|p\|_{S^p},$$

and by Lemma 3.1, we can get that $\|\varphi^0\|_* \leqslant \lambda$.

Now we are in a position to give our main result.

**Theorem 3.2** *Suppose that* $\mathbb{T} \neq \mathbb{R}$ *and the assumptions* $(H_1)$-$(H_3)$ *hold, and* $\frac{\lambda}{1-\theta_1} < 1$, *then* (10) *has a unique pseudo almost periodic solution in* $E_1^*$ *and* (5) *has a unique pseudo almost periodic solution u satisfying that* $(u, u^{\Delta} + \delta_1 u) \in E_1^*$.

***Proof*** It is easy to see that $u(t)$ is a solution of (5) if and only if $(u(t), u^{\Delta}(t) + \delta_1(t)u(t))$ is a solution of (10). Consider the following system:

$$\begin{cases} x^{\Delta}(t) = -\delta_1(t)x(t) + \varphi_2(t), \\ y^{\Delta}(t) = -\delta_2(t)y(t) + \beta(t)\varphi_1(t) - b(t)\varphi_1^m(t-\tau(t)) + p(t), \end{cases} \tag{13}$$

for $t \in \mathbb{T}$, $\varphi = (\varphi_1, \varphi_2) \in PAP(\mathbb{T}; \mathbb{R}^2)$.

Let

$$A(t) = \begin{pmatrix} -\delta_1(t) & 0 \\ 0 & -\delta_2(t) \end{pmatrix}, \tag{14}$$

then the homogeneous equation of (13) is

$$z^{\Delta}(t) = A(t)z(t), \ t \in \mathbb{T},$$

and we can get that

$$\|e_A(t,s)\| \leqslant 2e_{-\bar{\delta}}(t,s) \leqslant 2e_{\ominus\bar{\delta}}(t,s), \ t \geqslant s,$$

since

$$e_{-\bar{\delta}}(t,s) = \exp\left(\int_s^t \frac{\mathrm{Log}(1-\bar{\delta}\mu(\tau))}{\mu(\tau)}\Delta\tau\right) \leqslant \exp\left(\int_s^t \frac{\mathrm{Log}\frac{1}{1+\bar{\delta}\mu(\tau)}}{\mu(\tau)}\Delta\tau\right) = e_{\ominus\bar{\delta}}(t,s).$$

By Lemma 2.8, we have $\varphi_1(t-\tau(t))$ is pseudo almost periodic, and by Lemma 2.11 and 2.12, we derive that

$$\varphi_2, \ \beta\varphi_1 - b\varphi_1^m(\cdot - \tau(\cdot)) + p \in S^1PAP(\mathbb{T}; \mathbb{R}).$$

Denote

$$z(t) = (x(t), y(t)), \ F(t) = (\varphi_2(t), \beta(t)\varphi_1(t) - b(t)\varphi_1^m(t-\tau(t)) + p(t)).$$

We can rewrite (13) as

$$z^{\Delta}(t) = A(t)z(t) + F(t), t \in \mathbb{T}, \tag{15}$$

where $A(t)$ is given by (14). By $(H_1)$ and $(H_2)$, it is easy to see that all conditions in Theorem 3.1 are satisfied with (15) instead of (11). Thus, we obtain that (15) has a unique pseudo almost periodic solution $z^{\varphi}(t) = (x^{\varphi}(t), y^{\varphi}(t))$, which is expressed as follows:

$$\begin{cases} x^{\varphi}(t) = \int_{-\infty}^t e_{-\delta_1}(t,\sigma(s))\varphi_2(s)\Delta s, \\ y^{\varphi}(t) = \int_{-\infty}^t e_{-\delta_2}(t,\sigma(s))\big(\beta(s)\varphi_1(s) - b(s)\varphi_1^m(s-\tau(s)) + p(s)\big)\Delta s. \end{cases} \tag{16}$$

For $\varphi \in E_1^*$, we have

$$\|\varphi\|_* \leqslant \|\varphi - \varphi^0\|_* + \|\varphi^0\|_* \leqslant \frac{\theta_1\lambda}{1-\theta_1} + \lambda = \frac{\lambda}{1-\theta_1} < 1.$$

Define a nonlinear operator:

$$T : E_1^* \mapsto PAP(\mathbb{T};\mathbb{R}^2),\ \varphi = (\varphi_1, \varphi_2) \mapsto z^\varphi = (x^\varphi, y^\varphi).$$

Then (10) has a unique pseudo almost periodic solution in $E_1^*$ if and only if $T$ has a fixed point in $E_1^*$. So we only need to prove that $T$ has a fixed point in $E_1^*$.

First, we show that for any $\varphi \in E_1^*$, $T\varphi \in E_1^*$, we have

$$\begin{aligned} \|T\varphi - \varphi_0\|_* &= \max\left\{ \sup_{t\in\mathbb{T}} \left| \int_{-\infty}^t e_{-\delta_1}(t,\sigma(s))\varphi_2(s)\Delta s \right|, \right. \\ &\qquad \left. \sup_{t\in\mathbb{T}} \left| \int_{-\infty}^t e_{-\delta_2}(t,\sigma(s)) \left(\beta(s)\varphi_1(s) - b(s)\varphi_1^m(s-\tau(s))\right)\Delta s \right| \right\} \\ &\leqslant \max\left\{ \sup_{t\in\mathbb{T}} \int_{-\infty}^t e_{-\delta_1}(t,\sigma(s))\Delta s \|\varphi\|_*, \right. \\ &\qquad \left. \sup_{t\in\mathbb{T}} \int_{-\infty}^t e_{-\delta_2}(t,\sigma(s))\left(|\beta(s)| + |b(s)|\right)\Delta s \|\varphi\|_* \right\}. \end{aligned}$$

By Lemma 2.11 (iii) and 2.14 (iv), we can get that

$$\begin{aligned} &\int_{\infty}^t e_{-\delta_2}(t,\sigma(s))(|\beta(s)| + |b(s)|)\Delta s \qquad (17) \\ &= \sum_{n=1}^{\infty} \int_{t-n\mathcal{K}}^{t-(n-1)\mathcal{K}} e_{-\delta_2}(t,\sigma(s))(|\beta(s)| + |b(s)|)\Delta s \\ &\leqslant \sum_{n=1}^{\infty} \int_{t-n\mathcal{K}}^{t-(n-1)\mathcal{K}} e_{-\delta_2^-}(t,\sigma(s))(|\beta(s)| + |b(s)|)\Delta s \\ &\leqslant \sum_{n=1}^{\infty} e_{-\delta_2^-}(t,\sigma(t)-(n-1)\mathcal{K}) \int_{t-n\mathcal{K}}^{t-(n-1)\mathcal{K}} (|\beta(s)| + |b(s)|)\Delta s \\ &\leqslant \lambda_{\delta_2^-}\mathcal{K}(\|\beta(s)\|_{S^p} + \|b\|_{S^p}) \leqslant \lambda_{\delta_2^-}\mathcal{K}(\|\beta\|_{S^p} + m\|b\|_{S^p}). \qquad (18) \end{aligned}$$

Then we obtain that

$$\|T\varphi - \varphi_0\|_* \leqslant \max\left\{ \frac{1}{\delta_1^-}, \lambda_{\delta_2^-}\mathcal{K}(\|\beta\|_{S^p} + m\|b\|_{S^p}) \right\} \|\varphi\|_* = \theta_1\|\varphi\|_* \leqslant \frac{\theta_1\lambda}{1-\theta_1},$$

that is $T\varphi \in E_1^*$. Next, we will prove that $T$ is a contraction. In fact, for any $\varphi = (\varphi_1, \varphi_2)$, $\psi = (\psi_1, \psi_2) \in E_1^*$, by Lemma 2.11 (iii), 2.14 (iv) and the same calculation in (18), we can get

$$\begin{aligned} &\|T\varphi - T\psi\|_* \\ &= \max\left\{ \sup_{t\in\mathbb{T}} \left| \int_{-\infty}^t e_{-\delta_1}(t,\sigma(s))(\varphi_2(s) - \psi_2(s))\Delta s \right|, \sup_{t\in\mathbb{T}} \left| \int_{-\infty}^t e_{-\delta_2}(t,\sigma(s)) \right.\right. \\ &\qquad \left.\left. \cdot\Big(\beta(s)(\varphi_1(s) - \psi_1(s)) - b(s)\big(\varphi_1^m(s-\tau(s)) - \psi_1^m(s-\tau(s))\big)\Big)\Delta s \right| \right\} \\ &\leqslant \max\left\{ \sup_{t\in\mathbb{T}} \int_{-\infty}^t e_{-\delta_1}(t,\sigma(s))\Delta s\|\varphi - \psi\|_*, \sup_{t\in\mathbb{T}} \int_{-\infty}^t e_{-\delta_2}(t,\sigma(s)) \right. \\ &\qquad \left. \cdot\left( |\beta(s)| + |b(s)| \sum_{i+j=m-1} \|\varphi\|^i \|\psi\|^j \right)\Delta s\|\varphi - \psi\|_* \right\} \end{aligned}$$

$$\leqslant \max\left\{\frac{1}{\delta_1^-}, \lambda_{\delta_2^-}\mathcal{K}(\|\beta\|_{S^p} + m\|b\|_{S^p})\right\}\|\varphi - \psi\|_* = \theta_1\|\varphi - \psi\|_*.$$

Thus, $T$ is a contraction mapping, and by the Banach fixed point theorem, $T$ has a unique fixed point in $E_1^*$. □

For $\mathbb{T} = \mathbb{R}$, the following conditions will be useful.

(H$_4$) $\delta_i \in BC(\mathbb{R};\mathbb{R}) \cap S^pAP(\mathbb{R};\mathbb{R}), i = 1, 2$ and denote $\delta_i^+ = \sup_{t\in\mathbb{R}} \delta_i(t)$, $\delta_i^- = \inf_{t\in\mathbb{R}} \delta_i(t), i = 1, 2,\ \bar{\delta} = \min\{\delta_1^-, \delta_2^-\} > 0$;

(H$_5$) $\beta,\ b,\ p \in S^pPAP(\mathbb{R};\mathbb{R}) \cap BC(\mathbb{R};\mathbb{R}),\ \tau \in PAP(\mathbb{R};\mathbb{R})$;

(H$_6$) $\theta_2 = \max\left\{\frac{1}{\delta_1^-}, \frac{\|\beta\| + m\|b\|}{\delta_2^-}\right\} < 1.$

Let

$$E_2^* = \left\{\varphi \in PAP(\mathbb{R};\mathbb{R}^2) : \|\varphi - \varphi_0\|_* \leqslant \frac{\theta_2\lambda}{1-\theta_2} \text{ and } \varphi \text{ is uniformly continuous}\right\},$$

where $\varphi^0(t) = (0, \varphi_2^0(t)),\ \varphi_2^0(t) = \int_{-\infty}^t e_{-\delta_2}(t, s)p(s)ds,\ \lambda = \frac{\|p\|}{\delta_2^-}$. It is easy to verify that $\|\varphi_0\|_* \leqslant \lambda$.

**Lemma 3.2** (Liu and Tunç (2015)) *$E_2^*$ is a closed subset of $PAP(\mathbb{R};\mathbb{R}^n)$.*

**Theorem 3.3** *Suppose that $\mathbb{T} = \mathbb{R}$ and assumptions* (H$_4$)-(H$_6$) *hold, and $\frac{\lambda}{1-\theta_2} < 1$, then* (10) *has a unique pseudo almost periodic solution in $E_2^*$ and* (5) *has a unique pseudo almost periodic solution $u$ satisfying that $(u, u' + \delta_1 u) \in E_2^*$.*

***Proof*** We replace $\varphi = (\varphi_1, \varphi_2) \in PAP(\mathbb{R};\mathbb{R}^2) \cap BUC(\mathbb{R};\mathbb{R}^2)$ in (3.4), then we get the following system:

$$\begin{cases} x'(t) = -\delta_1(t)x(t) + \varphi_2(t), \\ y'(t) = -\delta_2(t)y(t) + \beta(t)\varphi_1(t) - b(t)\varphi_1^m(t - \tau(t)) + p(t). \end{cases} \tag{19}$$

Let

$$A(t) = \begin{pmatrix} -\delta_1(t) & 0 \\ 0 & -\delta_2(t) \end{pmatrix}, \tag{20}$$

and the homogeneous equation of (19) is

$$z'(t) = A(t)z(t),\ t \in \mathbb{R}.$$

We can check that $\|e_A(t, s)\| \leqslant 2e_{-\bar{\delta}}(t, s)$ for $t \geqslant s$. By Lemma 2.7, we have $\varphi_1(t - \tau(t))$ is pseudo almost periodic, and by Lemma 2.11 (i), we derive that

$$\varphi_2,\ \beta\varphi_1 - b\varphi_1^m(\cdot - \tau(\cdot)) + p \in S^1PAP(\mathbb{T};\mathbb{R}).$$

Denote

$$z(t) = (x(t), y(t)),\ F(t) = (\varphi_2(t), \beta(t)\varphi_1(t) - b(t)\varphi_1^m(t - \tau(t)) + p(t)).$$

We can rewrite (19) as

$$z'(t) = A(t)z(t) + F(t),\ t \in \mathbb{R}, \tag{21}$$

where $A(t)$ is given by (20). By ($H_4$) and ($H_5$), it is easy to see that all conditions in Theorem 3.1 are satisfied with (21) instead of (11). Thus, we obtain that (21) has a unique pseudo almost periodic solution $z^{\varphi}(t) = (x^{\varphi}(t), y^{\varphi}(t))$, which is expressed as (16).

For $\varphi \in E_2^*$, we have

$$\|\varphi\|_* \leqslant \|\varphi - \varphi^0\|_* + \|\varphi^0\|_* \leqslant \frac{\theta_2\lambda}{1-\theta_2} + \lambda = \frac{\lambda}{1-\theta_2} < 1.$$

Define a nonlinear operator

$$T : E_2^* \mapsto PAP(\mathbb{T}; \mathbb{R}^2),\ \varphi = (\varphi_1, \varphi_2) \mapsto z^{\varphi} = (x^{\varphi}, y^{\varphi}).$$

Then (10) has a unique pseudo almost periodic solution in $E_2^*$ if and only if $T$ has a fixed point in $E_2^*$. So we only need to prove that $T$ has a fixed point in $E_2^*$.

We first prove that $x^{\varphi}, y^{\varphi}$ are uniformly continuous. For $\varepsilon > 0$, let $0 < \eta < \min\left\{\frac{-\ln(1-\varepsilon)}{\delta_1^+}, \varepsilon\right\}$. For $t_1, t_2 \in \mathbb{R}$, $|t_1 - t_2| < \eta$, without loss generality we assume that $t_1 > t_2$, we have

$$\begin{aligned}
&|x^{\varphi}(t_1) - x^{\varphi}(t_2)| \\
&\quad = \left|\int_{-\infty}^{t_1} e_{-\delta_1}(t_1, s)\varphi_2(s)ds - \int_{-\infty}^{t_2} e_{-\delta_1}(t_2, s)\varphi_2(s)ds\right| \\
&\quad = \left|\int_{-\infty}^{t_2} (e_{-\delta_1}(t_1, s) - e_{-\delta_1}(t_2, s))\varphi_2(s)ds + \int_{t_2}^{t_1} e_{-\delta_1}(t_1, s)\varphi_2(s)ds\right| \\
&\quad \leqslant \left|\int_{-\infty}^{t_2} (e_{-\delta_1}(t_1, t_2) - 1)e_{-\delta_1}(t_2, s)\varphi_2(s)ds\right| + \left|\int_{t_2}^{t_1} e_{-\delta_1}(t_1, s)\varphi_2(s)ds\right| \\
&\quad \leqslant (1 - e_{-\delta_1}(t_1, t_2)) \int_{-\infty}^{t_2} e^{-\delta_1^-(t_2 - s)}|\varphi_2(s)|ds + \varepsilon\|\varphi_2\| \\
&\quad \leqslant \varepsilon \cdot \frac{\|\varphi_2\|}{\delta_1^-} + \varepsilon\|\varphi_2\| = C\varepsilon,
\end{aligned}$$

where $C = \frac{1+\delta_1^-}{\delta_1^-}\|\varphi_2\|$. Hence, $x^{\varphi}$ is uniformly continuous. Similarly, we can prove that $y^{\varphi}$ is uniformly continuous and we omit the details here. Next, we show that for any $\varphi \in E_2^*$, $T\varphi \in E_2^*$.

$$\begin{aligned}
\|T\varphi - \varphi_0\|_* &= \max\Big\{\sup_{t\in\mathbb{R}}\Big|\int_{-\infty}^{t} e_{-\delta_1}(t, s)\varphi_2(s)ds\Big|,\ \sup_{t\in\mathbb{R}}\Big|\int_{-\infty}^{t} e_{-\delta_2}(t, s)\big(\beta(s)\varphi_1(s) \\
&\qquad - b(s)\varphi_1^m(s - \tau(s))\big)ds\Big|\Big\} \\
&\leqslant \max\Big\{\sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e_{-\delta_1}(t, s)ds \cdot \|\varphi\|_*,\ \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e_{-\delta_2}(t, s) \\
&\qquad \cdot \big(|\beta(s)| + |b(s)|\big)ds \cdot \|\varphi\|_*\Big\}
\end{aligned}$$

$$\leqslant \max\left\{\frac{1}{\delta_1^-}, \frac{\|\beta\| + \|b\|}{\delta_2^-}\right\} \cdot \|\varphi\|_*$$
$$\leqslant \max\left\{\frac{1}{\delta_1^-}, \frac{\|\beta\| + m\|b\|}{\delta_2^-}\right\} \cdot \|\varphi\|_*$$
$$= \theta_2\|\varphi\|_* \leqslant \frac{\theta_2\lambda}{1-\theta_2},$$

that is $T\varphi \in E_2^*$. At the last, we will prove that $T$ is a contraction. In fact, for any $\varphi = (\varphi_1, \varphi_2), \psi = (\psi_1, \psi_2) \in E_2^*$, we can get

$$\begin{aligned}
&\|T\varphi - T\psi\|_* \\
&= \max\Big\{ \sup_{t\in\mathbb{R}} \Big| \int_{-\infty}^t e_{-\delta_1}(t,s)(\varphi_2(s) - \psi_2(s))ds \Big|,\ \sup_{t\in\mathbb{R}} \Big| \int_{-\infty}^t e_{-\delta_2}(t,s)\big(\beta(s)(\varphi_1(s) \\
&\quad - \psi_1(s)) - b(s)(\varphi_1^m(s-\tau(s)) - \psi_1^m(s-\tau(s)))\big)ds \Big| \Big\} \\
&\leqslant \max\Big\{ \sup_{t\in\mathbb{R}} \int_{-\infty}^t e_{-\delta_1}(t,s)ds \cdot \|\varphi - \psi\|_*,\ \sup_{t\in\mathbb{R}} \int_{-\infty}^t e_{-\delta_2}(t,s)\Big( |\beta(s)| + |b(s)| \\
&\quad \cdot \sum_{i+j=m-1} \|\varphi\|^i \|\psi\|^j \Big) ds \cdot \|\varphi - \psi\|_* \Big\} \\
&\leqslant \max\left\{\frac{1}{\delta_1^-}, \frac{\|\beta\| + m\|b\|)}{\delta_2^-}\right\} \cdot \|\varphi - \psi\|_* \\
&\leqslant \theta_2 \cdot \|\varphi - \psi\|_*.
\end{aligned}$$

Thus, $T$ is a contraction mapping, and by the Banach fixed point theorem, $T$ has a unique fixed point in $E_2^*$. □

***Remark 3.1*** We note that $\delta_i, i = 1, 2, \beta, b$ and $p$ are not assumed to be bounded in ($H_1$) and ($H_2$), but in ($H_4$) and ($H_5$), the boundedness is needed. In fact, for $\mathbb{T} \neq \mathbb{R}$, under the conditions ($H_1$) and ($H_2$), we can get the pseudo almost periodicity of $\varphi_1(\cdot - \tau(\cdot)),\ x^\varphi$ and $y^\varphi$ in Theorem 3.2 by Lemma 2.8 without the uniform continuity of $\varphi$. On the other hand, for $\mathbb{T} = \mathbb{R}$, to ensure $\varphi_1(\cdot - \tau(\cdot)) \in PAP(\mathbb{R}; \mathbb{R})$, we have to prove that the uniform continuity of $\varphi$, where the boundedness of the parameters is essential. There exists counterexamples showing that $\varphi(\cdot - \tau(\cdot)) \notin PAP(\mathbb{R}; \mathbb{R}^2)$ if $\varphi$ is not uniformly continuous. For more details of this problem, readers may refer to Zhang (2003). Moreover, assumption ($H_3$) is replaced by a simple form ($H_6$).

Let us end this work with two examples.

***Example 3.1*** Let

$$\mathbb{T} = \bigcup_{k\in\mathbb{Z}} \left( \left[2k, 2k+\frac{1}{2}\right] \cup \bigcup_{n=2}^{\infty} \left\{2k+1-\frac{1}{2^n}\right\} \cup \{2k+1\} \cup \left\{2k+\frac{3}{2}\right\} \right),$$

denote $g_{nl} = 10^n \cdot l + 1 - \frac{1}{2^{10^n - 1}}, n \geqslant 2, l$ is odd. Consider the following Duffing equation on $\mathbb{T}$.

$$(x^\Delta)^\Delta(t) + c(t)x^\Delta(t) - a(t)x(t) + b(t)x^3(t-\tau(t)) = p(t), \tag{22}$$

where

$$c(t)=\begin{cases}0.1\sin 2\pi t+3, & t\in[2k,2k+\frac{1}{2}];\\ 0.1n+1.5, & t=g_{nl}, n\geqslant 2, l \text{ is odd};\\ 3, & \text{otherwise,}\end{cases}$$

$$a(t)=\begin{cases}-0.2\pi\cos 2\pi t-1.5(0.1\sin 2\pi t+1.5)+\frac{1}{60}(\sin 2\pi t+h(t)), & t\in[2k,2k+\frac{1}{2});\\ -1.5n+\frac{h(t)}{60}, & t=g_{nl}, n\geqslant 2, l \text{ is odd};\\ -2.25+\frac{h(t)}{60}, & \text{otherwise ,}\end{cases}$$

$$b(t)=\begin{cases}\frac{1}{60}(\sin 2\pi t+h(t)), & t\in[2k,2k+\frac{1}{2}];\\ \frac{h(t)}{60}, & \text{otherwise,}\end{cases}$$

$$h(t)=\begin{cases}1, & t\in[-1,1]_{\mathbb{T}};\\ \frac{1}{\sqrt{|t|}}, & \text{otherwise,}\end{cases}$$

$$p(t)=\begin{cases}\frac{12}{49}\sin 2\pi t, & t\in\left[2k,2k+\frac{1}{2}\right];\\ 0, & \text{otherwise,}\end{cases}$$

$$\tau(t)=\begin{cases}8+\tau_0(t), & t\in\left[4k,4k+\frac{1}{2}\right];\\ 2+\tau_0(t), & t\in\left[4k+2,4k+\frac{5}{2}\right];\\ 0, & \text{otherwise,}\end{cases}$$

where

$$\tau_0(t)=\begin{cases}4, & t\in\left[2^n,2^n+\frac{1}{2}\right],\ n\in\mathbb{N}^+;\\ 0, & \text{otherwise.}\end{cases}$$

It is easy to see that $\tau\in PAP(\mathbb{T};\Pi)$. Let $y(t)=x^{\Delta}(t)+\delta_1(t)x(t)$ where

$$\delta_1(t)=\begin{cases}0.1\sin 2\pi t+1.5, & t\in\left[2k,2k+\frac{1}{2}\right];\\ 1.5, & \text{otherwise.}\end{cases}$$

Then we transform (22) into the following system:

$$\begin{cases}x^{\Delta}(t)=-\delta_1(t)x(t)+y(t),\\ y^{\Delta}(t)=-\delta_2(t)y(t)+\beta(t)x(t)-b(t)x^3(t-\tau(t))+p(t),\end{cases}$$

where

$$\delta_2(t)=\begin{cases}n, & t=g_{nl},\ n\geqslant 2,\ l \text{ is odd};\\ 1.5, & \text{otherwise,}\end{cases}$$

$$\beta(t)=\begin{cases}\frac{1}{60}(\sin 2\pi t+h(t)), & t\in\left[2k,2k+\frac{1}{2}\right];\\ \frac{h(t)}{60}, & \text{otherwise.}\end{cases}$$

Obviously, $\delta_1\in S^1AP(\mathbb{T};\mathbb{R})$, $b,\ \beta\in PAP(\mathbb{T};\mathbb{R})$, and use the same calculation in Example 3.2 in Yang and Li (2022), we can get that $\delta_2\in S^1AP(\mathbb{T};\mathbb{R})$. From Lemma 2.14 (iv), we have $\lambda_{\delta_2^-}=\frac{49}{12}$, and we derive that $\|\beta\|_{S^1}=\|b\|_{S^1}=\frac{1}{120}\left(2+\frac{1}{\pi}\right)$, $\lambda=\lambda_{\delta_2^-}\mathcal{K}\|p\|_{S^1}=\frac{1}{\pi}$, $\theta=\max(\delta_1^-,\lambda_{\delta_2^-}\mathcal{K}(\|\beta\|_{S^1}+3\|b\|_{S^1}))=\frac{2}{3}$, $\frac{\lambda}{1-\theta}=\frac{3}{\pi}<1$. Let

$$E_1^*=\left\{\phi\in PAP(\mathbb{T};\mathbb{R}^2):\|\phi-\phi_0\|\leqslant\frac{2}{\pi}\right\},$$

where $\phi_0(t) = (0, \phi_2^0(t))$, $\phi_2^0(t) = \int_{-\infty}^t e_{-\delta_2}(t,s)p(s)ds$. Thus, all conditions in Theorem 3.2 are satisfied, (22) has a unique pseudo almost periodic solution $u$ satisfying that $(u, u^\Delta + \delta_1 u) \in E_1^*$.

***Example 3.2*** Consider the following Duffing equation on $\mathbb{R}$ with time-varying coefficients:

$$x''(t) + (6 + \sin t + \cos g(t))x'(t) + (9 + \cos t + 3\cos g(t) + 3\sin t + \cos g(t)\sin t$$
$$- \frac{1}{24}(\sin t + \sin\sqrt{2}t))x(t) + \frac{1}{24}(\cos t + \cos\sqrt{2}t)x^3(t - \cos t) = \frac{1}{10}(3\sin t + h(t)), \tag{23}$$

where

$$g(t) = \frac{1}{2 + \cos t + \cos\sqrt{2}t}, \quad h(t) = \begin{cases} 1, & -1 < t < 1; \\ \frac{1}{\sqrt{|t|}}, & \text{otherwise.} \end{cases}$$

From the result in Levitan (1959) (see page 212–213), we know that $\cos g \in S^1AP(\mathbb{R};\mathbb{R})$, and it is easy to see that $h \in PAP_0(\mathbb{R};\mathbb{R})$. Let

$$y = x'(t) + (3 + \sin t)x(t),$$

then we can transform (23) into the following system:

$$\begin{cases} x'(t) = -(3 + \sin t)x(t) + y(t) \\ y'(t) = -(3 + \cos g(t))y(t) + \frac{1}{24}(\sin t + \sin\sqrt{2}t)x(t) - \frac{1}{24}(\cos t + \cos\sqrt{2}t) \\ \qquad\qquad \cdot x^3(t - \cos t) + \frac{1}{10}(2\sin t + \sin\sqrt{3}t + h(t)). \end{cases}$$

Denote $\delta_1(t) = 3 + \sin t$, $\delta_2(t) = 3 + \cos g(t)$, $\beta(t) = \frac{1}{24}(\sin t + \sin\sqrt{2}t)$, $b(t) = \frac{1}{24}(\cos t + \cos\sqrt{2}t)$, $p(t) = \frac{1}{10}(2\sin t + \sin\sqrt{3}t + h(t))$. Let

$$E^* = \left\{\varphi \in PAP(\mathbb{R};\mathbb{R}^2) : \|\varphi - \varphi^0\|_* \leqslant \frac{1}{5} \text{ and } \varphi \text{ is uniformly continuous}\right\},$$

where

$$\varphi^0(t) = (0, \varphi_2^0(t)), \ \varphi_2^0(t) = \int_{-\infty}^t e_{-\delta_2}(t,s)p(s)ds.$$

It is easy to see that $\delta_1^- = \delta_2^- = 2$, $\|\beta\| = \|b\| = \frac{1}{12}$, and $p \in PAP(\mathbb{R};\mathbb{R})$, and consequently $p(t) \in S^1PAP(\mathbb{R};\mathbb{R})$ $\|p\| \leqslant \frac{2}{5}$. Moreover,

$$\theta = \max\left\{\frac{1}{\delta_1^-}, \frac{\|\beta\| + 3\|b\|}{\delta_2^-}\right\} = \max\left\{\frac{1}{2}, \frac{1}{6}\right\} = \frac{1}{2} < 1; \quad \lambda = \frac{\|p\|}{\delta_2^-} \leqslant \frac{1}{5}; \ \frac{\lambda}{1-\theta} \leqslant \frac{2}{5} < 1.$$

Now it is easy to check that all conditions in Theorem 3.3 are satisfied. Hence, (23) has a unique pseudo almost periodic solution $u$ satisfying that $(u, u' + \delta_1 u) \in E^*$.

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## Declarations

**Conflict of interest** The authors declare that there is no conflict of interest regarding the publication of this paper.

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