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Decision-making method under the interval-valued complex fuzzy soft environment

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Abstract

In this paper, we introduce some set-theoretic operations and laws of the IV-CFSSs, such as interval-valued complex fuzzy soft complement, union, intersection, t-norm, s-norm, simple product, Cartesian product, probabilistic sum, simple difference, and the convex linear sum of min and max operators. We define the distance measure of two IV-CFSSs. This distance measure is then used to define the δ-equality of IV-CFSSs. We establish some particular examples and basic results of these operations and laws. Moreover, we use IV-CFSSs in decision-making problems. We develop a new decision-making method using the interval-valued complex fuzzy distance measures under the environments of IV-CFSSs. We discuss the real-life case based on the proposed decision-making method. A real-life example demonstrates that the decision-making method developed in the paper can be utilized to deal with problems of uncertainty. Further, the comparative study of IV-CFSSs with complex fuzzy soft sets, interval-valued fuzzy soft sets, and fuzzy soft sets is established.

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1 Introduction

Many theories are proposed to cope with uncertainty and imprecision that manage in almost all the real-life problems, such as the theory of fuzzy sets (FSs) (Zadeh 1965), theory of rough sets (Pawlak 1982), theory of intuitionistic fuzzy sets (IFSs) (Atanassov 2016), theory of Pythagorean fuzzy sets (Peng and Yang 2015), theory of complex fuzzy sets (CFSs) (Ramot et al. 2002), theory of picture fuzzy sets (Hussain et al. 2022), theory of soft sets (Molodtsov 1999), theory of fuzzy soft sets (FSSs) (Maji 2001), and theory of complex fuzzy soft sets (SSs) (Thirunavukarasu et al. 2017). All these models have their own limitations, advantages, and characteristics. These models are used in many situations of uncertainties, such as engineering, computer science, decision-making problems, networking, pattern recognition, and many other fields of science.

FSs, IFSs, and PFSs, these three types of sets all other a more flexible and nuanced representation of uncertainty and imprecision than classical set theory and have been applied in D–M, Pa–Re, and image processing. However, these three models cannot handle two-dimensional problems. To discuss two-dimensional phenomena, Ramot et al. (2002) gave the idea of complex fuzzy sets (CFSs). A CFS is the generalization of an FS whose membership function is a complex-valued function, that is, its range is the unit disk in a complex plane. The complex-valued function describes the degree of membership of a member in the CFS using both magnitude and phase information. CFSs have found applications in various fields such as image processing, Pa–Re, control systems, and D–M. They provide a powerful tool for modeling and analyzing complex systems with uncertainty and imprecision. Liu et al. (2020) discussed a D–M method based on complex fuzzy Di-Me and complex fuzzy entropy measure. Hu et al. (2018) introduced some Di-Mes in the environment of CFSs. They discussed their applications to continuity problems. Zhang et al. (2009) developed a new complex fuzzy Di-Me based on the maximum operator. They proposed an algorithm for signal processing in the environment of complex fuzzy Di-Mes. Zeeshan and Khan (2022) defined new Di-Mes in the environment of CFSs. They constructed an algorithm for the high resemblance of a signal based on complex fuzzy Di-Mes. A new algorithm for signal processing method in the environment of CFSs was proposed by Ma et al. (2019). A new self-learning complex neuro-fuzzy system based on CFSs was developed by Li et al. (2012). They utilized them in the adaptive image noise canceling process.

Bipolar fuzzy sets (BFSs) are introduced by Zhang (1994). BFS is the generalization of FSs and IFSs. Bipolar fuzzy sets find applications in various fields, including decision-making, control systems, and artificial intelligence, where uncertainty and imprecision need to be considered in a more nuanced manner than traditional fuzzy sets allow (Akram et al. 2018; Sarwar and Akram 2017; Patrascu 2015; Zararsız and Riaz 2022; Sakr et al. 2023). Alkouri et al. (2020) gave the idea of bipolar complex fuzzy sets (BCFSs). They discussed their applications in multiple attributes decision-making (MADM) problems. BCFSSs have many applications in various field of science (Gwak et al. 2023; Qiyas et al. 2024; Akram et al. 2023).

Maji (2001) developed the notion of the FSSs by associating the ideas of SSs and FSs. FSS is the generalization of FSs. The FSSs have considerable application potential in engineering, decision-making problems, computer science, and optimization (Atef et al. 2021; Petchimuthu et al. 2020; Beg et al. 2018). Močkoř and Hurtík (2021) introduced fuzzy soft relations and fuzzy soft approximations. They discussed an image processing application based on fuzzy soft relations and fuzzy soft approximations. Močkoř (2020) proposed the powerset theory of FSSs. Bhardwaj and Sharma (2021) developed a decision-making method under the environments of FSSs. Memis et al. utilized Euclidean pseudo-similarity to propose a new classification algorithm, i.e., fuzzy parameterized fuzzy soft Euclidean classifier (Memiş et al. 2021a). A new fuzzy soft expert system to predict lung cancer disease was proposed by Khalil et al. (2020). Begam et al. (2020) introduced similarity measures between lattice ordered multi-fuzzy soft sets. They discussed the applications of similarity measures of lattice ordered multi-fuzzy soft sets in decision-making problems.

Yang et al. (2009) introduced the concept of interval-valued soft set (IV-FSS) by combining the interval-valued fuzzy set and soft set models. The IV-FSS is more reasonable to give interval-valued data to describe membership degree. The IV-FSSs have desirable applications in engineering, decision-making problems, computer science, and optimization. Ma et al. (2021) proposed a decision-making method under the environments of IV-FSSs. A novel interval-valued fuzzy soft decision-making method based on Combined Compromise Solution (CoCoSo) and Criteria Importance Through Inter-criteria Correlation (CRITIC) for intelligent healthcare management evaluation was developed by Peng et al. (2021). Mohanty and Tripathy (2021) proposed an algorithm that is used to recommend the best variety of turmeric under the environments of IV-FSSs. Qin et al. (2021) discussed a new decision-making method based on an interval-valued fuzzy soft set by means of the contrast table.

Maji (2009) gave the concept of intuitionistic fuzzy soft set (IFSS) by combining the intuitionistic fuzzy set and soft set models. The IFSS is a powerful tool that is extremely useful in multicriteria decision-making (MCDM) problems. Hayat et al. (2021) proposed new aggregation operators on generalized intuitionistic fuzzy soft sets. They developed a decision-making method based on the generalized intuitionistic fuzzy soft aggregation operators. A TOPSIS method based on the correlation coefficient of interval-valued intuitionistic fuzzy soft sets (IV-IFSSs) was proposed by Zulqarnain et al. (2021). They defined interval-valued IFS weighted average and interval-valued IFS weighted geometric operators and proposed decision-making techniques based on the defined operators. The complex intuitionistic fuzzy soft prioritized weighted averaging operator, the complex intuitionistic fuzzy soft prioritized ordered weighted averaging operator, the complex intuitionistic fuzzy soft prioritized weighted geometric operator, and complex intuitionistic fuzzy soft prioritized ordered weighted geometric operator were developed by Ali et al. (2021). They discussed a decision-making method on the developed operators. Memiş et al. (2021b) defined a classification method based on Hamming pseudo-similarity of intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices. Ghosh et al. (2021) proposed a new hybrid soft computing entrenched segmentation method for the detection of breast cancer in early stages. Singh discussed the non-linear programming (NLP) methods under the environment of an IV-IFSS for solving multiattribute decision-making (MADM) problems (Singh 2021). To improve the visual quality and highlight the local details in enhanced images, Ghosh and Ghosh discussed a colonogram enhancement approach utilizing IFSSs (Ghosh and Ghosh 2021). Ali introduced complex intuitionistic fuzzy power interaction aggregation operators and discussed their applications (Ali 2022).

FSSs, IV-FSSs, IFSSs, and IV-IFSSs cannot handle imprecise, inconsistent, and incomplete information of two-dimensional phenomena. These models are applicable to different areas of science, but there is one major deficiency in these models, that is, a lack of capability to model periodic nature. To overcome this difficulty, Thirunavukarasu et al. (2017) introduced the concept of a complex fuzzy soft set (CFSS). They discussed an application in decision-making under the complex fuzzy soft information. Zeeshan et al. (2022) defined new distance measure based on CFSSs and applied them in signal processing. Mahmood et al. (2021) explored the concept of complex fuzzy N-soft sets. They developed a decision-making method under the environments of complex fuzzy N-soft sets. Akram et al. (2021a) discussed an application of Complex Spherical Fuzzy N-Soft Sets for solving a multiattribute group decision-making (MAGDM) problem, namely, the selection of a firm for participation in a Saudi oil refinery project in Pakistan. Ahsan et al. (2021) proposed a new mathematical model for the treatment of HIV using complex fuzzy hypersoft mapping. A multiattribute group decision-making method under complex spherical fuzzy N-soft sets was proposed by Akram et al. (2021b).

Selvachandran and Salleh gave the notion of IV-CFSSs by combining complex fuzzy sets with type-2 fuzzy sets and soft sets (Selvachandran and Salleh 2017). They discussed the application of the proposed model in economic problems. A generalized interval-valued complex fuzzy soft set model was proposed by Selvachandran and Salleh (2017). They discussed an application of the proposed model in multiattribute decision-making problems. Rahman et al. (2021) presented the novel concept of a complex interval-valued fuzzy hypersoft set. The interval-valued complex fuzzy soft weighted arithmetic averaging operator and the interval-valued complex fuzzy soft weighted geometric averaging operator was defined by Fan et al. (2019). They developed a decision-making method based on the proposed operators. Greenfield et al. (2016) gave the notions of interval-valued complex fuzzy logic. Dai et al. (2019) introduced distance measures between the interval-valued complex fuzzy sets. They discussed their applications in decision-making problems.

1.1 Background and motivation

IV-CFSSs are an extension of classical fuzzy soft sets that incorporate the concept of intervals and complex numbers. Here are some motivations for considering IV-CFSSs and interval-valued complex fuzzy soft distance measure in decision-making problems:

(1) In many real-world situations, information is uncertain or vague. IV-CFSS provide a framework to represent uncertainty using interval-valued fuzzy sets, allowing for a more flexible representation of imprecision in the data.
(2) Interval-valued fuzzy sets are a generalization of classical fuzzy sets that allow for a range of possible membership values rather than a single point value. IV-CFSS extend this idea to complex fuzzy sets, enabling a more versatile representation of uncertainty in both the real and imaginary components.
(3) Fuzzy soft sets, including their interval-valued complex counterparts, find applications in decision-making processes. The flexibility of representation allows decision-makers to consider various sources of uncertainty and vagueness when making decisions.
(4) Soft sets are well suited for modeling situations where information is incomplete or partial. IVCFSS enhance this modeling capability by introducing intervals and complex numbers, providing a richer representation of the uncertainty associated with the elements in the set.
(5) The mathematical formalism of IV-CFSS provides a systematic way to handle uncertainty through the use of intervals and complex numbers. This allows researchers and practitioners to develop algorithms and methodologies for analysis, manipulation, and decision-making based on this formalism.

1.2 Contribution of interval-valued complex fuzzy soft sets in decision-making problems

The application of IV-CFSS in decision-making problems offers several contributions, mainly due to their ability to handle complex and uncertain information. Here are some key contributions of IVCFSS in decision-making problems:

(1) IVCFSS provide a flexible and comprehensive representation of uncertainty. By incorporating intervals and complex numbers, these sets can express not only imprecision but also capture the relationships between real and imaginary components of uncertain information. This representation is valuable in decision-making scenarios where uncertainty is inherent.
(2) IVCFSS allow the integration of both quantitative and qualitative information through the use of complex numbers. Decision-making often involves multiple criteria or attributes, and IV-CFSS enable the representation of these multidimensional aspects in a unified framework. This is particularly beneficial in cases where decisions are based on a combination of numerical and non-numerical factors.
(3) Real-world decision-making problems are often characterized by complex relationships and uncertainty. IV-CFSS provide a more accurate modeling approach by considering the interplay of interval-valued and complex fuzzy information, offering a richer representation of the underlying complexities in the decision environment.
(4) Decision-making problems may involve inconsistent or ambiguous information. IV-CFSS can accommodate such inconsistencies and ambiguities, allowing decision-makers to explicitly model and manage situations where the available information may be conflicting or unclear.

1.3 Significance of interval-valued complex fuzzy soft sets

IV-CFSSs hold significance in various fields and applications due to their ability to effectively model and handle complex, uncertain, and imprecise information. Here are some key aspects of the significance of IV-CFSSs:

(1) IVCFSS provide a powerful framework for modeling uncertainty and imprecision in information. The combination of intervals and complex numbers allows for a more nuanced representation of uncertain data, addressing real-world scenarios where precise information may be unavailable or difficult to obtain.
(2) IVCFSS contribute to decision-making processes by providing a soft computing approach that accommodates the uncertainty inherent in decision environments. The soft set framework allows decision-makers to model their subjective judgments, and the interval-valued complex nature enhances the realism of decision models.
(3) IVCFSS seamlessly integrate with interval-valued fuzzy sets, offering a unified approach to handling uncertainty in decision-making problems. This integration is valuable in applications where both interval-valued and complex fuzzy information need to be considered simultaneously.
(4) Decision problems frequently require the consideration of multiple criteria or attributes. IVCFSS enable the construction of multidimensional decision models, where both real and imaginary components can be used to represent diverse aspects of the decision-making criteria. This is especially useful in complex decision environments.
(5) IVCFSS distance measures and similarity measures enable a quantitative assessment of the relationships between decision alternatives. This allows decision-makers to compare

and evaluate alternatives based on a more comprehensive understanding of the similarities and differences within the decision space.

In this paper, we introduce some set-theoretic operations and laws of the IV-CFSSs, such as interval-valued complex fuzzy soft complement, union, intersection, t-norm, s-norm, simple product, Cartesian product, probabilistic sum, simple difference, and the convex linear sum of min and max operators. We propose the distance measure of two IV-CFSSs. This distance measure is then used to define the δ-equality of IV-CFSSs. We establish some particular examples and basic results of these operations and laws. Moreover, we develop a decision-making method utilizing the interval-valued complex fuzzy distance measures under the environments of IV-CFSSs. We discuss the real-life case and comparison analysis based on the proposed decision-making method.

2 Preliminaries

In this section, we will recall some basic notions of fuzzy sets theory.

Definition 1 (Ramot et al. 2002) A CFS χ is characterized by a grade value $\Im_{\mathcal{L}}(x) = \ell_{\mathcal{L}}(x)e^{iw_{\mathcal{L}}(x)}$ on a universe of discourse $\mathcal{L}$. In a CFS $\mathcal{L}$, the GV $\Im_{\mathcal{L}}(x)$ of an element $x \in \mathcal{L}$ is determined by a complex-valued function rather than a simple grade value. In the grade value of CFS $\mathcal{L}$, the term $\ell_{\mathcal{L}}(x)$ is said to be an amplitude term and $w_{\mathcal{L}}(x)$ is called a phase term.

Note that both these functions are real-valued and $\ell_{\mathcal{L}}(x) \in [0, 1]$.
Mathematically;
It can be written as

$$\mathcal{L} = \{(\varkappa; \Im_{\mathcal{L}}(x)) : x \in \chi\},$$
$$= \ell_{\mathcal{L}}(x)e^{iw_{\mathcal{L}}(\varkappa)} : x \in \chi.$$

Motivation

In many real-world scenarios, information is not only uncertain but also possesses a complex nature, characterized by both magnitude and phase. Complex fuzzy sets provide a mathematical framework to represent this complex information, allowing for a more accurate modeling of the relationships between elements. Complex fuzzy sets are built on a mathematically rigorous foundation. The use of complex numbers allows for precise and systematic representation, making it suitable for applications where a high level of mathematical rigor is required. Complex fuzzy sets can model cognitive and linguistic aspects of decision-making and reasoning. The qualitative and quantitative aspects of information in complex fuzzy sets can represent human judgments and linguistic expressions in a more nuanced manner. In summary, the motivation for complex fuzzy sets arises from the need to represent complex, multidimensional, and uncertain information in a mathematically rigorous manner. Their application spans various domains, ranging from engineering and physics to decision-making and cognitive modeling, making them a valuable tool in addressing real-world problems characterized by both complexity and uncertainty.

Definition 2 (Molodtsov 1999) Let χ be a universe of discourse, and $\mathcal{F}$ be a set of parameters. Let $P(\chi)$ denotes the power set of χ. Let $\mathcal{L} \subset \mathcal{F}$. A pair $(\Im, \mathcal{L})$ is called a soft set over χ, where $\Im$ is a mapping given by $\Im : \mathcal{L} \rightarrow P(\chi)$.

Definition 3 (Maji 2001) Let $\Im(\chi)$ denotes the set of all fuzzy sets of χ. A pair $(\Im, \mathcal{L})$ is called a fuzzy soft set over $\Im(\chi)$, where $\Im$ is a mapping given by $\Im : \tau \to P(\Im(\chi))$.

Definition 4 (Selvachandran and Salleh 2017) Let χ be a universe of discourse, and $\mathcal{F}$ be a set of parameters. Let $\mathcal{L} \subset \mathcal{F}$ and $P(\chi)$ represent the power set of χ. Then, a pair $(\Im, \mathcal{L})$ is called an interval-valued complex fuzzy soft set (IV-CFSS) over χ, where $\Im$ is a mapping defined by $\Im : \mathcal{L} \to P(\chi)$, and is specified by

$$\begin{aligned}\mathcal{L}(x) &= \left\{\left(x, \left(\mathbb{Z}^-_{\Im_\alpha}(x), \mathbb{Z}^+_{\Im_\alpha}(x)\right)\right) : x \in \chi\right\}\\ &= \left\{\left(x, \left([\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^+_{\Im_\alpha}(x)].e^{i[\circledS^-_{\Im_\alpha}(x), \circledS^+_{\Im_\alpha}(x)]}\right)\right) : x \in \chi\right\} \qquad (1)\end{aligned}$$

for all $\alpha \in \mathcal{L}$ and $i = \sqrt{-1}$. Both the amplitude terms $\mathbb{C}^-_{\Im_\alpha}(x)$ and $\mathbb{C}^+_{\Im_\alpha}(x)$ and the phase terms $\circledS^-_{\Im_\alpha}(x)$ and $\circledS^+_{\Im_\alpha}(x)$ are real-valued with $\mathbb{C}^-_{\Im_\alpha}(x) < \mathbb{C}^+_{\Im_\alpha}(x)$ and the phase terms $\circledS^-_{\Im_\alpha}(x) < \circledS^+_{\Im_\alpha}(x)$, and $\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^+_{\Im_\alpha}(x) \in [0, 1]$, $\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^+_{\Im_\alpha}(x) \in [0, 2\pi]$.

Note that complex fuzzy soft set is a special case of IV-CFSS by taking $\mathbb{C}^-_{\Im_\alpha}(x) = \mathbb{C}^+_{\Im_\alpha}(x)$ and $\circledS^-_{\Im_\alpha}(x) = \circledS^+_{\Im_\alpha}(x)$.

Definition 5 (Selvachandran and Salleh 2017) Let $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ be two IV-CFSSs over χ. Then,

(i) $(\Im, \mathcal{L})$ is said to be a subset of $(\Xi, \aleph)$, denoted by $(\Im, \mathcal{L}) \subset (\Xi, \aleph)$ if and only if $\mathbb{C}^-_{\Im_\alpha}(x) \le \mathbb{C}^-_{\Xi_\alpha}(x)$ and $\mathbb{C}^+_{\Im_\alpha}(x) \le \mathbb{C}^+_{\Xi_\alpha}(x)$ for the amplitude terms, and $\circledS^-_{\Im_\alpha}(x) \le \circledS^-_{\Xi_\alpha}(x)$ and $\circledS^+_{\Im_\alpha}(x) \le \circledS^+_{\Xi_\alpha}(x)$ for the phase terms for all $x \in \chi$.

(ii) $(\Im, \mathcal{L})$ is said to be a equal to $(\Xi, \aleph)$, denoted by $(\Im, \mathcal{L}) = (\Xi, \aleph)$ if and only if $\mathbb{C}^-_{\Im_\alpha}(x) = \mathbb{C}^-_{\Xi_\alpha}(x)$ and $\mathbb{C}^+_{\Im_\alpha}(x) = \mathbb{C}^+_{\Xi_\alpha}(x)$ for the amplitude terms, and $\circledS^-_{\Im_\alpha}(x) = \circledS^-_{\Xi_\alpha}(x)$ and $\circledS^+_{\Im_\alpha}(x) = \circledS^+_{\Xi_\alpha}(x)$ for the phase terms for all $x \in \chi$.

Definition 6 (Selvachandran and Salleh 2017) The complement of $(\Im, \mathcal{L})$ denoted by $(\Im, \mathcal{L})^c$ is defined as

$$\begin{aligned}(\Im, \mathcal{L})^c &= \left\{\left(x, \left([(\mathbb{C}^-_{\Im_\alpha}(x))^c, (\mathbb{C}^+_{\Im_\alpha}(x))^c].e^{i[(\circledS^-_{\Im_\alpha}(x))^c, (\circledS^+_{\Im_\alpha}(x))^c]}\right)\right) : x \in \chi\right\}\\ &= \left\{\left(x, \left([1 - \mathbb{C}^+_{\Im_\alpha}(x), 1 - \mathbb{C}^-_{\Im_\alpha}(x)].e^{i[2\pi - \circledS^+_{\Im_\alpha}(x), 2\pi - \circledS^-_{\Im_\alpha}(x)]}\right)\right) : x \in \chi\right\}.\end{aligned}$$

Definition 7 (Selvachandran and Salleh 2017) Let $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ be two IV-CFSSs over χ. Then, the union of $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ is an IV-CFSSs $(\Re, \Pi)$, defined as $(\Im, \mathcal{L}) \cup (\Xi, \aleph) = (\Re, \Pi)$, where $\Pi = \mathcal{L} \cup \aleph$, and

$$(\Re, \Pi) = \left\{\begin{array}{c}\left(x, \left(\begin{array}{c}[\max(\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^-_{\Xi_\alpha}(x)), \max(\mathbb{C}^+_{\Im_\alpha}(x), \mathbb{C}^+_{\Xi_\alpha}(x))].\\ e^{i[\max(\circledS^-_{\Im_\alpha}(x), \circledS^-_{\Xi_\alpha}(x)), \max(\circledS^+_{\Im_\alpha}(x), \circledS^+_{\Xi_\alpha}(x))]}\end{array}\right)\right) : x \in \chi\\ ; \Pi \in \mathcal{L} \cap \aleph.\end{array}\right\}.$$

Definition 8 (Selvachandran and Salleh 2017) Let $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ be two IV-CFSSs over χ. Then, the intersection of $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ is an IV-CFSSs $(\Re, \Pi)$, defined as $(\Im, \mathcal{L}) \cap (\Xi, \aleph) = (\Re, \Pi)$, where $\Pi = \mathcal{L} \cap \aleph$, and

$$(\Re, \Pi) = \left\{\begin{array}{c}\left(x, \left(\begin{array}{c}[\min(\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^-_{\Xi_\alpha}(x)), \min(\mathbb{C}^+_{\Im_\alpha}(x), \mathbb{C}^+_{\Xi_\alpha}(x))].\\ e^{i[\min(\circledS^-_{\Im_\alpha}(x), \circledS^-_{\Xi_\alpha}(x)), \min(\circledS^+_{\Im_\alpha}(x), \circledS^+_{\Xi_\alpha}(x))]}\end{array}\right)\right) : x \in \chi\\ ; \Pi \in \mathcal{L} \cap \aleph\end{array}\right\}.$$

3 Interval-valued complex fuzzy soft sets

In this section, we will introduce some new operations on IV-CFSSs.

Definition 9 1). A function $\top : (0, 1] \times (0, 1] \to [0, 1]$ is a quasi-triangular norm that satisfies the following conditions:

$i). \top(1, 1) = 1;$
$ii). \top(a, b) = \top(b, a);$
$iii). \top(a, b) \leq \top(c, d); a \leq c, b \leq d.$
$iv). \top(\top(a, b), c) = \top(a, \top(b, c)).$

2). A function $\top : [0, 1] \times [0, 1] \to [0, 1]$ is a triangular norm that satisfies the conditions (i)–(iv) and the following condition:
$v). \top(0, 0) = 0.$
We said $\top$ is an s-norm, if a triangular norm $\top$ satisfies
$vi). \top(a, 0) = a.$
We said $\top$ is an t-norm, if a triangular norm $\top$ satisfies
$vii). \top(a, 1) = a.$
We said a binary function $\dot{\top}$:

$$\dot{\top} : \mathcal{L}^*(\chi) \times \mathcal{L}^*(\chi) \to \mathcal{L}^*(\chi),$$

$$\dot{\top}((\Im, \mathcal{L}), (\Xi, \aleph)) \to \sup_{x\in\chi} \Big[\top_1(\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^-_{\Xi_\alpha}(x)), \top_1(\mathbb{C}^+_{\Im_\alpha}(x), \mathbb{C}^+_{\Xi_\alpha}(x))\Big]$$
$$e^{i \sup_{x\in\chi}[\top_2(\circledS^-_{\Im_\alpha}(x), \circledS^-_{\Xi_\alpha}(x)), \top_2(\circledS^+_{\Im_\alpha}(x), \circledS^+_{\Xi_\alpha}(x))]},$$

where $\mathcal{L}(\chi)$ is a collection of IV-CFSSs.

The operator $\dot{\top}$ is a triangular norm if $\top_1$ is a triangular norm and $\top_2$ is a quasi-triangular norm. $\dot{\top}$ is an s-norm if $\top_1$ is an s-norm and $\dot{\top}$ is a t-norm if $\top_1$ is a t-norm.

Definition 10 Let $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ be two IV-CFSSs over χ. Then, the IV-CFS product of $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ is an IV-CFSSs $(\Re, \Pi)$, denoted as $(\Im, \mathcal{L}) \bullet (\Xi, \aleph) = (\Re, \Pi)$, and is specified by a function

$$(\Re, \Pi) = \left\{ \begin{array}{c} \left(x, \left(\begin{array}{c} [\mathbb{C}^-_{\Im_\alpha}(x) \bullet \mathbb{C}^-_{\Xi_\alpha}(x)), \bullet\mathbb{C}^+_{\Im_\alpha}(x), \mathbb{C}^+_{\Xi_\alpha}(x))]. \\ e^{i[2\pi(\frac{\circledS^-_{\Im_\alpha}(x)}{2\pi} \bullet \frac{\circledS^-_{\Xi_\alpha}(x)}{2\pi}), 2\pi(\frac{\circledS^+_{\Im_\alpha}(x)}{2\pi} \bullet \frac{\circledS^+_{\Xi_\alpha}(x)}{2\pi})]} \end{array}\right)\right) : x \in \chi \\ ; \Pi \in \mathcal{L} \cap \aleph \end{array} \right\}.$$

Example 1 Let $\chi = \{x_1, x_2\}$ be a universe of discourse and $\mathcal{L} = \aleph = \{\alpha_1 = beautiful, \alpha_2 = big\}$ be sets of parameters, and

$$\Im(\alpha_1) = \left\{(x_1, [0.2, 0.6]e^{i[0.2\pi, 1.4\pi]}), (x_2, [0.4, 1]e^{i[0.6\pi, \pi]})\right\}.$$
$$\Im(\alpha_2) = \left\{(x_1, [0.5, 0.9]e^{i[0.4\pi, 1.2\pi]}), (x_2, [0.1, 0.8]e^{i[\pi, 1.4\pi]})\right\}.$$
$$\Xi(\alpha_1) = \left\{(x_1, [0.3, 0.5]e^{i[0.8\pi, 1.8\pi]}), (x_2, [0.7, 0.9]e^{i[0.4\pi, 1.6\pi]})\right\}.$$
$$\Xi(\alpha_2) = \left\{(x_1, [0.1, 1]e^{i[0.6\pi, 2\pi]}), (x_2, [0.2, 0.7]e^{i[0.8\pi, 2\pi]})\right\}.$$

Then, the IV-CFS product of $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ is

$$\Re(\alpha_1) = \left\{(x_1, [0.06, 0.3]e^{i[0.08\pi, 1.26\pi]}), (x_2, [0.28, 0.9]e^{i[0.12\pi, 0.8\pi]})\right\},$$
$$\Re(\alpha_2) = \left\{(x_1, [0.05, 0.9]e^{i[0.12\pi, 1.2\pi]}), (x_2, [0.02, 0.56]e^{i[0.4\pi, 1.4\pi]})\right\}.$$

Definition 11 Let $(\Im_n, \mathcal{L}_n)$; $n = 1, 2, \ldots, N$ be N IV-CFSSs over χ. Then, the IV-CFS Cartesian product of $(\Im_n, \mathcal{L}_n)$, denoted $(\Im_1, \mathcal{L}_1) \times (\Im_2, \mathcal{L}_2) \times \cdots \times (\Im_n, \mathcal{L}_n) = (\Im, \mathcal{L})$, and is specified by a function

$$(\Im, \mathcal{L}) = \left\{ \begin{array}{c} \left(x, \left(\begin{array}{c} [\min(\mathbb{C}^-_{\Im_{1\alpha}}(x), \mathbb{C}^-_{\Im_{2\alpha}}(x), \ldots, \mathbb{C}^-_{\Im_{n\alpha}}(x)), \\ \min(\mathbb{C}^+_{\Im_{1\alpha}}(x), \mathbb{C}^+_{\Im_{2\alpha}}(x), \ldots, \mathbb{C}^+_{\Im_{n\alpha}}(x))]. \\ e^{i[\min(\circledS^-_{\Im_{1\alpha}}(x), \circledS^-_{\Im_{2\alpha}}(x), \ldots, \circledS^-_{\Im_{n\alpha}}(x)), \min(\circledS^+_{\Im_{1\alpha}}(x), \circledS^+_{\Im_{2\alpha}}(x), \ldots, \circledS^+_{\Im_{n\alpha}}(x))]} \end{array} \right) \right) \\ : x \in \chi; \mathcal{L} \in \mathcal{L}_1 \cap \mathcal{L}_2 \cap \cdots \cap \mathcal{L}_n \end{array} \right\}.$$

Example 2 Consider $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ in Example 1. Then, the Cartesian product of $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ is

$$\Re(\alpha_1) = \left\{ \begin{array}{c} ((x_1, x_1), [0.2, 0.5]e^{i[0.2\pi, 1.4\pi]}), ((x_1, x_2), [0.2, 0.6]e^{i[0.2\pi, 1.4\pi]}), \\ ((x_2, x_1), [0.3, 0.5]e^{i[0.6\pi, \pi]}), ((x_2, x_2), [0.4, 0.9]e^{i[0.4\pi, \pi]}) \end{array} \right\},$$
$$\Re(\alpha_2) = \left\{ \begin{array}{c} ((x_1, x_1), [0.1, 0.9]e^{i[0.4\pi, 1.2\pi]}), ((x_1, x_2), [0.2, 0.7]e^{i[0.4\pi, 1.2\pi]}), \\ ((x_2, x_1), [0.1, 0.8]e^{i[0.6\pi, 1.4\pi]}), ((x_2, x_2), [0.1, 0.7]e^{i[0.8\pi, 1.4\pi]}) \end{array} \right\}.$$

Definition 12 Let $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ be two IV-CFSSs over χ. Then, the IV-CFS probabilistic sum of $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ is an IV-CFSSs $(\Re, \Pi)$, denoted as $(\Im, \mathcal{L}) \circledast (\Xi, \aleph) = (\Re, \Pi)$, and is specified by a function

$$(\Re, \Pi) = \left\{ \begin{array}{c} \left(x, \left(\begin{array}{c} [\mathbb{C}^-_{\Im_\alpha}(x) + \mathbb{C}^-_{\Xi_\alpha}(x) - \mathbb{C}^-_{\Im_\alpha}(x) \bullet \mathbb{C}^-_{\Xi_\alpha}(x), \\ \mathbb{C}^+_{\Im_\alpha}(x) + \mathbb{C}^+_{\Xi_\alpha}(x) - \mathbb{C}^+_{\Im_\alpha}(x) \bullet \mathbb{C}^+_{\Xi_\alpha}(x)]. \\ e^{i[2\pi(\frac{\circledS^-_{\Im_\alpha}(x)}{2\pi} + \frac{\circledS^-_{\Xi_\alpha}(x)}{2\pi} - \frac{\circledS^-_{\Im_\alpha}(x)}{2\pi} \bullet \frac{\circledS^-_{\Xi_\alpha}(x)}{2\pi}), 2\pi(\frac{\circledS^+_{\Im_\alpha}(x)}{2\pi} + \frac{\circledS^+_{\Xi_\alpha}(x)}{2\pi} - \frac{\circledS^+_{\Im_\alpha}(x)}{2\pi} \bullet \frac{\circledS^+_{\Xi_\alpha}(x)}{2\pi})]} \end{array} \right) \right) : x \in \chi \\ ; \Pi \in \mathcal{L} \cap \aleph \end{array} \right\}.$$

Example 3 Consider $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ in Example 1. Then, the probabilistic sum of $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ is

$$\Re(\alpha_1) = \left\{(x_1, [0.44, 0.8]e^{i[0.92\pi, 1.88\pi]}), (x_2, [0.82, 1]e^{i[0.88\pi, 1.8\pi]})\right\},$$
$$\Re(\alpha_2) = \left\{(x_1, [0.55, 0.1]e^{i[0.88\pi, 1.28\pi]}), (x_2, [0.28, 0.94]e^{i[1.4\pi, 2\pi]})\right\}.$$

Definition 13 Let $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ be two IV-CFSSs over χ. Then, the IV-CFS simple difference of $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ is an IV-CFSSs $(\Re, \Pi)$, denoted as $(\Im, \mathcal{L}) \diagdown (\Xi, \aleph) = (\Re, \Pi)$, and is specified by a function

$$\begin{aligned} (\Re, \Pi) &= (\Im, \mathcal{L}) \cap (\Xi, \aleph)^c \\ &= \left\{ \begin{array}{c} \left(x, \left(\begin{array}{c} [\min(\mathbb{C}^-_{\Im_\alpha}(x), 1 - \mathbb{C}^+_{\Xi_\alpha}(x)), \min(\mathbb{C}^+_{\Im_\alpha}(x), 1 - \mathbb{C}^-_{\Xi_\alpha}(x))]. \\ e^{i[\min(\circledS^-_{\Im_\alpha}(x), 2\pi - \circledS^+_{\Xi_\alpha}(x)), \min(\circledS^+_{\Im_\alpha}(x), 2\pi - \circledS^-_{\Xi_\alpha}(x))]} \end{array} \right) \right) \\ : x \in \chi; \Pi \in \mathcal{L} \cap \aleph \end{array} \right\}. \end{aligned}$$

Example 4 Consider $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ in Example 1. Then, the complement of $(\Xi, \aleph)$ is

$$(\Xi(\alpha_1))^c = \left\{(x_1, [0.5, 0.7]e^{i[0.2\pi, 1.2\pi]}), (x_2, [0.1, 0.3]e^{i[0.4\pi, 1.6\pi]})\right\}.$$
$$(\Xi(\alpha_2))^c = \left\{(x_1, [0, 0.9]e^{i[0.\pi, 1.4\pi]}), (x_2, [0.3, 0.8]e^{i[0.\pi, 1.2\pi]})\right\}.$$

The simple difference of $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ is

$$\Re(\alpha_1) = \left\{(x_1, [0.2, 0.6]e^{i[0.2\pi, 1.2\pi]}), (x_2, [0.1, 0.3]e^{i[0.4\pi, \pi]})\right\},$$
$$\Re(\alpha_2) = \left\{(x_1, [0, 0.9]e^{i[0\pi, 1.2\pi]}), (x_2, [0.1, 0.8]e^{i[0.\pi, 1.2\pi]})\right\}.$$

Definition 14 Let $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ be two IV-CFSSs over χ. Then, the IV-CFS convex linear sum of min and max of $(\Im, \mathcal{L})$ and $(\Xi, \aleph)$ is an IV-CFSSs $(\Re, \Pi)$, denoted as $(\Im, \mathcal{L}) \oplus_\lambda (\Xi, \aleph) = (\Re, \Pi)$, and is specified by a function

$$(\Re, \Pi) = \left\{ \begin{array}{c} \left(x, \left(\begin{array}{c} [\lambda \min(\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^-_{\Xi_\alpha}(x)) + (1-\lambda)\max(\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^-_{\Xi_\alpha}(x)), \\ \lambda \min(\mathbb{C}^+_{\Im_\alpha}(x), \mathbb{C}^+_{\Xi_\alpha}(x)) + (1-\lambda)\max(\mathbb{C}^+_{\Im_\alpha}(x), \mathbb{C}^+_{\Xi_\alpha}(x))]. \\ e^{\begin{array}{c} i[\lambda \min(\circledS^-_{\Im_\alpha}(x), \circledS^-_{\Xi_\alpha}(x)) + (1-\lambda)\max(\circledS^-_{\Im_\alpha}(x), \circledS^-_{\Xi_\alpha}(x)), \\ \lambda \min(\circledS^+_{\Im_\alpha}(x), \circledS^+_{\Xi_\alpha}(x)) + (1-\lambda)\max(\circledS^+_{\Im_\alpha}(x), \circledS^+_{\Xi_\alpha}(x))] \end{array}} \end{array} \right) \right) : \\ x \in \chi;\ \Pi \in \mathcal{L} \cap \aleph \end{array} \right\}.$$

Definition 15 A distance measure of IV-CFSSs is a function $\Gamma : \dot{\top}(\chi) \times \dot{\top}(\chi) \to [0, 1]$ with the property: for any $\xi_1 = (\Im, \mathcal{L}), \xi_2 = (\Xi, \aleph), \xi_3 = (\Re, \Pi) \in \dot{\top}(\chi)$ are three IV-CFSSs over χ

$i)$. $\Gamma(\xi_1, \xi_2) \geq 0$,
$ii)$. $\Gamma(\xi_1, \xi_2) = 0$ if and only if $\xi_1 = \xi_2$,
$iii)$. $\Gamma(\xi_1, \xi_2) = \Gamma(\xi_2, \xi_1)$,
$iv)$. $\Gamma(\xi_1, \xi_3) \leq \Gamma(\xi_1, \xi_2) + \Gamma(\xi_2, \xi_3)$.

In the following, the distance measure Γ is defined as follows:

$$\Gamma(\xi, \eta) = \max \left\{ \begin{array}{c} \sup_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi} \sup_x |l([\circledS^-_{\xi_\alpha}(x), \circledS^+_{\xi_\alpha}(x)]) - l([\circledS^-_{\eta_\alpha}(x), \circledS^+_{\eta_\alpha}(x)])| \end{array} \right\}, \tag{2}$$

where l denotes the length of the interval.

Example 5 Consider $\xi = (\Im, \mathcal{L})$ and $\eta = (\Xi, \aleph)$ in Example 1. Then, the IV-CFS distance measure is

$$\Gamma(\xi, \eta) = \max \left\{0.5, \frac{1}{2\pi}(0.8\pi)\right\}$$
$$= 0.5.$$

Definition 16 Let $\xi = (\Im, \mathcal{L})$ and $\eta = (\Xi, \aleph)$ be two IV-CFSSs over χ. Then, the IV-CFSSs ξ and η is said to be δ-equal if and only if

$$\Gamma(\xi, \eta) \leq 1 - \delta; 0 \leq \delta \leq 1.$$

4 Main results

Proposition 1 *The IV-CFS union on $\dot{\top}(\chi)$ is an s-norm.*

Proof The conditions (i), (ii), (v), and (vi) can be easily verified from definition 6. Here, we only prove the condition (iii) and (iv).

$iii)$. Let $(\Im, \mathcal{L})$, $(\Xi, \aleph)$, $(\Re, \Pi)$ be three IV-CFSSs over χ, and $\mu_{(\Im,\mathcal{L})} = [\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^+_{\Im_\alpha}(x)].e^{i[\circledS^-_{\Im_\alpha}(x), \circledS^+_{\Im_\alpha}(x)]}$, $\mu_{(\Xi,\aleph)} = [\mathbb{C}^-_{\Xi_\alpha}(x), \mathbb{C}^+_{\Xi_\alpha}(x)].e^{i[\circledS^-_{\Xi_\alpha}(x), \circledS^+_{\Xi_\alpha}(x)]}$, and $\mu_{(\Re,\Pi)} = [\mathbb{C}^-_{\Pi_\alpha}(x), \mathbb{C}^+_{\Pi_\alpha}(x)].e^{i[\circledS^-_{\Pi_\alpha}(x), \circledS^+_{\Pi_\alpha}(x)]}$ their membership functions, respectively. We suppose $\mathbb{C}^-_{\Im_\alpha}(x) \leq \mathbb{C}^-_{\Xi_\alpha}(x)$, $\mathbb{C}^+_{\Im_\alpha}(x) \leq \mathbb{C}^+_{\Xi_\alpha}(x)$, $\circledS^-_{\Im_\alpha}(x) \leq \circledS^-_{\Xi_\alpha}(x)$, $\circledS^+_{\Im_\alpha}(x) \leq \circledS^+_{\Xi_\alpha}(x)$, $\forall x \in \chi$. Then,

$$\begin{aligned}
\mathbb{C}^-_{(\Im,\mathcal{L})\cup(\Re,\Pi)}(x) &= \max(\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^-_{\Pi_\alpha}(x)) \\
&\leq \max(\mathbb{C}^-_{\Xi_\alpha}(x), \mathbb{C}^-_{\Pi_\alpha}(x)) \\
&= \mathbb{C}^-_{(\Xi,\aleph)\cup(\Re,\Pi)}, \ \forall x \in \chi. \\
\mathbb{C}^+_{(\Im,\mathcal{L})\cup(\Re,\Pi)}(x) &= \max(\mathbb{C}^+_{\Im_\alpha}(x), \mathbb{C}^+_{\Pi_\alpha}(x)) \\
&\leq \max(\mathbb{C}^+_{\Xi_\alpha}(x), \mathbb{C}^+_{\Pi_\alpha}(x)) \\
&= \mathbb{C}^+_{(\Xi,\aleph)\cup(\Re,\Pi)}, \ \forall x \in \chi.
\end{aligned}$$

$$\begin{aligned}
\circledS^-_{(\Im,\mathcal{L})\cup(\Re,\Pi)}(x) &= \max(\circledS^-_{\Im_\alpha}(x), \circledS^-_{\Pi_\alpha}(x)) \\
&\leq \max(\circledS^-_{\Xi_\alpha}(x), \circledS^-_{\Pi_\alpha}(x)) \\
&= \circledS^-_{(\Xi,\aleph)\cup(\Re,\Pi)}, \ \forall x \in \chi. \\
\circledS^+_{(\Im,\mathcal{L})\cup(\Re,\Pi)}(x) &= \max(\circledS^+_{\Im_\alpha}(x), \circledS^+_{\Pi_\alpha}(x)) \\
&\leq \max(\circledS^+_{\Xi_\alpha}(x), \circledS^+_{\Pi_\alpha}(x)) \\
&= \circledS^+_{(\Xi,\aleph)\cup(\Re,\Pi)}, \ \forall x \in \chi.
\end{aligned}$$

$iv)$.

$$\begin{aligned}
\mu_{(\Im,\mathcal{L})\cup((\Xi,\aleph)\cup(\Re,\Pi))}(x) &= \begin{matrix} [\max(\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^-_{((\Xi,\aleph)\cup(\Re,\Pi)}(x)), \\ \max(\mathbb{C}^+_{\Im_\alpha}(x), \mathbb{C}^+_{((\Xi,\aleph)\cup(\Re,\Pi)}(x))]. \\ e^{\substack{i[\max(\circledS^-_{\Im_\alpha}(x), \circledS^-_{((\Xi,\aleph)\cup(\Re,\Pi)}(x)), \\ \max(\circledS^+_{\Im_\alpha}(x), \circledS^+_{((\Xi,\aleph)\cup(\Re,\Pi)}(x))]}} \end{matrix} \\
&= \begin{bmatrix} \max(\mathbb{C}^-_{\Im_\alpha}(x), \max(\mathbb{C}^-_{\Xi_\alpha}(x), \mathbb{C}^-_{\Pi_\alpha}(x))), \\ \max(\mathbb{C}^+_{\Im_\alpha}(x), \max(\mathbb{C}^+_{\Xi_\alpha}(x), \mathbb{C}^+_{\Pi_\alpha}(x))) \end{bmatrix} \\
&\quad e^{\substack{i[\max(\circledS^-_{\Im_\alpha}(x), \max(\circledS^-_{\Xi_\alpha}(x), \circledS^-_{\Pi_\alpha}(x))), \\ \max(\circledS^+_{\Im_\alpha}(x), \max(\circledS^+_{\Xi_\alpha}(x), \circledS^+_{\Pi_\alpha}(x)))]}} \\
&= \begin{bmatrix} \max(\max(\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^-_{\Xi_\alpha}(x)), \mathbb{C}^-_{\Pi_\alpha}(x))), \\ \max(\max(\mathbb{C}^+_{\Im_\alpha}(x), \mathbb{C}^+_{\Xi_\alpha}(x)), \mathbb{C}^+_{\Pi_\alpha}(x))) \end{bmatrix} \\
&\quad e^{\substack{i[\max(\max(\circledS^-_{\Im_\alpha}(x), \circledS^-_{\Xi_\alpha}(x)), \circledS^-_{\Pi_\alpha}(x))), \\ \max(\max(\circledS^+_{\Im_\alpha}(x), \circledS^+_{\Xi_\alpha}(x)), \circledS^+_{\Pi_\alpha}(x)))]}} \\
&= \mu_{((\Im,\mathcal{L})\cup(\Xi,\aleph))\cup(\Re,\Pi)}(x).
\end{aligned}$$

□

Proposition 2 *The IV-CFS intersection on* $\overset{\div}{\top}(\chi)$ *is a t-norm.*

Proof The conditions (i), (ii), (v), and (vi) can be easily verified from definition 6. Here, we only prove the condition (iii) and (iv).

$iii)$. Let $(\Im,\mathcal{L})$, $(\Xi,\aleph)$, $(\Re,\Pi)$ be three IV-CFSSs over χ, and $\mu_{(\Im,\mathcal{L})} = [\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^+_{\Im_\alpha}(x)].e^{i[\circledS^-_{\Im_\alpha}(x),\circledS^+_{\Im_\alpha}(x)]}$, $\mu_{(\Xi,\aleph)} = [\mathbb{C}^-_{\Xi_\alpha}(x), \mathbb{C}^+_{\Xi_\alpha}(x)].e^{i[\circledS^-_{\Xi_\alpha}(x),\circledS^+_{\Xi_\alpha}(x)]}$, and $\mu_{(\Re,\Pi)} = [\mathbb{C}^-_{\Pi_\alpha}(x), \mathbb{C}^+_{\Pi_\alpha}(x)].e^{i[\circledS^-_{\Pi_\alpha}(x),\circledS^+_{\Pi_\alpha}(x)]}$ their membership functions, respectively. We suppose $\mathbb{C}^-_{\Im_\alpha}(x) \le \mathbb{C}^-_{\Xi_\alpha}(x)$, $\mathbb{C}^+_{\Im_\alpha}(x) \le \mathbb{C}^+_{\Xi_\alpha}(x)$, $\circledS^-_{\Im_\alpha}(x) \le \circledS^-_{\Xi_\alpha}(x)$, $\circledS^+_{\Im_\alpha}(x) \le \circledS^+_{\Xi_\alpha}(x)$, $\forall x \in \chi$. Then

$$\begin{aligned}
\mathbb{C}^-_{(\Im,\mathcal{L})\cap(\Re,\Pi)}(x) &= \min(\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^-_{\Pi_\alpha}(x))\\
&\le \min(\mathbb{C}^-_{\Xi_\alpha}(x), \mathbb{C}^-_{\Pi_\alpha}(x))\\
&= \mathbb{C}^-_{(\Xi,\aleph)\cap(\Re,\Pi)},\ \forall x \in \chi.\\
\mathbb{C}^+_{(\Im,\mathcal{L})\cap(\Re,\Pi)}(x) &= \min(\mathbb{C}^+_{\Im_\alpha}(x), \mathbb{C}^+_{\Pi_\alpha}(x))\\
&\le \min(\mathbb{C}^+_{\Xi_\alpha}(x), \mathbb{C}^+_{\Pi_\alpha}(x))\\
&= \mathbb{C}^+_{(\Xi,\aleph)\cap(\Re,\Pi)},\ \forall x \in \chi.
\end{aligned}$$

$$\begin{aligned}
\circledS^-_{(\Im,\mathcal{L})\cap(\Re,\Pi)}(x) &= \min(\circledS^-_{\Im_\alpha}(x), \circledS^-_{\Pi_\alpha}(x))\\
&\le \min(\circledS^-_{\Xi_\alpha}(x), \circledS^-_{\Pi_\alpha}(x))\\
&= \circledS^-_{(\Xi,\aleph)\cap(\Re,\Pi)},\ \forall x \in \chi.\\
\circledS^+_{(\Im,\mathcal{L})\cap(\Re,\Pi)}(x) &= \min(\circledS^+_{\Im_\alpha}(x), \circledS^+_{\Pi_\alpha}(x))\\
&\le \min(\circledS^+_{\Xi_\alpha}(x), \circledS^+_{\Pi_\alpha}(x))\\
&= \circledS^+_{(\Xi,\aleph)\cap(\Re,\Pi)},\ \forall x \in \chi.
\end{aligned}$$

$iv)$.

$$\begin{aligned}
\mu_{(\Im,\mathcal{L})\cap((\Xi,\aleph)\cap(\Re,\Pi))}(x) &= \begin{array}{c}[\min(\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^-_{((\Xi,\aleph)\cap(\Re,\Pi)}(x)),\\ \min(\mathbb{C}^+_{\Im_\alpha}(x), \mathbb{C}^+_{((\Xi,\aleph)\cap(\Re,\Pi)}(x))].\\ e^{\substack{i[\min(\circledS^-_{\Im_\alpha}(x),\circledS^-_{((\Xi,\aleph)\cap(\Re,\Pi)}(x)),\\ \min(\circledS^+_{\Im_\alpha}(x),\circledS^+_{((\Xi,\aleph)\cap(\Re,\Pi)}(x))]}}\end{array}\\
&= \begin{bmatrix}\min(\mathbb{C}^-_{\Im_\alpha}(x), \min(\mathbb{C}^-_{\Xi_\alpha}(x), \mathbb{C}^-_{\Pi_\alpha}(x))),\\ \min(\mathbb{C}^+_{\Im_\alpha}(x), \min(\mathbb{C}^+_{\Xi_\alpha}(x), \mathbb{C}^+_{\Pi_\alpha}(x)))\end{bmatrix}\\
&\quad e^{\substack{i[\min(\circledS^-_{\Im_\alpha}(x),\min(\circledS^-_{\Xi_\alpha}(x),\circledS^-_{\Pi_\alpha}(x))),\\ \min(\circledS^+_{\Im_\alpha}(x),\min(\circledS^+_{\Xi_\alpha}(x),\circledS^+_{\Pi_\alpha}(x)))]}}\\
&= \begin{bmatrix}\min(\min(\mathbb{C}^-_{\Im_\alpha}(x), \mathbb{C}^-_{\Xi_\alpha}(x)), \mathbb{C}^-_{\Pi_\alpha}(x))),\\ \min(\min(\mathbb{C}^+_{\Im_\alpha}(x), \mathbb{C}^+_{\Xi_\alpha}(x)), \mathbb{C}^+_{\Pi_\alpha}(x)))\end{bmatrix}\\
&\quad e^{\substack{i[\min(\min(\circledS^-_{\Im_\alpha}(x),\circledS^-_{\Xi_\alpha}(x)),\circledS^-_{\Pi_\alpha}(x))),\\ \min(\min(\circledS^+_{\Im_\alpha}(x),\circledS^+_{\Xi_\alpha}(x)),\circledS^+_{\Pi_\alpha}(x)))]}}\\
&= \mu_{((\Im,\mathcal{L})\cap(\Xi,\aleph))\cap(\Re,\Pi)}(x).
\end{aligned}$$

□

Proposition 3 *The IV-CFS product on* $\overset{\div}{\top}(\chi)$ *is a t-norm.*

Proof The conditions (i), (ii), (v), and (vi) can be easily verified from definition 6. Here, we only prove the condition (iii) and (iv).

$iii)$. Let $(\Im,\mathcal{L})$, $(\Xi,\aleph)$, $(\Re,\Pi)$ be three IV-CFSSs over χ, and $\mu_{(\Im,\mathcal{L})} = [\mathbb{C}^{-}_{\Im_\alpha}(x), \mathbb{C}^{+}_{\Im_\alpha}(x)].e^{i[\circledS^{-}_{\Im_\alpha}(x),\circledS^{+}_{\Im_\alpha}(x)]}$, $\mu_{(\Xi,\aleph)} = [\mathbb{C}^{-}_{\Xi_\alpha}(x), \mathbb{C}^{+}_{\Xi_\alpha}(x)].e^{i[\circledS^{-}_{\Xi_\alpha}(x),\circledS^{+}_{\Xi_\alpha}(x)]}$, and $\mu_{(\Re,\Pi)} = [\mathbb{C}^{-}_{\Pi_\alpha}(x), \mathbb{C}^{+}_{\Pi_\alpha}(x)].e^{i[\circledS^{-}_{\Pi_\alpha}(x),\circledS^{+}_{\Pi_\alpha}(x)]}$ their membership functions, respectively. We suppose $\mathbb{C}^{-}_{\Im_\alpha}(x) \le \mathbb{C}^{-}_{\Xi_\alpha}(x)$, $\mathbb{C}^{+}_{\Im_\alpha}(x) \le \mathbb{C}^{+}_{\Xi_\alpha}(x)$, $\circledS^{-}_{\Im_\alpha}(x) \le \circledS^{-}_{\Xi_\alpha}(x)$, $\circledS^{+}_{\Im_\alpha}(x) \le \circledS^{+}_{\Xi_\alpha}(x)$, $\forall x \in \chi$. Then

$$\begin{aligned}
\mathbb{C}^{-}_{(\Im,\mathcal{L})\bullet(\Re,\Pi)}(x) &= \mathbb{C}^{-}_{\Im_\alpha}(x) \bullet \mathbb{C}^{-}_{\Pi_\alpha}(x)\\
&\le \mathbb{C}^{-}_{\Xi_\alpha}(x) \bullet \mathbb{C}^{-}_{\Pi_\alpha}(x)\\
&= \mathbb{C}^{-}_{(\Xi,\aleph)\bullet(\Re,\Pi)},\ \forall x \in \chi.\\
\mathbb{C}^{+}_{(\Im,\mathcal{L})\bullet(\Re,\Pi)}(x) &= \mathbb{C}^{+}_{\Im_\alpha}(x) \bullet \mathbb{C}^{+}_{\Pi_\alpha}(x)\\
&\le \mathbb{C}^{+}_{\Xi_\alpha}(x) \bullet \mathbb{C}^{+}_{\Pi_\alpha}(x)\\
&= \mathbb{C}^{+}_{(\Xi,\aleph)\bullet(\Re,\Pi)},\ \forall x \in \chi.
\end{aligned}$$

$$\begin{aligned}
\circledS^{-}_{(\Im,\mathcal{L})\bullet(\Re,\Pi)}(x) &= 2\pi(\frac{\circledS^{-}_{\Im_\alpha}(x)}{2\pi} \bullet \frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi})\\
&\le 2\pi(\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi} \bullet \frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi})\\
&= \circledS^{-}_{(\Xi,\aleph)\bullet(\Re,\Pi)},\ \forall x \in \chi.\\
\circledS^{+}_{(\Im,\mathcal{L})\cap(\Re,\Pi)}(x) &= 2\pi(\frac{\circledS^{+}_{\Im_\alpha}(x)}{2\pi} \bullet \frac{\circledS^{+}_{\Pi_\alpha}(x)}{2\pi})\\
&\le 2\pi(\frac{\circledS^{+}_{\Xi_\alpha}(x)}{2\pi} \bullet \frac{\circledS^{+}_{\Pi_\alpha}(x)}{2\pi})\\
&= \circledS^{+}_{(\Xi,\aleph)\bullet(\Re,\Pi)},\ \forall x \in \chi.
\end{aligned}$$

$iv)$.

$$\begin{aligned}
\mu_{(\Im,\mathcal{L})\bullet((\Xi,\aleph)\bullet(\Re,\Pi))}(x) &= \begin{matrix} [\mathbb{C}^{-}_{\Im_\alpha}(x) \bullet \mathbb{C}^{-}_{((\Xi,\aleph)\bullet(\Re,\Pi)}(x)),\\ \mathbb{C}^{+}_{\Im_\alpha}(x), \mathbb{C}^{+}_{((\Xi,\aleph)\bullet(\Re,\Pi)}(x))].\\ e^{\begin{matrix} i[2\pi(\frac{\circledS^{-}_{\Im_\alpha}(x)}{2\pi}\bullet\frac{\circledS^{-}_{((\Xi,\aleph)\bullet(\Re,\Pi)}(x)}{2\pi}),\\ 2\pi(\frac{\circledS^{+}_{\Im_\alpha}(x)}{2\pi}\bullet\frac{\circledS^{+}_{((\Xi,\aleph)\bullet(\Re,\Pi)}(x)}{2\pi})]\end{matrix}} \end{matrix},\\
&= \left[\begin{matrix} \mathbb{C}^{-}_{\Im_\alpha}(x) \bullet \mathbb{C}^{-}_{\Xi_\alpha}(x) \bullet \mathbb{C}^{-}_{\Pi_\alpha}(x),\\ \mathbb{C}^{+}_{\Im_\alpha}(x) \bullet \mathbb{C}^{+}_{\Xi_\alpha}(x) \bullet \mathbb{C}^{+}_{\Pi_\alpha}(x))) \end{matrix}\right]\\
&\quad e^{\begin{matrix} i[2\pi(\frac{\circledS^{-}_{\Im_\alpha}(x)}{2\pi}\bullet\frac{2\pi(\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}\bullet\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi})}{2\pi}),\\ 2\pi(\frac{\circledS^{+}_{\Im_\alpha}(x)}{2\pi}\bullet\frac{2\pi(\frac{\circledS^{+}_{\Xi_\alpha}(x)}{2\pi}\bullet\frac{\circledS^{+}_{\Pi_\alpha}(x)}{2\pi})}{2\pi})]\end{matrix}}
\end{aligned}$$

$$\mu_{(\Im,\mathcal{L})\bullet((\Xi,\aleph)\bullet(\Re,\Pi))}(x) = \begin{bmatrix} (\mathbb{C}^{-}_{\Im_\alpha}(x) \bullet \mathbb{C}^{-}_{\Xi_\alpha}(x)) \bullet \mathbb{C}^{-}_{\Pi_\alpha}(x), \\ (\mathbb{C}^{+}_{\Im_\alpha}(x) \bullet \mathbb{C}^{+}_{\Xi_\alpha}(x)) \bullet \mathbb{C}^{+}_{\Pi_\alpha}(x)) \end{bmatrix}$$

$$e^{\substack{i[2\pi(\frac{2\pi(\frac{\circledS^{-}_{\Im_\alpha}(x)}{2\pi}\bullet\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi})}{2\pi}\bullet\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi}),\\ 2\pi(\frac{2\pi(\frac{\circledS^{+}_{\Im_\alpha}(x)}{2\pi}\bullet\frac{\circledS^{+}_{\Xi_\alpha}(x)}{2\pi})}{2\pi}\bullet\frac{\circledS^{+}_{\Pi_\alpha}(x)}{2\pi})]}}$$

$$= \mu_{((\Im,\mathcal{L})\bullet(\Xi,\aleph))\bullet(\Re,\Pi)}(x).$$

□

Proposition 4 *The IV-CFS probabilistic sum on* $\dot{\top}(\chi)$ *is an s-norm.*

Proof The conditions (i), (ii), (v), and (vi) can be easily verified from definition 6. Here, we only prove the condition (iii) and (iv).

$iii)$. Let $(\Im,\mathcal{L})$, $(\Xi,\aleph)$, $(\Re,\Pi)$ be three IV-CFSSs over χ, and $\mu_{(\Im,\mathcal{L})} = [\mathbb{C}^{-}_{\Im_\alpha}(x), \mathbb{C}^{+}_{\Im_\alpha}(x)].e^{i[\circledS^{-}_{\Im_\alpha}(x),\circledS^{+}_{\Im_\alpha}(x)]}$, $\mu_{(\Xi,\aleph)} = [\mathbb{C}^{-}_{\Xi_\alpha}(x), \mathbb{C}^{+}_{\Xi_\alpha}(x)].e^{i[\circledS^{-}_{\Xi_\alpha}(x),\circledS^{+}_{\Xi_\alpha}(x)]}$, and $\mu_{(\Re,\Pi)} = [\mathbb{C}^{-}_{\Pi_\alpha}(x), \mathbb{C}^{+}_{\Pi_\alpha}(x)].e^{i[\circledS^{-}_{\Pi_\alpha}(x),\circledS^{+}_{\Pi_\alpha}(x)]}$ their membership functions, respectively. We suppose $\mathbb{C}^{-}_{\Im_\alpha}(x) \le \mathbb{C}^{-}_{\Xi_\alpha}(x)$, $\mathbb{C}^{+}_{\Im_\alpha}(x) \le \mathbb{C}^{+}_{\Xi_\alpha}(x)$, $\circledS^{-}_{\Im_\alpha}(x) \le \circledS^{-}_{\Xi_\alpha}(x)$, $\circledS^{+}_{\Im_\alpha}(x) \le \circledS^{+}_{\Xi_\alpha}(x)$, $\forall x \in \chi$. Then

$$\begin{aligned} \mathbb{C}^{-}_{(\Im,\mathcal{L})\circledast(\Re,\Pi)}(x) &= \mathbb{C}^{-}_{\Im_\alpha}(x) + \mathbb{C}^{-}_{\Pi_\alpha}(x) - \mathbb{C}^{-}_{\Im_\alpha}(x).\mathbb{C}^{-}_{\Pi_\alpha}(x), \\ &\le \mathbb{C}^{-}_{\Xi_\alpha}(x) + \mathbb{C}^{-}_{\Pi_\alpha}(x) - \mathbb{C}^{-}_{\Xi_\alpha}(x).\mathbb{C}^{-}_{\Pi_\alpha}(x), \\ &= \mathbb{C}^{-}_{(\Xi,\aleph)\circledast(\Re,\Pi)},\ \forall x \in \chi. \end{aligned}$$

$$\begin{aligned} \mathbb{C}^{+}_{(\Im,\mathcal{L})\circledast(\Re,\Pi)}(x) &= \mathbb{C}^{+}_{\Im_\alpha}(x) + \mathbb{C}^{+}_{\Pi_\alpha}(x) - \mathbb{C}^{+}_{\Im_\alpha}(x).\mathbb{C}^{+}_{\Pi_\alpha}(x), \\ &\le \mathbb{C}^{+}_{\Xi_\alpha}(x) + \mathbb{C}^{+}_{\Pi_\alpha}(x) - \mathbb{C}^{+}_{\Xi_\alpha}(x).\mathbb{C}^{+}_{\Pi_\alpha}(x), \\ &= \mathbb{C}^{+}_{(\Xi,\aleph)\circledast(\Re,\Pi)},\ \forall x \in \chi. \end{aligned}$$

$$\begin{aligned} \circledS^{-}_{(\Im,\mathcal{L})\bullet(\Re,\Pi)}(x) &= 2\pi(\frac{\circledS^{-}_{\Im_\alpha}(x)}{2\pi} + \frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi} - \frac{\circledS^{-}_{\Im_\alpha}(x)}{2\pi}.\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi}), \\ &\le 2\pi(\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi} + \frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi} - \frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}.\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi}), \\ &= \circledS^{-}_{(\Xi,\aleph)\circledast(\Re,\Pi)},\ \forall x \in \chi. \end{aligned}$$

$$\begin{aligned} \circledS^{+}_{(\Im,\mathcal{L})\circledast(\Re,\Pi)}(x) &= 2\pi(\frac{\circledS^{+}_{\Im_\alpha}(x)}{2\pi} + \frac{\circledS^{+}_{\Pi_\alpha}(x)}{2\pi} - \frac{\circledS^{+}_{\Im_\alpha}(x)}{2\pi}.\frac{\circledS^{+}_{\Pi_\alpha}(x)}{2\pi}), \\ &\le 2\pi(\frac{\circledS^{+}_{\Xi_\alpha}(x)}{2\pi} + \frac{\circledS^{+}_{\Pi_\alpha}(x)}{2\pi} - \frac{\circledS^{+}_{\Xi_\alpha}(x)}{2\pi}.\frac{\circledS^{+}_{\Pi_\alpha}(x)}{2\pi}), \\ &= \circledS^{+}_{(\Xi,\aleph)\circledast(\Re,\Pi)},\ \forall x \in \chi. \end{aligned}$$

$iv)$.

$$\mu_{(\mathfrak{I},\mathcal{L})\circledast((\Xi,\aleph)\circledast(\mathfrak{R},\Pi))}(x)=\begin{bmatrix}[\mathbb{C}^{-}_{\mathfrak{I}_\alpha}(x)+\mathbb{C}^{-}_{((\Xi,\aleph)\circledast(\mathfrak{R},\Pi)}(x)-\mathbb{C}^{-}_{\mathfrak{I}_\alpha}(x).\mathbb{C}^{-}_{((\Xi,\aleph)\circledast(\mathfrak{R},\Pi)}(x)\\ ,\mathbb{C}^{+}_{\mathfrak{I}_\alpha}(x)+\mathbb{C}^{+}_{((\Xi,\aleph)\circledast(\mathfrak{R},\Pi)}(x)-\mathbb{C}^{+}_{\mathfrak{I}_\alpha}(x).\mathbb{C}^{+}_{((\Xi,\aleph)\circledast(\mathfrak{R},\Pi)}(x)]\end{bmatrix} e^{\substack{i[2\pi(\frac{\circledS^{-}_{\mathfrak{I}_\alpha}(x)}{2\pi}+\frac{\circledS^{-}_{((\Xi,\aleph)\circledast(\mathfrak{R},\Pi)}(x)}{2\pi}-\frac{\frac{\circledS^{-}_{\mathfrak{I}_\alpha}(x)}{2\pi}\cdot\frac{\circledS^{-}_{((\Xi,\aleph)\circledast(\mathfrak{R},\Pi)}(x)}{2\pi}}{2\pi}),\\ 2\pi(\frac{\circledS^{+}_{\mathfrak{I}_\alpha}(x)}{2\pi}+\frac{\circledS^{+}_{((\Xi,\aleph)\circledast(\mathfrak{R},\Pi)}(x)}{2\pi}-\frac{\frac{\circledS^{+}_{\mathfrak{I}_\alpha}(x)}{2\pi}\cdot\frac{\circledS^{+}_{((\Xi,\aleph)\circledast(\mathfrak{R},\Pi)}(x)}{2\pi}}{2\pi})]}}$$

$$=\begin{bmatrix}[\mathbb{C}^{-}_{\mathfrak{I}_\alpha}(x)+(\mathbb{C}^{-}_{\Xi_\alpha}(x)+\mathbb{C}^{-}_{\Pi_\alpha}(x)-\mathbb{C}^{-}_{\Xi_\alpha}(x).\mathbb{C}^{-}_{\Pi_\alpha}(x))-\\ \mathbb{C}^{-}_{\mathfrak{I}_\alpha}(x).(\mathbb{C}^{-}_{\Xi_\alpha}(x)+\mathbb{C}^{-}_{\Pi_\alpha}(x)-\mathbb{C}^{-}_{\Xi_\alpha}(x).\mathbb{C}^{-}_{\Pi_\alpha}(x)),\\ \mathbb{C}^{+}_{\mathfrak{I}_\alpha}(x)+(\mathbb{C}^{+}_{\Xi_\alpha}(x)+\mathbb{C}^{+}_{\Pi_\alpha}(x)-\mathbb{C}^{+}_{\Xi_\alpha}(x).\mathbb{C}^{+}_{\Pi_\alpha}(x))-\\ \mathbb{C}^{+}_{\mathfrak{I}_\alpha}(x).(\mathbb{C}^{+}_{\Xi_\alpha}(x)+\mathbb{C}^{+}_{\Pi_\alpha}(x)-\mathbb{C}^{+}_{\Xi_\alpha}(x).\mathbb{C}^{+}_{\Pi_\alpha}(x))]\end{bmatrix} e^{\substack{i[2\pi(\frac{\circledS^{-}_{\mathfrak{I}_\alpha}(x)}{2\pi}+\frac{2\pi(\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}+\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi}-\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}\cdot\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi})}{2\pi}-\\ \frac{\frac{\circledS^{-}_{\mathfrak{I}_\alpha}(x)}{2\pi}.2\pi(\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}+\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi}-\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}\cdot\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi})}{2\pi}),\\ 2\pi(\frac{\circledS^{+}_{\mathfrak{I}_\alpha}(x)}{2\pi}+\frac{2\pi(\frac{\circledS^{+}_{\Xi_\alpha}(x)}{2\pi}+\frac{\circledS^{+}_{\Pi_\alpha}(x)}{2\pi}-\frac{\circledS^{+}_{\Xi_\alpha}(x)}{2\pi}\cdot\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi})}{2\pi}-\\ \frac{\frac{\circledS^{-}_{\mathfrak{I}_\alpha}(x)}{2\pi}.2\pi(\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}+\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi}-\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}\cdot\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi})}{2\pi}]}}$$

$$=\begin{bmatrix}[(\mathbb{C}^{-}_{\mathfrak{I}_\alpha}(x)+\mathbb{C}^{-}_{\Xi_\alpha}(x)-\mathbb{C}^{-}_{\mathfrak{I}_\alpha}(x).\mathbb{C}^{-}_{\Xi_\alpha}(x))+\mathbb{C}^{-}_{\Pi_\alpha}(x)-\\ (\mathbb{C}^{-}_{\mathfrak{I}_\alpha}(x)+\mathbb{C}^{-}_{\Xi_\alpha}(x)-\mathbb{C}^{-}_{\mathfrak{I}_\alpha}(x).\mathbb{C}^{-}_{\Xi_\alpha}(x)).\mathbb{C}^{-}_{\Pi_\alpha}(x),\\ (\mathbb{C}^{+}_{\mathfrak{I}_\alpha}(x)+\mathbb{C}^{+}_{\Xi_\alpha}(x)-\mathbb{C}^{+}_{\mathfrak{I}_\alpha}(x).\mathbb{C}^{+}_{\Xi_\alpha}(x))+\mathbb{C}^{+}_{\Pi_\alpha}(x)-\\ (\mathbb{C}^{+}_{\mathfrak{I}_\alpha}(x)+\mathbb{C}^{+}_{\Xi_\alpha}(x)-\mathbb{C}^{+}_{\mathfrak{I}_\alpha}(x).\mathbb{C}^{+}_{\Xi_\alpha}(x)).\mathbb{C}^{+}_{\Pi_\alpha}(x)]\end{bmatrix}$$

$$e^{\substack{i[2\pi(\frac{\circledS^{-}_{\mathfrak{I}_\alpha}(x)}{2\pi}+\frac{2\pi(\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}+\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi}-\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}\cdot\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi})}{2\pi}-\\ \frac{\frac{\circledS^{-}_{\mathfrak{I}_\alpha}(x)}{2\pi}.2\pi(\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}+\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi}-\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}\cdot\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi})}{2\pi}),\\ 2\pi(\frac{\circledS^{+}_{\mathfrak{I}_\alpha}(x)}{2\pi}+\frac{2\pi(\frac{\circledS^{+}_{\Xi_\alpha}(x)}{2\pi}+\frac{\circledS^{+}_{\Pi_\alpha}(x)}{2\pi}-\frac{\circledS^{+}_{\Xi_\alpha}(x)}{2\pi}\cdot\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi})}{2\pi}-\\ \frac{\frac{\circledS^{-}_{\mathfrak{I}_\alpha}(x)}{2\pi}.2\pi(\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}+\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi}-\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}\cdot\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi})}{2\pi}]}}$$

$$=\begin{bmatrix}[(\mathbb{C}^{-}_{\mathfrak{I}_\alpha}(x)+\mathbb{C}^{-}_{\Xi_\alpha}(x)-\mathbb{C}^{-}_{\mathfrak{I}_\alpha}(x).\mathbb{C}^{-}_{\Xi_\alpha}(x))+\mathbb{C}^{-}_{\Pi_\alpha}(x)-\\ (\mathbb{C}^{-}_{\mathfrak{I}_\alpha}(x)+\mathbb{C}^{-}_{\Xi_\alpha}(x)-\mathbb{C}^{-}_{\mathfrak{I}_\alpha}(x).\mathbb{C}^{-}_{\Xi_\alpha}(x)).\mathbb{C}^{-}_{\Pi_\alpha}(x),\\ (\mathbb{C}^{+}_{\mathfrak{I}_\alpha}(x)+\mathbb{C}^{+}_{\Xi_\alpha}(x)-\mathbb{C}^{+}_{\mathfrak{I}_\alpha}(x).\mathbb{C}^{+}_{\Xi_\alpha}(x))+\mathbb{C}^{+}_{\Pi_\alpha}(x)-\\ (\mathbb{C}^{+}_{\mathfrak{I}_\alpha}(x)+\mathbb{C}^{+}_{\Xi_\alpha}(x)-\mathbb{C}^{+}_{\mathfrak{I}_\alpha}(x).\mathbb{C}^{+}_{\Xi_\alpha}(x)).\mathbb{C}^{+}_{\Pi_\alpha}(x)]\end{bmatrix}$$

$$e^{\substack{i[2\pi(\frac{2\pi(\frac{\circledS^{-}_{\mathfrak{I}_\alpha}(x)}{2\pi}+\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}-\frac{\circledS^{-}_{\mathfrak{I}_\alpha}(x)}{2\pi}\cdot\frac{\Xi^{-}_{\Pi_\alpha}(x)}{2\pi})}{2\pi}+\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi}-\\ \frac{2\pi(\frac{\circledS^{-}_{\mathfrak{I}_\alpha}(x)}{2\pi}+\frac{\circledS^{-}_{\Xi_\alpha}(x)}{2\pi}-\frac{\circledS^{-}_{\mathfrak{I}_\alpha}(x)}{2\pi}\cdot\frac{\Xi^{-}_{\Pi_\alpha}(x)}{2\pi}).\frac{\circledS^{-}_{\Pi_\alpha}(x)}{2\pi}}{2\pi}),\\ 2\pi(\frac{2\pi(\frac{\circledS^{+}_{\mathfrak{I}_\alpha}(x)}{2\pi}+\frac{\circledS^{+}_{\Xi_\alpha}(x)}{2\pi}-\frac{\circledS^{+}_{\mathfrak{I}_\alpha}(x)}{2\pi}\cdot\frac{\Xi^{+}_{\Pi_\alpha}(x)}{2\pi})}{2\pi}+\frac{\circledS^{+}_{\Pi_\alpha}(x)}{2\pi}-\\ \frac{2\pi(\frac{\circledS^{+}_{\mathfrak{I}_\alpha}(x)}{2\pi}+\frac{\circledS^{+}_{\Xi_\alpha}(x)}{2\pi}-\frac{\circledS^{+}_{\mathfrak{I}_\alpha}(x)}{2\pi}\cdot\frac{\Xi^{+}_{\Pi_\alpha}(x)}{2\pi}).\frac{\circledS^{+}_{\Pi_\alpha}(x)}{2\pi}}{2\pi}]}}$$

$$=\mu_{((\mathfrak{I},\mathcal{L})\circledast(\Xi,\aleph))\circledast(\mathfrak{R},\Pi)}(x).$$

□

Proposition 5 *The IV-CFS simple difference on* $\dot{\top}(\chi)$ *is an s-norm.*

Proof The proof is similar to the Proof of Proposition 2. □

Proposition 6 $\Gamma(\xi, \eta)$ *defined by the equality* (12.1) *is a distance function of IV-CFSSs on* χ.

Proof (i). The condition $\Gamma(\xi, \eta) \geq 0$ obviously holds true. Next, consider

$$\Gamma(\xi,\eta) = \max\left\{\begin{array}{c} \sup\limits_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi}\sup\limits_x |l([\mathbb{S}^-_{\xi_\alpha}(x), \mathbb{S}^+_{\xi_\alpha}(x)]) - l([\mathbb{S}^-_{\eta_\alpha}(x), \mathbb{S}^+_{\eta_\alpha}(x)])| \end{array}\right\}$$
$$= \max\left\{1, \frac{1}{2\pi}(2\pi)\right\} = 1.$$

Therefore, $0 \leq \Gamma(\xi, \eta) \leq 1$.
Also

$$\Gamma(\xi,\xi) = \max\left\{\begin{array}{c} \sup\limits_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi}\sup\limits_x |l([\mathbb{S}^-_{\xi_\alpha}(x), \mathbb{S}^+_{\xi_\alpha}(x)]) - l([\mathbb{S}^-_{\xi_\alpha}(x), \mathbb{S}^+_{\xi_\alpha}(x)])| \end{array}\right\}$$
$$= \max\{0, 0\} = 0.$$

The condition (ii) is a straight forward. To prove (iii), we have

$$\Gamma(\xi,\eta) = \max\left\{\begin{array}{c} \sup\limits_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi}\sup\limits_x |l([\mathbb{S}^-_{\xi_\alpha}(x), \mathbb{S}^+_{\xi_\alpha}(x)]) - l([\mathbb{S}^-_{\eta_\alpha}(x), \mathbb{S}^+_{\eta_\alpha}(x)])| \end{array}\right\}$$
$$= \max\left\{\begin{array}{c} \sup\limits_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\beta_{\alpha_i}}(x), \mathbb{C}^+_{\beta_{\alpha_i}}(x)]) + \\ l([\mathbb{C}^-_{\beta_{\alpha_i}}(x), \mathbb{C}^+_{\beta_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi}\sup\limits_x |l([\mathbb{S}^-_{\xi_\alpha}(x), \mathbb{S}^+_{\xi_\alpha}(x)]) - l([\mathbb{S}^-_{\beta_\alpha}(x), \mathbb{S}^+_{\beta_\alpha}(x)]) + \\ l([\mathbb{S}^-_{\beta_\alpha}(x), \mathbb{S}^+_{\beta_\alpha}(x)]) - l([\mathbb{S}^-_{\eta_\alpha}(x), \mathbb{S}^+_{\eta_\alpha}(x)])| \end{array}\right\}$$
$$\leq \max\left\{\begin{array}{c} \sup\limits_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\beta_{\alpha_i}}(x), \mathbb{C}^+_{\beta_{\alpha_i}}(x)])| + \\ |l([\mathbb{C}^-_{\beta_{\alpha_i}}(x), \mathbb{C}^+_{\beta_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi}\sup\limits_x |l([\mathbb{S}^-_{\xi_\alpha}(x), \mathbb{S}^+_{\xi_\alpha}(x)]) - l([\mathbb{S}^-_{\beta_\alpha}(x), \mathbb{S}^+_{\beta_\alpha}(x)])| + \\ |l([\mathbb{S}^-_{\beta_\alpha}(x), \mathbb{S}^+_{\beta_\alpha}(x)]) - l([\mathbb{S}^-_{\eta_\alpha}(x), \mathbb{S}^+_{\eta_\alpha}(x)])| \end{array}\right\}$$
$$= \max\left\{\begin{array}{c} \sup\limits_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\beta_{\alpha_i}}(x), \mathbb{C}^+_{\beta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi}\sup\limits_x |l([\mathbb{S}^-_{\xi_\alpha}(x), \mathbb{S}^+_{\xi_\alpha}(x)]) - l([\mathbb{S}^-_{\beta_\alpha}(x), \mathbb{S}^+_{\beta_\alpha}(x)])| \end{array}\right\} +$$
$$\max\left\{\begin{array}{c} \sup\limits_x |l([\mathbb{C}^-_{\beta_{\alpha_i}}(x), \mathbb{C}^+_{\beta_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi}\sup\limits_x |l([\mathbb{S}^-_{\beta_\alpha}(x), \mathbb{S}^+_{\beta_\alpha}(x)]) - l([\mathbb{S}^-_{\eta_\alpha}(x), \mathbb{S}^+_{\eta_\alpha}(x)])| \end{array}\right\}$$
$$= \Gamma(\xi, \beta) + \Gamma(\beta, \eta).$$

Therefore

$$\Gamma(\xi, \eta) \leq \Gamma(\xi, \beta) + \Gamma(\beta, \eta).$$

□

Proposition 7 *Let $\Gamma(\xi, \eta)$ be a distance function on IV-CFSSs. Then, the following holds.*

$i). \ \Gamma(\xi, \eta^c) = \Gamma(\xi^c, \eta) = \Gamma(\xi, \eta),$
$ii). \ \Gamma(\xi^c, \eta^c) = \Gamma(\xi, \eta),$
$iii). \ \Gamma(\xi, \eta) = \Gamma(\xi \cap \eta, \xi \cup \eta).$

Proof (i). For $\Gamma(\xi, \eta) = \max \left\{ \begin{array}{l} \sup_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi} \sup_x |l([\circledS^-_{\xi_\alpha}(x), \circledS^+_{\xi_\alpha}(x)]) - l([\circledS^-_{\eta_\alpha}(x), \circledS^+_{\eta_\alpha}(x)])| \end{array} \right\},$

we have the following:

$$\begin{aligned} \Gamma(\xi, \eta^c) &= \max \left\{ \begin{array}{c} \sup_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([1 - \mathbb{C}^-_{\eta_{\alpha_i}}(x), 1 - \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi} \sup_x |l([\circledS^-_{\xi_\alpha}(x), \circledS^+_{\xi_\alpha}(x)]) - l([2\pi - \circledS^-_{\eta_\alpha}(x), 2\pi - \circledS^+_{\eta_\alpha}(x)])| \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} \sup_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi} \sup_x |l([\circledS^-_{\xi_\alpha}(x), \circledS^+_{\xi_\alpha}(x)]) - l([\circledS^-_{\eta_\alpha}(x), \circledS^+_{\eta_\alpha}(x)])| \end{array} \right\} \\ &\left(\begin{array}{l} \because l([1 - \mathbb{C}^-_{\eta_{\alpha_i}}(x), 1 - \mathbb{C}^+_{\eta_{\alpha_i}}(x)]) = l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)]) \ \& \\ l([2\pi - \circledS^-_{\eta_\alpha}(x), 2\pi - \circledS^+_{\eta_\alpha}(x)]) = l([\circledS^-_{\eta_\alpha}(x), \circledS^+_{\eta_\alpha}(x)]) \end{array} \right) \\ &= \Gamma(\xi, \eta). \end{aligned}$$

Similarly, $\Gamma(\xi^c, \eta) = \Gamma(\xi, \eta)$.
$ii)$.

$$\begin{aligned} \Gamma(\xi^c, \eta^c) &= \max \left\{ \begin{array}{c} \sup_x |l([1 - \mathbb{C}^-_{\xi_{\alpha_i}}(x), 1 - \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - \\ l([1 - \mathbb{C}^-_{\eta_{\alpha_i}}(x), 1 - \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi} \sup_x |l([2\pi - \circledS^-_{\xi_\alpha}(x), 2\pi - \circledS^+_{\xi_\alpha}(x)]) - \\ l([2\pi - \circledS^-_{\eta_\alpha}(x), 2\pi - \circledS^+_{\eta_\alpha}(x)])| \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} \sup_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi} \sup_x |l([\circledS^-_{\xi_\alpha}(x), \circledS^+_{\xi_\alpha}(x)]) - l([\circledS^-_{\eta_\alpha}(x), \circledS^+_{\eta_\alpha}(x)])| \end{array} \right\} \\ &= \Gamma(\xi, \eta). \end{aligned}$$

$iii)$. To prove (iii), two cases arise here.
Case 1.

$$\mathbb{C}^-_{\xi_{\alpha_i}}(x) \leq \mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x) \leq \mathbb{C}^+_{\eta_{\alpha_i}}(x), \circledS^-_{\xi_\alpha}(x) \leq \circledS^-_{\eta_\alpha}(x), \circledS^+_{\xi_\alpha}(x) \leq \circledS^+_{\eta_\alpha}(x).$$

$$\begin{aligned} \Gamma(\xi \cap \eta, \xi \cup \eta) &= \max \left\{ \begin{array}{c} \sup_x |l([\min(\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^-_{\eta_{\alpha_i}}(x)), \min(\mathbb{C}^+_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x))]) \\ -l([\max(\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^-_{\eta_{\alpha_i}}(x)), \max(\mathbb{C}^+_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x))]), \\ \frac{1}{2\pi} \sup_x |l([\min(\circledS^-_{\xi_\alpha}(x), \circledS^-_{\eta_\alpha}(x)), \min(\circledS^+_{\xi_\alpha}(x), \circledS^+_{\eta_\alpha}(x))]) - \\ l([\max(\circledS^-_{\xi_\alpha}(x), \circledS^-_{\eta_\alpha}(x)), \max(\circledS^+_{\xi_\alpha}(x), \circledS^+_{\eta_\alpha}(x))])| \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} \sup_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi} \sup_x |l([\circledS^-_{\xi_\alpha}(x), \circledS^+_{\xi_\alpha}(x)]) - l([\circledS^-_{\eta_\alpha}(x), \circledS^+_{\eta_\alpha}(x)])| \end{array} \right\} \\ &= \Gamma(\xi, \eta). \end{aligned}$$

Case 2.

$$\mathbb{C}^{-}_{\xi_{\alpha_i}}(x) \geq \mathbb{C}^{-}_{\eta_{\alpha_i}}(x), \mathbb{C}^{+}_{\xi_{\alpha_i}}(x) \geq \mathbb{C}^{+}_{\eta_{\alpha_i}}(x), \circledS^{-}_{\xi_\alpha}(x) \geq \circledS^{-}_{\eta_\alpha}(x), \circledS^{+}_{\xi_\alpha}(x) \geq \circledS^{+}_{\eta_\alpha}(x).$$

$$\begin{aligned}
\Gamma(\xi \cap \eta, \xi \cup \eta) &= \max \left\{ \begin{array}{c} \sup_x |l([\min(\mathbb{C}^{-}_{\xi_{\alpha_i}}(x), \mathbb{C}^{-}_{\eta_{\alpha_i}}(x)), \min(\mathbb{C}^{+}_{\xi_{\alpha_i}}(x), \mathbb{C}^{+}_{\eta_{\alpha_i}}(x))]) \\ -l([\max(\mathbb{C}^{-}_{\xi_{\alpha_i}}(x), \mathbb{C}^{-}_{\eta_{\alpha_i}}(x)), \max(\mathbb{C}^{+}_{\xi_{\alpha_i}}(x), \mathbb{C}^{+}_{\eta_{\alpha_i}}(x))]), \\ \frac{1}{2\pi} \sup_x |l([\min(\circledS^{-}_{\xi_\alpha}(x), \circledS^{-}_{\eta_\alpha}(x)), \min(\circledS^{+}_{\xi_\alpha}(x), \circledS^{+}_{\eta_\alpha}(x))]) - \\ l([\max(\circledS^{-}_{\xi_\alpha}(x), \circledS^{-}_{\eta_\alpha}(x)), \max(\circledS^{+}_{\xi_\alpha}(x), \circledS^{+}_{\eta_\alpha}(x))])| \end{array} \right\} \\
&= \max \left\{ \begin{array}{c} \sup_x |l([\mathbb{C}^{-}_{\eta_{\alpha_i}}(x), \mathbb{C}^{+}_{\eta_{\alpha_i}}(x)]) - l([\mathbb{C}^{-}_{\xi_{\alpha_i}}(x), \mathbb{C}^{+}_{\xi_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi} \sup_x |l([\circledS^{-}_{\eta_\alpha}(x), \circledS^{+}_{\eta_\alpha}(x)]) - l([\circledS^{-}_{\xi_\alpha}(x), \circledS^{+}_{\xi_\alpha}(x)])| \end{array} \right\} \\
&= \max \left\{ \begin{array}{c} \sup_x |l([\mathbb{C}^{-}_{\xi_{\alpha_i}}(x), \mathbb{C}^{+}_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^{-}_{\eta_{\alpha_i}}(x), \mathbb{C}^{+}_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi} \sup_x |l([\circledS^{-}_{\xi_\alpha}(x), \circledS^{+}_{\xi_\alpha}(x)]) - l([\circledS^{-}_{\eta_\alpha}(x), \circledS^{+}_{\eta_\alpha}(x)])| \end{array} \right\} \\
&= \Gamma(\xi, \eta).
\end{aligned}$$

□

Proposition 8 *Let* $\xi = (\Im, \mathcal{L})$ *and* $\eta = (\Xi, \aleph)$ *be two IV-CFSSs on* χ. *Then*

$i)$. $\zeta = (0)\eta$.
$ii)$. $\zeta = (1)\eta$ for $\zeta = \eta$.
$iii)$. $\zeta = (\delta)\eta \iff \eta = (\delta)\zeta$.
$iv)$. $\zeta = (\delta_1)\eta$ and $\delta_2 \leq \delta_1 \implies \zeta = (\delta_2)\eta$.

$v)$. If, for all $\gamma \in I$, $\zeta = (\delta_\gamma)\eta$, where I is an index set, then $\zeta = (\sup_{\gamma \in I} \delta_\gamma)\eta$.

Proof (i). Since $\Gamma(\xi, \eta) \leq 1$. It can be written as $\Gamma(\xi, \eta) \leq 1 - 0$. So, by definition of IV-CFS δ-equality, we have $\zeta = (0)\eta$.

(ii). Since $\Gamma(\xi, \xi) = 0$. It can be written as $\Gamma(\xi, \xi) = 1 - 1$. Therefore, by definition of IV-CFS δ-equality, we have $\zeta = (1)\eta$.

(iii). Let $\zeta = (\delta)\eta$, then, by definition of IV-CFS δ-equality, we have

$$\begin{aligned} &\Gamma(\xi, \eta) \leq 1 - \delta \\ &\Gamma(\eta, \xi) \leq 1 - \delta \ (\because \Gamma(\xi, \eta) = \Gamma(\eta, \xi)). \end{aligned} \tag{3}$$

Therefore, (3) implies that $\eta = (\delta)\zeta$.
Similarly, if $\eta = (\delta)\zeta$, then $\zeta = (\delta)\eta$.
(iv). Let $\zeta = (\delta_1)\eta$, then

$$\begin{aligned} \Gamma(\xi, \eta) &= \max \left\{ \begin{array}{c} \sup_x |l([\mathbb{C}^{-}_{\xi_{\alpha_i}}(x), \mathbb{C}^{+}_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^{-}_{\eta_{\alpha_i}}(x), \mathbb{C}^{+}_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi} \sup_x |l([\circledS^{-}_{\xi_\alpha}(x), \circledS^{+}_{\xi_\alpha}(x)]) - l([\circledS^{-}_{\eta_\alpha}(x), \circledS^{+}_{\eta_\alpha}(x)])| \end{array} \right\} \\ &\leq 1 - \delta_1. \end{aligned}$$

Since for $\delta_2 \leq \delta_1$, the above inequality also holds, that is

$$\Gamma(\xi,\eta) = \max\left\{\begin{array}{l} \sup\limits_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi}\sup\limits_x |l([\mathbb{S}^-_{\xi_\alpha}(x), \mathbb{S}^+_{\xi_\alpha}(x)]) - l([\mathbb{S}^-_{\eta_\alpha}(x), \mathbb{S}^+_{\eta_\alpha}(x)])| \end{array}\right\} \\ \leq 1 - \delta_2.$$

Thus, $\zeta = (\delta_2)\eta$.
(v). Let $\zeta = (\delta_\gamma)\eta$, then

$$\Gamma(\xi,\eta) = \max\left\{\begin{array}{l} \sup\limits_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi}\sup\limits_x |l([\mathbb{S}^-_{\xi_\alpha}(x), \mathbb{S}^+_{\xi_\alpha}(x)]) - l([\mathbb{S}^-_{\eta_\alpha}(x), \mathbb{S}^+_{\eta_\alpha}(x)])| \end{array}\right\} \\ \leq 1 - \delta_\gamma, \text{ for all } \gamma \in I.$$

Thus, it is easy to see that

$$\sup_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])| \leq 1 - \sup_{\gamma\in I}\delta_\gamma,$$
$$\frac{1}{2\pi}\sup_x |l([\mathbb{S}^-_{\xi_\alpha}(x), \mathbb{S}^+_{\xi_\alpha}(x)]) - l([\mathbb{S}^-_{\eta_\alpha}(x), \mathbb{S}^+_{\eta_\alpha}(x)])| \leq 1 - \sup_{\gamma\in I}\delta_\gamma.$$

Therefore

$$\max\left\{\begin{array}{l} \sup\limits_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi}\sup\limits_x |l([\mathbb{S}^-_{\xi_\alpha}(x), \mathbb{S}^+_{\xi_\alpha}(x)]) - l([\mathbb{S}^-_{\eta_\alpha}(x), \mathbb{S}^+_{\eta_\alpha}(x)])| \end{array}\right\} \leq 1 - \sup_{\gamma\in I}\delta_\gamma.$$

This implies that $\Gamma(\xi,\eta) \leq 1 - \sup\limits_{\gamma\in I}\delta_\gamma$, and hence, $\xi = (\sup\limits_{\gamma\in I}\delta_\gamma)\eta$. □

Proposition 9 *If $\zeta = (\delta_1)\eta$ and $\eta = (\delta_2)\psi$, then $\zeta = (\delta)\psi$, where $\delta = \delta_1 * \delta_2$.*

Proof Let $\zeta = (\delta_1)\eta$ and $\eta = (\delta_2)\psi$, then we have

$$\Gamma(\xi,\eta) = \max\left\{\begin{array}{l} \sup\limits_x |l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi}\sup\limits_x |l([\mathbb{S}^-_{\xi_\alpha}(x), \mathbb{S}^+_{\xi_\alpha}(x)]) - l([\mathbb{S}^-_{\eta_\alpha}(x), \mathbb{S}^+_{\eta_\alpha}(x)])| \end{array}\right\} \\ \leq 1 - \delta_1,$$

and

$$\Gamma(\eta,\psi) = \max\left\{\begin{array}{l} \sup\limits_x |l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\psi_{\alpha_i}}(x), \mathbb{C}^+_{\psi_{\alpha_i}}(x)])|, \\ \frac{1}{2\pi}\sup\limits_x |l([\mathbb{S}^-_{\eta_\alpha}(x), \mathbb{S}^+_{\eta_\alpha}(x)]) - l([\mathbb{S}^-_{\psi_\alpha}(x), \mathbb{S}^+_{\psi_\alpha}(x)])| \end{array}\right\} \\ \leq 1 - \delta_2.$$

Therefore

$$\Gamma(\xi,\psi)=\max\left\{\begin{array}{c}\sup_x|l([\mathbb{C}^-_{\xi_{\alpha_i}}(x),\mathbb{C}^+_{\xi_{\alpha_i}}(x)])-l([\mathbb{C}^-_{\psi_{\alpha_i}}(x),\mathbb{C}^+_{\psi_{\alpha_i}}(x)])|,\\ \frac{1}{2\pi}\sup_x|l([\mathbb{S}^-_{\xi_\alpha}(x),\mathbb{S}^+_{\xi_\alpha}(x)])-l([\mathbb{S}^-_{\psi_\alpha}(x),\mathbb{S}^+_{\psi_\alpha}(x)])|\end{array}\right\}$$

$$=\max\left\{\begin{array}{c}\sup_x|l([\mathbb{C}^-_{\xi_{\alpha_i}}(x),\mathbb{C}^+_{\xi_{\alpha_i}}(x)])-l([\mathbb{C}^-_{\eta_{\alpha_i}}(x),\mathbb{C}^+_{\eta_{\alpha_i}}(x)])+\\ l([\mathbb{C}^-_{\eta_{\alpha_i}}(x),\mathbb{C}^+_{\eta_{\alpha_i}}(x)])-l([\mathbb{C}^-_{\psi_{\alpha_i}}(x),\mathbb{C}^+_{\psi_{\alpha_i}}(x)])|,\\ \frac{1}{2\pi}\sup_x|l([\mathbb{S}^-_{\xi_\alpha}(x),\mathbb{S}^+_{\xi_\alpha}(x)])-l([\mathbb{S}^-_{\eta_\alpha}(x),\mathbb{S}^+_{\eta_\alpha}(x)])+\\ l([\mathbb{S}^-_{\eta_\alpha}(x),\mathbb{S}^+_{\eta_\alpha}(x)])-l([\mathbb{S}^-_{\psi_\alpha}(x),\mathbb{S}^+_{\psi_\alpha}(x)])|\end{array}\right\}$$

$$\leq\max\left\{\begin{array}{c}\sup_x|l([\mathbb{C}^-_{\xi_{\alpha_i}}(x),\mathbb{C}^+_{\xi_{\alpha_i}}(x)])-l([\mathbb{C}^-_{\eta_{\alpha_i}}(x),\mathbb{C}^+_{\eta_{\alpha_i}}(x)])|,\\ \frac{1}{2\pi}\sup_x|l([\mathbb{S}^-_{\xi_\alpha}(x),\mathbb{S}^+_{\xi_\alpha}(x)])-l([\mathbb{S}^-_{\eta_\alpha}(x),\mathbb{S}^+_{\eta_\alpha}(x)])|\end{array}\right\}+$$

$$\max\left\{\begin{array}{c}\sup_x|l([\mathbb{C}^-_{\eta_{\alpha_i}}(x),\mathbb{C}^+_{\eta_{\alpha_i}}(x)])-l([\mathbb{C}^-_{\psi_{\alpha_i}}(x),\mathbb{C}^+_{\psi_{\alpha_i}}(x)])|,\\ \frac{1}{2\pi}\sup_x|l([\mathbb{S}^-_{\eta_\alpha}(x),\mathbb{S}^+_{\eta_\alpha}(x)])-l([\mathbb{S}^-_{\psi_\alpha}(x),\mathbb{S}^+_{\psi_\alpha}(x)])|\end{array}\right\}$$

$$\leq(1-\delta_1)+(1-\delta_2)$$
$$=1-(\delta_1+\delta_2-1)=1-(\delta_1*\delta_2)=1-\delta.$$

Thus, $\xi=(\delta)\psi$. □

5 Applications of interval-valued complex fuzzy soft sets

In this section, we will discuss a real-life application of IV-CFSSs. Especially, the IV-CFSS explains how to get a better and clear choice in decision-making problems.

We propose the following definitions utilized in decision-making algorithm taking the idea of an IV-CFSSs into account.

Definition 17 Let $\xi=(\Im,\mathcal{L})\in\dot{\top}(\chi)$ be an IV-CFSSs. Then, the cardinality set of ξ, represented by ξ_C, is defined by

$$\xi_C=\left\{(x,\mu_{\xi_C}(x)):x\in\chi\right\},$$

is a fuzzy set over $\mathcal{L}$, and the membership function $\mu_{\xi_C}(x)$ of ξ_C is a mapping $\mu_{\xi_C}(x):\mathcal{L}\to[0,1]$, defined by

$$\mu_{\xi_C^{(\alpha)}}(x)=[\frac{\mathbb{C}^-_{\Im_\alpha}(x)}{|\chi|},\frac{\mathbb{C}^+_{\Im_\alpha}(x)}{|\chi|}]).e^{i[\frac{\mathbb{S}^-_{\Im_\alpha}(x)}{|\chi|},\frac{\mathbb{S}^+_{\Im_\alpha}(x)}{|\chi|}]};\ x\in\chi,$$
$$=$$

where $|\chi|$ is the cardinality of universe χ, and $|l([\mathbb{C}^-_{\Im_\alpha}(x),\mathbb{C}^+_{\Im_\alpha}(x)]).e^{il([\mathbb{S}^-_{\Im_\alpha}(x),\mathbb{S}^+_{\Im_\alpha}(x)])}|$ is the scalar cardinality of a fuzzy set.

Note that the set of all cardinality sets is denoted by $\dot{\top}_C(\chi)$.

Definition 18 Let $\xi=(\Im,\mathcal{L})\in\dot{\top}(\chi)$ be an IV-CFSSs and ξ_C be the cardinality of ξ. Assume that $\mathcal{L}\subset\mathcal{F}$ be a subset of parameters, such that $\mathcal{L}=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$. Then, ξ_C can be denoted by the following table.

$\mathcal{L}$	α_1	α_2	.	.	.	α_n
$\mu_{\xi_C^{(\alpha)}}(x)$	$\mu_{\xi_C^{(\alpha_1)}}(x)$	$\mu_{\xi_C^{(\alpha_2)}}(x)$	.	.	.	$\mu_{\xi_C^{(\alpha_3)}}(x)$

The cardinal set $\mu_{\xi_C^{(\alpha)}}(x)$ is uniquely represented by a matrix, $[a_{1j}]_{1\times n} = [a_{11}\, a_{12}\, \ldots a_{1n}]$, and is called the cardinal matrix of the cardinal set $\mu_{\xi_C^{(\alpha)}}(x)$ over $\mathcal{L}$.

Definition 19 Let $\xi_1 = (\Im, \mathcal{L}), \xi_2 = (\Xi, \aleph), \xi_3 = (\Re, \Pi) \in \dot{\top}(\chi)$ be any three IV-CFSSs over χ. Then, the weighted distance measure of IV-CFSSs is defined as follows:

$$\Gamma(\xi, \eta) = \max \left\{ \begin{array}{l} \sup\limits_x |l([w_{(\alpha)}\mathbb{C}^-_{\xi_\alpha}(x), w_{(\alpha)}\mathbb{C}^+_{\xi_\alpha}(x)]) - l([w_{(\alpha)}\mathbb{C}^-_{\eta_\alpha}(x), w_{(\alpha)}\mathbb{C}^+_{\eta_\alpha}(x)])|, \\ \frac{1}{2\pi}\sup\limits_x |l([w_{(\alpha)}\circledS^-_{\xi_\alpha}(x), w_{(\alpha)}\circledS^+_{\xi_\alpha}(x)]) - l([w_{(\alpha)}\circledS^-_{\eta_\alpha}(x), w_{(\alpha)}\circledS^+_{\eta_\alpha}(x)])| \end{array} \right\}, \tag{4}$$

where l denotes the length of the interval and $w_{(\alpha)}$ is the weighted vector corresponding to the parameter α.

6 Algorithm

In this algorithm, we utilize the concepts of the cardinality of IV-CFSSs, interval-valued complex fuzzy soft distance measure, interval-valued complex fuzzy soft weighted distance measure, and cardinal matrix proposed in the previous section. Here, we solve the decision-making problems under the environments of IV-CFSSs. The detailed steps of the decision-making algorithm for finding the best choice in decision-making problems are described as follows:

Step 1.
Construct an IV-CFSSs $\xi = (\Im, \mathcal{L})$ over χ.
Step 2.
Compute the cardinal set ξ_C of ξ for amplitude term and phase term separately.
Step 3.
Compute the distance measure of IV-CFSSs $\xi(\alpha_i)$ and cardinal set ξ_C.
Step 4.
Rank the alternatives based on the distance measures between the parameters and hence select the most desirable one(s).

7 A case analysis

Suppose a businessman decides to make an investment in four football clubs (Alternatives). Let $\chi = \{x_1 =$*Real Madrid*$, x_2 = Barcelona, x_3 = Manchester\ United, x_4 = Juventus\}$ be a set of alternatives. Let $C = \{\alpha_1 = Sponsorship\ Deal, \alpha_2 = Social\ Media\ Income, \alpha_2 = Players\ Worth, Annual\ Tax\}$ be a set of criteria. Football clubs are organizations that field teams to compete in football (soccer) competitions. They are typically associated with a specific city or region and have a fan base that supports them passionately. Investing in football clubs can take various forms, and individuals or entities may choose different strategies based on their goals and interests.

Table 1 Evaluated values of parameters in the environment of IV-CFSSs

	x_1	x_2	x_3	x_4
α_1	$[0.2, 0.9]e^{i[0.4\pi,1.5\pi]}$	$[0.4, 0.7]e^{i[0.1\pi,1.8\pi]}$	$[0.1, 0.8]e^{i[0.2\pi,1.2\pi]}$	$[0.3, 0.6]e^{i[0.3\pi,1.9\pi]}$
α_2	$[0.1, 1]e^{i[0.5\pi,1.1\pi]}$	$[0.3, 0.8]e^{i[0.5\pi,1.4\pi]}$	$[0.4, 1]e^{i[0.2\pi,2\pi]}$	$[0.2, 0.9]e^{i[0.8\pi,1.6\pi]}$
α_3	$[0.7, 0.9]e^{i[0.3\pi,\pi]}$	$[0.5, 1]e^{i[0.6\pi,1.5\pi]}$	$[0.1, 0.8]e^{i[0.7\pi,2\pi]}$	$[0.3, 0.7]e^{i[0.6\pi,1.6\pi]}$
α_4	$[0.2, 0.6]e^{i[0.1\pi,1.2\pi]}$	$[0.1, 0.9]e^{i[0.9\pi,1.8\pi]}$	$[0.5, 0.8]e^{i[0.3\pi,1.3\pi]}$	$[0.4, 1]e^{i[0.3\pi,2\pi]}$

Sponsorship deal

Companies often invest in football clubs through sponsorship deals. These deals involve the company's logo appearing on the club's jerseys, stadium, or other promotional materials. Sponsorship agreements can be lucrative and provide visibility for the sponsoring brand.

Media and broadcasting rights

Investment in media and broadcasting rights is crucial for the financial success of football clubs. Investors may negotiate lucrative broadcasting deals, streaming rights, or digital media partnerships, contributing to the club's revenue streams.

Partnerships and collaborations

Investors may seek strategic partnerships and collaborations with football clubs. This could involve joint ventures, co-branding opportunities, or collaborative initiatives that benefit both the investor and the club.

Here are four well-known football clubs from different parts of the world:

Real Madrid CF (Spain)

One of the most successful football clubs globally based in Madrid, Spain. Real Madrid has won numerous domestic and international titles, including the UEFA Champions League.

FC Barcelona (Spain)

Another Spanish powerhouse, based in Barcelona. FC Barcelona has a rich history and is known for its commitment to an attractive, possession-based style of play.

Manchester United FC (England)

Located in Manchester, England, Manchester United is one of the most popular and successful football clubs in the world. The club has a massive global fan base.

Juventus FC (Italy)

Located in Turin, Italy, Juventus is one of the most successful and popular football clubs in Italy. The club has a strong tradition of success in Serie A.

The expert team investigates and takes part in the evaluation of these three enterprises under the environments of IV-CFSSs. The information of the decision-makers is given in Table 1.

Step 1. The IV-CFSSs $\xi = (\Im, \mathcal{L})$ over χ is written by

$$\xi(x_1) = \left\{ \begin{array}{c} (\alpha_1, [0.2, 0.9]e^{i[0.4\pi,1.5\pi]}), (\alpha_2, [0.1, 1]e^{i[0.5\pi,1.1\pi]}), (\alpha_3, [0.7, 0.9]e^{i[0.3\pi,\pi]}), \\ (\alpha_4, [0.2, 0.6]e^{i[0.1\pi,1.2\pi]}) \end{array} \right\}.$$

$$\xi(x_2) = \left\{ \begin{array}{c} (\alpha_1, [0.4, 0.7]e^{i[0.1\pi,1.8\pi]}), (\alpha_2, [0.3, 0.8]e^{i[0.5\pi,1.4\pi]}), (\alpha_3, [0.5, 1]e^{i[0.6\pi,1.5\pi]}), \\ (\alpha_4, [0.1, 0.9]e^{i[0.9\pi,1.8\pi]}) \end{array} \right\}.$$

$$\xi(x_3) = \left\{ \begin{array}{c} (\alpha_1, [0.1, 0.8]e^{i[0.2\pi,1.2\pi]}), (\alpha_2, [0.4, 1]e^{i[0.2\pi,2\pi]}), (\alpha_3, [0.1, 0.8]e^{i[0.7\pi,2\pi]}), \\ (\alpha_4, [0.5, 0.8]e^{i[0.3\pi,1.3\pi]}) \end{array} \right\}.$$

$$\xi(x_4) = \left\{ \begin{array}{c} (\alpha_1, [0.3, 0.6]e^{i[0.3\pi,1.9\pi]}), (\alpha_2, [0.2, 0.9]e^{i[0.8\pi,1.6\pi]}), (\alpha_3, [0.3, 0.7]e^{i[0.6\pi,1.6\pi]}), \\ (\alpha_4, [0.4, 1]e^{i[0.3\pi,2\pi]}) \end{array} \right\}.$$

Table 2 Values of IV-CFS distance measure

Γ	$\Gamma(\xi(x_1), \xi_C)$	$\Gamma(\xi(x_2), \xi_C)$	$\Gamma(\xi(x_3), \xi_C)$	$\Gamma(\xi(x_4), \xi_C)$
	0.25	0.315	0.3875	0.275

Table 3 Values of IV-CFS weighted distance measure

Γ	$\Gamma(\xi(x_1), \xi_C)$	$\Gamma(\xi(x_2), \xi_C)$	$\Gamma(\xi(x_3), \xi_C)$	$\Gamma(\xi(x_4), \xi_C)$
	0.05	0.0945	0.11625	0.055

Step 2.

The cardinal set ξ_C of ξ for amplitude term and phase term is

$$\xi_C = \left\{ \begin{array}{l} (\alpha_1, [0.25, 0.75]e^{i[0.25\pi, 1.6\pi]}), (\alpha_2, [0.11, 0.925]e^{i[0.5\pi, 1.525\pi]}), \\ (\alpha_3, [0.4, 0.85]e^{i[0.55\pi, 1.525\pi]}), (\alpha_4, [0.3, 0.825]e^{i[0.4\pi, 1.575\pi]}) \end{array} \right\}.$$

Step 3.

The distance measure of $\xi(\alpha_i)$ and cardinal set ξ_C are given in Table 2.

If $w = (0.2, 0.3, 0.2, 0.3)$ is the weightage assigned to each parameter α_i; $i = 1, 2, 3, 4$, then

From the distance measure of $\xi(\alpha_i)$ and ξ_C, we conclude that $x_3 > x_2 > x_4 > x_1$. Therefore, $x_3 = Manchester\ United$ is the best football club for investment.

If we assigned the weighted vector $w = (0.2, 0.3, 0.2, 0.3)$ to the parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, then the rank of alternatives is $x_3 > x_2 > x_4 > x_1$. Therefore, we get the same order of rank.

8 Comparison analysis

Here, we discussed the comparison of the proposed distance measure of the interval-valued complex fuzzy soft set with the complex fuzzy soft set and fuzzy soft set. The interval-valued complex fuzzy soft distance measure reduces the environments of complex fuzzy soft set and fuzzy soft set. The comparison is given in cases 1 and 2. The numerical data in other fuzzy environments are also considered. Moreover, we discussed the advantages of the proposed distance measure.

Case 1.

The interval-valued complex fuzzy soft distance measure reduces to the environment of complex fuzzy soft sets if $\mathbb{C}^-_{\xi_{\alpha_i}}(x) = \mathbb{C}^+_{\xi_{\alpha_i}}(x)$, $\mathbb{C}^-_{\eta_{\alpha_i}}(x) = \mathbb{C}^+_{\eta_{\alpha_i}}(x)$, $\mathbb{S}^-_{\xi_\alpha}(x) = \mathbb{S}^+_{\xi_\alpha}(x)$, $\mathbb{S}^-_{\eta_\alpha}(x) = \mathbb{S}^+_{\eta_\alpha}(x)$, & remove length from the definition, then we have

$$\Gamma(\xi, \eta) = \max\left\{ \sup_x |\mathbb{C}_{\xi_{\alpha_i}}(x) - \mathbb{C}_{\eta_{\alpha_i}}(x)|, \frac{1}{2\pi} \sup_x |\mathbb{S}_{\xi_\alpha}(x) - \mathbb{S}_{\eta_\alpha}(x)| \right\},$$

where $\mathbb{C}_{\xi_{\alpha_i}}(x), \mathbb{C}_{\eta_{\alpha_i}}(x) \in [0, 1]$, and $\mathbb{S}_{\xi_\alpha}(x), \mathbb{S}_{\eta_\alpha}(x) \in [0, 2\pi]$.

Case 2.

The interval-valued complex fuzzy soft distance measure reduces to the environment of interval-valued fuzzy soft sets if $\mathbb{S}^-_{\xi_\alpha}(x) = \mathbb{S}^+_{\xi_\alpha}(x) = \mathbb{S}^-_{\eta_\alpha}(x) = \mathbb{S}^+_{\eta_\alpha}(x) = 0$, then we have

$$\Gamma(\xi, \eta) = \sup_x (|l([\mathbb{C}^-_{\xi_{\alpha_i}}(x), \mathbb{C}^+_{\xi_{\alpha_i}}(x)]) - l([\mathbb{C}^-_{\eta_{\alpha_i}}(x), \mathbb{C}^+_{\eta_{\alpha_i}}(x)])|),$$

Table 4 Evaluated values of parameters in the environment of CFSSs

	x_1	x_2	x_3	x_4
α_1	$0.9e^{i0.4\pi}$	$0.4e^{i0.1\pi}$	$0.8e^{i1.2\pi}$	$0.3e^{i0.3\pi}$
α_2	$1e^{i1.1\pi}$	$0.3e^{i0.5\pi}$	$1e^{i2\pi}$	$0.2e^{i0.8\pi}$
α_3	$0.9e^{i\pi}$	$1e^{i1.5\pi}$	$0.1e^{i0.7\pi}$	$0.7e^{i1.6\pi}$
α_4	$0.2e^{i0.1\pi}$	$0.9e^{i1.8\pi}$	$0.5e^{i0.3\pi}$	$1e^{i2\pi}$

Table 5 Values of CFS distance measure

Γ	$\Gamma(\xi(x_1), \xi_C)$	$\Gamma(\xi(x_2), \xi_C)$	$\Gamma(\xi(x_3), \xi_C)$	$\Gamma(\xi(x_4), \xi_C)$
	0.475	0.325	0.575	0.475

Table 6 Values of CFS weighted distance measure

Γ	$\Gamma(\xi(x_1), \xi_C)$	$\Gamma(\xi(x_2), \xi_C)$	$\Gamma(\xi(x_3), \xi_C)$	$\Gamma(\xi(x_4), \xi_C)$
	0.095	0.0975	0.115	0.095

where $\mathbb{C}^{-}_{\xi_{\alpha_i}}(x), \mathbb{C}^{+}_{\xi_{\alpha_i}}(x), \mathbb{C}^{-}_{\eta_{\alpha_i}}(x), \mathbb{C}^{+}_{\eta_{\alpha_i}}(x) \in [0, 1]$.

Case 3.

The interval-valued complex fuzzy soft distance measure reduces to the environment of fuzzy soft sets if $\mathbb{C}^{-}_{\xi_{\alpha_i}}(x) = \mathbb{C}^{+}_{\xi_{\alpha_i}}(x)$, $\mathbb{C}^{-}_{\eta_{\alpha_i}}(x) = \mathbb{C}^{+}_{\eta_{\alpha_i}}(x)$, $Ⓢ^{-}_{\xi_\alpha}(x) = Ⓢ^{+}_{\xi_\alpha}(x) = Ⓢ^{-}_{\eta_\alpha}(x) = Ⓢ^{+}_{\eta_\alpha}(x) = 0$, & remove length from the definition, then we have

$$\Gamma(\xi, \eta) = \sup_{x} |\mathbb{C}_{\xi_{\alpha_i}}(x) - \mathbb{C}_{\eta_{\alpha_i}}(x)|,$$

where $\mathbb{C}_{\xi_{\alpha_i}}(x), \mathbb{C}_{\eta_{\alpha_i}}(x) \in [0, 1]$.

All the above distance measures can be applied to decision-making problems. Now, we consider the decision-making problem in the environment of complex fuzzy soft sets.

The complex fuzzy soft information about the index's evaluation is given in Table 4.

The IV-CFSSs $\xi = (\Im, \mathcal{L})$ over χ are written by

$$\xi(x_1) = \left\{(\alpha_1, 0.9e^{i0.4\pi}), (\alpha_2, 1e^{i1.1\pi}), (\alpha_3, 0.9e^{i\pi}), (\alpha_4, 0.2e^{i0.1\pi})\right\}.$$
$$\xi(x_2) = \left\{(\alpha_1, 0.4e^{i0.1\pi}), (\alpha_2, 0.3e^{i0.5\pi}), (\alpha_3, 1e^{i1.5\pi}), (\alpha_4, 0.9e^{i1.8\pi})\right\}.$$
$$\xi(x_3) = \left\{(\alpha_1, 0.8e^{i1.2\pi}), (\alpha_2, 1e^{i2\pi}), (\alpha_3, 0.1e^{i0.7\pi}), (\alpha_4, 0.5e^{i0.3\pi})\right\}.$$
$$\xi(x_4) = \left\{(\alpha_1, 0.3e^{i0.3\pi}), (\alpha_2, 0.2e^{i0.8\pi}), (\alpha_3, 0.7e^{i1.6\pi}), (\alpha_4, 1e^{i2\pi})\right\}.$$

The cardinal set ξ_C is

$$\xi_C = \left\{(\alpha_1, 0.6e^{i0.5\pi}), (\alpha_2, 0.625e^{i1.1\pi}), (\alpha_3, 0.675e^{i1.2\pi}), (\alpha_4, 0.65e^{i1.05\pi}), \right\}$$

Now, the complex fuzzy soft distance measure of $\xi(x)$ and ξ_C is given in Table 5.

The rank of the scheme is $x_3 > x_1 = x_4 > x_2$.

If the weighted vector w is assigned to each parameter $\alpha_i; i = 1, 2, 3, 4$, that is, $w = (0.2, 0.3, 0.2, 0.3)$. Then, the distance measure is

Therefore, the rank is of the scheme is $x_3 > x_2 > x_1 = x_4$.

Table 7 Evaluated values of parameters in the environment of IV-FSSs

	x_1	x_2	x_3	x_4
α_1	[0.2, 0.9]	[0.4, 0.7]	[0.1, 0.8]	[0.3, 0.6]
α_2	[0.1, 1]	[0.3, 0.8]	[0.4, 1]	[0.2, 0.9]
α_3	[0.7, 0.9]	[0.5, 1]	[0.1, 0.8]	[0.3, 0.7]
α_4	[0.2, 0.6]	[0.1, 0.9]	[0.5, 0.8]	[0.4, 1]

Table 8 Evaluated values of parameters in the environment of FSSs

	x_1	x_2	x_3	x_4
α_1	0.2	0.4	0.1	0.3
α_2	1	0.8	0.4	0.2
α_3	0.7	0.5	0.8	0.7
α_4	0.6	0.9	0.8	1

Similarly, we consider the decision-making problem in the environment of interval-valued fuzzy soft sets.

The interval-valued fuzzy soft information about the index's evaluation is given in Table 7.

The decision-making problem based on interval-valued fuzzy soft information can be solved using the distance measure proposed in case 2.

We consider the decision-making problem in the environment of fuzzy soft sets.

The fuzzy soft information about the index's evaluation is given in Table 8.

The decision-making problem based on fuzzy soft information can be solved using the distance measure proposed in case 3.

9 Key features and benefits of proposed distance measure

The key features and benefits of the proposed distance measure are given below:

i) The interval-valued complex fuzzy soft distance measure can solve the problems based on complex fuzzy soft sets, interval-valued fuzzy soft sets, and fuzzy soft sets. On the other hand, complex fuzzy soft distance measure, interval-valued fuzzy soft distance measure, and fuzzy soft distance measure cannot solve the decision-making problems in the form of CPFSs.

ii) The amplitude term and phase term are any interval numbers; it shows more probability of occurrence in the interval $D[0, 1]$. Therefore, the IV-CFSSs are more faithful than a complex fuzzy set and complex fuzzy soft set.

iii) Complex numbers in interval-valued complex fuzzy soft distance measures enable the modeling of information with both real and imaginary components. This is particularly valuable in applications where the phase of information is significant.

iv) Interval-valued complex fuzzy soft distance measures can assess the distance between sets in a multidimensional space. This is beneficial in decision-making problems involving multiple criteria or attributes, where the relationships between elements may have both real and imaginary components.

v) Interval-valued complex fuzzy soft distance measures seamlessly integrate with interval-valued fuzzy sets, allowing for a unified approach to handling uncertainty. This is particularly advantageous when dealing with applications that involve both interval-valued and complex fuzzy information.

10 Conclusion

In this paper, we explored the further development of the theory of IV-CFSSs (IV-CFSSs). Some set-theoretic operations and laws of the IV-CFSSs were proposed. We defined the distance measure in the environment of IV-CFSSs. The concept of δ-equality of IV-CFSSs was developed based on interval-valued complex fuzzy soft distance measures. We discussed some particular examples and basic results of these operations and laws. Moreover, we discussed a decision-making problem based on IV-CFSSs. We developed a new decision-making method using the interval-valued complex fuzzy distance measures under the environments of IV-CFSSs. We discussed the real-life case based on the proposed decision-making method. A real-life example demonstrated that the decision-making method developed in the paper could be utilized to deal with problems of uncertainty. Further, the comparative study of IV-CFSSs with complex fuzzy soft sets, interval-valued fuzzy soft sets, and fuzzy soft sets was established.

In the future, we will extend the elaborated work for interval-valued complex intuitionistic fuzzy soft sets, interval-valued complex Pythagorean fuzzy soft sets, interval-valued complex neutrosophic fuzzy soft sets, complex q-rung orthopair fuzzy sets, complex spherical fuzzy sets, and complex T-spherical fuzzy sets, etc., to improve the quality of the research works.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

Ahsan M, Saeed M, Mehmood A, Saeed MH, Asad J (2021) The study of HIV diagnosis using complex fuzzy hypersoft mapping and proposing appropriate treatment. IEEE Access 9:104405–104417

Akram M, Feng F, Borumand Saeid A, Leoreanu-Fotea V (2018) A new multiple criteria decision-making method based on bipolar fuzzy soft graphs. Iran J Fuzzy Syst 15(4):73–92

Akram M, Shabir M, Adeel A, Al-Kenani AN (2021a) A multiattribute decision-making framework: VIKOR method with complex spherical fuzzy-soft sets. Math Problems Eng 2021:1–25

Akram M, Shabir M, Al-Kenani AN, Alcantud JCR (2021b) Hybrid decision-making frameworks under complex spherical fuzzy-soft sets. J Math 2021:1–46

Akram M, Shumaiza, Rodríguez Alcantud JC (2023) Extended VIKOR method based with complex bipolar fuzzy sets. In: Multi-criteria decision making methods with bipolar fuzzy sets. Springer Nature Singapore, Singapore, pp 93–122

Ali Z (2022) Decision-making techniques based on complex intuitionistic fuzzy power interaction aggregation operators and their applications. J Innov Res Math Comput Sci 1(1):107–125

Ali Z, Mahmood T, Aslam M, Chinram R (2021) Another view of complex intuitionistic fuzzy soft sets based on prioritized aggregation operators and their applications to multiattribute decision making. Mathematics 9(16):1922

Alkouri AUM, Massa'deh MO, Ali M (2020) On bipolar complex fuzzy sets and its application. J Intell Fuzzy Syst 39(1):383–397

Atanassov K (2016) Intuitionistic fuzzy sets. Int J Bioautom 20:1

Atef M, Ali MI, Al-shami TM (2021) Fuzzy soft covering-based multi-granulation fuzzy rough sets and their applications. Comput Appl Math 40(4):115

Beg I, Rashid T, Jamil RN (2018) Human attitude analysis based on fuzzy soft differential equations with Bonferroni mean. Comput Appl Math 37:2632–2647

Begam SS, Selvachandran G, Ngan TT, Sharma R (2020) Similarity measure of lattice ordered multi-fuzzy soft sets based on set theoretic approach and its application in decision making. Mathematics 8(8):1255

Bhardwaj N, Sharma P (2021) An advanced uncertainty measure using fuzzy soft sets: application to decision-making problems. Big Data Min Anal 4(2):94–103

Dai S, Bi L, Hu B (2019) Distance measures between the interval-valued complex fuzzy sets. Mathematics 7(6):549

Fan JP, Cheng R, Wu MQ (2019) Extended EDAS methods for multi-criteria group decision-making based on IV-CFSWAA and IV-CFSWGA operators with interval-valued complex fuzzy soft information. IEEE Access 7:105546–105561

Ghosh SK, Ghosh A (2021) A novel intuitionistic fuzzy soft set based colonogram enhancement for polyps localization. Int J Imaging Syst Technol 31:1486–1502

Ghosh SK, Mitra A, Ghosh A (2021) A novel intuitionistic fuzzy soft set entrenched mammogram segmentation under multigranulation approximation for breast cancer detection in early stages. Expert Syst Appl 169:114329

Greenfield S, Chiclana F, Dick S (2016) Interval-valued complex fuzzy logic. In: 2016 IEEE international conference on fuzzy systems (FUZZ-IEEE). IEEE, pp 2014–2019

Gwak J, Garg H, Jan N (2023) Hybrid integrated decision-making algorithm for clustering analysis based on a bipolar complex fuzzy and soft sets. Alex Eng J 67:473–487

Hayat K, Tariq Z, Lughofer E, Aslam MF (2021) New aggregation operators on group-based generalized intuitionistic fuzzy soft sets. Soft Comput 25(21):13353–13364

Hu B, Bi L, Dai S, Li S (2018) Distances of complex fuzzy sets and continuity of complex fuzzy operations. J Intell Fuzzy Syst 35(2):2247–2255

Hussain A, Ali S, Ullah K (2022) A novel approach of picture fuzzy sets with unknown degree of weights based on Schweizer–Sklar aggregation operators. J Innov Res Math Comput Sci 1(2):18–39

Khalil AM, Li SG, Lin Y, Li HX, Ma SG (2020) A new expert system in prediction of lung cancer disease based on fuzzy soft sets. Soft Comput 24(18):14179–14207

Li C, Wu T, Chan FT (2012) Self-learning complex neuro-fuzzy system with complex fuzzy sets and its application to adaptive image noise canceling. Neurocomputing 94:121–139

Liu P, Ali Z, Mahmood T (2020) The distance measures and cross-entropy based on complex fuzzy sets and their application in decision making. J Intell Fuzzy Syst 39(3):3351–3374

Ma X, Zhan J, Khan M, Zeeshan M, Anis S, Awan AS (2019) Complex fuzzy sets with applications in signals. Comput Appl Math 38:1–34

Ma X, Fei Q, Qin H, Li H, Chen W (2021) A new efficient decision making algorithm based on interval-valued fuzzy soft set. Appl Intell 51(6):3226–3240

Mahmood T, Ur Rehman U, Ali Z (2021) A novel complex fuzzy N-soft sets and their decision-making algorithm. Complex Intell Syst 7:2255–2280

Maji PK (2009) More on intuitionistic fuzzy soft sets. In: International workshop on rough sets, fuzzy sets, data mining, and granular-soft computing. Springer, Berlin, pp 231–240

Maji PK, Roy AR, Biswas R (2001) Fuzzy Soft Sets. J Fuzzy Math 9: 589–602

Memiş S, Enginoğlu S, Erkan U (2021a) Numerical data classification via distance-based similarity measures of fuzzy parameterized fuzzy soft matrices. IEEE Access 9:88583–88601

Memiş S, Arslan B, Aydin T, Enginoğlu S, Camci Ç (2021b) A classification method based on Hamming pseudo-similarity of intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices. J New Results Sci 10(2):59–76

Močkoř J (2020) Powerset theory of fuzzy soft sets. Int J Fuzzy Log Intell Syst 20(4):298–315

Močkoř J, Hurtík P (2021) Approximations of fuzzy soft sets by fuzzy soft relations with image processing application. Soft Comput 25(10):6915–6925

Mohanty RK, Tripathy BK (2021) Recommending turmeric variety for higher production using interval-valued fuzzy soft set model and PSO. Int J Swarm Intell Res (IJSIR) 12(2):94–110

Molodtsov D (1999) Soft set theory—first results. Comput Math Appl 37(4–5):19–31

Patrascu V (2015) Similarity, cardinality and entropy for bipolar fuzzy set in the framework of penta-valued representation. arXiv preprint arXiv:1506.02060

Pawlak Z (1982) Rough sets. Int J Comput Inf Sci 11(5):341–356

Peng X, Yang Y (2015) Some results for Pythagorean fuzzy sets. Int J Intell Syst 30(11):1133–1160

Peng X, Krishankumar R, Ravichandran KS (2021) A novel interval-valued fuzzy soft decision-making method based on CoCoSo and CRITIC for intelligent healthcare management evaluation. Soft Comput 25(6):4213–4241

Petchimuthu S, Garg H, Kamacı H, Atagün AO (2020) The mean operators and generalized products of fuzzy soft matrices and their applications in MCGDM. Comput Appl Math 39:1–32

Qin H, Wang Y, Ma X, Wang J (2021) A novel approach to decision making based on interval-valued fuzzy soft set. Symmetry 13(12):2274

Qiyas M, Naeem M, Khan N, Khan S, Khan F (2024) Confidence levels bipolar complex fuzzy aggregation operators and their application in decision making problem. IEEE Access 12:6204–6214

Rahman AU, Saeed M, Khalid A, Ahmad MR, Ayaz S (2021) Decision-making application based on aggregations of complex fuzzy hypersoft set and development of interval-valued complex fuzzy hypersoft set. Neutrosophic Sets Syst 46(1):22

Ramot D, Milo R, Friedman M, Kandel A (2002) Complex fuzzy sets. IEEE Trans Fuzzy Syst 10(2):171–186

Sakr HH, Muse AH, Mohamed MS, Ateya SF (2023) Applications on bipolar vague soft sets. J Math 2023:1–25

Sarwar M, Akram M (2017) New applications of m-polar fuzzy matroids. Symmetry 9(12):319

Selvachandran G, Salleh AR (2017) Interval-valued complex fuzzy soft sets. In: AIP conference proceedings, vol 1830, no 1. AIP Publishing LLC, p 070009

Selvachandran G, Singh PK (2018) Interval-valued complex fuzzy soft set and its application. Int J Uncertain Quantif 8(2):101–117

Singh A (2021) Modified non-linear programming methodology for multi-attribute decision-making problem with interval-valued intuitionistic fuzzy soft sets information. In: 2021 2nd Global conference for advancement in technology (GCAT). IEEE, pp 1–9

Thirunavukarasu P, Suresh R, Ashokkumar V (2017) Theory of complex fuzzy soft set and its applications. Int J Innov Res Sci Technol 3(10):13–18

Yang X, Lin TY, Yang J, Li Y, Yu D (2009) Combination of interval-valued fuzzy set and soft set. Comput Math Appl 58(3):521–527

Zadeh LA (1965) Fuzzy sets. Inf Control 8(3):338–353

Zararsız Z, Riaz M (2022) Bipolar fuzzy metric spaces with application. Comput Math Appl 41(1):49

Zeeshan M, Khan M (2022) Complex fuzzy sets with applications in decision-making. Iran J Fuzzy Syst 19(4):147–163

Zeeshan M, Khan M, Iqbal S (2022) Distance function of complex fuzzy soft sets with application in signals. Comput Appl Math 41(3):96

Zhang WR (1994) Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. In: NAFIPS/IFIS/NASA'94. Proceedings of the first international joint conference of the North American fuzzy information processing society biannual conference. The Industrial Fuzzy Control and Intelligence. IEEE, pp 305–309

Zhang G, Dillon TS, Cai KY, Ma J, Lu J (2009) Operation properties and δ-equalities of complex fuzzy sets. Int J Approx Reason 50(8):1227–1249

Zulqarnain RM, Xin XL, Saqlain M, Khan WA (2021) TOPSIS method based on the correlation coefficient of interval-valued intuitionistic fuzzy soft sets and aggregation operators with their application in decision-making. J Math 2021:1–16