Computational and Applied Mathematics (2024) 43:8
https://doi.org/10.1007/s40314-023-02518-0

Asymptotical stability of the exact solutions and the numerical solutions for impulsive neutral differential equations

Gui-Lai Zhang[1] · **Yang Sun**[1] · **Zhi-Wei Wang**[1]

Received: 2 July 2023 / Revised: 25 September 2023 / Accepted: 27 October 2023 /
Published online: 7 December 2023

Abstract

In this paper, we not only study asymptotical stability of a class of linear impulsive neutral delay differential equations(INDDEs), but also study stability and asymptotical stability of nonlinear INDDEs. Asymptotical stability of zero solution of linear INDDEs is studied by the properties of simple autonomous linear neutral delay differential equations(NDDEs) without impulsive perturbations. Base on this idea, numerical methods of INDDEs are constructed. The constructed numerical methods preserve asymptotical stability of linear INDDEs if corresponding methods are A-stable. Moreover, some stability and asymptotical stability criteria are established for nonlinear INDDEs, respectively. The constructed numerical methods which can preserve stability and asymptotical stability of the exact solutions under these criteria are obtained. Some numerical examples are given to confirm the theoretical results.

Keywords Impulsive neutral delay differential equations · Runge–Kutta method · Stability · Asymptotical stability

Mathematics Subject Classification 65L03 · 65L05 · 65L07

1 Introduction

Impulsive differential equations arise widely in the study of medicine, biology, economics, engineering, and so forth. In recent years, INDDEs are attracting more and more attention. In papers Anguraj and Karthikeyan (2009) and Cuevasa et al. (2009), existence uniqueness and continuous dependence of INDDEs are investigated. In papers Li and Rogovchenko (2015, 2016) and Li and Deng (2017), oscillation of first-order, second-order, even-order of

Communicated by Hui Liang.

This work is supported by the NSF of PR China (No. 11701074).

✉ Gui-Lai Zhang
guilaizhang@126.com

1 College of sciences, Northeastern University, Shenyang 110819, China

INDDEs are studied, respectively. In papers Li and Deng (2017) and Li and Rogovchenko (2020), stability of first-order, third-order INDDEs are studied, respectively. Mean square exponential input-to state stability of stochastic Markovian reaction diffusion systems with impulsive perturbations has also been studied in paper Xue et al. (2023).

On the other hand, stability and asymptotical stability of numerical methods for NDDEs without impulsive perturbations have been widely studied (see Liu 1999; Enright and Hayashi 1998; Engelborghs et al. 2001; Wang and Li 2008 etc). Wang and Li (2008) studied stability and asymptotic stability of θ-methods for nonlinear NDDEs with constant delay and proportional delay. Enright and Hayashi (1998) estabilished sufficient conditions for order of convergence results about continuous Runge–Kutta methods for NDDEs with state dependent delays. But to the best of our knowledge, up to now, there are few articles referring to stability of numerical methods for INDDEs.

The aim of this paper is to provide asymptotical stability criteria for the exact solutions and the numerical solutions of INDDEs. Applying asymptotical stability of NDDEs without impulsive perturbations, asymptotical stability criteria are obtained for the exact solutions of linear INDDEs and nonlinear INDDEs, respectively. Numerical schemes for INDDEs are constructed based on the relationship between INDDEs and NDDEs. Moreover, we proved that some numerical methods can preserve asymptotical stability of linear and nonlinear INDDEs, respectively. The rest of this paper is organized as follows. In Sect. 2, asymptotical stability of zero solution of linear INDDEs is studied by the properties of simple linear NDDEs with constant coefficients. Base on this idea, numerical methods of INDDEs are constructed. The constructed numerical methods furnished A-stable Runge–Kutta methods can preserve asymptotical stability of linear INDDEs. In Sect. 3, some stability and asymptotical stability criteria are established for nonlinear INDDEs by the properties of nonlinear NDDEs. Under these stability and asymptotical stability criteria, the constructed numerical methods furnished by implicit Euler method or 2-stage Lobatto IIIC method are stability and asymptotical stability. In Sect. 4, we provide some numerical examples to confirm our theoretical results.

2 Linear INDDEs

In this section, we consider the following scalar linear INDDE:

$$\begin{cases} x'(t) = ax(t) + bx(t-\tau) + cx'(t-\tau), & t \geq 0, t \neq k\tau, k \in \mathbb{N}, \\ x(t) = \lambda x(t^-), & t = k\tau, k \in \mathbb{N}, \\ x(t) = \phi(t), & t \in [-\tau, 0), \end{cases} \tag{2.1}$$

where τ is a positive constant, a, b, c and λ are complex constants, the initial function $\phi(t)$ is continuous differentiable on $[-\tau, 0)$, $x'(t)$ denotes the right-hand derivative of $x(t)$ and $\mathbb{N} = \{0, 1, 2, \cdots\}$.

In the following of this paper, always assume $\lambda \neq 1$ and $\lambda \neq 0$. When $\lambda = 1$, INDDE (2.1) is changed into NDDE without impulsive perturbations. When $\lambda = 0$, the solution $x(t)$ of INDDE (2.1) satisfies $x(k\tau) = 0$ for all $k \in \mathbb{N}$.

2.1 Relations between INDDEs and NDDEs

Theorem 2.1 *Assume that $x(t)$ is the solution of (2.1) and $y(t)=\lambda^{\{t/\tau\}}x(t)$, $t\in[-\tau,\infty)$, then $y(t)$ is the solution of the following equation without impulsive perturbations*

$$\begin{cases} y'(t)=\alpha y(t)+\beta y(t-\tau)+cy'(t-\tau), & t\geq 0,\\ y(t)=\Phi(t), & t\in[-\tau,0], \end{cases} \tag{2.2}$$

where $\alpha=a+\frac{\ln\lambda}{\tau}$, $\beta=b-\frac{c\ln\lambda}{\tau}$, $\{\frac{t}{\tau}\}=\frac{t}{\tau}-\lfloor\frac{t}{\tau}\rfloor$, $\lfloor\cdot\rfloor$ denotes the greatest integer function and

$$\Phi(t)=\begin{cases} \lambda^{(\frac{t}{\tau}+1)}\phi(t), & t\in[-\tau,0),\\ \lambda\phi(0^-), & t=0. \end{cases}$$

On the other hand, if $y(t)$ is the solution of (2.2) and $x(t)=\lambda^{-\{t/\tau\}}y(t)$, $t\in[-\tau,\infty)$, then $x(t)$ is the solution of (2.1).

Proof We will prove that $y(t)$ is continuous if $x(t)$ is the solution of (2.1). Obviously,

$$y(t)=\lambda^{\{t/\tau\}}x(t)=\lambda^{t/\tau-k}x(t) \tag{2.3}$$

is continuous on $[k\tau,(k+1)\tau)$, $k=-1,0,1,2,\ldots$. Therefore,

$$\begin{aligned} y(0^-)&=\lim_{t\to 0^-}\lambda^{t/\tau+1}x(t)=\lambda x(0^-)=\lambda\phi(0^-),\\ y(0)&=x(0)=\lambda x(0^-)=\lambda\phi(0^-). \end{aligned}$$

Hence $y(0)=y(0^-)$, which implies that $y(t)$ is continuous at $t=0$. It follows from

$$\begin{aligned} y(k\tau)&=\lambda^{\{\frac{k\tau}{\tau}\}}x(k\tau)=x(k\tau)=\lambda x(k\tau^-),\\ y(k\tau^-)&=\lim_{t\to k\tau^-}\lambda^{\{\frac{t}{\tau}\}}x(t)=\lambda x(k\tau^-), \end{aligned}$$

that $y(t)$ is continuous at $t=k\tau$, $k=1,2,\ldots$. Consequently, $y(t)$ is continuous on $[-\tau,\infty)$.

Next, we will prove that $y(t)$ is the solution of (2.2) if $x(t)$ is the solution of (2.1). For $t\in[k\tau,(k+1)\tau)$, $k=0,1,2,\ldots$, we can obtain that

$$\begin{aligned} y'(t)&=\lambda^{t/\tau-k}x(t)\frac{\ln\lambda}{\tau}+\lambda^{t/\tau-k}x'(t)\\ &=\frac{y(t)\ln\lambda}{\tau}+\lambda^{t/\tau-k}x'(t), \end{aligned}$$

and

$$\begin{aligned} y'(t-\tau)&=\lambda^{(t-\tau)/\tau-(k-1)}x(t-\tau)\frac{\ln\lambda}{\tau}+\lambda^{(t-\tau)/\tau-(k-1)}x'(t-\tau)\\ &=\frac{y(t-\tau)\ln\lambda}{\tau}+\lambda^{t/\tau-k}x'(t-\tau), \end{aligned}$$

which implies that

$$
\begin{aligned}
& y'(t) \\
&= \frac{y(t)\ln\lambda}{\tau} + \lambda^{t/\tau-k}\left[ax(t)+bx(t-\tau)+cx'(t-\tau)\right] \\
&= \frac{y(t)\ln\lambda}{\tau} + ay(t)+by(t-\tau)+c\lambda^{t/\tau-k}x'(t-\tau) \\
&= \left(a+\frac{\ln\lambda}{\tau}\right)y(t)+by(t-\tau)+c\left[y'(t-\tau)-\frac{y(t-\tau)\ln\lambda}{\tau}\right] \\
&= \left(a+\frac{\ln\lambda}{\tau}\right)y(t)+\left(b-\frac{c\ln\lambda}{\tau}\right)y(t-\tau)+cy'(t-\tau) \\
&= \alpha y(t)+\beta y(t-\tau)+cy'(t-\tau).
\end{aligned}
$$

Finally, we will prove that $x(t)$ is the solution of (2.1) if $y(t)$ is the solution of (2.2). For $t \in [k\tau, (k+1)\tau)$, $k = 0, 1, 2, \cdots$, we can obtain that

$$
\begin{aligned}
x'(t) &= -\lambda^{-(t/\tau-k)}y(t)\frac{\ln\lambda}{\tau}+\lambda^{-(t/\tau-k)}y'(t) \\
&= -\frac{x(t)\ln\lambda}{\tau}+\lambda^{-(t/\tau-k)}y'(t),
\end{aligned}
$$

and

$$
\begin{aligned}
x'(t-\tau) &= -\lambda^{-[(t-\tau)/\tau-(k-1)]}y(t-\tau)\frac{\ln\lambda}{\tau}+\lambda^{-[(t-\tau)/\tau-(k-1)]}y'(t-\tau) \\
&= -\frac{x(t-\tau)\ln\lambda}{\tau}+\lambda^{-(t/\tau-k)}y'(t-\tau),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& x'(t) \\
&= -\frac{x(t)\ln\lambda}{\tau}+\lambda^{-(t/\tau-k)}\left[\left(a+\frac{\ln\lambda}{\tau}\right)y(t)+\left(b-\frac{c\ln\lambda}{\tau}\right)y(t-\tau)+cy'(t-\tau)\right] \\
&= -\frac{x(t)\ln\lambda}{\tau}+\left(a+\frac{\ln\lambda}{\tau}\right)x(t)+\left(b-\frac{c\ln\lambda}{\tau}\right)x(t-\tau)+c\lambda^{-(t/\tau-k)}y'(t-\tau) \\
&= ax(t)+\left(b-\frac{c\ln\lambda}{\tau}\right)x(t-\tau)+c\left[x'(t-\tau)+\frac{x(t-\tau)\ln\lambda}{\tau}\right] \\
&= ax(t)+bx(t-\tau)+cx'(t-\tau).
\end{aligned}
$$

Obviously, it follows from

$$
\begin{aligned}
x(k\tau) &= \lambda^{-\{\frac{k\tau}{\tau}\}}y(k\tau) = y(k\tau), \\
x(k\tau^-) &= \lim_{t\to k\tau^-}\lambda^{\{\frac{x}{\tau}\}}x(t) = \lambda^{-1}y(k\tau^-),
\end{aligned}
$$

that $x(k\tau) = \lambda x(k\tau^-)$, $k = 0, 1, 2, \ldots$. Hence $x(t)$ is the solution of (2.1). □

2.2 Asmpotical stability of linear INDDEs

By (Bellen et al. 1988, Theorem 2.1) and Theorem 2.1 of present paper, we can obtain the following results.

Theorem 2.2 *Assume that*

$$|\alpha\bar{c}-\bar{\beta}|+|\alpha c+\beta|<-2\Re(\alpha).$$

Then the solution $y(t)$ *of NDDE* (2.2) *tends to zero as* $t\to\infty$. *Hence if*

$$\left|\left(a+\frac{\ln\lambda}{\tau}\right)\bar{c}-\overline{\left(b-\frac{c\ln\lambda}{\tau}\right)}\right|+\left|\left(a+\frac{\ln\lambda}{\tau}\right)c+\left(b-\frac{c\ln\lambda}{\tau}\right)\right|<-2\Re\left(a+\frac{\ln\lambda}{\tau}\right),$$

then the solution $x(t)$ *of INDDE* (2.1) *tends to zero as* $t\to\infty$.

Maybe some readers are interested in linear INDDE as the following form.

$$\begin{cases} x'(t)=ax(t)+bx(t-\tau)+cx'(t-\tau), & t\geq 0, t\neq k\tau, k\in\mathbb{Z}^+,\\ x(t)=\lambda x(t^-), & t=k\tau, k\in\mathbb{Z}^+,\\ x(t)=\phi(t), & t\in[-\tau,0], \end{cases} \tag{2.4}$$

where $\mathbb{Z}^+=\{1,2,\ldots\}$. In fact, on interval $[-\tau,\tau)$, the Eq. (2.4) is a delay differential equations without impulsive perturbations. If the solution $x(t)$, $t\in[0,\tau)$ is seen as initial function, the Eq. (2.4) can be seen the same as (2.1) for $t\geq\tau$. So the asymptotically stable results of Theorem 2.2 can be extended as follows.

Corollary 2.3 *If*

$$\left|\left(a+\frac{\ln\lambda}{\tau}\right)\bar{c}-\overline{\left(b-\frac{c\ln\lambda}{\tau}\right)}\right|+\left|\left(a+\frac{\ln\lambda}{\tau}\right)c+\left(b-\frac{c\ln\lambda}{\tau}\right)\right|<-2\Re\left(a+\frac{\ln\lambda}{\tau}\right),$$

then the solution $x(t)$ *of INDDE* (2.4) *tends to zero as* $t\to\infty$.

In fact, all the stable and asymptotically stable results of INDDE in this paper can be extended similarly, we do not introduce in detail for concise.

2.3 Stability analysis of numerical methods for linear INDDEs

Based on the relations between INDDE (2.1) and NDDE (2.2), the numerical method for linear INDDE (2.1) can be constructed as follows

$$\begin{cases} y_{n+1}=y_n+\sum_{i=1}^{s}w_i z_{n+1}^{(i)}, \quad n\in\mathbb{N}\\ z_{n+1}^{(i)}=\alpha\left(y_n+h\sum_{j=1}^{s}a_{ij}z_{n+1}^{(j)}\right)+\beta\left(y_{n-m}+h\sum_{j=1}^{s}b_{ij}z_{n-m+1}^{(j)}\right)\\ \qquad\qquad +c\sum_{j=1}^{s}c_{ij}z_{n-m+1}^{(j)}\\ x_n=\lambda^{-\{\frac{t_n}{\tau}\}}y_n, \end{cases} \tag{2.5}$$

where $t_n=nh$, $n\in\mathbb{N}$, $h=\frac{\tau}{m}$ and m is a positive integer. Here, the vector $w=[w_1,w_2,\ldots,w_s]^T$ and the matrix $A=[a_{ij}]_{i,j=1}^{s}$ define a Runge–Kutta method for ODEs. For $\forall n\in\mathbb{N}$, y_n is an approximation to $y(t_n)$ of (2.2), x_n is an approximation to $x(t_n)$ of (2.1), $y_n+h\sum_{j=1}^{s}a_{ij}z_{n+1}^{(j)}$ is an approximation to $y(t_n+c_ih)$, $y_{n-m}+h\sum_{j=1}^{s}b_{ij}z_{n-m+1}^{(j)}$ is an approximation to $y(t_{n-m}+c_ih)$, $\sum_{j=1}^{s}c_{ij}z_{n-m+1}^{(j)}$ is an approximation to $y'(t_{n-m}+c_ih)$, where $c_i=\sum_{j=1}^{s}a_{ij}$, $i=1,2,\ldots,s$.

Usually, $b_{ij}=b_j(c_i)$ and $c_{ij}=b'_j(c_i)$, where $b_j(\theta)$, $j=1,2,\ldots,s$, are polynomials which define the natural continuous extension of the Runge–Kutta method, i.e. polynomials such that the approximate solution y_h defined on the whole interval of integration is given by

$$y_h(t_n + \theta h) = y_h(t_n) + h \sum_{j=1}^{s} b_j(\theta) z_{n+1}^{(j)}, \ n \in \mathbb{N}, \theta \in [0, 1].$$

Put $B = [b_{ij}]_{i,j=1}^{s}$ and $C = [c_{ij}]_{i,j=1}^{s}$. Let $\tilde{\alpha} = h\alpha$, $\tilde{\beta} = h\beta$ and denote by $\{y_n(m; \alpha, \beta, c)\}_{n=0}^{\infty}$ can be described by quadrupple $\{w, A, B, C\}$. So the numerical method (2.5) for linear INDDE (2.1) can be rewritten as follows

$$\begin{cases} y_{n+1} = y_n + hw^T z_{n+1}, \quad n \in \mathbb{N} \\ z_{n+1} = \alpha(y_n e + hAz_{n+1}) + \beta(y_{n-m} e + hBz_{n-m+1}) + cCz_{n-m+1} \\ x_n = \lambda^{-\{\frac{t_n}{\tau}\}} y_n, \end{cases} \tag{2.6}$$

where z_n stands for $[z_n^{(1)}, z_n^{(2)}, \ldots, z_n^{(s)}]^T$ and $e = [1, 1, \ldots, 1]^T$.

A-stable Runge–Kutta methods for ODEs can be extended to asympotically stable numerical method for INDDEs. By (Bellen et al. 1988, Theorem 3.4), we can obtain the following theorem.

Theorem 2.4 *Assume that the Runge–Kutta method* $\{w, A\}$ *for ODE is A-stable. Then the corresponding method* (2.6) *for INDDE* (2.1) *with* $B = A$ *and* $C = I$*, which satisfies*

$$\left| \left(a + \frac{\ln \lambda}{\tau}\right)\bar{c} - \overline{\left(b - \frac{c \ln \lambda}{\tau}\right)} \right| + \left| \left(a + \frac{\ln \lambda}{\tau}\right)c + \left(b - \frac{c \ln \lambda}{\tau}\right) \right| < -2\Re\left(a + \frac{\ln \lambda}{\tau}\right).$$

Then the numerical solution x_n *of* (2.6) *tends to zero as* $n \to \infty$.

3 Nonlinear INDDEs

In this section, we will consider the following nonlinear INDDE:

$$\begin{cases} x'(t) = f(t, x(t), x(t-\tau), x'(t-\tau)), & t \ge 0, \ t \ne k\tau, \ k \in \mathbb{N} \\ x(t) = \lambda x(t^-), & t = k\tau, \ k \in \mathbb{N} \\ x(t) = \phi(t), & t \in [-\tau, 0), \end{cases} \tag{3.1}$$

and the same equation with another initial function:

$$\begin{cases} \tilde{x}'(t) = f(t, \tilde{x}(t), \tilde{x}(t-\tau), \tilde{x}'(t-\tau)), & t \ge 0, \ t \ne k\tau, \ k \in \mathbb{N} \\ \tilde{x}(t) = \lambda \tilde{x}(t^-), & t = k\tau, \ k \in \mathbb{N} \\ \tilde{x}(t) = \tilde{\phi}(t), & t \in [-\tau, 0), \end{cases} \tag{3.2}$$

where $\lambda \ne 1$, $\lambda \ne 0$, $\tau > 0$, ϕ and $\tilde{\phi}$ are continuous functions on $[-\tau, 0)$, $\lim_{t \to 0^-} \phi(t)$ and $\lim_{t \to 0^-} \tilde{\phi}(t)$ exist, $x'(t)$ denotes the right-hand derivative of $x(t)$. Let $\langle \cdot, \cdot \rangle$ be a given inner product on $\mathbb{C}^d$ and $\| \cdot \|$ the corresponding norm. Assume that the function $f : [0, \infty) \times \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$ is continuous in t and satisfies the following conditions: for arbitrary $x, u, v, x_1, x_2, u_1, u_2, v_1, v_2 \in \mathbb{C}^d$ and $\forall t \in [0, +\infty)$,

$$\Re(\langle x_1 - x_2, f(t, x_1, u, v) - f(t, x_2, u, v) \rangle) \le R(t) \|x_1 - x_2\|^2 \tag{3.3}$$

$$\|f(t,x,u_1,v)-f(t,x,u_2,v)\| \le \beta(t)\|u_1-u_2\| \tag{3.4}$$

$$\|f(t,x,u,v_1)-f(t,x,u,v_2)\| \le \gamma(t)\|v_1-v_2\| \tag{3.5}$$

$$\|U(t,x,u_1,v,w)-U(t,x,u_2,v,w)\| \le \sigma(t)\|u_1-u_2\| \tag{3.6}$$

where $R(t)$, $\beta(t)$, $\gamma(t)$ and $\sigma(t)$ are continuous real functions and

$$U(t,x,u,v,w) = f(t,x,u,f(t-\tau,u,v,w)).$$

3.1 Relations between INDDEs and NDDEs

Assume that the scalar function $\alpha : [-\tau,\infty) \to \mathbb{C}$ satisfies the following conditions:

(1) for any $t \in [0,\infty)$, $\alpha(t) = \alpha(t-\tau)$;
(2) $\alpha(t)$ is infinite smooth on $[0,\tau)$;
(3) $\alpha(0) = 1$ and $\alpha(0^-) = \lambda$;
(4) $\inf_{t\in[0,\tau)} |\alpha(t)| \ge m > 0$.

Theorem 3.1 *Assume that $x(t)$ is the solution of (2.1) and $y(t) = \alpha(t)x(t)$, $t \in [-\tau,+\infty)$. Then $y(t)$ is the solution of*

$$\begin{cases} y'(t) = F(t,y(t),y(t-\tau),y'(t-\tau)), & t \ge 0, \\ y(t) = \Phi(t), & t \in [-\tau,0], \end{cases} \tag{3.7}$$

where

$$F(t,y,u,v) = \frac{\alpha'(t)y}{\alpha(t)} + \alpha(t)f(t,\frac{y}{\alpha(t)},\frac{u}{\alpha(t)},\frac{v}{\alpha(t)} - \frac{\alpha'(t)u}{\alpha^2(t)})$$

and

$$\Phi(t) = \begin{cases} \alpha(t)\phi(t), & t \in [-\tau,0), \\ \alpha(0^-)\phi(0^-), & t = 0. \end{cases}$$

On the other hand, assume that $y(t)$ is the solution of (3.7) and $x(t) = \frac{y(t)}{\alpha(t)}$, $t \in [-\tau,+\infty)$. Then $x(t)$ is the solution of (2.1).

Proof (i) Because $\alpha(t)$ and $x(t)$ are continuous on $[k\tau,(k+1)\tau)$, $y(t)$ is continuous on $[k\tau,(k+1)\tau)$, where $k = 0,1,\ldots$. Obviously, we have

$$\begin{aligned} y(k\tau) &= y(k\tau^+) = \alpha(k\tau^+)x(k\tau^+) \\ &= \alpha(k\tau)\lambda x(k\tau^-) = \alpha(0)\lambda x(k\tau^-) \\ &= \lambda x(k\tau^-) \end{aligned}$$

and

$$y(k\tau^-) = \alpha(k\tau^-)x(k\tau^-) = \lambda x(k\tau^-).$$

So $y(k\tau) = y(k\tau^+) = y(k\tau^-)$, $k = 0,1,2,\ldots$. Hence $y(t)$ is continuous on $[-\tau,\infty)$.

Obviously, for $t \in [k\tau,(k+1)\tau)$, $k = 0,1,\ldots$, we can obtain that

$$\begin{aligned} x'(t) &= (\frac{y(t)}{\alpha(t)})' = \frac{y'(t)}{\alpha(t)} - \frac{\alpha'(t)y(t)}{\alpha^2(t)} \\ \Rightarrow x'(t-\tau) &= \frac{y'(t-\tau)}{\alpha(t)} - \frac{\alpha'(t)y(t-\tau)}{\alpha^2(t)}, \end{aligned}$$

which implies

$$\begin{aligned}
&y'(t) = \alpha'(t)x(t) + \alpha(t)x'(t)\\
&= \alpha'(t)x(t) + \alpha(t)f(t, x(t), x(t-\tau), x'(t-\tau))\\
&= \frac{\alpha'(t)y(t)}{\alpha(t)} + \alpha(t)f\left(t, \frac{y(t)}{\alpha(t)}, \frac{y(t-\tau)}{\alpha(t-\tau)}, \frac{y'(t-\tau)}{\alpha(t-\tau)} - \frac{\alpha'(t-\tau)y(t-\tau)}{\alpha^2(t-\tau)}\right)\\
&= \frac{\alpha'(t)y(t)}{\alpha(t)} + \alpha(t)f\left(t, \frac{y(t)}{\alpha(t)}, \frac{y(t-\tau)}{\alpha(t)}, \frac{y'(t-\tau)}{\alpha(t)} - \frac{\alpha'(t)y(t-\tau)}{\alpha^2(t)}\right)\\
&= F(t, y(t), y(t-\tau), y'(t-\tau)).
\end{aligned}$$

(ii) Assume that $y(t)$ is the solution of (3.7). For $t \in [k\tau, (k+1)\tau)$, $k = 0, 1, \ldots$,

$$\begin{aligned}
&x'(t) = \left(\frac{y(t)}{\alpha(t)}\right)' = \frac{y'(t)}{\alpha(t)} - \frac{\alpha'(t)y(t)}{\alpha^2(t)}\\
&= \frac{1}{\alpha(t)}\left(\frac{\alpha'(t)y(t)}{\alpha(t)} + \alpha(t)f\left(t, \frac{y(t)}{\alpha(t)}, \frac{y(t-\tau)}{\alpha(t)}, \frac{y'(t-\tau)}{\alpha(t)} - \frac{\alpha'(t)y(t-\tau)}{\alpha^2(t)}\right)\right) - \frac{\alpha'(t)y(t)}{\alpha^2(t)}\\
&= f\left(t, \frac{y(t)}{\alpha(t)}, \frac{y(t-\tau)}{\alpha(t)}, \frac{y'(t-\tau)}{\alpha(t)} - \frac{\alpha'(t)y(t-\tau)}{\alpha^2(t)}\right)\\
&= f(t, x(t), x(t-\tau), x'(t-\tau)).
\end{aligned}$$

Obviously,

$$x(k\tau) = \frac{y(k\tau)}{\alpha(k\tau)} = \frac{y(k\tau)}{\alpha(0)} = y(k\tau)$$

and

$$x(k\tau^-) = \lim_{t\to k\tau^-} \frac{y(t)}{\alpha(t)} = \frac{y(k\tau)}{\alpha(k\tau^-)} = \frac{y(k\tau)}{\alpha(\tau^-)} = \frac{y(k\tau)}{\lambda}.$$

So $x(k\tau) = \lambda x(k\tau^-)$, $k = 0, 1, \cdots$. Obviously, we have $x(t) = \frac{y(t)}{\alpha(t)} = \phi(t)$, $t \in [-\tau, 0)$. Hence $x(t)$ is the solution of (3.1). □

3.2 Asymptotical stability of INDDEs

Theorem 3.2 *Assume that IDDEs (3.1) and (3.2) fulfill the inequalities (3.3)-(3.6), and $R(t) \le 0$ for $\forall t \ge 0$,*

$$\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right) + \frac{R(t)}{\omega^2} < 0, \quad \frac{\omega\sigma(t) - \omega\gamma(t)\left[\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right) + \frac{R(t-\tau)}{\omega^2}\right]}{-\left[\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right) + \frac{R(t)}{\omega^2}\right]} \le 1, \tag{3.8}$$

then

$$\|x(t) - \tilde{x}(t)\| \le \max\{\frac{\lambda\|\phi(0^-) - \tilde{\phi}(0^-)\|}{m}, \kappa\}, \tag{3.9}$$

where $\omega = \frac{\sup_{t\in[-\tau,0)} |\alpha(t)|}{\inf_{t\in[-\tau,0)} |\alpha(t)|}$, $\zeta = \sup_{t\in[-\tau,0)} |\frac{\alpha'(t)}{\alpha(t)}|$, $\delta\phi(s) = \|\phi(s-\tau) - \tilde{\phi}(s-\tau)\|$, $\delta\dot{\phi}(s) = \|\phi'(s-\tau) - \tilde{\phi}'(s-\tau)\|$ *and*

$$\kappa = \sup_{0\le s<\tau} \frac{\omega^2(\beta(s) + 2\zeta\gamma(s))\delta\phi(s) + \omega^2\gamma(s)\delta\dot{\phi}(s)}{-\left(\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right) + \frac{R(t)}{\omega^2}\right)}.$$

Moreover, assume that

$$\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\frac{R(t)}{\omega^2}\le R_0<0,\ \forall t\ge 0, \tag{3.10}$$

and

$$\frac{\omega\gamma(t)\left(\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\frac{R(t-\tau)}{\omega^2}\right)}{\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\frac{R(t)}{\omega^2}}=r(t)\le\bar{\xi}<1,\ \forall t\ge 0, \tag{3.11}$$

$$\frac{\omega\sigma(t)}{-\left(\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\frac{R(t)}{\omega^2}\right)}\le k(1-r(t)),\ k<1,\ \forall t\ge 0, \tag{3.12}$$

then we have

$$\lim_{t\to\infty}\|x(t)-\tilde{x}(t)\|=0.$$

Proof We will apply inequalities (3.3)-(3.6) to prove that the function $F : [0,\infty)\times\mathbb{C}^d\times\mathbb{C}^d\times\mathbb{C}^d\to\mathbb{C}^d$ satisfies the following conditions respectively: for arbitrary $y,u,v,y_1,y_2,u_1,u_2,v_1,v_2\in\mathbb{C}^d$ and $\forall t\in[0,+\infty)$,

$$\Re(\langle y_1-y_2,F(t,y_1,u,v)-F(t,y_2,u,v)\rangle)\le\left(\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\frac{R(t)}{\omega^2}\right)\|y_1-y_2\|^2 \tag{3.13}$$

$$\|F(t,y,u_1,v)-F(t,y,u_2,v)\|\le\omega\left(\beta(t)+\zeta\gamma(t)\right)\|u_1-u_2\| \tag{3.14}$$

$$\|F(t,y,u,v_1)-F(t,y,u,v_2)\|\le\omega\gamma(t)\|v_1-v_2\| \tag{3.15}$$

$$\|V(t,y,u_1,v,w)-V(t,y,u_2,v,w)\|\le\omega\sigma(t)\|u_1-u_2\| \tag{3.16}$$

where $V(t,y,u,v,w)=F(t,y,u,F(t-\tau,u,v,w))$.

First of all, we will prove inequality (3.13) as follows

$$\begin{aligned}
&\Re\langle y_1-y_2,F(t,y_1,u,v)-F(t,y_2,u,v)\rangle\\
&=\Re\langle y_1-y_2,\frac{\alpha'(t)y_1}{\alpha(t)}+\alpha(t)f\left(t,\frac{y_1}{\alpha(t)},\frac{u}{\alpha(t)},\frac{v}{\alpha(t)}-\frac{\alpha'(t)u}{\alpha^2(t)}\right)-\frac{\alpha'(t)y_2}{\alpha(t)}\\
&\quad-\alpha(t)f\left(t,\frac{y_2}{\alpha(t)},\frac{u}{\alpha(t)},\frac{v}{\alpha(t)}-\frac{\alpha'(t)u}{\alpha^2(t)}\right)\rangle\\
&=\Re\langle y_1-y_2,\frac{\alpha'(t)}{\alpha(t)}(y_1-y_2)\rangle+|\alpha(t)|^2\Re\langle\frac{y_1}{\alpha(t)}-\frac{y_2}{\alpha(t)},f\left(t,\frac{y_1}{\alpha(t)},\frac{u}{\alpha(t)},\frac{v}{\alpha(t)}-\frac{\alpha'(t)u}{\alpha^2(t)}\right)\\
&\quad-f\left(t,\frac{y_2}{\alpha(t)},\frac{u}{\alpha(t)},\frac{v}{\alpha(t)}-\frac{\alpha'(t)u}{\alpha^2(t)}\right)\rangle\\
&\le\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)\|y_1-y_2\|^2+|\alpha(t)|^2R(t)\|\frac{y_1}{\alpha(t)}-\frac{y_2}{\alpha(t)}\|^2
\end{aligned}$$

which implies, if $R(t)\le 0$ for $\forall t$,

$$\Re\langle y_1-y_2,F(t,y_1,u,v)-F(t,y_2,u,v)\rangle\le[\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\frac{R(t)}{\omega^2}]\|y_1-y_2\|^2, \tag{3.17}$$

and if $R(t)\le\hat{R}$ and $\hat{R}>0$ for $\forall t$,

$$\Re\langle y_1-y_2,F(t,y_1,u,v)-F(t,y_2,u,v)\rangle\le[\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\omega^2\hat{R}]\|y_1-y_2\|^2. \tag{3.18}$$

Second, the inequality (3.14) can be proved as follows

$$
\begin{aligned}
&\|F(t,y,u_1,v)-F(t,y,u_2,v)\|\\
&=\|\frac{\alpha'(t)y}{\alpha(t)}+\alpha(t)f\left(t,\frac{y}{\alpha(t)},\frac{u_1}{\alpha(t)},\frac{v}{\alpha(t)}-\frac{\alpha'(t)u_1}{\alpha^2(t)}\right)\\
&\quad-\left[\frac{\alpha'(t)y}{\alpha(t)}+\alpha(t)f\left(t,\frac{y}{\alpha(t)},\frac{u_2}{\alpha(t)},\frac{v}{\alpha(t)}-\frac{\alpha'(t)u_2}{\alpha^2(t)}\right)\right]\|\\
&\le(\sup_{-\tau\le t<0}|\alpha(t)|)\|f\left(t,\frac{y}{\alpha(t)},\frac{u_1}{\alpha(t)},\frac{v}{\alpha(t)}-\frac{\alpha'(t)u_1}{\alpha^2(t)}\right)-f\left(t,\frac{y}{\alpha(t)},\frac{u_2}{\alpha(t)},\frac{v}{\alpha(t)}-\frac{\alpha'(t)u_2}{\alpha^2(t)}\right)\|\\
&\le(\sup_{-\tau\le t<0}|\alpha(t)|)\|f\left(t,\frac{y}{\alpha(t)},\frac{u_1}{\alpha(t)},\frac{v}{\alpha(t)}-\frac{\alpha'(t)u_1}{\alpha^2(t)}\right)-f\left(t,\frac{y}{\alpha(t)},\frac{u_2}{\alpha(t)},\frac{v}{\alpha(t)}-\frac{\alpha'(t)u_1}{\alpha^2(t)}\right)\|\\
&\quad+(\sup_{-\tau\le t<0}|\alpha(t)|)\|f\left(t,\frac{y}{\alpha(t)},\frac{u_2}{\alpha(t)},\frac{v}{\alpha(t)}-\frac{\alpha'(t)u_1}{\alpha^2(t)}\right)-f\left(t,\frac{y}{\alpha(t)},\frac{u_2}{\alpha(t)},\frac{v}{\alpha(t)}-\frac{\alpha'(t)u_2}{\alpha^2(t)}\right)\|\\
&\le\left(\sup_{-\tau\le t<0}|\alpha(t)|\right)\left[\beta(t)\|\frac{u_1}{\alpha(t)}-\frac{u_2}{\alpha(t)}\|+\gamma(t)\|\frac{v}{\alpha(t)}-\frac{\alpha'(t)u_1}{\alpha^2(t)}-\left(\frac{v}{\alpha(t)}-\frac{\alpha'(t)u_2}{\alpha^2(t)}\right)\|\right]\\
&\le\omega(\beta(t)+\gamma(t)\zeta)\|u_1-u_2\|
\end{aligned}
$$

Third, the inequality (3.15) can be proved as follows

$$
\begin{aligned}
&\|F(t,y,u,v_1)-F(t,y,u,v_2)\|\\
&=\|\frac{\alpha'(t)y}{\alpha(t)}+\alpha(t)f\left(t,\frac{y}{\alpha(t)},\frac{u}{\alpha(t)},\frac{v_1}{\alpha(t)}-\frac{\alpha'(t)u}{\alpha^2(t)}\right)\\
&\quad-\left[\frac{\alpha'(t)y}{\alpha(t)}+\alpha(t)f\left(t,\frac{y}{\alpha(t)},\frac{u}{\alpha(t)},\frac{v_2}{\alpha(t)}-\frac{\alpha'(t)u}{\alpha^2(t)}\right)\right]\|\\
&\le\left(\sup_{-\tau\le t<0}|\alpha(t)|\right)\gamma(t)\|\frac{v_1}{\alpha(t)}-\frac{\alpha'(t)u}{\alpha^2(t)}-\left(\frac{v_2}{\alpha(t)}-\frac{\alpha'(t)u}{\alpha^2(t)}\right)\|\\
&\le\omega\gamma(t)\|v_1-v_2\|
\end{aligned}
$$

Finally, the inequality (3.16) can be proved as follows

$$
\begin{aligned}
&\|V(t,y,u_1,v,w)-V(t,y,u_2,v,w)\|\\
&=\|F(t,y,u_1,F(t-\tau,u_1,v,w))-F(t,y,u_2,F(t-\tau,u_2,v,w))\|\\
&=\left(\sup_{-\tau\le t<0}|\alpha(t)|\right)\|f\left(t,\frac{y}{\alpha(t)},\frac{u_1}{\alpha(t)},f\left(t-\tau,\frac{u_1}{\alpha(t)},\frac{v}{\alpha(t)},\frac{w}{\alpha(t)}-\frac{\alpha'(t)v}{\alpha^2(t)}\right)\right)\\
&\qquad-f\left(t,\frac{y}{\alpha(t)},\frac{u_2}{\alpha(t)},f\left(t-\tau,\frac{u_2}{\alpha(t)},\frac{v}{\alpha(t)},\frac{w}{\alpha(t)}-\frac{\alpha'(t)v}{\alpha^2(t)}\right)\right)\|\\
&=\left(\sup_{-\tau\le t<0}|\alpha(t)|\right)\|U\left(t,\frac{y}{\alpha(t)},\frac{u_1}{\alpha(t)},\frac{v}{\alpha(t)},\frac{w}{\alpha(t)}-\frac{\alpha'(t)v}{\alpha^2(t)}\right)\\
&\qquad-U\left(t,\frac{y}{\alpha(t)},\frac{u_2}{\alpha(t)},\frac{v}{\alpha(t)},\frac{w}{\alpha(t)}-\frac{\alpha'(t)v}{\alpha^2(t)}\right)\|\\
&\le\left(\sup_{-\tau\le t<0}|\alpha(t)|\right)\sigma(t)\|\frac{u_1}{\alpha(t)}-\frac{u_2}{\alpha(t)}\|\\
&\le\omega\sigma(t)\|u_1-u_2\|
\end{aligned}
$$

By the condition (3.8) and (Wang et al. 2009, Theorem 3.1), (Wang and Li 2004, Theorem 1), we can obtain that

$$\|y(t)-\tilde{y}(t)\| \le \max\{\|\Phi(0)-\tilde{\Phi}(0)\|, \hat{\kappa}\},$$

which implies (3.9) holds, where

$$\hat{\kappa}=\sup_{0\le s\le\tau}\frac{(\omega\beta(s)+\omega\zeta\gamma(t))\|\Phi(s-\tau)-\tilde{\Phi}(s-\tau)\|+\omega\gamma(s)\|\Phi'(s-\tau)-\tilde{\Phi}'(s-\tau)\|}{-\left[\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\frac{R(t)}{\omega^2}\right]}.$$

Moreover, by the condition (3.10)-(3.12) and (Wang et al. 2009, Theorem 3.2), (Wang and Li 2004, Theorem 4), we can obtain that

$$\lim_{t\to\infty}\|y(t)-\tilde{y}(t)\|=0,$$

which implies

$$\lim_{t\to\infty}\|x(t)-\tilde{x}(t)\|=0.$$

□

Due to the difference in (3.17) and (3.18), similar to Theorem 3.2, by (Wang et al. 2009, Theorem 3.2) or (Wang and Li 2004, Theorem 4), we can obtain the following theorem.

Theorem 3.3 *Assume that IDDEs* (3.1) *and* (3.2) *fulfill the inequalities* (3.3)-(3.6)*,* $R(t)\le\hat{R}$*,* $\hat{R}>0$*, and for* $\forall t\ge 0$*,*

$$\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\omega^2\hat{R}<0,\quad \frac{\omega\sigma(t)-\omega\gamma(t)\left[\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\omega^2\hat{R}\right]}{-\left[\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\omega^2\hat{R}\right]}\le 1, \tag{3.19}$$

then

$$\|x(t)-\tilde{x}(t)\|\le \max\{\frac{\lambda\|\phi(0^-)-\tilde{\phi}(0^-)\|}{m},\bar{\kappa}\}, \tag{3.20}$$

where

$$\bar{\kappa}=\sup_{0\le s<\tau}\frac{\omega^2(\beta(s)+2\zeta\gamma(s))\delta\phi(s)+\omega^2\gamma(s)\delta\dot{\phi}(s)}{-\left[\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\omega^2\hat{R}\right]}.$$

Moreover, assume that

$$\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\omega^2\hat{R}\le\bar{R}<0,\ \omega\gamma(t)\le\bar{\xi}<1,\ \forall t\ge 0, \tag{3.21}$$

$$\frac{\omega\sigma(t)}{-\left[\Re\left(\frac{\alpha'(t)}{\alpha(t)}\right)+\omega^2\hat{R}\right]}\le\bar{k}(1-\omega\gamma(t)),\ \bar{k}<1,\ \forall t\ge 0, \tag{3.22}$$

then we have

$$\lim_{t\to\infty}\|x(t)-\tilde{x}(t)\|=0.$$

3.3 Special cases

In this subsection, we only introduce three special cases and some sufficient conditions for stability and asymptotical stability of (3.1) and (3.2) are obtained from them. Many different functions can be chosen as $\alpha(t)$ (see Zhang et al. 2015), which implies many special cases can be obtained.

Special case I. Obviously, $\alpha_1(t) = \lambda^{\{\frac{t}{\tau}\}}$, $t \in [-\tau, \infty)$. By Theorem 3.2, when $\alpha(t) = \alpha_1(t) = \lambda^{\{\frac{t}{\tau}\}}$, we can obtain the following theorem.

Theorem 3.4 *Assume that $x(t)$ is the solution of (2.1) and $y_1(t) = \lambda^{\{\frac{t}{\tau}\}}x(t),\ t \in [-\tau, +\infty)$. Then $y_1(t)$ is the solution of*

$$\begin{cases} y_1'(t) = F_1(t, y_1(t), y_1(t-\tau), y_1'(t-\tau)), & t \geq 0, \\ y_1(t) = \Phi_1(t), & t \in [-\tau, 0], \end{cases} \tag{3.23}$$

where

$$F_1(t, y, u, v) = \left(\frac{\ln\lambda}{\tau}\right) y + \lambda^{\{\frac{t}{\tau}\}} f\left(t, \lambda^{-\{\frac{t}{\tau}\}}y, \lambda^{-\{\frac{t}{\tau}\}}u, \lambda^{-\{\frac{t}{\tau}\}}\left(v - \frac{u\ln\lambda}{\tau}\right)\right)$$

and

$$\Phi_1(t) = \begin{cases} \lambda^{\frac{t}{\tau}+1}\phi(t), & t \in [-\tau, 0), \\ \lambda\phi(0^-), & t = 0. \end{cases}$$

On the other hand, assume that $y(t)$ is the solution of (3.23) and $x(t) = \lambda^{-\{\frac{t}{\tau}\}}y(t),\ t \in [-\tau, +\infty)$. Then $x(t)$ is the solution of (2.1).

Theorem 3.5 *Assume that IDDEs (3.1) and (3.2) fulfill the inequalities (3.3)-(3.6), and $R(t) \leq 0$ for $\forall t \geq 0$,*

$$\frac{\ln|\lambda|}{\tau} + \frac{R(t)}{\omega_1^2} < 0, \quad \frac{\omega_1\sigma(t) - \omega_1\gamma(t)\left(\frac{\ln|\lambda|}{\tau} + \frac{R(t-\tau)}{\omega_1^2}\right)}{-\left(\frac{\ln|\lambda|}{\tau} + \frac{R(t)}{\omega_1^2}\right)} \leq 1,$$

then

$$\|x(t) - \tilde{x}(t)\| \leq \max\{\frac{\lambda\|\phi(0^-) - \tilde{\phi}(0^-)\|}{m_1}, \kappa_1\},$$

where $\omega_1 = \max\{|\lambda|, \frac{1}{|\lambda|}\}$, $m_1 = \min\{1, |\lambda|\}$, $\zeta_1 = |\frac{\ln\lambda}{\tau}|$ and

$$\kappa_1 = \sup_{0\leq s<\tau} \frac{\omega_1^2(\beta(s) + 2\zeta\gamma(s))\delta\phi(s) + \omega_1^2\gamma(s)\delta\dot{\phi}(s)}{-\left(\frac{\ln|\lambda|}{\tau} + \frac{R(t)}{\omega_1^2}\right)}.$$

Moreover, assume that

$$\frac{\ln|\lambda|}{\tau} + \frac{R(t)}{\omega_1^2} \leq R_1 < 0,\ \forall t \geq 0,$$

and

$$\frac{\omega_1\gamma(t)\left(\frac{\ln|\lambda|}{\tau}+\frac{R(t-\tau)}{\omega_1^2}\right)}{\frac{\ln|\lambda|}{\tau}+\frac{R(t)}{\omega_1^2}}=r_1(t)\le\bar{\xi}_1<1,\ \forall t\ge 0,$$

$$\frac{\omega_1\sigma(t)}{-\left(\frac{\ln|\lambda|}{\tau}+\frac{R(t)}{\omega_1^2}\right)}\le k_1(1-r_1(t)),\ k_1<1,\ \forall t\ge 0,$$

then we have

$$\lim_{t\to\infty}\|x(t)-\tilde{x}(t)\|=0.$$

Theorem 3.6 *Assume that IDDEs* (3.1) *and* (3.2) *fulfill the inequalities* (3.3)–(3.6)*,* $R(t)\le\hat{R}$*,* $\hat{R}>0$*, and for* $\forall t\ge 0$*,*

$$\frac{\ln|\lambda|}{\tau}+\omega_1^2\hat{R}<0,\quad \frac{\omega_1\sigma(t)-\omega_1\gamma(t)\left(\frac{\ln|\lambda|}{\tau}+\omega_1^2\hat{R}\right)}{-\left(\frac{\ln|\lambda|}{\tau}+\omega_1^2\hat{R}\right)}\le 1,$$

then

$$\|x(t)-\tilde{x}(t)\|\le\max\{\frac{\lambda\|\phi(0^-)-\tilde{\phi}(0^-)\|}{m_1},\check{\kappa}_1\},$$

where

$$\check{\kappa}_1=\sup_{0\le s<\tau}\frac{\omega_1^2(\beta(s)+2\zeta_1\gamma(s))\delta\phi(s)+\omega_1^2\gamma(s)\delta\dot{\phi}(s)}{-\left(\frac{\ln|\lambda|}{\tau}+\omega_1^2\hat{R}\right)}.$$

Moreover, assume that

$$\frac{\ln|\lambda|}{\tau}+\omega_1^2\hat{R}<0,\ \omega_1\gamma(t)\le\bar{\xi}_1<1,\ \forall t\ge 0,$$

and

$$\frac{\omega_1\sigma(t)}{-\left(\frac{\ln|\lambda|}{\tau}+\omega_1^2\hat{R}\right)}\le\bar{k}_1(1-\omega_1\gamma(t)),\ \bar{k}_1<1,\ \forall t\ge 0,$$

then we have

$$\lim_{t\to\infty}\|x(t)-\tilde{x}(t)\|=0.$$

Special Case II: When $\lambda\in\mathbb{R}$ and $\lambda>0$, $\alpha_2(t)=1+(\lambda-1)\{\frac{t}{\tau}\}$, $t\in[-\tau,\infty)$. By Theorem 3.2, when $\alpha(t)=\alpha_2(t)$, we can obtain the following theorem.

Theorem 3.7 *Assume that* $x(t)$ *is the solution of* (2.1) *and* $y_2(t)=\alpha_2(t)x(t)$*,* $t\in[-\tau,+\infty)$*. Then* $y_2(t)$ *is the solution of*

$$\begin{cases} y_2'(t)=F_2(t,y_2(t),y_2(t-\tau),y_2'(t-\tau)), & t\ge 0,\\ y_2(t)=\Phi_2(t), & t\in[-\tau,0], \end{cases}\tag{3.24}$$

where

$$F_2(t, y, u, v) = \frac{(\lambda - 1)y}{\tau\alpha_2(t)} + \alpha_2(t) f\left(t, \frac{y}{\alpha_2(t)}, \frac{u}{\alpha_2(t)}, \frac{v}{\alpha_2(t)} - \frac{(\lambda - 1)u}{\tau\alpha_2^2(t)}\right)$$

and

$$\Phi_2(t) = \begin{cases} [1 + (\lambda - 1)(\frac{t}{\tau} + 1)]\phi(t), & t \in [-\tau, 0), \\ \lambda\phi(0^-), & t = 0. \end{cases}$$

On the other hand, assume that $y(t)$ *is the solution of* (3.24) *and* $x(t) = \frac{y_2(t)}{1+(\lambda-1)\{\frac{t}{\tau}\}}$, $t \in [-\tau, +\infty)$. *Then* $x(t)$ *is the solution of* (2.1).

Theorem 3.8 *When* $\lambda \in \mathbb{R}$, $\lambda > 0$ *and* $\lambda \neq 1$, *assume that IDDEs* (3.1) *and* (3.2) *fulfill the inequalities* (3.3)–(3.6), $R(t) \leq 0$, *for* $\forall t \geq 0$,

$$\frac{\lambda - 1}{\tau + (\lambda - 1)\tau\{\frac{t}{\tau}\}} + \frac{R(t)}{\omega_2^2} < 0, \quad \frac{\omega_2\sigma(t) - \omega_2\gamma(t)\left(\frac{\lambda-1}{\tau+(\lambda-1)\tau\{\frac{t}{\tau}\}} + \frac{R(t-\tau)}{\omega_2^2}\right)}{-\left(\frac{\lambda-1}{\tau+(\lambda-1)\tau\{\frac{t}{\tau}\}} + \frac{R(t)}{\omega_2^2}\right)} \leq 1,$$

then

$$\|x(t) - \tilde{x}(t)\| \leq \max\{\frac{\lambda\|\phi(0^-) - \tilde{\phi}(0^-)\|}{m_2}, \kappa_2\},$$

where $\omega_2 = \max\{\lambda, \frac{1}{\lambda}\}$, $m_2 = \min\{1, \lambda\}$, $\zeta_2 = \max\{|\frac{\lambda-1}{\tau}|, |\frac{\lambda-1}{\lambda\tau}|\}$ *and*

$$\kappa_2 = \sup_{0\leq s<\tau} \frac{\omega_2^2[\beta(s) + 2\zeta\gamma(s)]\delta\phi(s) + \omega_2^2\gamma(s)\delta\dot{\phi}(s)}{-\left(\frac{\lambda-1}{\tau+(\lambda-1)\tau\{\frac{t}{\tau}\}} + \frac{R(t)}{\omega_2^2}\right)}.$$

Moreover, assume that

$$\frac{\lambda - 1}{\tau + (\lambda - 1)\tau\{\frac{t}{\tau}\}} + \frac{R(t)}{\omega_2^2} \leq R_2 < 0, \ \forall t \geq 0,$$

and

$$\frac{\omega_1\gamma(t)\left(\frac{\lambda-1}{\tau+(\lambda-1)\tau\{\frac{t}{\tau}\}} + \frac{R(t-\tau)}{\omega_2^2}\right)}{\frac{\lambda-1}{\tau+(\lambda-1)\tau\{\frac{t}{\tau}\}} + \frac{R(t)}{\omega_2^2}} = r_2(t) \leq \xi_2 < 1, \ \forall t \geq 0,$$

$$\frac{\omega_1\sigma(t)}{-\left(\frac{\lambda-1}{\tau+(\lambda-1)\tau\{\frac{t}{\tau}\}} + \frac{R(t)}{\omega_2^2}\right)} \leq k_2(1 - r_2(t)), \ k_2 < 1, \ \forall t \geq 0,$$

then we have

$$\lim_{t\to\infty} \|x(t) - \tilde{x}(t)\| = 0.$$

Theorem 3.9 *When $\lambda \in \mathbb{R}$ and $\lambda > 0$, assume that IDDEs (3.1) and (3.2) fulfill the inequalities (3.3)–(3.6), $R(t) \le \hat{R}$, $\hat{R} > 0$, and for $\forall t \ge 0$,*

$$\frac{\lambda - 1}{\tau + (\lambda - 1)\tau\{\frac{t}{\tau}\}} + \omega_2^2 \hat{R} < 0,$$

and

$$\frac{\omega_2\sigma(t) - \omega_2\gamma(t)\left(\frac{\lambda-1}{\tau+(\lambda-1)\tau\{\frac{t}{\tau}\}} + \omega_2^2\hat{R}\right)}{-\left(\frac{\lambda-1}{\tau+(\lambda-1)\tau\{\frac{t}{\tau}\}} + \omega_2^2\hat{R}\right)} \le 1,$$

then

$$\|x(t) - \tilde{x}(t)\| \le \max\{\frac{\lambda\|\phi(0^-) - \tilde{\phi}(0^-)\|}{m_2}, \check{\kappa}_2\},$$

where

$$\check{\kappa}_2 = \sup_{0\le s<\tau} \frac{\omega_2^2\left(\beta(s) + 2\zeta_2\gamma(t)\right)\delta\phi(s) + \omega_2^2\gamma(s)\delta\dot{\phi}(s)}{-\left(\frac{\lambda-1}{\tau+(\lambda-1)\tau\{\frac{t}{\tau}\}} + \omega_2^2\hat{R}\right)}.$$

Moreover, assume that

$$\frac{\lambda - 1}{\tau + (\lambda - 1)\tau\{\frac{t}{\tau}\}} + \omega_2^2\hat{R} \le \bar{R}_2 < 0,\ \ \omega_2\gamma(t) \le \bar{\xi}_2 < 1,\ \ \forall t \ge 0,$$
$$\frac{\omega_2\sigma(t)}{-\left(\frac{\lambda-1}{\tau+(\lambda-1)\tau\{\frac{t}{\tau}\}} + \omega_2^2\hat{R}\right)} \le \bar{k}_2(1 - \omega_2\gamma(t)),\ \ \bar{k}_2 < 1,\ \ \forall t \ge 0,$$

then we have

$$\lim_{t\to\infty} \|x(t) - \tilde{x}(t)\| = 0.$$

Because $\frac{\lambda-1}{\tau+(\lambda-1)\tau\{\frac{t}{\tau}\}} \le \frac{\lambda-1}{\tau}$ for all $\lambda > 0$, $\forall t \in \mathbb{R}$, by Theorem 3.9, we can obtain the following corollary.

Corollary 3.10 *When $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\lambda \ne 1$, assume that IDDEs (3.1) and (3.2) fulfill the inequalities (3.3)–(3.6), $R(t) \le 0$, for $\forall t \ge 0$,*

$$\frac{\lambda - 1}{\tau} + \frac{R(t)}{\omega_2^2} < 0,\ \ \frac{\omega_2\sigma(t) - \omega_2\gamma(t)\left(\frac{\lambda-1}{\tau} + \frac{R(t-\tau)}{\omega_2^2}\right)}{-\left(\frac{\lambda-1}{\tau} + \frac{R(t)}{\omega_2^2}\right)} \le 1,$$

then

$$\|x(t) - \tilde{x}(t)\| \le \max\{\frac{\lambda\|\phi(0^-) - \tilde{\phi}(0^-)\|}{m_2}, \kappa_2\}.$$

Moreover, assume that

$$\frac{\lambda - 1}{\tau} + \frac{R(t)}{\omega_2^2} \le \check{R}_2 < 0,\ \ \forall t \ge 0,$$

and

$$\frac{\omega_1\gamma(t)\left(\frac{\lambda-1}{\tau}+\frac{R(t-\tau)}{\omega_2^2}\right)}{\frac{\lambda-1}{\tau}+\frac{R(t)}{\omega_2^2}}=\check{r}_2(t)\le\check{\xi}_2<1,\ \forall t\ge 0,$$
$$\frac{\omega_1\sigma(t)}{-\left(\frac{\lambda-1}{\tau}+\frac{R(t)}{\omega_2^2}\right)}\le\check{k}_2(1-\check{r}_2(t)),\ \check{k}_2<1,\ \forall t\ge 0,$$

then we have

$$\lim_{t\to\infty}\|x(t)-\tilde{x}(t)\|=0.$$

Corollary 3.11 *When* $\lambda\in\mathbb{R}$ *and* $\lambda>0$, *assume that IDDEs* (3.1) *and* (3.2) *fulfill the inequalities* (3.3)–(3.6), $R(t)\le\hat{R}$, $\hat{R}>0$, *and for* $\forall t\ge 0$,

$$\frac{\lambda-1}{\tau}+\omega_2^2\hat{R}<0,$$

and

$$\frac{\omega_2\sigma(t)-\omega_2\gamma(t)\left(\frac{\lambda-1}{\tau}+\omega_2^2\hat{R}\right)}{-\left(\frac{\lambda-1}{\tau}+\omega_2^2\hat{R}\right)}\le 1,$$

then

$$\|x(t)-\tilde{x}(t)\|\le\max\{\frac{\lambda\|\phi(0^-)-\tilde{\phi}(0^-)\|}{m_2},\kappa_2\}.$$

Moreover, assume that

$$\frac{\lambda-1}{\tau}+\omega_2^2\hat{R}<0,\ \omega_2\gamma(t)\le\tilde{\xi}_2<1,\quad\forall t\ge 0,$$
$$\frac{\omega_2\sigma(t)}{-\left(\frac{\lambda-1}{\tau}+\omega_2^2\hat{R}\right)}\le\tilde{k}_2(1-\omega_2\gamma(t)),\ \tilde{k}_2<1,\ \forall t\ge 0,$$

then we have

$$\lim_{t\to\infty}\|x(t)-\tilde{x}(t)\|=0.$$

Special Case III: When $\lambda\in\mathbb{R}$ and $\lambda>0, \alpha(t)=\alpha_3(t)=-\{\frac{t}{\tau}\}^2+\lambda\{\frac{t}{\tau}\}+1, t\in[-\tau,\infty)$. By Theorem 3.2, we can obtain the following theorem.

Theorem 3.12 *Assume that* $x(t)$ *is the solution of* (2.1) *and* $y_3(t)=(-\{\frac{t}{\tau}\}^2+\lambda\{\frac{t}{\tau}\}+1)x(t),\ t\in[-\tau,+\infty)$. *Then* $y_3(t)$ *is the solution of*

$$\begin{cases}y_3'(t)=F_3(t,y_3(t),y_3(t-\tau),y_3'(t-\tau)), & t\ge 0,\\ y_3(t)=\Phi_3(t), & t\in[-\tau,0],\end{cases}\tag{3.25}$$

where

$$F_3(t,y,u,v)=\frac{[-2\{\frac{t}{\tau}\}+\lambda]y}{\tau\alpha_3(t)}+\alpha_3(t)f(t,\frac{y}{\alpha_3(t)},\frac{u}{\alpha_3(t)},\frac{v}{\alpha_3(t)}-\frac{(\lambda-1)u}{\tau\alpha_3^2(t)})$$

and

$$\Phi_3(t) = \begin{cases} [-(\frac{t}{\tau}+1)^2+\lambda(\frac{t}{\tau}+1)+1]\phi(t), & t \in [-\tau, 0), \\ \lambda\phi(0^-), & t = 0. \end{cases}$$

On the other hand, assume that $y_3(t)$ *is the solution of* (3.25) *and* $x(t) = \frac{y_3(t)}{-\{\frac{t}{\tau}\}^2+\lambda\{\frac{t}{\tau}\}+1}$, $t \in [-\tau, +\infty)$. *Then* $x(t)$ *is the solution of* (2.1).

Theorem 3.13 *When* $\lambda \in \mathbb{R}$ *and* $\lambda > 0$, *assume that IDDEs* (3.1) *and* (3.2) *fulfill the inequalities* (3.3)–(3.6), $R(t) \le 0$, *for* $\forall t \ge 0$,

$$\frac{-2\{\frac{t}{\tau}\}+\lambda}{-\{\frac{t}{\tau}\}^2\tau+\lambda\tau\{\frac{t}{\tau}\}+\tau} + \frac{R(t)}{\omega_3^2} < 0,$$

and

$$\frac{\omega_3\sigma(t) - \omega_3\gamma(t)\left(\frac{-2\{\frac{t}{\tau}\}+\lambda}{-\{\frac{t}{\tau}\}^2\tau+\lambda\tau\{\frac{t}{\tau}\}+\tau} + \frac{R(t-\tau)}{\omega_3^2}\right)}{-\left(\frac{-2\{\frac{t}{\tau}\}+\lambda}{-\{\frac{t}{\tau}\}^2\tau+\lambda\tau\{\frac{t}{\tau}\}+\tau} + \frac{R(t)}{\omega_3^2}\right)} \le 1,$$

then

$$\|x(t) - \tilde{x}(t)\| \le \max\{\frac{\lambda\|\phi(0^-) - \tilde{\phi}(0^-)\|}{m_3}, \kappa_3\},$$

where $m_3 = \min\{1, \lambda\}$, $\zeta_3 = \sup_{t\in[-\tau,0)} |\frac{\alpha_3'(t)}{\alpha_3(t)}| = \max\{|\frac{\lambda-2}{\lambda\tau}|, \frac{\lambda}{\tau}\}$,

$$\omega_3 = \begin{cases} \frac{\lambda}{4} + \frac{1}{\lambda}, & 0 < \lambda \le 1, \\ \frac{\lambda^2}{4} + 1, & 1 < \lambda \le 2, \\ \lambda, & \lambda > 2, \end{cases}$$

and

$$\kappa_3 = \sup_{0\le s<\tau} \frac{\omega_3^2(\beta(s) + 2\zeta_3\gamma(s))\delta\phi(s) + \omega_3^2\gamma(s)\delta\dot{\phi}(s)}{-\left[\frac{-2\{\frac{t}{\tau}\}+\lambda}{-\{\frac{t}{\tau}\}^2\tau+\lambda\tau\{\frac{t}{\tau}\}+\tau} + \frac{R(s)}{\omega_3^2}\right]}.$$

Moreover, assume that

$$\frac{-2\{\frac{t}{\tau}\}+\lambda}{-\{\frac{t}{\tau}\}^2\tau+\lambda\tau\{\frac{t}{\tau}\}+\tau} + \frac{R(t)}{\omega_3^2} \le R_3 < 0, \ \forall t \ge 0,$$

and

$$\frac{\omega_3\gamma(t)\left(\frac{-2\{\frac{t}{\tau}\}+\lambda}{-\{\frac{t}{\tau}\}^2\tau+\lambda\tau\{\frac{t}{\tau}\}+\tau} + \frac{R(t-\tau)}{\omega_3^2}\right)}{\frac{-2\{\frac{t}{\tau}\}+\lambda}{-\{\frac{t}{\tau}\}^2\tau+\lambda\tau\{\frac{t}{\tau}\}+\tau} + \frac{R(t)}{\omega_3^2}} \le \bar{\xi}_3 < 1, \ \forall t \ge 0,$$

$$\frac{\omega_3\sigma(t)}{-\left(\frac{-2\{\frac{t}{\tau}\}+\lambda}{-\{\frac{t}{\tau}\}^2\tau+\lambda\tau\{\frac{t}{\tau}\}+\tau} + \frac{R(t)}{\omega_3^2}\right)} \le k_3(1 - \omega_3\gamma(t)), \ k_3 < 1, \ \forall t \ge 0,$$

then we have

$$\lim_{t\to\infty} \|x(t) - \tilde{x}(t)\| = 0.$$

Because $\frac{-2\{\frac{t}{\tau}\}+\lambda}{-\{\frac{t}{\tau}\}^2\tau+\lambda\tau\{\frac{t}{\tau}\}+\tau} \leq \frac{\lambda}{\tau}$ for all $\lambda > 0$, $\forall t \in \mathbb{R}$, by Theorem 3.13, we can obtain the following corollary.

Corollary 3.14 *When $\lambda \in \mathbb{R}$ and $\lambda > 0$, assume that IDDEs (3.1) and (3.2) fulfill the inequalities (3.3)–(3.6), $R(t) \leq 0$, for $\forall t \geq 0$,*

$$\frac{\lambda}{\tau} + \frac{R(t)}{\omega_3^2} < 0,$$

and

$$\frac{\omega_3\sigma(t) - \omega_3\gamma(t)\left(\frac{\lambda}{\tau} + \frac{R(t-\tau)}{\omega_3^2}\right)}{-\left(\frac{\lambda}{\tau} + \frac{R(t)}{\omega_3^2}\right)} \leq 1,$$

then

$$\|x(t) - \tilde{x}(t)\| \leq \max\{\frac{\lambda\|\phi(0^-) - \tilde{\phi}(0^-)\|}{m_3}, \kappa_3\}.$$

Moreover, assume that

$$\frac{\lambda}{\tau} + \frac{R(t)}{\omega_3^2} \leq R_3 < 0, \ \forall t \geq 0,$$

and

$$\frac{\omega_3\gamma(t)\left(\frac{\lambda}{\tau} + \frac{R(t-\tau)}{\omega_3^2}\right)}{\frac{\lambda}{\tau} + \frac{R(t)}{\omega_3^2}} \leq \bar{\xi}_3 < 1, \ \forall t \geq 0,$$

$$\frac{\omega_3\sigma(t)}{-\left(\frac{\lambda}{\tau} + \frac{R(t)}{\omega_3^2}\right)} \leq k_3(1 - \omega_3\gamma(t)), \ k_3 < 1, \ \forall t \geq 0,$$

then we have

$$\lim_{t\to\infty} \|x(t) - \tilde{x}(t)\| = 0.$$

3.4 Numerical methods for nonlinear INDDEs (3.1) and (3.2)

The numerical method for nonlinear INDDE (3.1) can be constructed as follows

$$\begin{cases} Y_{n+1}^i = \bar{y}(t_n) + h\sum_{j=1}^s a_{ij}F\left(t_{n+1}^{j-}, Y_{n+1}^j, Y_{n-m+1}^j, z'\left(t_{n-m+1}^{j-}\right)\right), & i = 1, 2, \ldots, s, \\ \bar{y}(t_n + \theta h) = \bar{y}(t_n) + h\sum_{i=1}^s b_i(\theta)F\left(t_{n+1}^{i-}, Y_{n+1}^i, Y_{n-m+1}^i, z'\left(t_{n-m+1}^{i-}\right)\right), & n \in \mathbb{N}, \end{cases} \tag{3.26}$$

and

$$\bar{x}_n = \frac{\bar{y}(t_n)}{\alpha(t_n)}, \quad n \in \mathbb{N}, \tag{3.27}$$

where $Y^i_{n-m+1} = \Phi(t^i_{n-m+1})$ if $t^i_{n-m+1} \le 0$, $i = 1, 2, \cdots, s$,

$$z'(t^i_{n-m+1}) = \begin{cases} \Phi'(t^i_{n-m+1}), & t^i_{n-m+1} \le 0, \\ F(t^i_{n-m+1}, Y^i_{n-m+1}, \Phi^i_{n-2m+1}, \Phi'(t^i_{n-2m+1})), & 0 \le t^i_{n-m+1} \le \tau, \\ F(t^i_{n-m+1}, Y^i_{n-m+1}, Y^i_{n-2m+1}, z'(t^i_{n-2m+1})), & \tau \le t^i_{n-m+1}, \end{cases}$$

$t_n = nh$, $n \in \mathbb{N}$, $h = \frac{\tau}{m}$ and m is a positive integer.

Similarly, the numerical method for (3.2) can be constructed as follows

$$\begin{cases} \bar{\tilde{Y}}^i_{n+1} = \bar{\tilde{y}}(t_n) + h\sum_{j=1}^s a_{ij} F(t^{j-}_{n+1}, \bar{\tilde{Y}}^j_{n+1}, \bar{\tilde{Y}}^j_{n-m+1}, \bar{\tilde{z}}'(t^{j-}_{n-m+1})), & i = 1, 2, \cdots, s, \\ \bar{\tilde{y}}(t_n + \theta h) = \bar{\tilde{y}}(t_n) + h\sum_{i=1}^s b_i(\theta) F(t^{i-}_{n+1}, \bar{\tilde{Y}}^i_{n+1}, \bar{\tilde{Y}}^i_{n-m+1}, \bar{\tilde{z}}'(t^{i-}_{n-m+1})), & n \in \mathbb{N}, \end{cases} \tag{3.28}$$

and

$$\bar{\tilde{x}}_n = \frac{\bar{\tilde{y}}(t_n)}{\alpha(t_n)}, \quad n \in \mathbb{N}, \tag{3.29}$$

where

$$\tilde{z}'(t^i_{n-m+1}) = \begin{cases} \tilde{\Phi}'(t^i_{n-m+1}), & t^i_{n-m+1} \le 0, \\ F(t^i_{n-m+1}, Y^i_{n-m+1}, \tilde{\Phi}^i_{n-2m+1}, \tilde{\Phi}'(t^i_{n-2m+1})), & 0 \le t^i_{n-m+1} \le \tau, \\ F(t^i_{n-m+1}, Y^i_{n-m+1}, Y^i_{n-2m+1}, \tilde{z}'(t^i_{n-2m+1})), & \tau \le t^i_{n-m+1}. \end{cases}$$

Theorem 3.15 *Under the condition of* (3.8)*, the constructed numerical methods* (3.26)–(3.27) *and* (3.28)–(3.29) *furnished by backward Euler method with linear interpolation (2-stage Lobatto IIIC method with linear interpolation) are stable, in the following sense,*

$$\|\bar{x}_n - \bar{\tilde{x}}_n\| \le \max\{\frac{\lambda\|\phi(0^-) - \tilde{\phi}(0^-)\|}{m}, \kappa\}.$$

Moreover, under the condition of Theorem 3.2, the constructed numerical method (3.26)–(3.27) *and* (3.28)–(3.29) *furnished by backward Euler method with linear interpolation (or 2-stage Lobatto IIIC method with linear interpolation) are also asymptotically stable, that is,*

$$\lim_{n\to+\infty} \|\bar{x}_n - \bar{\tilde{x}}_n\| = 0.$$

Proof By (Wang et al. 2009, Theorem 4.8), RVCRK formulae (3.26) and (3.28) furnished by backward Euler method with linear interpolation (or 2-stage Lobatto IIIC method with linear interpolation) are RN-stable, that is

$$\|\bar{y}(t_n) - \bar{\tilde{y}}(t_n)\| \le \max\{\|\Phi(0) - \tilde{\Phi}(0)\|, \kappa\}.$$

Moreover, (Wang et al. 2009, Theorem 4.9), under the condition of Theorem 3.2, the RVCRK formulae (3.26) and (3.28) furnished by backward Euler method with linear interpolation (or 2-stage Lobatto IIIC method with linear interpolation) are also asymptotically stable, that is,

$$\lim_{n\to+\infty} \|\bar{y}(t_n) - \bar{\tilde{y}}(t_n)\| = 0.$$

Because the relation between the numerical solutions INDDE and NDDE without impulsive perturbations, that is (3.27) and (3.29), the theorem holds. □

Similar to Theorem 3.15, we can obtain the following result.

Theorem 3.16 *Under the condition of* (3.19)*, the constructed numerical methods* (3.26)–(3.27) *and* (3.28)–(3.29) *furnished by backward Euler method with linear interpolation (or 2-stage Lobatto IIIC method with linear interpolation) are stable, in the following sense*

$$\|\bar{x}_n - \bar{\bar{x}}_n\| \leq \frac{\max\{\lambda\|\phi(0^-) - \tilde{\phi}(0^-)\|, \bar{\kappa}\}}{m}.$$

Moreover, under the condition of Theorem 3.3, the constructed numerical method (3.26)–(3.27) *and* (3.28)–(3.29) *furnished by backward Euler method with linear interpolation (or 2-stage Lobatto IIIC method with linear interpolation) are also asymptotically stable, that is*

$$\lim_{n\to+\infty} \|\bar{x}_n - \bar{\bar{x}}_n\| = 0.$$

The following results are provided to analysis the difference between the linear equations and nonlinear equations.

Remark 3.17 1. The meanings of asymptotic stability of the exact solutions of linear INDDE (2.1) and nonlinear INDDE (3.1) are different. The defintions of asymptotic stability of the exact solutions of linear INDDE (2.1) and nonlinear INDDE (3.1) are provided as follows.

(1) The exact solution $x(t)$ of linear INDDE (2.1) is said to be asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0.$$

(2) For $\forall\epsilon > 0$, if there exists a constant $\delta > 0$ such that $\|\Phi - \tilde{\Phi}\| \leq \delta$ and $\|\Phi' - \tilde{\Phi}'\| \leq \delta$ imply that

$$\|x(t) - \tilde{x}(t)\| < \epsilon,$$

then we call the exact solution $x(t)$ of nonlinear INDDE (3.1) and $\tilde{x}(t)$ of (3.2) are stable.

(3) The exact solution $x(t)$ of nonlinear INDDE (3.1) and $\tilde{x}(t)$ of (3.2) are said to be asymptotically stable, if they are stable and fulfil

$$\lim_{t\to\infty} \|x(t) - \tilde{x}(t)\| = 0.$$

2. The meanings of asymptotic stability of numerical method (2.5) for linear equation (2.1) and numerical method (3.26)–(3.27) for nonlinear equation (3.1) are also different. The asymptotic stability of numerical method (2.5) and numerical method (3.26)–(3.27) are given as follows.

(1) Numerical method (2.5) for linear INDDE (2.1) is said to be asymptotically stable if x_n obtained from (2.5) satisfies

$$\lim_{n\to\infty} x_n = 0.$$

(2) For $\forall\epsilon > 0$, if there exists a constant $\delta > 0$ such that $\|\Phi - \tilde{\Phi}\| \leq \delta$ and $\|\Phi' - \tilde{\Phi}'\| \leq \delta$ imply that $\bar{x}_n$ obtained from (3.26)–(3.27) and $\bar{\bar{x}}_n$ obtained from (3.28)–(3.29) satisfy

$$\|\bar{x}_n - \bar{\bar{x}}_n\| < \epsilon,$$

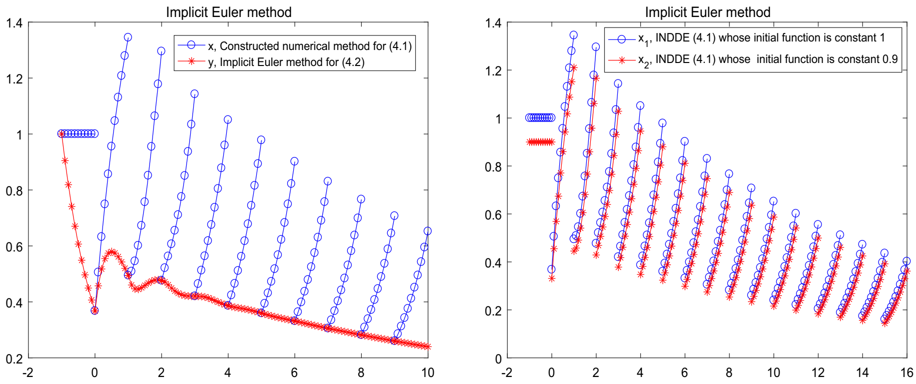

Fig. 1 The numerical methods (2.5) for (4.1) furnished by implicit Euler method with the stepsize $h = \frac{1}{10}$

then we call numerical methods (3.26)–(3.27) for nonlinear INDDE (3.1) and (3.28)-(3.29) for (3.2) are stable.

(3) Numerical methods (3.26)–(3.27) for nonlinear INDDE (3.1) and (3.28)–(3.29) for (3.2) are said to be asymptotically stable, if they are stable and fulfil

$$\lim_{n\to\infty} \|\bar{x}_n - \bar{\bar{x}}_n\| = 0.$$

3. For linear equations, the constructed numerical methods furnished by A-stable Runge–Kutta methods can preserve asymptotic stability of the exact solutions. But, for nonlinear equations, the constructed numerical methods furnished by implicit Euler method (or 2-stage Lobatto IIIC method) can preserve asymptotic stability of the exact solutions.

4 Numerical experiments

Example 4.1 Consider the following scalar linear INDDE:

$$\begin{cases} x'(t) = -x(t) + 2x(t-1) - \frac{1}{4}x'(t-1), & t \geq 0, t \neq k, k \in \mathbb{N}, \\ x(k) = \frac{x(k^-)}{\mathrm{e}}, \\ x(t) = \phi(t), & t \in [-1, 0). \end{cases} \tag{4.1}$$

By Theorem 2.1, assume that $y(t) = \mathrm{e}^{-\{t\}}x(t), t \geq -1$, then $x(t)$ is the solution of INDDE (4.1) if and only if $y(t)$ is the solotion of the following equation:

$$\begin{cases} y'(t) = -2y(t) + \frac{7}{4}y(t-1) - \frac{1}{4}y'(t-1), & t \geq 0, t \neq k, k \in \mathbb{N}, \\ x(t) = \Phi(t), & t \in [-1, 0], \end{cases} \tag{4.2}$$

where

$$\Phi(t) = \begin{cases} \mathrm{e}^{-(t+1)}\phi(t), & t \in [-1, 0), \\ \frac{\phi(0^-)}{\mathrm{e}}, & t = 0. \end{cases}$$

By Theorem 2.2, both the solution $x(t)$ of (4.1) and the solution $y(t)$ of (4.2) tend to zeros as $t \to \infty$. By Theorem 2.4, the numerical methods (2.5) for (4.1) furnished by A-stable Runge-Kutta methods are stable and asymptotically stable (see Figs. 1 and 2).

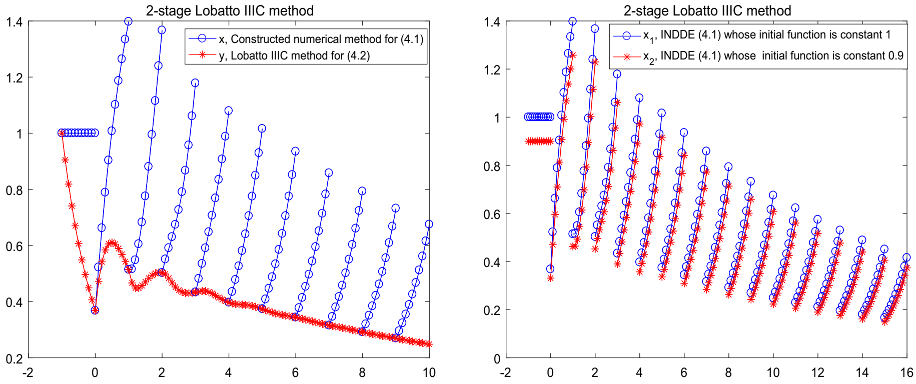

Fig. 2 The numerical methods (2.5) for (4.1) furnished by Lobatto IIIC method with the stepsize $h = \frac{1}{10}$

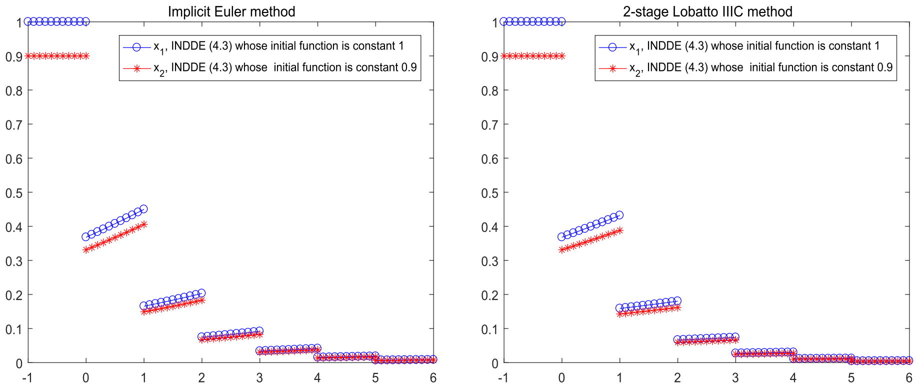

Fig. 3 The numerical methods (2.5) for (4.3) furnished by implicit Euler method and Lobatto IIIC method, respectively, with the stepsize $h = \frac{1}{10}$

Example 4.2 Consider the following scalar nonlinear INDDE:

$$\begin{cases} x'(t) = ax(t) + b\cos(x'(t-1))\sin^2(x(t-1)), & t \geq 0, t \neq k, k \in \mathbb{N}, \\ x(k) = \lambda x(k^-), & \\ x(t) = \phi(t), & t \in [-1, 0), \end{cases} \tag{4.3}$$

where a, b and λ are real constants. It is easy to prove that the first equation of (4.3) satisfies inequalities (3.3)–(3.6) with $R(t) = \Re(a)$, $\beta(t) = 2|b|$, $\gamma(t) = |b|$, $\sigma(t) = (|a| + 2)|b|$.

(i) When $a = 1/10$, $b = 1/30$, $\lambda = \frac{1}{e}$, by Theorem 3.6, the solution of (4.3) is stable and asymptotically stable. By Theorem 3.16, the numerical methods (3.26)–(3.27) for (4.3) furnished by backward Euler method (2-stage Lobatto IIIC method) are stable and asymptotically stable (See Fig. 3).

(ii) When $a = 1/50$, $b = 1/50$, $\lambda = \frac{1}{3}$, by Theorem 3.6, Theorem 3.9 or Corollary 3.11 the solution of (4.3) is stable and asymptotically stable. By Theorem 3.16, the numerical methods (3.26)–(3.27) for (4.3) furnished by backward Euler method (2-stage Lobatto IIIC method) are stable and asymptotically stable.

Table 1 The errors between the numerical solutions obtained from (2.5) and the exact solution of (4.1) at $t = 10$

m	The implicit Euler		2-Lobatto IIIC	
	AE	RE	AE	RE
10	0.0090740168	0.0337160014	1.3427387777e−04	4.9891649809e−04
20	0.0046057353	0.0171133667	3.5550581855e−05	1.3209398655e−04
40	0.0023229857	0.0086314352	9.1563539956e−06	3.4021927024e−05
80	0.0011669682	0.0043360623	2.3219666386e−06	8.6276458479e−06
Ratio	1.9811503945	1.9811503945	3.8676520087	3.8676520087

Table 2 The errors between the numerical solutions obtained from (3.26)–(3.27) and the exact solution of (4.3) when $a = 1/10$, $b = 1/30$, $\lambda = \frac{1}{e}$, $t = 6$

m	The implicit Euler		2-Lobatto IIIC	
	AE	RE	AE	RE
10	9.0752467510e−04	0.1988466569	2.4198037644e−05	0.0053022129
20	4.4587961799e−04	0.0976961551	6.2382599550e−06	0.0013669118
40	2.2092011816e−04	0.0484055455	1.5845179520e−06	3.4719557106e−04
80	1.0994864767e−04	0.0240907180	3.9931665023e−07	8.7497255698e−05
Ratio	2.0209817161	2.0209817161	3.9280180709	3.9280180709

(iii) When $a = -50$, $b = 1/50$, $\lambda = e$, by Theorem 3.5, Theorem 3.8, Corollary 3.10, Theorem 3.13 or Corollary 3.14, the solution of (4.3) is bound and asymptotically stable. By Theorem 3.15, the numerical methods (3.26)–(3.27) for (4.3) furnished by backward Euler method (2-stage Lobatto IIIC method) are stable and asymptotically stable.

Tables 1 and 2 roughly illuminate that the constructed method furnished by backward Euler method is convergent of order 1 and by 2-stage Lobatto IIIC method is convergent of order 2.

5 Future work

The special case of (2.1) and (3.1) (when $\lambda = 0$) and the general case of (2.1) and (3.1) (when the impulsive interval does not equal to the delay τ) will be studied in the future.

Data Availability The datasets generated during the current study are available from the corresponding author on reasonable request.

References

Anguraj A, Karthikeyan K (2009) Existence of solutions for impulsive neutral functional differential equations with nonlocal conditions. Nonlinear Anal 70:2717–2721

Bellen A, Jackiewicz Z, Zennaro M (1988) Stability analysis of one-step methods for neutral delay-differential equations. Numer Math 52:605–619

Cuevasa C, Hernándezb E, Rabelo M (2009) The existence of solutions for impulsive neutral functional differential equations. Comput Math Appl 58:744–757

Engelborghs K, Luzyanina T, Hout KJ, Roose D (2001) Collocation methods for the computation of periodic solutions of delay differential equations. SIAM J Sci Comput 22:1593–1609

Enright WH, Hayashi H (1998) Convergence analysis of the solution of retarded and neutral delay differential equations by continuous numerical methods. SIAM J Numer Anal 35:572–585

Li XD, Deng FQ (2017) Razumikhin method for impulsive functional differential equations of neutral type. Chaos, Solitons Fractals 101:41–49

Li TX, Rogovchenko YV (2015) Oscillation of second-order neutral differential equations. Math Nachr 288:1150–1162

Li TX, Rogovchenko YV (2016) Oscillation criteria for even-order neutral differential equations. Appl Math Lett 61:35–41

Li TX, Rogovchenko YV (2020) On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations. Appl Math Lett 105:1–7

Liu YK (1999) Numerical solution of implicit neutral functional differential equations. SIAM J Numer Anal 36:516–528

Wang WS, Li SF (2004) Stability analysis of nonlinear delay differential equations of neutral type. Math Numer Sin 26:303–314

Wang WS, Li SF (2008) Stability analysis of θ-methods for nonlinear neutral functional differential equations. SIAM J Sci Comput 30:2181–2205

Wang WS, Zhang Y, Li SF (2009) Stability of continuous Runge–Kutta-type methods for nonlinear neutral delay-differential equations. Appl Math Model 33:3319–3329

Xue Z, Han XX, Wu KN (2023) Mean square exponential input-to state stability of stochastic Markovian reaction diffusion systems with impulsive perturbations. J Franklin Inst 360:7085–7104

Zhang GL, Song MH, Liu MZ (2015) Asymptotical stability of the exact solutions and the numerical solutions for a class of impulsive differential equations. Appl Math Comput 258:12–21