Computational and Applied Mathematics (2023) 42:343
https://doi.org/10.1007/s40314-023-02483-8

Generalized bilateral inverses of tensors via Einstein product with applications to singular tensor equations

Ehsan Kheirandish[1] · **Abbas Salemi**[1]

Received: 2 August 2023 / Revised: 25 September 2023 / Accepted: 29 September 2023 /
Published online: 26 October 2023

Abstract

In this paper, a unified approach for various extended inverses of tensors, *the generalized bilateral inverse of tensors via Einstein products*, is introduced and we show that a number of known generalized tensor inverses can be regarded as special cases of this idea. Some characterizations of the *CMP*, *DMP*, and *MPD* inverse of tensors by using Einstein products are provided. The notion of generalized bilateral inverses' dual and self-duality are investigated. In addition, the bilateral inverse solutions for singular linear tensor equations are studied.

Keywords Tensor · Generalized bilateral inverse of tensor · Dual · CMP inverse · DMP inverse · Einstein product

Mathematics Subject Classification 15A09 · 15A69 · 65F20

1 Introduction

Tensors are higher-dimensional generalizations of matrices and can thus be viewed as multidimensional array (Weiyang and Yimin 2016; Wei et al. 2018). Tensors have various applications, such as data mining (Eldén 2007), machine learning (Rabanser et al. 2017), computer vision (Cyganek and Gruszczyński 2014), automation systems (Zhao et al. 2017), neuroscience (Beckmann and Smith 2005) etc.

Let $\mathbb{C}^{I_1\times\cdots\times I_M}$ denotes the set of all tensors of order M and their elements are denoted as $\mathcal{A} = (a_{i_1,i_2,\cdots,i_M})_{1\le i_j\le I_j}$, $j = 1,\ldots,M$. Suppose that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$. Then $\mathcal{A}^* \in \mathbb{C}^{J_1\times\cdots\times J_N\times I_1\times\cdots\times I_M}$ is a conjugate transpose of $\mathcal{A}$ and is defined as $(\mathcal{A}^*)_{j_1\cdots j_N i_1\cdots i_M} = \bar{a}_{i_1\cdots i_M j_1\cdots j_N}$, where the over-line stands for the conjugate of $a_{i_1\cdots i_M j_1\cdots j_N}$. If the tensor $\mathcal{A}$ is real, then its transpose is represented by $\mathcal{A}^T$.

Communicated by Yimin Wei.

✉ Ehsan Kheirandish
ehsankheirandish@math.uk.ac.ir

Abbas Salemi
salemi@uk.ac.ir

[1] Department of Applied Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran

Consider the Einstein product of two tensors, $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times K_1\times\cdots\times K_N}$ and $\mathcal{B} \in \mathbb{C}^{K_1\times\cdots\times K_N\times J_1\times\cdots\times J_M}$. The Einstein product $\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{K_1\times\cdots\times K_N\times J_1\times\cdots\times J_M}$ was defined as in Einstein (2007), using the operation via $*_N$

$$(\mathcal{A} *_N \mathcal{B})_{i_1\cdots i_N j_1\cdots j_M} = \sum_{k_1\cdots k_N} a_{i_1\cdots i_N k_1\cdots k_N} b_{k_1\cdots k_N j_1\cdots j_M}.$$

Suppose that $\mathcal{B} \in \mathbb{C}^{K_1\times\cdots\times K_N}$. Thus,

$$\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_N} \quad \& \quad (\mathcal{A} *_N \mathcal{B})_{i_1\cdots i_N} = \sum_{k_1\cdots k_N} a_{i_1\cdots i_N k_1\cdots k_N} b_{k_1\cdots k_N}.$$

Definition 1 Sun et al. (2016) Let $\mathcal{D} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then the tensor $\mathcal{D}$ is diagonal if $(\mathcal{D})_{i_1\cdots i_N\times j_1\cdots j_N} = 0$ for $(i_1,\ldots,i_N) \neq (j_1,\ldots,j_N)$.

Suppose that $\mathcal{I} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ is the identity tensor. Then the tensor $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ is considered the inverse of tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ if it satisfies the condition $\mathcal{X} *_N \mathcal{A} = \mathcal{A} *_N \mathcal{X} = \mathcal{I}$ and it is represented by $\mathcal{A}^{-1}$ (see Brazell et al. 2013).

Suppose that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$. If $\mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_M\times I_1\times\cdots\times I_N}$ satisfies $\mathcal{A} *_M \mathcal{X} *_N \mathcal{A} = \mathcal{A}$, then $\mathcal{X}$ is referred to as an inner inverse of tensor $\mathcal{A}$. Alternatively, if $\mathcal{X} *_N \mathcal{A} *_M \mathcal{X} = \mathcal{X}$, then $\mathcal{X}$ is referred to as an outer inverse of tensor $\mathcal{A}$. Throughout this paper, the following notations are established.

$$\mathcal{G}_i(\mathcal{A}) := \{\mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_M\times I_1\times\cdots\times I_N} : \mathcal{A} *_M \mathcal{X} *_N \mathcal{A} = \mathcal{A}\},$$
$$\mathcal{G}_o(\mathcal{A}) := \{\mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_M\times I_1\times\cdots\times I_N} : \mathcal{X} *_N \mathcal{A} *_M \mathcal{X} = \mathcal{X}\}.$$

Furthermore, if $\mathcal{X} \in \mathcal{G}_r(\mathcal{A}) := \mathcal{G}_i(\mathcal{A}) \cap \mathcal{G}_o(\mathcal{A})$, then $\mathcal{X}$ is represented as the reflexive inverse of $\mathcal{A}$.

Definition 2 Sun et al. (2016, Definition 2.2) Suppose that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$. The tensor $\mathcal{X} \in \mathcal{G}_r(\mathcal{A})$ that satisfies the following:

$$(\mathcal{A} *_M \mathcal{X})^* = \mathcal{A} *_M \mathcal{X} \quad \& \quad (\mathcal{X} *_N \mathcal{A})^* = \mathcal{X} *_N \mathcal{A},$$

is referred to as the Moore-Penrose inverse of the tensor $\mathcal{A}$.

For $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times K_1\times\cdots\times K_N}$, the null space $N(\mathcal{A})$ and the range $R(\mathcal{A})$ are defined by:

$$N(\mathcal{A}) = \{\mathcal{A} *_N \mathcal{X} = \mathcal{O} : \mathcal{X} \in \mathbb{C}^{K_1\times\cdots\times K_N}\} \quad \& \quad R(\mathcal{A}) = \{\mathcal{A} *_N \mathcal{X} : \mathcal{X} \in \mathbb{C}^{K_1\times\cdots\times K_N}\},$$

where $\mathcal{O}$ is the zero tensor (see Ji and Wei 2018).

Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Define $\mathcal{A}^e := \mathcal{A}^{e-1} *_N \mathcal{A}, \quad for \quad e \geq 2$.

Note that

$$\{0\} = N(\mathcal{I}) \subseteq N(\mathcal{A}) \subseteq N(\mathcal{A}^2) \subseteq \cdots \subseteq N(\mathcal{A}^e) \subseteq N(\mathcal{A}^{e+1}) \subseteq \cdots \subseteq \mathbb{C}^{I_1\times\cdots\times I_N},$$
$$\{0\} \subseteq \cdots \subseteq R(\mathcal{A}^{e+1}) \subseteq R(\mathcal{A}^e) \subseteq \cdots \subseteq R(\mathcal{A}^2) \subseteq R(\mathcal{A}) \subseteq R(\mathcal{I}) = \mathbb{C}^{I_1\times\cdots\times I_N}.$$

In Ji and Wei (2018), the index of a tensor $\mathcal{A}$ is represented by $index(\mathcal{A})$ is defined as the smallest non-negative integer e such that $R(\mathcal{A}^{e+1}) = R(\mathcal{A}^e)$ or $N(\mathcal{A}^{e+1}) = N(\mathcal{A}^e)$.

Definition 3 Ji and Wei (2018, Theorem 3.3) The Drazin inverse of $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ with index($\mathcal{A}$)=k, is the tensor $\mathcal{X} \in \mathcal{G}_o$, which satisfies:

$$\mathcal{A} *_N \mathcal{X} = \mathcal{X} *_N \mathcal{A} \quad \& \quad \mathcal{A}^{k+1} *_N \mathcal{X} = \mathcal{A}^k.$$

The Drazin inverse is represented by $\mathcal{A}^d$. For more information (see Sahoo et al. 2020; Du et al. 2019; Ma et al. 2019; Wang et al. 2023; Wang and Wei 2022; Sun et al. 2018; Bu et al. 2014).

Theorem 4 *Wang et al. (2020, Theorem 1.1) Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then $\mathcal{A}$ can be represented as the sum of two tensors $C_\mathcal{A}$ and $N_\mathcal{A}$, such that, $\mathcal{A} = C_\mathcal{A} + N_\mathcal{A}$, where $index(C_\mathcal{A}) \le 1$, $N_\mathcal{A}$ is nilpotent and $C_\mathcal{A} *_N N_\mathcal{A} = N_\mathcal{A} *_N C_\mathcal{A} = \mathcal{O}$.*

The tensors $C_\mathcal{A}$ and $N_\mathcal{A}$ are referred to as the *core* part and the *nilpotent* part of $\mathcal{A}$, respectively. It is readily seen that $C_\mathcal{A} = \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A}$.

Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. If the following conditions hold, the unique matrix $\mathcal{X} \in \mathcal{G}_o(\mathcal{A})$ is referred to as the *DMP inverse* of $\mathcal{A}$ and is represented by $\mathcal{A}^{d,\dagger}$ Wang et al. (2020, Theorem 2.2).

$$\mathcal{A}^k *_N \mathcal{X} = \mathcal{A}^k *_N \mathcal{A}^\dagger \quad \& \quad \mathcal{X} *_N \mathcal{A} = \mathcal{A}^d *_N \mathcal{A}.$$

Note that $\mathcal{A}^{d,\dagger} = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger$.

By employing the same approach as in Wang et al. (2020, Theorem 2.2), the following holds.

Proposition 1 *Suppose that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ with index($\mathcal{A}$) $= k$. Then $\mathcal{X} = \mathcal{A}^{\dagger,d} = \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d$ is the unique solution of the following:*

$$\mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X} \quad \& \quad \mathcal{A} *_N \mathcal{X} = \mathcal{A} *_N \mathcal{A}^d \quad \& \quad \mathcal{X} *_N \mathcal{A}^k = \mathcal{A}^\dagger *_N \mathcal{A}^k. \tag{1}$$

Definition 5 Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ with index($\mathcal{A}$) $= k$. Then The MPD inverse of $\mathcal{A}$, represented by $\mathcal{A}^{\dagger,d}$, the definition is as follows

$$\mathcal{A}^{\dagger,d} := \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d. \tag{2}$$

Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. If the following conditions hold, the unique matrix $\mathcal{X} \in \mathcal{G}_o(\mathcal{A})$ is referred to as the *CMP inverse* of $\mathcal{A}$ and is represented by $\mathcal{A}^{c,\dagger} = \mathcal{A}^\dagger *_N C_\mathcal{A} *_N \mathcal{A}^\dagger$ Wang et al. (2020).

$$\mathcal{A} *_N \mathcal{X} = C_\mathcal{A} *_N \mathcal{A}^\dagger \quad \& \quad \mathcal{X} *_N \mathcal{A} = \mathcal{A}^\dagger *_N C_\mathcal{A} \quad \& \quad \mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = C_\mathcal{A}. \tag{3}$$

2 CMP and DMP generalized inverses of tensors

This section introduces novel characterizations of CMP, DMP, and MPD inverses of tensors.

The theorem below demonstrates that one of the conditions in Wang et al. (2020, Theorem 2.7) is unnecessary.

Theorem 6 *Suppose that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then $\mathcal{X} = \mathcal{A}^{c,\dagger}$ is the unique solution of the following:*

$$\mathcal{A} *_N \mathcal{X} = C_\mathcal{A} *_N \mathcal{A}^\dagger \quad \& \quad \mathcal{X} *_N \mathcal{A} = \mathcal{A}^\dagger *_N C_\mathcal{A} \quad \& \quad \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}. \tag{4}$$

Proof It is obvious that the tensor $\mathcal{X} = \mathcal{A}^{c,\dagger}$ satisfies the system (4). Assume that two tensors $\mathcal{X}_1$ and $\mathcal{X}_2$ satisfy (4), then

$$\begin{aligned}\mathcal{X}_1 &= \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_1 = \mathcal{A}^\dagger *_N C_\mathcal{A} *_N \mathcal{X}_1 = \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{X}_1\\ &= \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d *_N C_\mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{X}_2\\ &= \mathcal{A}^\dagger *_N C_\mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2.\end{aligned}$$

□

A novel characterization of DMP inverses of tensors, which does not rely on the index of $\mathcal{A}$, is presented in the following (see Wang et al. (2020, Theorem 2.2)).

Theorem 7 *Suppose that* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. *Then* $\mathcal{X} = \mathcal{A}^{d,\dagger}$ *is the unique solution of the following:*

$$\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = C_{\mathcal{A}} *_N \mathcal{A}^{\dagger} \ \ \& \ \ \mathcal{X} *_N \mathcal{A} = \mathcal{A}^d *_N \mathcal{A} \ \ \& \ \ \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}. \tag{5}$$

Proof It is evident that the tensor $\mathcal{X} = \mathcal{A}^d *_N A *_N A^{\dagger}$ satisfies the system (5). Assume that two tensors $\mathcal{X}_1$ and $\mathcal{X}_2$ satisfy (5), then

$$\begin{aligned}
\mathcal{X}_1 &= \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_1 = \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_1 = \mathcal{X}_1 *_N C_{\mathcal{A}} *_N \mathcal{A}^{\dagger}\\
&= \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger} = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger}\\
&= \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger} = \mathcal{X}_2 *_N C_{\mathcal{A}} *_N \mathcal{A}^{\dagger}\\
&= \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2.
\end{aligned}$$

□

By employing the same approach as in the proof of Theorem 7, the following holds.

Corollary 8 *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. *Then* $\mathcal{X} = \mathcal{A}^{\dagger,d}$ *is the unique solution of the following:*

$$\mathcal{A} *_N \mathcal{X} = \mathcal{A} *_N \mathcal{A}^d \ \ \& \ \ \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A}^{\dagger} *_N \mathcal{C}_{\mathcal{A}} \ \ \& \ \ \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}.$$

In the following theorem we state a new characterization of $A^{c,\dagger}$.

Theorem 9 *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. *Then* $\mathcal{X} = \mathcal{A}^{c,\dagger}$ *is the unique solution satisfies in 6.*

$$\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = C_{\mathcal{A}} \ \ \& \ \ \mathcal{R}(\mathcal{X}) \subseteq \mathcal{R}(\mathcal{A}^*) \ \ \& \ \ \mathcal{R}(\mathcal{X}^*) \subseteq \mathcal{R}(\mathcal{A}), \tag{6}$$

Proof By (3),

$$\begin{aligned}
&\mathcal{A} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{A} = C_{\mathcal{A}},\\
\mathcal{A}^{c,\dagger} &= \mathcal{A}^{c,\dagger} *_N \mathcal{A} *_N \mathcal{A}^{c,\dagger} = (\mathcal{A}^{\dagger} *_N \mathcal{A})^* *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger} *_N \mathcal{A} *_N \mathcal{A}^{c,\dagger}\\
&= \mathcal{A}^* *_N (\mathcal{A}^{\dagger})^* *_N \mathcal{A}^{d,\dagger},\\
\mathcal{A}^{c,\dagger} &= \mathcal{A}^{c,\dagger} *_N \mathcal{A} *_N \mathcal{A}^{c,\dagger} = \mathcal{A}^{c,\dagger} *_N \mathcal{A} *_N \mathcal{A}^{\dagger} *_N \mathcal{A} *_N \mathcal{A}^d *_N (\mathcal{A} *_N \mathcal{A}^{\dagger})^*\\
&= \mathcal{A}^{\dagger,d} *_N (\mathcal{A}^{\dagger})^* *_N \mathcal{A}^*.
\end{aligned} \tag{7}$$

where $\mathcal{U} = (\mathcal{A}^{\dagger})^* *_N \mathcal{A}^{d,\dagger} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ and $\mathcal{V} = \mathcal{A}^{\dagger,d} *_N (\mathcal{A}^{\dagger})^* \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Therefore, by Stanimirović et al. (2020, Lemma 2.2 (a)), we obtain that $R(\mathcal{X}) \subseteq R(\mathcal{A}^*)$ and $R(\mathcal{X}^*) \subseteq R(\mathcal{A})$ are equivalent to $\mathcal{A}^{c,\dagger} = \mathcal{A}^* *_N \mathcal{U}$ and $\mathcal{A}^{c,\dagger} = \mathcal{V} *_N \mathcal{A}^*$, respectively. By the Eq. (7), it is clear to see that $\mathcal{A}^{c,\dagger}$ satisfies (6). Assume possible, there exist $\mathcal{X}_1$ and $\mathcal{X}_2$ such that $\mathcal{X}_1 \neq \mathcal{X}_2$, we have that

$$\mathcal{A} *_N \mathcal{X}_1 *_N \mathcal{A} = C_{\mathcal{A}} \ \ \& \ \ \mathcal{X}_1 = \mathcal{A}^* *_N \mathcal{U}_1 \ \ \& \ \ \mathcal{X}_1 = \mathcal{V}_1 *_N \mathcal{A}^*, \tag{8}$$

$$\mathcal{A} *_N \mathcal{X}_2 *_N \mathcal{A} = C_{\mathcal{A}} \ \ \& \ \ \mathcal{X}_2 = \mathcal{A}^* *_N \mathcal{U}_2 \ \ \& \ \ \mathcal{X}_2 = \mathcal{V}_2 *_N \mathcal{A}^*, \tag{9}$$

where $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1, \mathcal{V}_2 \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Let

$$\mathcal{X} = \mathcal{X}_2 - \mathcal{X}_1, \quad \mathcal{U} = \mathcal{U}_2 - \mathcal{U}_1, \quad \mathcal{V} = \mathcal{V}_2 - \mathcal{V}_1. \tag{10}$$

It then follows from (6), (8), (9) and (10),

$$\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = 0 \quad \& \quad \mathcal{X} = \mathcal{A}^* *_N \mathcal{U} \quad \& \quad \mathcal{X} = \mathcal{V} *_N \mathcal{A}^*.$$

By Panigrahy et al. (2020, Lemma 3.7), we have

$$\begin{aligned}(\mathcal{X} *_N \mathcal{A})^* *_N \mathcal{X} *_N \mathcal{A} &= \mathcal{A}^* *_N (\mathcal{X})^* *_N \mathcal{X} *_N \mathcal{A} \\ &= \mathcal{A}^* *_N (\mathcal{A}^* *_N \mathcal{U})^* *_N \mathcal{X} *_N \mathcal{A} \\ &= \mathcal{A}^* *_N (\mathcal{U})^* *_N (\mathcal{A} *_N \mathcal{X} *_N \mathcal{A}) = \mathcal{O}.\end{aligned}$$

Therefore, $\mathcal{X} *_N \mathcal{A} = \mathcal{O}$. Meanwhile,

$$\mathcal{X} *_N \mathcal{X}^* = \mathcal{X} *_N (\mathcal{V} *_N \mathcal{A}^*)^* = \mathcal{X} *_N \mathcal{A} *_N \mathcal{V}^* = \mathcal{O},$$

by Panigrahy et al. (2020, Remark 3.8), yields that $\mathcal{X} = \mathcal{O}$, and hence $\mathcal{X}_1 = \mathcal{X}_2$. Therefore, we conclude that unique tensor $\mathcal{X} = \mathcal{A}^{c,\dagger}$ satisfying (6). □

Corollary 10 *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. If there exist $\mathcal{X}$ and $\mathcal{Z}$ in $\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ satisfying*

$$\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{C}_{\mathcal{A}} \quad \& \quad \mathcal{X} = \mathcal{A}^* *_N \mathcal{Z} *_N \mathcal{A}^*,$$

then $\mathcal{X} = \mathcal{A}^{c,\dagger}$.

By employing the same approach as in the proof of Theorem 9, the following holds.

Corollary 11 *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then $\mathcal{X} = \mathcal{O}$ is the unique solution satisfies in 11.*

$$\mathcal{A} *_N \mathcal{X} = \mathcal{O} \quad \& \quad R(\mathcal{X}) \subseteq R(\mathcal{A}^*). \tag{11}$$

By using Corollary 11, we characterize $\mathcal{A}^{c,\dagger}$ by two relations.

Theorem 12 *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then $\mathcal{X} = \mathcal{A}^{c,\dagger}$ is the unique solution satisfies in 12.*

$$\mathcal{A} *_N \mathcal{X} = C_{\mathcal{A}} *_N \mathcal{A}^\dagger \quad \& \quad R(\mathcal{X}) \subseteq R(\mathcal{A}^*). \tag{12}$$

In the following theorem, we characterize $\mathcal{A}^{d,\dagger}$ by the relations in 13.

Theorem 13 *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then $\mathcal{X} = \mathcal{A}^{d,\dagger}$ is the unique solution satisfies in 13.*

$$\mathcal{A}^\dagger *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A}^{\dagger,d} \quad \& \quad R(\mathcal{X}) \subseteq R(\mathcal{A}) \quad \& \quad R(\mathcal{X}^*) \subseteq R(\mathcal{A}). \tag{13}$$

Proof It is clear that $\mathcal{A}^\dagger *_N \mathcal{A}^{d,\dagger} *_N \mathcal{A} = \mathcal{A}^{\dagger,d}$, $R(\mathcal{A}^{d,\dagger}) = R(\mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A}^\dagger) \subseteq R(\mathcal{A})$, and $R((\mathcal{A}^{d,\dagger})^*) = R((\mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger)^*) = R((\mathcal{A}^\dagger)^* *_N \mathcal{A}^* *_N (\mathcal{A}^d)^*) \subseteq R((\mathcal{A}^\dagger)^*) = R(\mathcal{A})$. That is, we have proved that $\mathcal{A}^{d,\dagger}$ satisfies (13). By Stanimirović et al. (2020, Lemma 2.2 (a)), from (13), we can assume that $\mathcal{X} = (\mathcal{A}^\dagger)^* *_N \mathcal{U}$ and $\mathcal{X} = \mathcal{V} *_N \mathcal{A}^*$ for some $\mathcal{U}, \mathcal{V} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$.

Assume possible, there exist $\mathcal{X}_1$ and $\mathcal{X}_2$ such that $\mathcal{X}_1 \neq \mathcal{X}_2$ and

$$\mathcal{A}^\dagger *_N \mathcal{X}_1 *_N \mathcal{A} = \mathcal{A}^{\dagger,d} \quad \& \quad \mathcal{X}_1 = (\mathcal{A}^\dagger)^* *_N \mathcal{U}_1 \quad \& \quad \mathcal{X}_1 = \mathcal{V}_1 *_N \mathcal{A}^*, \tag{14}$$

$$\mathcal{A}^\dagger *_N \mathcal{X}_2 *_N \mathcal{A} = \mathcal{A}^{\dagger,d} \quad \& \quad \mathcal{X}_2 = (\mathcal{A}^\dagger)^* *_N \mathcal{U}_2 \quad \& \quad \mathcal{X}_2 = \mathcal{V}_2 *_N \mathcal{A}^*, \tag{15}$$

where $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1, \mathcal{V}_2 \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Let

$$\mathcal{X} = \mathcal{X}_2 - \mathcal{X}_1, \quad \mathcal{U} = \mathcal{U}_2 - \mathcal{U}_1, \quad \mathcal{V} = \mathcal{V}_2 - \mathcal{V}_1. \tag{16}$$

It then follows from (13), (14), (15) and (16),

$$\mathcal{A}^{\dagger} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{O} \quad \& \quad \mathcal{X} = (\mathcal{A}^{\dagger})^* *_N \mathcal{U} \quad \& \quad \mathcal{X} = \mathcal{V} *_N \mathcal{A}^*.$$

By Panigrahy et al. (2020, Lemma 3.7), we have that

$$\begin{aligned}(\mathcal{X} *_N \mathcal{A})^* *_N \mathcal{X} *_N \mathcal{A} &= \mathcal{A}^* *_N (\mathcal{X})^* *_N \mathcal{X} *_N \mathcal{A} \\ &= \mathcal{A}^* *_N ((\mathcal{A}^{\dagger})^* *_N \mathcal{U})^* *_N \mathcal{X} *_N \mathcal{A} \\ &= \mathcal{A}^* *_N (\mathcal{U})^* *_N (\mathcal{A}^{\dagger} *_N \mathcal{X} *_N \mathcal{A}) = \mathcal{O}.\end{aligned}$$

We obtain $\mathcal{X} *_N \mathcal{A} = \mathcal{O}$. Meanwhile, we find

$$\mathcal{X} *_N \mathcal{X}^* = \mathcal{X} *_N (\mathcal{V} *_N \mathcal{A}^*)^* = (\mathcal{X} *_N \mathcal{A}) *_N \mathcal{V}^* = \mathcal{O},$$

by Panigrahy et al. (2020, Remark 3.8), we obtain $\mathcal{X} = \mathcal{O}$., and hence $\mathcal{X}_1 = \mathcal{X}_2$. Therefore, we conclude that unique tensor $\mathcal{X} = \mathcal{A}^{d,\dagger}$ satisfying (13). □

Corollary 14 *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. If there exist $\mathcal{X}, \mathcal{Z} \in M_n(\mathbb{C})$ satisfying*

$$\mathcal{A}^{\dagger} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A}^{\dagger,d} \quad \& \quad \mathcal{X} = \mathcal{A} *_N \mathcal{Z} *_N \mathcal{A}^*. \tag{17}$$

then $\mathcal{X} = \mathcal{A}^{d,\dagger}$.

By using Corollary 11, we characterize $\mathcal{A}^{d,\dagger}$ by two relations.

Theorem 15 *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then $\mathcal{X} = \mathcal{A}^{d,\dagger}$ is the unique solution satisfies in 18.*

$$\mathcal{A}^{\dagger} *_N \mathcal{X} = \mathcal{A}^{\dagger,d} *_N \mathcal{A}^{\dagger} \quad \& \quad R(\mathcal{X}) \subseteq R(\mathcal{A}). \tag{18}$$

By employing the same approach as in the proof of Theorem 13, the following hold.

Theorem 16 *Suppose that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then the solution satisfies in 19.*

$$\mathcal{A} *_N \mathcal{X} *_N \mathcal{A}^{\dagger} = \mathcal{A}^{d,\dagger} \quad \& \quad R(\mathcal{X}) \subseteq R(\mathcal{A}^*) \quad \& \quad R(\mathcal{X}^*) \subseteq R(\mathcal{A}^*). \tag{19}$$

is unique and is given by $\mathcal{X} = \mathcal{A}^{\dagger,d}$.

Corollary 17 *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. If there exist $\mathcal{X}, \mathcal{Z} \in M_n(\mathbb{C})$ satisfying*

$$\mathcal{A} *_N \mathcal{X} *_N \mathcal{A}^{\dagger} = \mathcal{A}^{d,\dagger} \quad \& \quad \mathcal{X} = \mathcal{A}^* *_N \mathcal{Z} *_N \mathcal{A}. \tag{20}$$

then $\mathcal{X} = \mathcal{A}^{\dagger,d}$.

By using Corollary 11, we characterize $\mathcal{A}^{\dagger,d}$ by two relations.

Theorem 18 *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then the solution satisfies in 21.*

$$\mathcal{A} *_N \mathcal{X} = \mathcal{A} *_N \mathcal{A}^{d} \quad \& \quad R(\mathcal{X}) \subseteq R(\mathcal{A}^*). \tag{21}$$

is unique and is given by $\mathcal{X} = \mathcal{A}^{\dagger,d}$.

First, we obtain the null space and the range of the outer inverse of the tensor $\mathcal{A}$.

Lemma 1 *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$ and $\mathcal{X} \in \mathcal{G}_o(\mathcal{A})$. Then*

$$\begin{aligned}R(\mathcal{I} - \mathcal{A} *_M \mathcal{X}) &= N(\mathcal{A} *_M \mathcal{X}) = N(\mathcal{X}), \\ N(\mathcal{I} - \mathcal{X} *_N \mathcal{A}) &= R(\mathcal{X} *_N \mathcal{A}) = R(\mathcal{X}).\end{aligned}$$

Proof Given that $\mathcal{X} *_N \mathcal{A}$ and $\mathcal{A} *_M \mathcal{X}$ are projections, we can conclude that:

$$R(\mathcal{I} - \mathcal{A} *_M \mathcal{X}) = N(\mathcal{A} *_M \mathcal{X}) \subseteq N(\mathcal{X} *_N \mathcal{A} *_M \mathcal{X}) = N(\mathcal{X}) \subseteq N(\mathcal{A} *_M \mathcal{X}),$$
$$N(\mathcal{I} - \mathcal{X} *_N \mathcal{A}) = R(\mathcal{X} *_N \mathcal{A}) \subseteq R(\mathcal{X}) = R(\mathcal{X} *_N \mathcal{A} *_M \mathcal{X}) \subseteq R(\mathcal{X} *_N \mathcal{A}).$$

□

Lemma 2 *(Panigrahy and Mishra (2022, Lemma 2.3)) If $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ is a Hermitian idempotent tensor, then $\mathcal{A}^\dagger = \mathcal{A}$.*

Remark 1 Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ is Hermitian idempotent tensor. Then $C_\mathcal{A} = \mathcal{A}^{d,\dagger} = \mathcal{A}^{\dagger,d}$.

Theorem 19 *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ with index($\mathcal{A}$) $= k$. The solution to the system of following:*

$$\mathcal{A}^k *_N \mathcal{X} = \mathcal{A}^{k+1} \quad \& \quad \mathcal{A} *_N \mathcal{X} = \mathcal{X} *_N \mathcal{A} \quad \& \quad \mathcal{X} *_N \mathcal{A}^d *_N \mathcal{X} = \mathcal{X} \tag{22}$$

is unique and is given by $\mathcal{X} = C_\mathcal{A}$.

Proof It is evident that the tensor $\mathcal{X} = C_\mathcal{A}$ satisfies the system (22). Assume that two tensors $\mathcal{X}_1$ and $\mathcal{X}_2$ satisfy (22), then by Behera et al. (2020, Lemma 3.1), we have

$$\begin{aligned}
\mathcal{X}_1 &= \mathcal{X}_1 *_N \mathcal{A}^d *_N \mathcal{X}_1 = \mathcal{X}_1 *_N (\mathcal{A}^d)^2 *_N \mathcal{A} *_N \mathcal{X}_1 \\
&= \mathcal{X}_1 *_N (\mathcal{A}^d)^2 *_N \mathcal{X}_1 *_N \mathcal{A} = \mathcal{X}_1 *_N (\mathcal{A}^d)^{k+2} *_N A^k *_N \mathcal{X}_1 *_N \mathcal{A} \\
&= \mathcal{X}_1 *_N (\mathcal{A}^d)^{k+2} *_N \mathcal{A}^{k+1} *_N \mathcal{A} = \mathcal{X}_1 *_N \mathcal{A}^{k+1} *_N (\mathcal{A}^d)^{k+2} *_N \mathcal{A} \\
&= \mathcal{A}^k *_N \mathcal{X}_1 *_N \mathcal{A} *_N (\mathcal{A}^d)^{k+2} *_N \mathcal{A} = \mathcal{A}^{k+1} *_N \mathcal{A} *_N (\mathcal{A}^d)^{k+2} *_N \mathcal{A} \\
&= \mathcal{A}^k *_N \mathcal{X}_2 *_N \mathcal{A} *_N (\mathcal{A}^d)^{k+2} *_N \mathcal{A} = \mathcal{X}_2 *_N \mathcal{A}^{k+1} *_N (\mathcal{A}^d)^{k+2} *_N \mathcal{A} \\
&= \mathcal{X}_2 *_N (\mathcal{A}^d)^{k+2} *_N \mathcal{A}^{k+1} *_N \mathcal{A} = \mathcal{X}_2 *_N (\mathcal{A}^d)^{k+2} *_N \mathcal{A}^k *_N \mathcal{X}_2 *_N \mathcal{A} \\
&= \mathcal{X}_2 *_N (\mathcal{A}^d)^2 *_N \mathcal{X}_2 *_N \mathcal{A} = \mathcal{X}_2 *_N (\mathcal{A}^d)^2 *_N \mathcal{A} *_N \mathcal{X}_2 \\
&= \mathcal{X}_2 *_N \mathcal{A}^d *_N \mathcal{X}_2 = \mathcal{X}_2.
\end{aligned}$$

□

Next result gives the aforementioned relationships in terms of mainly the core part of the tensor $\mathcal{A}$.

Theorem 20 *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ with index($\mathcal{A}$) $= k$. Then*

(i) $\mathcal{A}^{d,\dagger} *_N C_\mathcal{A} = C_\mathcal{A} *_N \mathcal{A}^{d,\dagger}$ *if and only if* $\mathcal{A}^{k+1} *_N \mathcal{A}^\dagger = \mathcal{A}^k$.
(ii) $\mathcal{A}^{\dagger,d} *_N C_\mathcal{A} = C_\mathcal{A} *_N \mathcal{A}^{\dagger,d}$ *if and only if* $\mathcal{A}^\dagger *_N \mathcal{A}^{k+1} = \mathcal{A}^k$.
(iii) $C_\mathcal{A} = \mathcal{A}^{d,\dagger} *_N \mathcal{A}$ *if and only if* $\mathcal{A}^k = \mathcal{A}^{k+1}$.
(iv) $C_\mathcal{A} = \mathcal{A}^{\dagger,d} *_N \mathcal{A}$ *if and only if* $\mathcal{A}^\dagger *_N \mathcal{A}^k = \mathcal{A}^k$.

Proof (i) By Ji and Wei (2018, Theorem 3.4 (1)) and Lemma 1, we have

$$\begin{aligned}
&\mathcal{A}^{d,\dagger} *_N C_\mathcal{A} = C_\mathcal{A} *_N \mathcal{A}^{d,\dagger} \\
&\Leftrightarrow \mathcal{A}^d *_N \mathcal{A} *_N (\mathcal{I} - \mathcal{A} *_N \mathcal{A}^\dagger) = \mathcal{O} \\
&\Leftrightarrow N(\mathcal{A}^\dagger) = N(\mathcal{A} *_N \mathcal{A}^\dagger) = R(\mathcal{I} - \mathcal{A} *_N \mathcal{A}^\dagger) \subseteq N(\mathcal{A}^d *_N \mathcal{A}) = N(\mathcal{A}^d) = N(\mathcal{A}^k) \\
&\Leftrightarrow \mathcal{A}^{k+1} *_N \mathcal{A}^\dagger = \mathcal{A}^k.
\end{aligned}$$

(ii) and (iii) are similar to part (i).

(iv)

$$
\begin{aligned}
C_{\mathcal{A}} = \mathcal{A}^{\dagger,d} *_N \mathcal{A} &\Leftrightarrow C_{\mathcal{A}} = \mathcal{A}^{\dagger} *_N C_{\mathcal{A}} \\
&\Leftrightarrow (\mathcal{I} - \mathcal{A}^{\dagger}) *_N C_{\mathcal{A}} = \mathcal{O} \\
&\Leftrightarrow R(C_{\mathcal{A}}) \subseteq N(\mathcal{I} - \mathcal{A}^{\dagger}).
\end{aligned}
$$

By Ji and Wei (2018, Theorem 3.4 (1)), we can conclude that

$$
\begin{aligned}
R(C_{\mathcal{A}}) &= R(\mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}) \subseteq R(\mathcal{A}^d) = R(\mathcal{A}^k) \\
&= R(C_{\mathcal{A}} *_N \mathcal{A}^d *_N \mathcal{A}^k) \subseteq R(C_{\mathcal{A}}).
\end{aligned}
$$

Therefore, $R(C_{\mathcal{A}}) = R(\mathcal{A}^k)$. We obtain $R(\mathcal{A}^k) \subseteq N(\mathcal{I} - \mathcal{A}^{\dagger}) \Leftrightarrow \mathcal{A}^{\dagger} *_N \mathcal{A}^k = \mathcal{A}^k$. □

Hartwig and Spindelböck decomposition of tensor $\mathcal{A}$ arrived at the following lemma.

Lemma 3 *Wang et al. (2020, Lemma 1.3) Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then there exist unitary $\mathcal{U} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ such that*

$$
\mathcal{A} = \mathcal{U} *_N \begin{pmatrix} \Sigma *_N \mathcal{K} & \Sigma *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*, \tag{23}
$$

where $\Sigma \in \mathbb{C}^{R_1\times\cdots\times R_N\times R_1\times\cdots\times R_N}$ is a diagonal tensor of singular values of tensor $\mathcal{A}$, and the tensors $\mathcal{K} \in \mathbb{C}^{R_1\times\cdots\times R_N\times R_1\times\cdots\times R_N}$, $\mathcal{L} \in \mathbb{C}^{R_1\times\cdots\times R_N\times(I_1-R_1)\times\cdots\times(I_N-R_N)}$ satisfy:

$$
\mathcal{K} *_N \mathcal{K}^* + \mathcal{L} *_N \mathcal{L}^* = \mathcal{I}. \tag{24}
$$

Using the same approach as described in the proof of Wang et al. (2020, Theorem 2.3), the following holds.

Corollary 21 *Suppose that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ is in the form of (23). Then*

$$
\mathcal{A}^{\dagger,d} = \mathcal{U} *_N \begin{pmatrix} \mathcal{K}^* *_N \tilde{\Sigma} & \mathcal{K}^* *_N \tilde{\Sigma} *_N (\Sigma *_N \mathcal{K})^d *_N \Sigma *_N \mathcal{L} \\ \mathcal{L}^* *_N \tilde{\Sigma} & \mathcal{L}^* *_N \tilde{\Sigma} *_N (\Sigma *_N \mathcal{K})^d *_N \Sigma *_N \mathcal{L} \end{pmatrix} *_N \mathcal{U}^*, \tag{25}
$$

where $\tilde{\Sigma} = \mathcal{K} *_N (\Sigma *_N \mathcal{K})^d$.

We extend the recently obtained properties by using CMP inverse to the tensor (see Mehdipour and Salemi (2018, p. 4 (9))).

Theorem 22 *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ be of the form (23). Then*

$$
(\mathcal{A}^{c,\dagger})^{\dagger} = \mathcal{U} *_N \begin{pmatrix} (\tilde{\Sigma})^{\dagger} *_N \mathcal{K} & (\tilde{\Sigma})^{\dagger} *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*. \tag{26}
$$

Proof Suppose that $\mathcal{A}$ is expressed as shown in (23) and

$$
\mathcal{X} = \mathcal{U} *_N \begin{pmatrix} (\tilde{\Sigma})^{\dagger} *_N \mathcal{K} & (\tilde{\Sigma})^{\dagger} *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.
$$

By Wang et al. (2020, p. 7(2.6)) and (24), we have that

$$
\begin{aligned}
&\mathcal{X} *_N \mathcal{A}^{c,\dagger} \\
&= \mathcal{U} *_N \begin{pmatrix} (\tilde{\Sigma})^{\dagger} *_N \mathcal{K} & (\tilde{\Sigma})^{\dagger} *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* *_N \mathcal{U} *_N \begin{pmatrix} \mathcal{K}^* *_N \tilde{\Sigma} & \mathcal{O} \\ \mathcal{L}^* *_N \tilde{\Sigma} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* *_N
\end{aligned}
$$

$$= \mathcal{U} *_N \begin{pmatrix} (\tilde{\Sigma})^\dagger *_N \tilde{\Sigma} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

According to Definition 2, it is straightforward to calculate the first equation, which states that.

$$\begin{aligned}
&\mathcal{A}^{c,\dagger} *_N \mathcal{X} *_N \mathcal{A}^{c,\dagger} \\
&= \mathcal{U} *_N \begin{pmatrix} \mathcal{K}^* *_N \tilde{\Sigma} & \mathcal{O} \\ \mathcal{L}^* *_N \tilde{\Sigma} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* *_N \mathcal{U} *_N \begin{pmatrix} (\tilde{\Sigma})^\dagger *_N \tilde{\Sigma} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\
&= \mathcal{U} *_N \begin{pmatrix} \mathcal{K}^* *_N \tilde{\Sigma} & \mathcal{O} \\ \mathcal{L}^* *_N \tilde{\Sigma} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* = \mathcal{A}^{c,\dagger}.
\end{aligned}$$

Moreover, the second equation

$$\begin{aligned}
&\mathcal{X} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{X} \\
&= \mathcal{U} *_N \begin{pmatrix} (\tilde{\Sigma})^\dagger *_N \tilde{\Sigma} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* *_N \mathcal{U} *_N \begin{pmatrix} (\tilde{\Sigma})^\dagger *_N \mathcal{K} & (\tilde{\Sigma})^\dagger *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\
&= \mathcal{U} *_N \begin{pmatrix} (\tilde{\Sigma})^\dagger *_N \mathcal{K} & (\tilde{\Sigma})^\dagger *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* = \mathcal{X}.
\end{aligned}$$

The third equation follows from

$$\begin{aligned}
&(\mathcal{A}^{c,\dagger} *_N \mathcal{X})^* \\
&= \left(\mathcal{U} *_N \begin{pmatrix} \mathcal{K}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{K} & \mathcal{K}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{L} \\ \mathcal{L}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{K} & \mathcal{L}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{L} \end{pmatrix} *_N \mathcal{U}^* \right)^* \\
&= \mathcal{U} *_N \begin{pmatrix} \mathcal{K}^* *_N \left(\tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger\right)^* *_N \mathcal{K} & \mathcal{K}^* *_N \left(\tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger\right)^* *_N \mathcal{L} \\ \mathcal{L}^* *_N \left(\tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger\right)^* *_N \mathcal{K} & \mathcal{L}^* *_N \left(\tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger\right)^* *_N \mathcal{L} \end{pmatrix} *_N \mathcal{U}^* \\
&= \mathcal{U} *_N \begin{pmatrix} \mathcal{K}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{K} & \mathcal{K}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{L} \\ \mathcal{L}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{K} & \mathcal{L}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{L} \end{pmatrix} *_N \mathcal{U}^* \\
&= \mathcal{A}^{c,\dagger} *_N \mathcal{X}.
\end{aligned}$$

The fourth equation follows from

$$\begin{aligned}
(\mathcal{X} *_N \mathcal{A}^{c,\dagger})^* &= \left(\mathcal{U} *_N \begin{pmatrix} (\tilde{\Sigma})^\dagger *_N \tilde{\Sigma} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \right)^* \\
&= \mathcal{U} *_N \begin{pmatrix} \left((\tilde{\Sigma})^\dagger *_N \tilde{\Sigma}\right)^* & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\
&= \mathcal{X} *_N \mathcal{A}^{c,\dagger}.
\end{aligned}$$

The tensor $\mathcal{X}$ fulfills four equations. Assume that both $\mathcal{W}$ and $\mathcal{Z}$ also satisfy four equations each. In order to demonstrate the uniqueness, we need to show that

$$\begin{aligned}
\mathcal{W} &= \mathcal{W} *_N (\mathcal{A} *_N \mathcal{W})^* = \mathcal{W} *_N \mathcal{W}^* *_N \mathcal{A}^* = \mathcal{W} *_N \mathcal{W}^* *_N \mathcal{A}^* *_N \mathcal{Z}^* *_N \mathcal{A}^* \\
&= \mathcal{W} *_N (\mathcal{A} *_N \mathcal{W})^* *_N (\mathcal{A} *_N \mathcal{Z})^* = \mathcal{W} *_N \mathcal{A} *_N \mathcal{Z} \\
&= \mathcal{W} *_N \mathcal{A} *_N \mathcal{Z} *_N \mathcal{A} *_N \mathcal{Z} = (\mathcal{W} *_N \mathcal{A})^* *_N (\mathcal{Z} *_N \mathcal{A})^* *_N \mathcal{Z} \\
&= \mathcal{A}^* *_N \mathcal{W}^* *_N \mathcal{A}^* *_N \mathcal{Z}^* *_N \mathcal{Z} = (\mathcal{Z} *_N \mathcal{A})^* *_N \mathcal{Z} = \mathcal{Z}.
\end{aligned}$$

□

By using Theorem 22, we conclude the following.

Theorem 23 *Suppose that* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *be as in (23). Then* $\mathcal{A}^{\dagger,d} *_N (\mathcal{A}^{c,\dagger})^\dagger = (\mathcal{A}^{c,\dagger})^\dagger *_N \mathcal{A}^{d,\dagger}$ *if and only if the following conditions hold.*

1. $\mathcal{K}^* *_N \tilde{\Delta} *_N \mathcal{K} = (\tilde{\Sigma})^\dagger *_N \tilde{\Sigma}$,
2. $\mathcal{L}^* *_N \tilde{\Delta} = 0$,

where $\tilde{\Delta} = \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger$.

Proof By Wang et al. (2020, Theorem 2.3), (24), (25) and (26), we have

$$\mathcal{A}^{\dagger,d} *_N (\mathcal{A}^{c,\dagger})^\dagger = \mathcal{U} *_N \begin{pmatrix} \mathcal{K}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{K} & \mathcal{K}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{L} \\ \mathcal{L}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{K} & \mathcal{L}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{L} \end{pmatrix} *_N \mathcal{U}^*$$

$$(\mathcal{A}^{c,\dagger})^\dagger *_N \mathcal{A}^{d,\dagger} = \mathcal{U} *_N \begin{pmatrix} (\tilde{\Sigma})^\dagger *_N \tilde{\Sigma} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

Then $\mathcal{A}^{\dagger,d} *_N (\mathcal{A}^{c,\dagger})^\dagger = (\mathcal{A}^{c,\dagger})^\dagger *_N \mathcal{A}^{d,\dagger}$ if and only if the following conditions hold.

$$\mathcal{K}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{K} = (\tilde{\Sigma})^\dagger *_N \tilde{\Sigma}, \tag{27}$$

$$\mathcal{K}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{L} = \mathcal{O}, \tag{28}$$

$$\mathcal{L}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{L} = \mathcal{O}. \tag{29}$$

Note that the Eq. (27) and the Part 1 of Theorem 23 are equivalent. Since using (24), by left-multiplying the Eqs. (28) and (29) by $\mathcal{K}$ and $\mathcal{L}$, respectively, we obtain $\tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger *_N \mathcal{L} = \mathcal{O}$, equivalent to $\mathcal{L}^* *_N \tilde{\Sigma} *_N (\tilde{\Sigma})^\dagger = \mathcal{O}$, that is the Part 2 of Theorem 23. □

3 Generalized bilateral inverse of tensor via Einstein product

In this section, we expand upon the recently introduced concept of a *generalized bilateral inverse for a tensor* $\mathcal{A}$ *using the Einstein product*. Furthermore, we demonstrate that certain well-known generalized inverses can be viewed as specific instances of the generalized bilateral inverses for tensors (see Kheirandish and Salemi 2023).

Definition 24 Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$ and let $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{G}_i(\mathcal{A}) \cup \mathcal{G}_o(\mathcal{A})$. Then $\mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2$ is referred to as *generalized bilateral inverse of tensor* $\mathcal{A}$.

We will now present a theorem that characterizes the generalized bilateral inverses of tensors.

Theorem 25 *Suppose that* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$ *and suppose that* $\mathcal{X}_1 \in \mathcal{G}_o(\mathcal{A})$ *and* $\mathcal{X}_2 \in \mathcal{G}_i(\mathcal{A})$. *The unique solution to the system of following:*

$$\mathcal{X} *_N \mathcal{A} *_M \mathcal{X} = \mathcal{X},\ \mathcal{A} *_M \mathcal{X} *_N \mathcal{A} *_M \mathcal{X} = \mathcal{A} *_M \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2,\ \mathcal{X} *_N \mathcal{A} = \mathcal{X}_1 *_N \mathcal{A}. \tag{30}$$

is given by $\mathcal{X} = \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2$.

Proof Assume that $\mathcal{X} = \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2$ is a solution. Then

$$\mathcal{X} *_N \mathcal{A} *_M \mathcal{X} = \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2 *_N \mathcal{A} *_M \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2$$

$$\begin{aligned} &= \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2 = \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2 \\ \mathcal{A} *_M \mathcal{X} *_N \mathcal{A} *_M \mathcal{X} &= \mathcal{A} *_M \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2 *_N \mathcal{A} *_M \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2 \\ &= \mathcal{A} *_M \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2 \\ &= \mathcal{A} *_M \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2, \\ \mathcal{X} *_N \mathcal{A} &= \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2 *_N \mathcal{A} = \mathcal{X}_1 *_N \mathcal{A}. \end{aligned}$$

Suppose that two tensors $\mathcal{W}$ and $\mathcal{Z}$ satisfy (30), then

$$\begin{aligned} \mathcal{W} =& \mathcal{W} *_N \mathcal{A} *_M \mathcal{W} = \mathcal{W} *_N \mathcal{A} *_M \mathcal{W} *_N \mathcal{A} *_M \mathcal{W} \\ =& \mathcal{W} *_N \mathcal{A} *_M \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2 = \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2 \\ =& \mathcal{Z} *_N \mathcal{A} *_M \mathcal{X}_1 *_N \mathcal{A} *_M \mathcal{X}_2 = \mathcal{Z} *_N \mathcal{A} *_M \mathcal{Z} = \mathcal{Z}. \end{aligned}$$

□

Using the same approach as described in the proof of Theorem 25, the following holds.

Corollary 26 *Suppose that* $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$ *and suppose that* $\mathcal{X}_1 \in \mathcal{G}_o(\mathcal{A})$ *and* $\mathcal{X}_2 \in \mathcal{G}_i(\mathcal{A})$. *The unique solution to the system of following:*

$$\mathcal{X} *_N \mathcal{A} *_M \mathcal{X} = \mathcal{X}, \quad \mathcal{A} *_M \mathcal{X} = \mathcal{A} *_M \mathcal{X}_1, \quad \mathcal{X} *_N \mathcal{A} *_M \mathcal{X} *_N \mathcal{A} = \mathcal{X}_2 *_N \mathcal{A} *_M \mathcal{X}_1 *_N \mathcal{A}.$$

is given by $\mathcal{X} = \mathcal{X}_2 *_N \mathcal{A} *_M \mathcal{X}_1$.

The following proposition demonstrates that certain well-known generalized inverses of tensors can be regarded as generalized bilateral inverses of tensors.

Proposition 2 *Suppose that* $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$. *Then*

$$\begin{aligned} &(i)\ \mathcal{A}^{\dagger,d} = \mathcal{A}^{\dagger} *_N \mathcal{A} *_N \mathcal{A}^d, && (\mathcal{A}^{\dagger} \in \mathcal{G}_i(\mathcal{A}) \quad \&\ \mathcal{A}^d \in \mathcal{G}_o(\mathcal{A})). \\ &(ii)\ \mathcal{A}^{d,\dagger} = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger}, && (\mathcal{A}^d \in \mathcal{G}_o(\mathcal{A}) \quad \&\ \mathcal{A}^{\dagger} \in \mathcal{G}_i(\mathcal{A})). \\ &(iii)\ \mathcal{A}^{c,\dagger} = \mathcal{A}^{\dagger} *_N \mathcal{A} *_N \mathcal{A}^{d,\dagger}, && (\mathcal{A}^{\dagger} \in \mathcal{G}_i(\mathcal{A}) \quad \&\ \mathcal{A}^{d,\dagger} \in \mathcal{G}_o(\mathcal{A})). \\ &(iv)\ \mathcal{A}^{c,\dagger} = \mathcal{A}^{\dagger,d} *_N \mathcal{A} *_N \mathcal{A}^{\dagger}, && (\mathcal{A}^{\dagger,d} \in \mathcal{G}_o(\mathcal{A}) \quad \&\ \mathcal{A}^{\dagger} \in \mathcal{G}_i(\mathcal{A})). \end{aligned}$$

Next, will define the dual of the generalized bilateral inverse for tensors in the following manner:

Definition 27 Suppose that $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ and suppose that $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{G}_i(\mathcal{A}) \cup \mathcal{G}_o(\mathcal{A})$. Then the *dual of generalized bilateral inverse of tensor* $\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2$ is denoted by

$$(\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2)' := \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1,$$

and $\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2$ is called self dual, if $\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1$.

Now, we extend the recently obtained properties in Kheirandish and Salemi (2023) for tensors. Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$, $\mathcal{X}_1 \in \mathcal{G}_o(\mathcal{A})$ and $\mathcal{X}_2 \in \mathcal{G}_i(\mathcal{A})$. The following theorem presents the necessary and sufficient conditions for a generalized bilateral inverse of tensors $\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2$ to be self-dual.

Theorem 28 *Suppose that* $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$, $\mathcal{X}_1 \in \mathcal{G}_o(\mathcal{A})$ *and* $\mathcal{X}_2 \in \mathcal{G}_i(\mathcal{A})$. *Then, the following statements are equivalent.*

(i) $\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2$ *is self dual,*
(ii) $\mathcal{X}_1 = \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1$,

(iii) $N(\mathcal{A} *_N \mathcal{X}_2) \subseteq N(\mathcal{X}_1)$ *and* $R(\mathcal{X}_1) \subseteq R(\mathcal{X}_2 *_N \mathcal{A})$.

Proof $((i) \to (ii))$ Assume $\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1$. Since $\mathcal{A} *_N \mathcal{X}_2 *_N \mathcal{A} = \mathcal{A}$ and $\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_1 = \mathcal{X}_1$, we obtain that

$$\begin{aligned}\mathcal{X}_1 =& \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_1 = \mathcal{X}_1 *_N (\mathcal{A} *_N \mathcal{X}_2 *_N \mathcal{A}) *_N \mathcal{X}_1 \\ =&(\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2) *_N \mathcal{A} *_N \mathcal{X}_1 = (\mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1) *_N \mathcal{A} *_N \mathcal{X}_1 \\ =&\mathcal{X}_2 *_N \mathcal{A} *_N (\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_1) = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1.\end{aligned}$$

Then $\mathcal{X}_1 = \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1$.

$((ii) \to (iii))$ Since $\mathcal{X}_1 = \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2$, we obtain that $N(\mathcal{A} *_N \mathcal{X}_2) \subseteq N(\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2) = N(\mathcal{X}_1)$. Also, since $\mathcal{X}_1 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1$, we obtain that $R(\mathcal{X}_1) = R(\mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1) \subseteq R(\mathcal{X}_2 *_N \mathcal{A})$.

$((iii) \to (i))$ Using Lemma 1, we can see that $R(I - \mathcal{A} *_N \mathcal{X}_2) = N(\mathcal{A} *_N \mathcal{X}_2)$ and $N(\mathcal{A} *_N \mathcal{X}_2) \subseteq N(\mathcal{X}_1)$. Therefore, $R(I - \mathcal{A} *_N \mathcal{X}_2) \subseteq N(\mathcal{X}_1)$ which implies that $\mathcal{X}_1 *_N (I - \mathcal{A} *_N \mathcal{X}_2) = 0$. Hence, we have $\mathcal{X}_1 = \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2$. Similarly, using Lemma 1, we arrive $R(\mathcal{X}_2 *_N \mathcal{A}) = N(I - \mathcal{X}_2 *_N \mathcal{A})$ and $R(\mathcal{X}_1) \subseteq R(\mathcal{X}_2 *_N \mathcal{A}) = N(I - \mathcal{X}_2 *_N \mathcal{A})$. This implies that $(I - \mathcal{X}_2 *_N \mathcal{A}) *_N \mathcal{X}_1 = 0$. Therefore we have $\mathcal{X}_1 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1$, which completes the proof. □

Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$, $\mathcal{X}_1 = \mathcal{A}^D$, $\mathcal{X}_2 = \mathcal{A}^\dagger$. By Theorem 28, Lemma 1, Ji and Wei (2018, Theorem 3.4 (1)) and Sahoo et al. (2020, Theorem 3.7 a(i)), we deduce the following.

Proposition 3 *Suppose that* $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ *with index(*$\mathcal{A}$*)* $= k$*. Then, the following statements are equivalent.*

(i) $\mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d$,
(ii) $\mathcal{A}^d = \mathcal{A}^{d,\dagger} = \mathcal{A}^{\dagger,d}$,
(iii) $N(\mathcal{A}^*) \subseteq N(\mathcal{A}^k)$ & $R(\mathcal{A}^k) \subseteq R(\mathcal{A}^*)$.

The following theorem states the necessary and sufficient conditions for the generalized bilateral inverse of tensors to be self-dual.

Theorem 29 *Suppose that* $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$, $\mathcal{X}_2 \in \mathcal{G}_o(\mathcal{A})$ *and* $\mathcal{X}_1 \in \mathcal{G}_i(\mathcal{A}) \cup \mathcal{G}_o(\mathcal{A})$*. Then* $\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2$ *is self dual,* $\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2 = (\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2)' = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1$*, if and only if* $N(\mathcal{X}_2) \subseteq N(\mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1)$ *and* $R(\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2) \subseteq R(\mathcal{X}_2)$.

Proof Assume $\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1$. Since $\mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2$ and by Lemma 1, we obtain the following relations:

$$\mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1 = (\mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_2) *_N \mathcal{A} *_N \mathcal{X}_1 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2,$$

$$\mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1 *_N (\mathcal{I} - \mathcal{A} *_N \mathcal{X}_2) = 0, \tag{31}$$

$$N(\mathcal{X}_2) = N(\mathcal{A} *_N \mathcal{X}_2) = R(\mathcal{I} - \mathcal{A} *_N \mathcal{X}_2) \subseteq N(\mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1), \tag{32}$$

$$\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_1 *_N \mathcal{A} *_N (\mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_2) = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2$$

$$(I - \mathcal{X}_2 *_N \mathcal{A}) *_N \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2 = 0, \tag{33}$$

$$R(\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2) \subseteq N(\mathcal{I} - \mathcal{X}_2 *_N \mathcal{A}) = R(\mathcal{X}_2 *_N \mathcal{A}) = R(\mathcal{X}_2). \tag{34}$$

Therefore,

$$N(\mathcal{X}_2) \subseteq N(\mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1), \quad R(\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2) \subseteq R(\mathcal{X}_2).$$

Conversely, we know that Eqs. (31), (32), (33), and (34) are equivalent. Therefore, $\mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2$ and $\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2$. Therefore, $\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1$. □

Theorem 30 *Suppose that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then*

(i) $\mathcal{A}^d = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger$ *if and only if* $(\mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger)' = \mathcal{A}^{c,\dagger}$.
(ii) $\mathcal{A}^d = \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d$ *if and only if* $(\mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d)' = \mathcal{A}^{c,\dagger}$.

Proof (i)

$$\begin{aligned}&\mathcal{A}^d = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger,\\ &\Leftrightarrow \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d = \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger,\\ &\Leftrightarrow (\mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger)' = \mathcal{A}^{c,\dagger}.\end{aligned}$$

(ii)

$$\begin{aligned}&\mathcal{A}^d = \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d,\\ &\Leftrightarrow \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger\\ &\Leftrightarrow (\mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d)' = \mathcal{A}^{c,\dagger}.\end{aligned}$$

□

The remark below demonstrates that the dual of a generalized bilateral inverse $\mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_2$ is closely linked to $\mathcal{X}_1$ and $\mathcal{X}_2$.

Remark 2 Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Using Definition 27 and Proposition 2(iii)-(iv), it follows that $\mathcal{A}^{c\dagger} = \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^{d,\dagger} = \mathcal{A}^{\dagger,d} *_N \mathcal{A} *_N \mathcal{A}^\dagger$. But

$$\begin{aligned}(\mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^{d,\dagger})' = \mathcal{A}^{d,\dagger} *_N \mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{A}^{d,\dagger}.\\ (\mathcal{A}^{\dagger,d} *_N \mathcal{A} *_N \mathcal{A}^\dagger)' = \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^{\dagger,d} = \mathcal{A}^{\dagger,d},\end{aligned}$$

The theorem below presents the necessary and sufficient conditions for the generalized bilateral inverses of tensor $\mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^{d,\dagger}$ to be self dual.

Theorem 31 *Suppose that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ with index($\mathcal{A}$) $= k$. The following statements are equivalent:*

(i) $\mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^{d,\dagger} = \mathcal{A}^{d,\dagger} *_N \mathcal{A} *_N \mathcal{A}^\dagger$.
(ii) $R(\mathcal{A}^k) \subseteq R(\mathcal{A}^*)$ & $N(\mathcal{A}^*) \subseteq N(\mathcal{A}^k *_N \mathcal{A}^\dagger)$.

Proof From Theorem 28, we can conclude that $\mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^{d,\dagger} = \mathcal{A}^{d,\dagger} *_N \mathcal{A} *_N \mathcal{A}^\dagger$ if and only if $R(\mathcal{A}^{d,\dagger}) \subseteq R(\mathcal{A}^\dagger *_N \mathcal{A})$ and $N(\mathcal{A} *_N \mathcal{A}^\dagger) \subseteq N(\mathcal{A}^{d,\dagger})$. By applying Lemma 1, $R(\mathcal{A}^\dagger *_N \mathcal{A}) = R(\mathcal{A}^\dagger)$ and $N(\mathcal{A}^\dagger) = N(\mathcal{A} *_N \mathcal{A}^\dagger)$. Furthermore, according to Sahoo et al. (2020, the first part Theorem 3.7), we can conclude that $R(\mathcal{A}^\dagger) = R(\mathcal{A}^*)$ and $N(\mathcal{A}^\dagger) = N(\mathcal{A}^*)$. Moreover, by Ji and Wei (2018, Theorem 3.4 (1)) and Behera et al. (2020, Lemma 3.1), we have that

$$\begin{aligned}R(\mathcal{A}^{d,\dagger}) &= R(\mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger) \subseteq R(\mathcal{A}^d) = R(\mathcal{A}^k)\\ &= R(\mathcal{A}^{d,\dagger} *_N \mathcal{A} *_N \mathcal{A}^k) \subseteq R(\mathcal{A}^{d,\dagger})\\ N(\mathcal{A}^{d,\dagger}) &\subseteq N(\mathcal{A}^k *_N \mathcal{A}^{d,\dagger}) = N(\mathcal{A}^k *_N \mathcal{A}^\dagger) \subseteq N((\mathcal{A}^d)^k *_N \mathcal{A}^k *_N \mathcal{A}^\dagger)\\ &= N(\mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger) = N(\mathcal{A}^{d,\dagger}).\end{aligned}$$

Then, we have $R(\mathcal{A}^{d,\dagger}) = R(\mathcal{A}^k)$ and $N(\mathcal{A}^{d,\dagger}) = N(\mathcal{A}^k *_N \mathcal{A}^\dagger)$. Therefore, $R(\mathcal{A}^k) \subseteq R(\mathcal{A}^*)$ and $N(\mathcal{A}^*) \subseteq N(\mathcal{A}^k *_N \mathcal{A}^\dagger)$. □

4 Bilateral inverse solutions of singular tensor equations

Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ be a tensor with $index(\mathcal{A}) \geq 1$ and $\mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_N}$. As an application of the DMP, MPD and CMP inverses of tensor, we consider the following equation

$$\mathcal{A} *_N \mathcal{X} = \mathcal{B}. \tag{35}$$

First, we state the following theorem.

Theorem 32 *Suppose that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ with index$(\mathcal{A}) = k$. Then*

(i) The Eq. (35) has a solution $\mathcal{A}^{d,\dagger} *_N \mathcal{B}$ if and only if $\mathcal{B} \in R(\mathcal{A}^k)$.
(ii) The Eq. (35) has a solution $\mathcal{A}^{\dagger,d} *_N \mathcal{B}$ if and only if $\mathcal{B} \in R(\mathcal{A}^k)$.
(iii) The Eq. (35) has a solution $\mathcal{A}^{c,\dagger} *_N \mathcal{B}$ if and only if $\mathcal{B} \in R(\mathcal{A}^k)$.

Proof (i) Let $\mathcal{A}^{d,\dagger} *_N \mathcal{B}$ is a solution of (35). By Ji and Wei (2018, Theorem 3.4 (1)), we have

$$\mathcal{B} = \mathcal{A} *_N \mathcal{A}^{d,\dagger} *_N \mathcal{B} = \mathcal{A}^d *_N \mathcal{A}^2 *_N \mathcal{A}^\dagger *_N \mathcal{B} \in R(\mathcal{A}^d) = R(\mathcal{A}^k).$$

Suppose that $\mathcal{B} \in R(\mathcal{A}^k)$, by Stanimirović et al. (2020, Lemma 2.2 (a)), we can conclude that is a tensor $\mathcal{U} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ such that $\mathcal{B} = \mathcal{A}^k *_N \mathcal{U}$. Set $\mathcal{X}_1 = \mathcal{A}^{d,\dagger} *_N \mathcal{B}$. Thus,

$$\mathcal{A} *_N \mathcal{X}_1 = \mathcal{A} *_N \mathcal{A}^{d,\dagger} *_N \mathcal{B} = \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A}^k *_N \mathcal{U} = \mathcal{B},$$

implying that $\mathcal{A}^{d,\dagger} *_N \mathcal{B}$ is a solution of (35).

(ii) and (iii) have similar proofs to that of (i). □

Using the same approach as described in the proof of Theorem 32, the following holds.

Remark 3 Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ with index$(\mathcal{A}) = k$. Then

(i) $\mathcal{A}^{d,\dagger} *_N \mathcal{B} = \mathcal{A}^d *_N \mathcal{B}$, if $\mathcal{B} \in R(\mathcal{A})$.
(ii) $\mathcal{A}^{\dagger,d} *_N \mathcal{B} = \mathcal{A}^\dagger *_N \mathcal{B}$, if $\mathcal{B} \in R(\mathcal{A}^k)$.
(iii) $\mathcal{A}^{c,\dagger} *_N \mathcal{B} = \mathcal{A}^\dagger *_N \mathcal{B}$, if $\mathcal{B} \in R(\mathcal{A}^k)$.

Theorem 33 *Assume that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ with index$(\mathcal{A}) = k$ and assume that $\mathcal{B} \in R(\mathcal{A}^k)$. Then*

(i) The general solution of (35) takes of the form

$$\mathcal{X} = \mathcal{A}^{d,\dagger} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^\dagger *_N \mathcal{A}) * \mathcal{Y} \tag{36}$$

for any tensor $\mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N}$.
(ii) The Eq. (35) has the unique solution $\mathcal{A}^{d,\dagger} *_N \mathcal{B} \in R(\mathcal{A}^k)$.

Proof (i) By Theorem 32(i), $\mathcal{A}^{d,\dagger} *_N \mathcal{B}$ is a solution (35). Assume that $\mathcal{X} = \mathcal{A}^{d,\dagger} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^\dagger *_N \mathcal{A}) * (\mathcal{Y}_1 + \mathcal{Y}_2)$, where $\mathcal{Y}_1 \in \mathbb{C}^{I_1\times\cdots\times I_N}$ and $\mathcal{A} *_N \mathcal{Y}_2 = \mathcal{O}$. Then

$$\mathcal{A} *_N \mathcal{X} = \mathcal{A} *_N \mathcal{A}^{d,\dagger} *_N \mathcal{B} + (\mathcal{A} - \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A}) * (\mathcal{Y}_1 + \mathcal{Y}_2) = \mathcal{B},$$

that is $\mathcal{X}$ is a solution (35). Assume that $\mathcal{W}$ is any arbitrary solution of (35). It is clear that $R(\mathcal{I} - \mathcal{A}^\dagger *_N \mathcal{A}) \subseteq N(\mathcal{A})$ and $\mathcal{W} - \mathcal{A}^{d,\dagger} *_N \mathcal{B} \in N(\mathcal{A})$. Because

$$N(\mathcal{A}) = R(\mathcal{I} - \mathcal{A}^\dagger *_N \mathcal{A}) + \left(N(\mathcal{A}) \cap R(\mathcal{I} - \mathcal{A}^\dagger *_N \mathcal{A})^\perp\right),$$

we have that $\mathcal{W} - \mathcal{A}^{d,\dagger} *_N \mathcal{B} = (\mathcal{I} - \mathcal{A}^{\dagger} *_N \mathcal{A}) *_N \mathcal{W}_1 + \mathcal{W}_2$, where $\mathcal{W}_2 \in N(\mathcal{A}) \cap R(\mathcal{I} - \mathcal{A}^{\dagger} *_N \mathcal{A})^{\perp}$. Because $\mathcal{W}_2 \in N(\mathcal{A})$, we obtain $\mathcal{A} *_N \mathcal{W}_2 = \mathcal{O}$. Moreover,

$$\mathcal{W}_2 = \mathcal{W}_2 - \mathcal{A}^{\dagger} *_N \mathcal{A} *_N \mathcal{W}_2 = (\mathcal{I} - \mathcal{A}^{\dagger} *_N \mathcal{A}) *_N \mathcal{W}_2.$$

Therefore, $\mathcal{W} - \mathcal{A}^{d,\dagger} *_N \mathcal{B} = (\mathcal{I} - \mathcal{A}^{\dagger} *_N \mathcal{A}) * (\mathcal{W}_1 + \mathcal{W}_2)$, where $\mathcal{W}_1 \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ and $\mathcal{W}_2 \in N(\mathcal{A})$.

(ii) Let $\mathcal{X}$ be a solution in $R(\mathcal{A}^k)$. By Theorem 32(i), $\mathcal{A}^{d,\dagger} *_N \mathcal{B}$ is a solution in $R(\mathcal{A}^k)$. By the proof of Theorem 31, we have $R(\mathcal{A}^{d,\dagger}) = R(\mathcal{A}^k)$. We have that $\mathcal{X} - \mathcal{A}^{d,\dagger} *_N \mathcal{B} \in R(\mathcal{A}^k)$. Moreover, as stated in Part (i) of this theorem, we have that $\mathcal{X} - \mathcal{A}^{d,\dagger} *_N \mathcal{B} = (\mathcal{I} - \mathcal{A}^{\dagger} *_N \mathcal{A}) * \mathcal{Y}$ for some $\mathcal{Y}$. Now $\mathcal{A}^k *_N (\mathcal{X} - \mathcal{A}^{d,\dagger} *_N \mathcal{B}) = (\mathcal{A}^k - \mathcal{A}^k *_N \mathcal{A}^{\dagger} *_N \mathcal{A}) * \mathcal{Y} = \mathcal{O}$. Hence $\mathcal{X} - \mathcal{A}^{d,\dagger} *_N \mathcal{B} \in N(\mathcal{A}^k)$. Thus, $\mathcal{X} - \mathcal{A}^{d,\dagger} *_N \mathcal{B} \in R(\mathcal{A}^k) \cap N(\mathcal{A}^k) = \{\mathcal{O}\}$, that is $\mathcal{X} = \mathcal{A}^{d,\dagger} *_N \mathcal{B}$. □

Theorem 34 *Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ with index$(\mathcal{A}) = k$ and assume that $\mathcal{B} \in R(\mathcal{A}^k)$. Then*

(i) The general solution of (35) is of the form

$$\mathcal{X} = \mathcal{A}^{\dagger,d} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^{\dagger} *_N \mathcal{A}) * \mathcal{Y} \tag{37}$$

for any tensor $\mathcal{Y} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$.

(ii) The Eq. (35) has the unique solution $\mathcal{A}^{\dagger,d} *_N \mathcal{B} \in R(\mathcal{A}^{\dagger} *_N \mathcal{A}^k)$.

Proof (i) By using a method similar to the one employed in the proof of Theorem 33(i).

(ii) Suppose that $\mathcal{X}$ is a solution in $R(\mathcal{A}^{\dagger} *_N \mathcal{A}^k)$. Using a similar method as in the proof of Theorem 31, we obtain $R(\mathcal{A}^{\dagger,d}) = R(\mathcal{A}^{\dagger} *_N \mathcal{A}^k)$. This implies $\mathcal{X} - \mathcal{A}^{\dagger,d} *_N \mathcal{B} \in R(\mathcal{A}^{\dagger} *_N \mathcal{A}^k)$. Moreover, as stated in Part (i) of this theorem, we have that $\mathcal{X} - \mathcal{A}^{\dagger,d} *_N \mathcal{B} = (\mathcal{I} - \mathcal{A}^{\dagger} *_N \mathcal{A}) * \mathcal{Y}$ for some $\mathcal{Y}$. Now $\mathcal{A}^k *_N (\mathcal{X} - \mathcal{A}^{\dagger,d} *_N \mathcal{B}) = (\mathcal{A}^k - \mathcal{A}^k *_N \mathcal{A}^{\dagger} *_N \mathcal{A}) * \mathcal{Y} = \mathcal{O}$. Hence $\mathcal{X} - \mathcal{A}^{\dagger,d} *_N \mathcal{B} \in N(\mathcal{A}^k)$. Therefore,

$$\mathcal{X} - \mathcal{A}^{\dagger,d} *_N \mathcal{B} \in R(\mathcal{A}^{\dagger} *_N \mathcal{A}^k) \cap N(\mathcal{A}^k) \subseteq R(\mathcal{A}^{\dagger} *_N \mathcal{A}^k) \cap N(\mathcal{A}^{\dagger} *_N \mathcal{A}^k) = \{\mathcal{O}\},$$

that is $\mathcal{X} = \mathcal{A}^{\dagger,d} *_N \mathcal{B}$. □

Using the same approach as described in the proof of Theorem 34, the following holds.

Corollary 35 *Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ with index$(\mathcal{A}) = k$ and assume that $\mathcal{B} \in R(\mathcal{A}^k)$. Then*

(i) The general solution of (35) takes of the form

$$\mathcal{X} = \mathcal{A}^{c,\dagger} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^{\dagger} *_N \mathcal{A}) * \mathcal{Y} \tag{38}$$

for any tensor $\mathcal{Y} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$.

(ii) The Eq. (35) has the unique solution $\mathcal{A}^{c,\dagger} *_N \mathcal{B} \in R(\mathcal{A}^{\dagger} *_N \mathcal{A}^k)$.

As an application of the DMP, MPD and CMP inverses of tensor, we consider the following equation

$$C_{\mathcal{A}} *_N \mathcal{X} = \mathcal{B}, \tag{39}$$

where $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ and $\mathcal{X}, \mathcal{B} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$. Using the same approach as described in the proof of Theorem 32, the following holds.

Corollary 36 *Suppose that $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ with index$(\mathcal{A}) = k$. Then*

(i) *The Eq. (39) has a solution* $\mathcal{A}^{d,\dagger} *_N \mathcal{B}$ *if and only if* $\mathcal{B} \in R(\mathcal{A}^k)$.
(ii) *The Eq. (39) has a solution* $\mathcal{A}^{\dagger,d} *_N \mathcal{B}$ *if and only if* $\mathcal{B} \in R(\mathcal{A}^k)$.
(iii) *The Eq. (39) has a solution* $\mathcal{A}^{c,\dagger} *_N \mathcal{B}$ *if and only if* $\mathcal{B} \in R(\mathcal{A}^k)$.

Using the same approach as described in the proof of Theorems 33 and 34, the following hold.

Corollary 37 *Assume that* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *with index*$(\mathcal{A}) = k$ *and assume that* $\mathcal{B} \in R(\mathcal{A}^k)$. *Then*

(i) *The general solution of (39) takes of the form*

$$\mathcal{X} = \mathcal{A}^{d,\dagger} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^d *_N \mathcal{A}) *_N \mathcal{Y},$$

for any tensor $\mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N}$.
(ii) *The Eq. (39) has the unique solution* $\mathcal{A}^{d,\dagger} *_N \mathcal{B} \in R(\mathcal{A}^k)$.

Corollary 38 *Assume that* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *with index*$(\mathcal{A}) = k$ *and assume that* $\mathcal{B} \in R(\mathcal{A}^k)$. *Then*

(i) *The general solution of (39) is of the form*

$$\mathcal{X} = \mathcal{A}^{\dagger,d} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^d *_N \mathcal{A}) *_N \mathcal{Y},$$

for any tensor $\mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N}$.
(ii) *The Eq. (39) has the unique solution* $\mathcal{A}^{\dagger,d} *_N \mathcal{B} \in R(\mathcal{A}^\dagger *_N \mathcal{A}^k)$.

Corollary 39 *Assume that* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *with index*$(\mathcal{A}) = k$ *and assume that* $\mathcal{B} \in R(\mathcal{A}^k)$. *Then*

(i) *The general solution of (39) takes of the form*

$$\mathcal{X} = \mathcal{A}^{c,\dagger} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^d *_N \mathcal{A}) *_N \mathcal{Y},$$

for any tensor $\mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N}$.
(ii) *The Eq. (39) has the unique solution* $\mathcal{A}^{c,\dagger} *_N \mathcal{B} \in R(\mathcal{A}^\dagger *_N \mathcal{A}^k)$.

Suppose that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ and $\mathcal{X}, \mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_N}$. As an application of the core-part of $\mathcal{A}$, we consider the following equation:

$$\mathcal{A}^\dagger *_N \mathcal{X} = \mathcal{B}. \tag{40}$$

Theorem 40 *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *and* $\mathcal{B} \in R(\mathcal{A}^\dagger *_N \mathcal{A}^d)$. *Then the general solution of (40) takes of the form*

$$\mathcal{X} = C_{\mathcal{A}} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A} *_N \mathcal{A}^\dagger) *_N \mathcal{Y},$$

for any tensor $\mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N}$.

Proof Using a similar method as in the proof of Theorem 32, we obtain that $\mathcal{X}_0 = C_{\mathcal{A}} *_N \mathcal{B}$ is a solution of Eq. (40) if and only if $\mathcal{B} \in R(\mathcal{A}^\dagger *_N \mathcal{A}^d)$. Also, by Lemma 1, it is clear that $R(\mathcal{I} - \mathcal{A} *_N \mathcal{A}^\dagger) = N(\mathcal{A} *_N \mathcal{A}^\dagger) = N(\mathcal{A}^\dagger)$. Thus,

$$\mathcal{A}^\dagger *_N [C_{\mathcal{A}} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A} *_N \mathcal{A}^\dagger) *_N \mathcal{Y}] = \mathcal{A}^\dagger *_N C_{\mathcal{A}} *_N \mathcal{B} + \mathcal{O} = \mathcal{B}.$$

□

As an application of the DMP, MPD and CMP inverses of tensor, we consider the following equation

$$\mathcal{A}^{k+1} *_N \mathcal{X} = \mathcal{A}^k *_N \mathcal{B}, \tag{41}$$

where $index(\mathcal{A}) = k$ and $\mathcal{B} \in R(\mathcal{A}^k)$. If $\mathcal{B} \in R(\mathcal{A}^k)$ and $index(\mathcal{A}) = k$, then each member of the set $\{\mathcal{A}^d *_N \mathcal{B}, \mathcal{A}^{d,\dagger} *_N \mathcal{B}, \mathcal{A}^{\dagger,d} *_N \mathcal{B}, \mathcal{A}^{c,\dagger} *_N \mathcal{B}\}$ is a solution of Eqs. (35) and (41) (see Behera et al. (2020, P. 21)).

Theorem 41 *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ and $\mathcal{B} \in R(\mathcal{A}^k)$ with index$(\mathcal{A}) = k$. Then, the set of all solutions of (41) can be represented as*

$$\mathcal{X} = \mathcal{A}^{d,\dagger} *_N \mathcal{B} + N(\mathcal{A}^k).$$

Furthermore, the Eq. (41) has the unique solution $\mathcal{X} = \mathcal{A}^{d,\dagger} *_N \mathcal{B} \in R(\mathcal{A}^k)$.

Proof Assume $\mathcal{B} \in R(\mathcal{A}^k)$. By Stanimirović et al. (2020, Lemma 2.2 (a)), we can conclude that there is a tensor $\mathcal{U} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ such that $\mathcal{B} = \mathcal{A}^k *_N \mathcal{U}$.

$$\begin{aligned}\mathcal{A}^{k+1} *_N (\mathcal{X} - \mathcal{A}^{d,\dagger} *_N \mathcal{B}) &= \mathcal{A}^{k+1} *_N \mathcal{X} - \mathcal{A}^{k+1} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{B}\\ &= \mathcal{A}^k *_N \mathcal{B} - \mathcal{A}^k *_N \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A}^k *_N \mathcal{U}\\ &= \mathcal{A}^k *_N \mathcal{B} - \mathcal{A}^k *_N \mathcal{B} = \mathcal{O}.\end{aligned}$$

From Ji and Wei (2018, Theorem 3.2), we have that $\mathcal{X} - \mathcal{A}^{d,\dagger} *_N \mathcal{B} \in N(\mathcal{A}^{k+1}) = N(\mathcal{A}^k)$. Therefore, $\mathcal{X} = \mathcal{A}^{d,\dagger} *_N \mathcal{B} + N(\mathcal{A}^k)$. Let $\mathcal{X}$ be a solution in $R(\mathcal{A}^k)$. Moreover, by Theorem 32(i) and the proof of Theorem 31, we arrive $\mathcal{X} - \mathcal{A}^{d,\dagger} *_N \mathcal{B} \in R(\mathcal{A}^k)$. For the uniqueness in $R(\mathcal{A}^k)$, let $\mathcal{V} \in R(\mathcal{A}^k)$ be any solution of (41). Now $\mathcal{V} - \mathcal{A}^{d,\dagger} *_N \mathcal{B} \in R(\mathcal{A}^k)$, we have $\mathcal{A}^{k+1} *_N \mathcal{V} - \mathcal{A}^{k+1} *_N \mathcal{A}^{d,\dagger} *_N \mathcal{B} = \mathcal{O}$. So, $\mathcal{V} - \mathcal{A}^{d,\dagger} *_N \mathcal{B} \in N(\mathcal{A}^k)$. Hence, $\mathcal{V} - \mathcal{A}^{d,\dagger} *_N \mathcal{B} \in R(\mathcal{A}^k) \cap N(\mathcal{A}^k) = \{\mathcal{O}\}$. i.e., $\mathcal{V} = \mathcal{A}^{d,\dagger} *_N \mathcal{B}$. □

Using the same approach as described in the proof of Theorems 34(ii) and 41, the following hold.

Corollary 42 *Suppose that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ and $\mathcal{B} \in R(\mathcal{A}^k)$ with index$(\mathcal{A}) = k$. Then, the set of all solutions of (41) can be represented as*

$$\mathcal{X} = \mathcal{A}^{\dagger,d} *_N \mathcal{B} + N(\mathcal{A}^k).$$

Furthermore, the Eq. (41) has the unique solution $\mathcal{X} = \mathcal{A}^{\dagger,d} *_N \mathcal{B} \in R(\mathcal{A}^\dagger *_N \mathcal{A}^k)$.

Corollary 43 *Suppose that $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ and $\mathcal{B} \in R(\mathcal{A}^k)$ with index$(\mathcal{A}) = k$. Then, the set of all solutions of (41) can be represented as*

$$\mathcal{X} = \mathcal{A}^{c,\dagger} *_N \mathcal{B} + N(\mathcal{A}^k).$$

Furthermore, the Eq. (41) has the unique solution $\mathcal{X} = \mathcal{A}^{c,\dagger} *_N \mathcal{B} \in R(\mathcal{A}^\dagger *_N \mathcal{A}^k)$.

As an application of the DMP, MPD and CMP inverses of tensor, we consider the following equation

$$\mathcal{A}^k *_N \mathcal{X} = \mathcal{A}^k *_N \mathcal{A}^\dagger *_N \mathcal{B}. \tag{42}$$

where $index(\mathcal{A}) = k$ and $\mathcal{B} \in R(\mathcal{A}^k)$. If $\mathcal{B} \in R(\mathcal{A}^k)$ and $index(\mathcal{A}) = k$, then each member of the set $\{\mathcal{A}^{d,\dagger} *_N \mathcal{B}, \mathcal{A}^{\dagger,d} *_N \mathcal{B}, \mathcal{A}^{c,\dagger} *_N \mathcal{B}\}$ is a solution of Eqs. (41) and (42).

Theorem 44 *Suppose that* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *and* $\mathcal{X}, \mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ *with* $index(\mathcal{A}) = k$. *Then* $\mathcal{X} = \mathcal{A}^{c,\dagger} *_N \mathcal{B}$ *is a solution of Eq. (42). Moreover,* $\mathcal{X} = \mathcal{A}^{c,\dagger} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^d *_N \mathcal{A}) *_N \mathcal{Y}$ *is the general solution of Eq. (42), where* $\mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ *is an arbitrary tensor.*

Proof Set $\mathcal{X} = \mathcal{A}^{c,\dagger} *_N \mathcal{B}$. Then,

$$\mathcal{A}^k *_N \mathcal{X} = \mathcal{A}^k *_N \mathcal{A}^{c,\dagger} *_N \mathcal{B} = \mathcal{A}^k *_N \mathcal{A}^{\dagger} *_N \mathcal{B}.$$

By Ji and Wei (2018, Theorem 3.4), we have $R(\mathcal{I} - \mathcal{A}^d *_N \mathcal{A}) = N(\mathcal{A}^d *_N \mathcal{A}) = N(\mathcal{A}^d) = N(\mathcal{A}^k)$.

Thus, $\mathcal{A}^k *_N [\mathcal{A}^{c,\dagger} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^d *_N \mathcal{A}) *_N \mathcal{Y}] = \mathcal{A}^k *_N \mathcal{A}^{c,\dagger} *_N \mathcal{B} + \mathcal{O} = \mathcal{A}^k *_N \mathcal{A}^{\dagger} *_N \mathcal{B}$. □

Using the same approach as described in the proof of Theorem 44, the following holds.

Corollary 45 *Suppose that* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *and* $\mathcal{X}, \mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ *with* $index(\mathcal{A}) = k$. *Then* $\mathcal{X} = \mathcal{A}^{d,\dagger} *_N \mathcal{B}$ *is a solution of the Eq. (42). Moreover,* $\mathcal{X} = \mathcal{A}^{d,\dagger} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A} *_N \mathcal{A}^d) *_N \mathcal{Y}$ *is the general solution of Eq. (42), where* $\mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ *is an arbitrary tensor.*

Using the same approach as described in the proof of Theorems 34(ii) and 41, the following holds.

Remark 4 Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ and $\mathcal{B} \in R(\mathcal{A}^k)$ with $index(\mathcal{A}) = k$. Then

(i) The Eq. (42) has a unique solution $\mathcal{X} = \mathcal{A}^{d,\dagger} *_N \mathcal{B} \in R(\mathcal{A}^k)$ and its general solution $\mathcal{X} = \mathcal{A}^{d,\dagger} *_N \mathcal{B} + N(\mathcal{A}^k)$.
(ii) The Eq. (42) has a unique solution $\mathcal{X} = \mathcal{A}^{\dagger,d} *_N \mathcal{B} \in R(\mathcal{A}^{\dagger} *_N \mathcal{A}^k)$ and its general solution $\mathcal{X} = \mathcal{A}^{\dagger,d} *_N \mathcal{B} + N(\mathcal{A}^k)$.
(iii) The Eq. (42) has a unique solution $\mathcal{X} = \mathcal{A}^{c,\dagger} *_N \mathcal{B} \in R(\mathcal{A}^{\dagger} *_N \mathcal{A}^k)$ and its general solution $\mathcal{X} = \mathcal{A}^{c,\dagger} *_N \mathcal{B} + N(\mathcal{A}^k)$.

Declarations

Conflict of interest There is no conflict of interest in the manuscript.

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