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On the multiplicities of distance Laplacian eigenvalues

Rosário Fernandes[1] · Maria A. A. de Freitas[2] · Celso M. da Silva Jr.[3] · Renata R. Del-Vecchio[4]

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Abstract
In this work, connected graphs of order n and largest eigenvalue of the distance Laplacian matrix with multiplicity equal to $n-4$ are investigated. A complete characterization is presented if n is one of its distance Laplacian eigenvalues with multiplicity one. We also present a conjecture about forbidden subgraphs of G when the multiplicity of its largest eigenvalue is $n-4$, and we analyze the case where G has diameter four.

Keywords Distance Laplacian matrix · Laplacian matrix · Multiplicity of eigenvalues

Mathematics Subject Classification 05C12 · 05C50 · 15A18

1 Introduction

Let $G=(V,E)$ be a connected graph of order n and let $d_{i,j}$ be the distance (the length of the shortest path) between vertices v_i and v_j of G. The diameter of a connected graph G is $\max_{v_i,v_j \in V} d_{i,j}$. The distance matrix of G, denoted by $\mathcal{D}(G)$, is the $n \times n$ matrix whose

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✉ Celso M. da Silva Jr.
celso.silva@cefet-rj.br

Rosário Fernandes
mrff@fct.unl.pt

Maria A. A. de Freitas
maguieiras@im.ufrj.br

Renata R. Del-Vecchio
rrdelvecchio@id.uff.br

1 CMA and Department of Mathematics, FCT, UNL, 2829-516 Caparica, Portugal

2 IM and COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil

3 DEMET, Centro Federal de Educação Tecnológica Celso Suckow da Fonseca, Rio de Janeiro, Brazil

4 IME, Universidade Federal Fluminense, Niterói, Brazil

(i, j)-entry is equal to $d_{i,j}$, for $i, j = 1, 2, \ldots, n$. For $1 \leq i \leq n$, the sum of the distances from v_i to all other vertices in G is known as the transmission of the vertex v_i and is denoted by $Tr(v_i)$. Let $Tr(G)$ be the transmission matrix of G, the diagonal matrix of order n whose (i, i)-entry is equal to $Tr(v_i)$. The distance Laplacian matrix of G, $\mathcal{D}^L(G)$, is the difference between the transmission matrix and the distance matrix, that is, $\mathcal{D}^L(G) = Tr(G) - \mathcal{D}(G)$ (Aouchiche and Hansen 2013). Let $Spec_{\mathcal{D}^L}(G) = (\partial_1^L(G), \partial_2^L(G), \ldots, \partial_n^L(G) = 0)$ be the distance Laplacian spectrum of the connected graph G, denoted by $\mathcal{D}^L(G)$-spectrum, where $\partial_1^L(G) \geq \partial_2^L(G) \geq \ldots \geq \partial_n^L(G) = 0$. The multiplicity of the eigenvalue $\partial_i^L(G)$, $i = 1, \ldots, n$, is denoted by $m(\partial_i^L(G))$. We often use exponents to exhibit the multiplicity of the distance Laplacian eigenvalues, when we write the $\mathcal{D}^L$-spectrum. We recall that $\partial_{n-1}^L(G) = n$ if and only if $\overline{G}$, the complement of G, is disconnected (Aouchiche and Hansen 2013). Moreover, $\partial_{n-1}^L(G) \geq n$ and the multiplicity of n as an eigenvalue of $\mathcal{D}^L(G)$ is one less than the number of components of $\overline{G}$ (Aouchiche and Hansen 2013).

In recent years, several works investigated the connected graphs on n vertices in which one of its distance Laplacian eigenvalues has a high multiplicity, $n - r$. The characterization of such graphs is completely made for r equal to one (Aouchiche and Hansen 2014), two (Fernandes et al. 2018; Lin et al. 2016; da Silva et al. 2016), or three (Fernandes et al. 2018; Lu et al. 2017; Ma et al. 2018; da Silva et al. 2016). In a recent paper (Khan et al. 2023), the case $r = 4$, under the condition that n is a distance Laplacian eigenvalue with multiplicity two or three, was studied. In this work, we consider the remaining cases where the largest distance Laplacian eigenvalue has multiplicity equal to $n - 4$. In Sect. 3, we describe all possible graphs with such multiplicity that also has n in its distance Laplacian spectrum, with multiplicity one, i.e., $\partial_{n-1}^L(G) = n$ and $m(\partial_{n-1}^L(G)) = 1$. For the case $\partial_{n-1}^L(G) \neq n$, we first recall the following central result for $r = 2$ or $r = 3$, where P_{r+2} denotes the path on $r + 2$ vertices.

Proposition 1 (da Silva et al. 2016) *If G is a connected graph on n vertices such that $m(\partial_1^L(G)) = n - r$, $1 \leq r \leq 3$, then G has no P_{r+2} as an induced subgraph.*

Thus, based on Proposition 1, a natural way of trying to characterize the connected graphs such that $m(\partial_1^L(G)) = n - 4$ is investigating its relation with the existence of P_6 as an induced subgraph. Computationally, using the software nauty and Traces (McKay and Piperno 2014), and Graph6Java (Mohammad et al. 2019), we looked for graphs G on n vertices, $6 \leq n \leq 11$, $m(\partial_1^L(G)) = n - 4$ and $\partial_{n-1}^L(G) \neq n$. In addition to C_6, all other obtained graphs, for $6 \leq n \leq 8$, are presented in Fig. 1. No graphs were found if $9 \leq n \leq 11$. Note that, besides C_6, we have two graphs and their complements, with the following spectra: $Spec_{D^L}(C_6) = (13^{(2)}, 10, 9^{(2)}, 0)$, $Spec_{D^L}(G_1) = (14.16^{(2)}, 10, 7.84^{(2)}, 0)$, $Spec_{D^L}(\overline{G_1}) = (10.3^{(2)}, 8, 6.7^{(2)}, 0)$, $Spec_{D^L}(G_2) = (12.41^{(3)}, 9.59^{(3)}, 0)$, $Spec_{D^L}(\overline{G_2}) = (11.41^{(3)}, 8.59^{(3)}, 0)$. In any case, P_6 is a forbidden subgraph. Considering these facts, in Sect. 4 we focus our attention to investigate if there could be a connected graph G with at least six vertices, P_6 as a forbidden subgraph and $m(\partial_1^L(G)) = n - 4$. Such graph has diameter at most four. We conclude that this condition is not feasible if diameter of G is equal to four.

2 Preliminaries

In what follows, $G = (V, E)$, or just G, denotes a connected graph with n vertices and $\overline{G}$ denotes its complement. The diameter of a connected graph G is denoted by $diam(G)$.

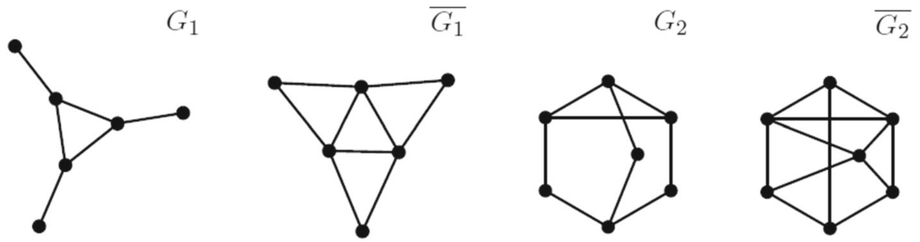

Fig. 1 Graphs with $m(\partial_1^L(G)) = n - 4$ and $\partial_{n-1}^L(G) \neq n$

As usual, we write, respectively, P_n, C_n, S_n, W_n and K_n, for the path, the cycle, the star, the wheel and the complete graph, all with n vertices. We denote by $K_{l_1,l_2,\ldots,l_k}$ the complete k-partite graph. If $e \in E$, the graph obtained from G by deleting the edge e is denoted $G - e$. If $e \notin E$, the graph obtained from G by adding the edge e is denoted $G + e$. Sometimes, we write $G + 2e$ meaning that we have added two edges in G.

Now, we recall the definitions of some operations with graphs that will be used. For this, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be vertex disjoint graphs. The *union* of G_1 and G_2 is the graph $G_1 \cup G_2$, whose vertex set is $V_1 \cup V_2$ and whose edge set is $E_1 \cup E_2$. The union of r copies of G_1 will be denoted by rG_1. The *join* of G_1 and G_2 is the graph $G_1 \vee G_2$ obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with every vertex of G_2.

The Laplacian matrix of G is the $n \times n$ matrix $L(G) = Deg(G) - A(G)$, where $Deg(G)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is its adjacency matrix. We denote by $(\mu_1(G), \mu_2(G), \ldots, \mu_n(G))$ the L-spectrum of G, i.e., the spectrum of the Laplacian matrix of G, and we assume that the eigenvalues are labelled such that $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$. It is well known that the multiplicity of the Laplacian eigenvalue 0 is equal to the number of components of G and that $\mu_{n-i}(\overline{G}) = n - \mu_i(G)$, $i = 1, \ldots, n-1$ (see Merris 1994 for more details).

The following result relates the spectra of the matrices L and D^L.

Theorem 2 (Aouchiche and Hansen 2013) *Let G be a connected graph on n vertices with $diam(G) \leq 2$. Let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) > \mu_n(G) = 0$ be the Laplacian spectrum of G. Then, the distance Laplacian spectrum of G is $2n - \mu_{n-1}(G) \geq 2n - \mu_{n-2}(G) \geq \cdots \geq 2n - \mu_1(G) > \partial_n^L(G) = 0$. Moreover, for every $i \in \{1, 2, \ldots, n-1\}$ the eigenspaces corresponding to $\mu_i(G)$ and to $2n - \mu_i(G)$ are the same.*

Propositions 3, 4, 5 provide the L-spectrum of some graphs that will be analyzed later. Proposition 5 can be easily checked.

Proposition 3 (Fernandes et al. 2018) *Let G be a connected graph of order $n \geq 4$. Then, $G \cong K_{n-2} \vee \overline{K_2}$ if and only if the L-spectrum of G is $(\mu_1^{(n-2)}(G), \mu_2(G), 0)$, with $\mu_1(G) > \mu_2(G) > 0$.*

Proposition 4 (Mohammadian and Tayfeh-Rezaie 2011) *Let G be a connected graph on $n \geq 5$ vertices whose distinct Laplacian eigenvalues are $0 < \alpha < \beta < \gamma$. Then, the multiplicity of γ is $n - 3$ if and only if $G \cong K_{n-3} \vee \overline{K_{1,2}}$.*

Proposition 5 *Let G be a connected graph on $n = 4$ vertices whose distinct Laplacian eigenvalues are $0 < \alpha < \beta < \gamma$. Then, $G \cong P_4$ or $G \cong K_{3,1} + e$.*

For proving the next result, we recall that a connected graph G has at least $diam(G) + 1$ distinct Laplacian eigenvalues (Brouwer and Haemers 2011, Proposition 1.3.3).

Table 1 L-spectrum of some graphs

Graph	L-spectrum	Graph	L-spectrum
P_4	$(3.4, 2, 0.6, 0)$	$K_{2,2}$	$(4, 2^{(2)}, 0)$
$K_{3,1}$	$(4, 1^{(2)}, 0)$	C_5	$(3.62^{(2)}, 1.38^{(2)}, 0)$
$K_{3,1} + e$	$(4, 3, 1, 0)$	K_n	$(n^{(n-1)}, 0)$
$K_n - e$	$(n^{(n-2)}, n-2, 0)$	$K_{n-3} \vee \overline{K_{2,1}}$	$(n^{(n-3)}, n-1, n-3, 0)$
$K_{n-4} \vee K_{2,2}$	$(n^{(n-3)}, (n-2)^{(2)}, 0)$	$K_{n-3} \vee 3K_1$	$(n^{(n-3)}, (n-3)^{(2)}, 0)$

Proposition 6 *Let G be a connected graph on $n \geq 4$ vertices whose distinct Laplacian eigenvalues are $\gamma > \alpha > 0$. Then, the multiplicity of γ is $n-3$ if and only if $G \cong K_{n-4} \vee K_{2,2}$, or $G \cong K_{n-3} \vee 3K_1$, or $G \cong C_5$, for $n \geq 5$, or $G \cong K_{2,2}$, or $G \cong K_{3,1}$, for $n = 4$.*

Proof As G has three distinct Laplacian eigenvalues, so $diam(G) = 2$ and its D^L-spectrum is $((2n-\alpha)^{(2)}, (2n-\gamma)^{(n-3)}, 0)$. For $n \geq 5$, these graphs are precisely determined in Theorems 4.4 and 4.5 of Fernandes et al. (2018) and in Theorem 1.2 of Ma et al. (2018). For $n = 4$, by Theorem 3.5 in da Silva et al. (2016), we get the result. □

In Table 1 are presented the L-spectra of some connected graphs that are well known and will be useful in this work. Also, in Proposition 7 we provide the D^L-spectra of the complete k-partite graph and of graphs obtained from it by adding edges. As these graphs have diameter two, each D^L-spectrum can be easily determined by considering the relation between the Laplacian spectrum of a graph and its complement, the spectra contained in Table 1 and Theorem 2.

Proposition 7 *Let $l_1 \geq l_2 \geq \ldots \geq l_k \geq 1$, $k \geq 2$, and n be integers such that $l_1 + l_2 + \cdots + l_k = n$. If $G = K_{l_1,l_2,\ldots,l_k}$ and $p = |\{i : l_i \geq 2\}|$, then:*

- *The $\mathcal{D}^L$-spectrum of G is $((n+l_1)^{(l_1-1)}, (n+l_2)^{(l_2-1)}, \ldots, (n+l_p)^{(l_p-1)}, n^{(k-1)}, 0)$.*
- *The $\mathcal{D}^L$-spectrum of the graph G plus one extra edge in the class with l_j vertices, if possible, is obtained from $Spec_{\mathcal{D}^L}(G)$, by replacing one eigenvalue $n+l_j$ by $n+l_j-2$.*
- *The $\mathcal{D}^L$-spectrum of the graph G plus two extra edges, one in the class with l_j vertices and other in the class with l_f vertices, if possible, is obtained from $Spec_{\mathcal{D}^L}(G)$, by replacing one eigenvalue $n+l_j$ and one $n+l_f$ by $n+l_j-2$ and $n+l_f-2$.*
- *The $\mathcal{D}^L$-spectrum of the graph G plus two extra edges sharing a common vertex in the class with l_j, if possible, is obtained from $Spec_{\mathcal{D}^L}(G)$, by replacing two eigenvalues $n+l_j$ by $n+l_j-3$ and $n+l_j-1$.*
- *The $\mathcal{D}^L$-spectrum of the graph G plus two extra independent edges in the class with l_j vertices, if possible, is obtained from $Spec_{\mathcal{D}^L}(G)$, by replacing two eigenvalues $n+l_j$ by $n+l_j-2$.*
- *The $\mathcal{D}^L$-spectrum of the graph G plus three extra edges determining a K_3 in the class with l_j vertices, if possible, is obtained from $Spec_{\mathcal{D}^L}(G)$, by replacing two eigenvalues $n+l_j$ by $n+l_j-3$.*

3 On graphs with *n* as a distance Laplacian eigenvalue

In this section, we completely characterize the graphs for which $m(\partial_1^L(G)) = n-4$, $\partial_{n-1}^L(G) = n$ and $m(n) = 1$. In this case, G has diameter of two and its distance Laplacian spectrum is related with the Laplacian spectrum (Theorem 2).

Theorem 8 *Let G be a connected graph with $n \geq 6$ vertices such that $m(\partial_1^L(G)) = n-4$ and $\partial_{n-1}^L(G) = n$. Then, $m(\partial_{n-1}^L(G)) = 1$ if and only if $G \cong W_6$, or G is isomorphic to one of the following graphs:*

- *for $n \geq 6$, $S_n + 2e$ (where the extra edges can share a vertex or they are independent), $S_n + 3e$ (where the extra edges induce a K_3), $K_{2,n-2} + e$ (where the extra edge is incident to vertices of the largest class) and $K_{p,p} + 2e$ (where the extra edges are in different classes);*
- *in addition to the previous graphs, for $n \geq 7$, $K_{3,n-3}$ and $K_{3,n-3} + e$ (where the extra edge is incident to vertices of the smallest class);*
- *in addition to the previous graphs, for $n \geq 8$, $K_{p,p} + 2e$ (where the extra edges are in the same class and they can share a vertex or they are independent) and $G \cong K_{p,p} + 3e$ (where the extra edges induce a K_3).*

Proof Since $m(\partial_{n-1}^L(G)) = 1$, the graph $\overline{G}$ has two components, say $\overline{G} \cong F_1 \cup F_2$. So, $diam(G) = 2$ and the L-spectrum of $\overline{G}$ is written as

$$(\partial_1^L(G) - n, \ldots, \partial_1^L(G) - n, \partial_{n-3}^L(G) - n, \partial_{n-2}^L(G) - n, 0, 0),$$

that is, the largest Laplacian eigenvalue of $\overline{G}$ has multiplicity $n-4$. Suppose that $|V(F_1)| \leq |V(F_2)|$. We have the following possibilities:

- $|V(F_1)| = 1$, then $F_1 = K_1$ and $(\partial_1^L(G) - n, \ldots, \partial_1^L(G) - n, \partial_{n-3}^L(G) - n, \partial_{n-2}^L(G) - n, 0)$ is the L-spectrum of F_2. If $\partial_{n-3}^L(G) > \partial_{n-2}^L(G)$, from Proposition 4, $G \cong S_n + 2e, n \geq 6$, where the extra edges share a vertex. If $\partial_{n-3}^L(G) = \partial_{n-2}^L(G)$, from Proposition 6, $G \cong W_6$ or, for $n \geq 6$, $G \cong S_n + 2e$, where the extra edges are independent, or $G \cong S_n + 3e$, where the extra edges induce a K_3.
- $|V(F_1)| = 2$, then $F_1 = K_2$ and its L-spectrum is $(2, 0)$. From Propositions 4, 5, 6, $\partial_1^L(G) - n > 2$. So, the L-spectrum of F_2 is $(\partial_1^L(G) - n, \ldots, \partial_1^L(G) - n, \alpha, 0)$. Then, from Proposition 3, for $n \geq 6$, $G \cong K_{2,n-2} + e$, where the extra edge is incident to vertices of the largest class.
- $|V(F_1)| = 3$, then $F_1 = K_3$ or $F_1 = P_3$ with L-spectrum, respectively, equal to $(3, 3, 0)$ and $(3, 1, 0)$. If $n = 6$, then $G \cong K_{3,3} + 2e$, where the extra edges are in different classes. For $n \geq 7$, from Propositions 3, 4, 5, 6, as $|V(F_2)| \geq 4$, it follows that $\partial_1^L(G) - n > 3$. So, the L-spectrum of F_2 is $(\partial_1^L(G) - n, \ldots, \partial_1^L(G) - n, 0)$. Thus, if $F_1 = K_3$, then $G \cong K_{3,n-3}, n \geq 7$. If $F_1 = P_3$, then $G = K_{3,n-3} + e$, $n \geq 7$, where the extra edge is incident to vertices of the smallest class.
- $|V(F_1)| = p > 3$,
  Case I. the L-spectrum of F_1 is $(\partial_1^L(G) - n, \ldots, \partial_1^L(G) - n, \partial_{n-2}^L(G) - n, \partial_{n-3}^L(G) - n, 0)$ and of F_2 is $(\partial_1^L(G) - n, \ldots, \partial_1^L(G) - n, 0)$. If $\partial_{n-2}^L(G) > \partial_{n-3}^L(G)$, and $|V(F_1)| = 4$, it follows, from Proposition 5, that $G \cong K_{4,4} + 2e$, where the extra edges share a vertex. If $\partial_{n-2}^L(G) > \partial_{n-3}^L(G)$, and $|V(F_1)| > 4$, then, from Proposition 4, $G \cong K_{p,p} + 2e$, $p \geq 5$, where the extra edges share a vertex. If $\partial_{n-2}^L(G) = \partial_{n-3}^L(G)$, from Proposition 6, $G \cong K_{p,p} + 2e$, $p \geq 5$, where the extra edges are in the same class and they are independent, $G \cong K_{p,p} + 3e$, $p \geq 5$, where the extra edges induce a K_3, $G \cong K_{4,4} + 2e$,

Table 2 $\mathcal{D}^L$-spectrum of some graphs

Graph	$\mathcal{D}^L$-spectrum
W_6	$(9.62^{(2)}, 7.38^{(2)}, 6, 0)$
$S_n + 2e$, $n \geq 6$, extra edges share a vertex	$((2n-1)^{(n-4)}, 2n-2, 2n-4, n, 0)$
$S_n + 2e$, $n \geq 6$, extra edges are independent	$((2n-1)^{(n-4)}, (2n-3)^{(2)}, n, 0)$
$S_n + 3e$, $n \geq 6$, extra edges induce a K_3	$((2n-1)^{(n-4)}, (2n-4)^{(2)}, n, 0)$
$K_{2,n-2} + e$, $n \geq 6$, extra edge is in the largest class	$((2n-2)^{(n-4)}, 2n-4, n+2, n, 0)$
$K_{p,p} + 2e$, $p \geq 3$, extra edges are in different classes	$\left(\left(\frac{3n}{2}\right)^{(n-4)}, \left(\frac{3n-4}{2}\right)^{(2)}, n, 0\right)$
$K_{3,n-3}$, $n \geq 7$,	$((2n-3)^{(n-4)}, (n+3)^{(2)}, n, 0)$
$K_{3,n-3} + e$, $n \geq 7$, extra edge is in the smallest class	$((2n-3)^{(n-4)}, n+3, n+1, n, 0)$
$K_{p,p} + 2e$, $p \geq 4$ extra edges share a vertex	$\left(\left(\frac{3n}{2}\right)^{(n-4)}, \left(\frac{3n-2}{2}\right), \left(\frac{3n-6}{2}\right), n, 0\right)$
$K_{p,p} + 2e$, $p \geq 4$, extra edges are independent in the same class	$\left(\frac{3n}{2}^{(n-4)}, \left(\frac{3n-4}{2}\right)^{(2)}, n, 0\right)$
$K_{p,p} + 3e$, $p \geq 4$, extra edges induce a K_3	$\left(\frac{3n}{2}^{(n-4)}, \left(\frac{3n-6}{2}\right)^{(2)}, n, 0\right)$

where the extra edges are in the same class and they are independent, or $G \cong K_{4,4} + 3e$, where the extra edges induce a K_3.
Case II. the L-spectrum of F_1 is $(\partial_1^L(G) - n, \ldots, \partial_1^L(G) - n, \partial_{n-3}^L(G) - n, 0)$ and the L-spectrum of F_2 is $(\partial_1^L(G) - n, \ldots, \partial_1^L(G) - n, \partial_{n-2}^L(G) - n, 0)$. From Proposition 3, it follows that $G \cong K_{p,p} + 2e$, $p \geq 4$, where the extra edges are in different classes.

By Proposition 7 we can explicit the D^L-spectra presented in Table 2. □

4 Diameter four graphs with forbidden P_6

In this section, we focus on connected graphs with at least six vertices, having P_6 as a forbidden subgraph. In particular, this condition implies investigating graphs with a maximum diameter equal to four and we will consider, specifically, graphs with a diameter four. So, from now on, we denote by v_1, v_2, v_3, v_4, v_5 the vertices inducing a P_5, with $d(v_1, v_5) = 4$ and $P = v_1v_2v_3v_4v_5$ been a shortest path between v_1 and v_5. In Fig. 2 are presented all possible graphs on six vertices having P_5 as an induced subgraph with $d(v_1, v_5) = 4$.

Besides, let G be a connected graph on n vertices, $n \geq 8$, $m(\partial_1^L(G)) = n - 4$ and let M be a principal submatrix of $\mathcal{D}^L(G)$, of order $k \in \{6, 7\}$, with largest eigenvalue λ. By Cauchy Interlacing, we get $\lambda = \partial_1^L(G)$ and $m(\lambda) \geq k - 4$. So, as **1**, the all ones vector

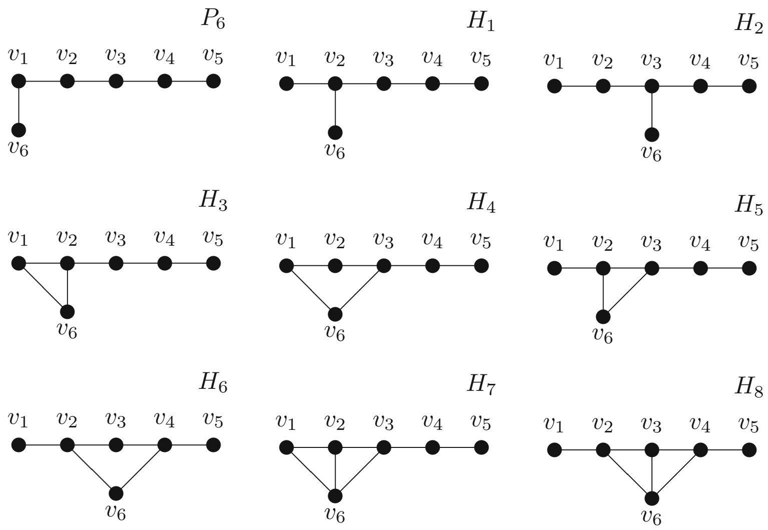

Fig. 2 Graphs on six vertices and P_5 as an induced subgraph

with appropriate order, is an eigenvector for $\mathcal{D}^L(G)$, it is possible to get a vector z of M corresponding to λ with at least $k-5$ entries equal to zero, which can be arbitrarily chosen (Proposition 3.1, Fernandes et al. 2018), such that $z \perp \mathbf{1}$. This fact will be fundamental for what follows in this work.

Theorem 9 (da Silva et al. 2016) *If G is a connected graph then* $\partial_1^L(G) \geq \max_{v_i \in V} Tr(v_i) + 1$. *Equality is attained if and only if* $G \cong K_n$.

The next propositions are similar to the results that appeared in Lu et al. (2017). Since they have analogous proofs we omit them.

Proposition 10 *Let G be a connected graph with* $n \geq 8$ *vertices. If* $m(\partial_1^L) = n-4$ *then* ∂_1^L *is an integer number.*

Proposition 11 *Let G be a connected graph with n vertices such that* $G \ncong K_n$ *and* ∂_1^L *is an integer number. Then,* $\partial_1^L \geq \max_{v \in V} Tr(v) + 2$. *Moreover, if there exists* $v_0 \in V$ *such that* $\partial_1^L = Tr(v_0) + 2$, *then* $Tr(v_0) = \max_{v \in V} Tr(v)$.

We will now state two results from matrix theory that will be useful in what follows. We denote by $\mathbf{0}$ and $\mathbf{1}$ vectors with a given size and all entries equal to zero and all entries equal to one, respectively.

Lemma 12 *Let*

$$M = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -a \\ -1 & t_2 & -1 & -2 & -3 & -b \\ -2 & -1 & t_3 & -1 & -2 & -c \\ -3 & -2 & -1 & t_4 & -1 & -d \\ -4 & -3 & -2 & -1 & t_5 & -e \\ -a & -b & -c & -d & -e & t_6 \end{bmatrix}$$

and $a, b, c, d, e, t_1, t_2, t_3, t_4, t_5, t_6 \in \mathbb{N}$. *Let* λ_1 *be an eigenvalue of* M *and* $z = (z_1, z_2, z_3, z_4, 0, z_6)$ *be an eigenvector of* M *associated to* λ_1 *such that* $z \perp \mathbf{1}$.

1. *If* $e = d+1$ *and* $t_4 \neq \lambda_1$, *then* $z = (z_1, z_2, z_3, 0, 0, z_6)$ *and* $-2z_1 - z_2 - (d-1)z_6 = 0$.
2. *If* $e = d+1 = c+2$, $t_4 \neq \lambda_1$ *and* $t_3 \neq \lambda_1$, *then* $z = (z_1, z_2, 0, 0, 0, z_6)$ *and* $-z_1 - (c-1)z_6 = 0$.
3. *If* $e = d+1 = c+2 = 3$, $t_4 \neq \lambda_1$ *and* $t_3 \neq \lambda_1$, *then* $z = (0, z_2, 0, 0, 0, z_6)$. *Moreover,* $a = 1$.
4. *If* $e = d+1 = c+2 = b+3 = 4$, $t_4 \neq \lambda_1$, $t_2 \neq \lambda_1$ *and* $t_3 \neq \lambda_1$, *then* $z = (z_1, 0, 0, 0, 0, z_6)$.

Proof 1. Using the fifth entry of both sides of $Mz = \lambda_1 z$ we get $-4z_1 - 3z_2 - 2z_3 - z_4 - ez_6 = 0$. Since $z \perp \mathbf{1}$, we obtain $-3z_1 - 2z_2 - z_3 - (e-1)z_6 = 0$. Using the fourth entry of $Mz = \lambda_1 z$ we get $-3z_1 - 2z_2 - z_3 + t_4 z_4 - dz_6 = \lambda_1 z_4$. As $d = e - 1$, we conclude that $t_4 z_4 = \lambda_1 z_4$. So, $z_4 = 0$, because $t_4 \neq \lambda_1$, $z = (z_1, z_2, z_3, 0, 0, z_6)$ and $-3z_1 - 2z_2 - z_3 - dz_6 = 0$. This implies that $-2z_1 - z_2 - (d-1)z_6 = 0$.
2. From Item 1 and the third entry of $Mz = \lambda_1 z$, we get $-2z_1 - z_2 + t_3 z_3 - cz_6 = \lambda_1 z_3$. As $c = d - 1$, we conclude that $t_3 z_3 = \lambda_1 z_3$. So, $z_3 = 0$, because $t_3 \neq \lambda_1$, $z = (z_1, z_2, 0, 0, 0, z_6)$ and $-2z_1 - z_2 - cz_6 = 0$. This implies that $-z_1 - (c-1)z_6 = 0$.
3. From Item 2,
   since $c = 1$, then $z_1 = 0$ and $z = (0, z_2, 0, 0, 0, z_6)$. If $a \neq 1$, using the first entry of $Mz = \lambda_1 z$ we get $-z_2 - az_6 = 0$. Consequently, $z_2 = z_6 = 0$ and $z = \mathbf{0}$, which is impossible.
4. The result is immediate from Item 2. □

Using a similar arguments as before, we get the next proposition.

Lemma 13 *Let*

$$M = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -a \\ -1 & t_2 & -1 & -2 & -3 & -b \\ -2 & -1 & t_3 & -1 & -2 & -c \\ -3 & -2 & -1 & t_4 & -1 & -d \\ -4 & -3 & -2 & -1 & t_5 & -e \\ -a & -b & -c & -d & -e & t_6 \end{bmatrix}$$

and $a, b, c, d, e, t_1, t_2, t_3, t_4, t_5, t_6 \in \mathbb{N}$. *Let* λ_1 *be an eigenvalue of* M *and* $z = (0, z_2, z_3, z_4, z_5, z_6)$ *be an eigenvector of* M *associated to* λ_1 *such that* $z \perp \mathbf{1}$.

1. *If* $a = b+1$, *and* $t_2 \neq \lambda_1$, *then* $z = (0, 0, z_3, z_4, z_5, z_6)$ *and* $-z_4 - 2z_5 - (b-1)z_6 = 0$.
2. *If* $a = b+1 = c+2$, $t_2 \neq \lambda_1$ *and* $t_3 \neq \lambda_1$, *then* $z = (0, 0, 0, z_4, z_5, z_6)$ *and* $-z_5 - (c-1)z_6 = 0$.
3. *If* $a = b+1 = c+2 = 3$, $t_2 \neq \lambda_1$ *and* $t_3 \neq \lambda_1$, *then* $z = (0, 0, 0, z_4, 0, z_6)$. *Moreover,* $e = 1$.

Proposition 14 *Let* G *be a connected graph with* $n \geq 6$ *vertices,* $diam(G) = 4$ *and* $m(\partial_1^L) = n - 4$. *Then,* H_7 *and* H_8 *are not induced subgraphs of* G.

Proof The principal submatrices of $\mathcal{D}^L(G)$ with respect to H_7 and H_8 are, respectively,

$$M_1 = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -1 \\ -1 & t_2 & -1 & -2 & -3 & -1 \\ -2 & -1 & t_3 & -1 & -2 & -1 \\ -3 & -2 & -1 & t_4 & -1 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -3 \\ -1 & -1 & -1 & -2 & -3 & t_6 \end{bmatrix}, M_2 = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -2 \\ -1 & t_2 & -1 & -2 & -3 & -1 \\ -2 & -1 & t_3 & -1 & -2 & -1 \\ -3 & -2 & -1 & t_4 & -1 & -1 \\ -4 & -3 & -2 & -1 & t_5 & -2 \\ -2 & -1 & -1 & -1 & -2 & t_6 \end{bmatrix}.$$

For each of the principal submatrices, there exists an eigenvector associated with ∂_1^L, $z = (z_1, z_2, z_3, z_4, 0, z_6)$, $z \perp \mathbf{1}$.

Case 1: If H_7 is an induced subgraph of G, by Lemma 12, $z = (0, z_2, 0, 0, 0, z_6)$. Considering the second entry of $M_1 z = \partial_1^L z$, it follows that $(t_2 + 1 - \partial_1^L)z_2 = 0$. Since $t_2 + 1 < \partial_1^L$, then $z_2 = 0$. Consequently, $z = \mathbf{0}$, which is impossible.

Case 2: If H_8 is an induced subgraph of G, by Lemma 12, $z = (z_1, z_2, z_3, 0, 0, z_6)$ and $-2z_1 - z_2 = 0$. From this, considering the third and sixth entries of $M_2 z = \partial_1^L z$, as $t_3 \neq \partial_1^L$, it follows, respectively, that $z_3 = \frac{1}{t_3 - \partial_1^L} z_6$, and $z_3 = (t_6 - \partial_1^L)z_6$. Consequently, $\frac{1}{t_3 - \partial_1^L} z_6 = (t_6 - \partial_1^L)z_6$.

If $z_6 = 0$, then $z_3 = 0$, implying $z_1 + z_2 = 0$ and $-2z_1 - z_2 = 0$. In this case, $z = \mathbf{0}$, which is impossible. So, $z_6 \neq 0$ and $\frac{1}{t_3 - \partial_1^L} = (t_6 - \partial_1^L)$. Therefore, $1 = (t_6 - \partial_1^L)(t_3 - \partial_1^L)$. This is impossible because $t_3 + 1 < \partial_1^L$ and $t_6 + 1 < \partial_1^L$. □

Proposition 15 *Let G be a connected graph with $n \geq 8$ vertices, $diam(G) = 4$ and $m(\partial_1^L) = n - 4$. Then, H_3 is not an induced subgraph of G.*

Proof Suppose that H_3 is an induced subgraph of G. Then, the principal submatrix of $\mathcal{D}^L(G)$ with respect to H_3 is

$$M_1 = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -1 \\ -1 & t_2 & -1 & -2 & -3 & -1 \\ -2 & -1 & t_3 & -1 & -2 & -2 \\ -3 & -2 & -1 & t_4 & -1 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -3 \\ -1 & -1 & -2 & -2 & -3 & t_6 \end{bmatrix} \quad \text{or} \quad M_2 = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -1 \\ -1 & t_2 & -1 & -2 & -3 & -1 \\ -2 & -1 & t_3 & -1 & -2 & -2 \\ -3 & -2 & -1 & t_4 & -1 & -3 \\ -4 & -3 & -2 & -1 & t_5 & -4 \\ -1 & -1 & -2 & -3 & -4 & t_6 \end{bmatrix}$$

$$\text{or } M_3 = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -1 \\ -1 & t_2 & -1 & -2 & -3 & -1 \\ -2 & -1 & t_3 & -1 & -2 & -2 \\ -3 & -2 & -1 & t_4 & -1 & -3 \\ -4 & -3 & -2 & -1 & t_5 & -3 \\ -1 & -1 & -2 & -3 & -3 & t_6 \end{bmatrix}.$$

- Suppose it is M_1. Let $z = (z_1, z_2, z_3, z_4, 0, z_6)$ be a vector satisfying $z \perp \mathbf{1}$ and $M_1 z = \partial_1^L z$. By Lemma 12, $z = (z_1, z_2, z_3, 0, 0, z_6)$ and $-2z_1 - z_2 - z_6 = 0$. Considering the second entry of $M_1 z = \partial_1^L z$, since $t_2 + 1 \neq \partial_1^L$, it follows that $z_2 = 0$ and, then, $z_1 = z_3$. Considering the first entry of both sides of $M_1 z = \partial_1^L z$, it follows that $z_1 = 0$. Consequently, $z_6 = 0$ and $z = \mathbf{0}$, which is impossible.
- In case it is M_2, the proof is analogous to the case M_1, by taking $z = (z_1, 0, 0, 0, 0, z_6)$ and considering the first entry of $M_2 z = \partial_1^L z$.

- Suppose it is M_3. Let $z = (z_1, z_2, z_3, z_4, z_5, 0)$ be a vector satisfying $z \perp \mathbf{1}$ and $M_3 z = \partial_1^L z$. Considering, respectively, the sixth and fifth entries of $M_3 z = \partial_1^L z$ and, then, looking at the first four lines of this eigenequation, we get:

$$z_1 = \frac{1}{t_1 + 1 - \partial_1^L} z_5, \quad z_2 = \frac{t_5 + 2 - \partial_1^L}{t_2 - \partial_1^L} z_5,$$
$$z_3 = \frac{t_5 + 2 - \partial_1^L}{t_3 + 1 - \partial_1^L} z_5, \quad z_4 = \frac{t_5 + 2 - \partial_1^L}{t_4 - \partial_1^L} z_5.$$

Since $z_1 + z_2 + z_3 + z_4 = -z_5$, then

$$\left[\frac{1}{t_1 + 1 - \partial_1^L} + \frac{t_5 + 2 - \partial_1^L}{t_2 - \partial_1^L} + \frac{t_5 + 2 - \partial_1^L}{t_3 + 1 - \partial_1^L} + \frac{t_5 + 2 - \partial_1^L}{t_4 - \partial_1^L}\right] z_5 = -z_5$$

and, as z is an eigenvector, then $z_5 \neq 0$ and

$$\left[\frac{1}{t_1 + 1 - \partial_1^L} + \frac{t_5 + 2 - \partial_1^L}{t_2 - \partial_1^L} + \frac{t_5 + 2 - \partial_1^L}{t_3 + 1 - \partial_1^L} + \frac{t_5 + 2 - \partial_1^L}{t_4 - \partial_1^L}\right] = -1.$$

This is impossible because

$$(t_1 + 1 - \partial_1^L)(t_4 - \partial_1^L)(t_2 - \partial_1^L)(t_3 + 1 - \partial_1^L) > 0,$$
$$\left[(t_2 - \partial_1^L)(t_3 + 1 - \partial_1^L) + (t_2 - \partial_1^L)(t_4 - \partial_1^L) + (t_3 + 1 - \partial_1^L)(t_4 - \partial_1^L)\right] > 0,$$
$$(t_1 + 1 - \partial_1^L)(t_5 + 2 - \partial_1^L) \geq 0$$

and

$$0 > \frac{(t_2 - \partial_1^L)(t_3 + 1 - \partial_1^L)(t_4 - \partial_1^L)}{(t_1 + 1 - \partial_1^L)(t_4 - \partial_1^L)(t_2 - \partial_1^L)(t_3 + 1 - \partial_1^L)} > -1.$$

□

Proposition 16 *Let G be a connected graph with $n \geq 8$ vertices, $diam(G) = 4$ and $m(\partial_1^L) = n - 4$. Then, H_5 is not an induced subgraph of G.*

Proof Suppose that H_5 is an induced subgraph of G. Then, the principal submatrix of $\mathcal{D}^L(G)$ with respect to H_5 is

$$N_1 = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -2 \\ -1 & t_2 & -1 & -2 & -3 & -1 \\ -2 & -1 & t_3 & -1 & -2 & -1 \\ -3 & -2 & -1 & t_4 & -1 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -3 \\ -2 & -1 & -1 & -2 & -3 & t_6 \end{bmatrix} \quad \text{or} \quad N_2 = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -2 \\ -1 & t_2 & -1 & -2 & -3 & -1 \\ -2 & -1 & t_3 & -1 & -2 & -1 \\ -3 & -2 & -1 & t_4 & -1 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -2 \\ -2 & -1 & -1 & -2 & -2 & t_6 \end{bmatrix}.$$

- Suppose it is N_1 and let $z = (z_1, z_2, z_3, z_4, 0, z_6)$ be a vector satisfying $z \perp \mathbf{1}$ and $N_1 z = \partial_1^L z$. By Lemma 12, we get a contradiction.
- Suppose it is N_2. Since $d(v_6, v_5) = 2$ and $v_1 v_2 v_3 v_4 v_5$ is a path between v_1 and v_5, then there is a vertex v_7 adjacent to v_5 and v_6 such that there is neither adjacent to v_2 nor v_1. If v_7 is adjacent to v_4 and it is not adjacent to v_3, then the subgraph of G induced by vertices $v_1, v_2, v_3, v_4, v_5, v_7$ is isomorphic to H_3. In case v_7 is adjacent to both, v_4 and

v_3, the subgraph of G induced by vertices $v_1, v_2, v_3, v_4, v_5, v_7$ is isomorphic to H_8. In any of these cases we have a contradiction.

If v_7 is adjacent to v_3 and it is not adjacent to v_4, then the principal submatrix of $\mathcal{D}^L(G)$ with respect to the subgraph of G induced by vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ is

$$F_1 = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -2 & -3 \\ -1 & t_2 & -1 & -2 & -3 & -1 & -2 \\ -2 & -1 & t_3 & -1 & -2 & -1 & -1 \\ -3 & -2 & -1 & t_4 & -1 & -2 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -2 & -1 \\ -2 & -1 & -1 & -2 & -2 & t_6 & -1 \\ -3 & -2 & -1 & -2 & -1 & -1 & t_7 \end{bmatrix}.$$

Let $z = (0, z_2, z_3, 0, z_5, z_6, z_7)$ be a vector satisfying $z \perp \mathbf{1}$ and $F_1 z = \partial_1^L z$. Considering, respectively, the equations from first, second, fourth and seventh entries of $F_1 z = \partial_1^L z$, we get $z = \mathbf{0}$, a contradiction.

If v_7 is neither adjacent to v_3 nor to v_4, then the principal submatrix of $\mathcal{D}^L(G)$ associated with vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ is

$$F_2 = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -2 & -3 \\ -1 & t_2 & -1 & -2 & -3 & -1 & -2 \\ -2 & -1 & t_3 & -1 & -2 & -1 & -2 \\ -3 & -2 & -1 & t_4 & -1 & -2 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -2 & -1 \\ -2 & -1 & -1 & -2 & -2 & t_6 & -1 \\ -3 & -2 & -2 & -2 & -1 & -1 & t_7 \end{bmatrix}.$$

Analogously to the case of F_1, we have a contradiction. □

Proposition 17 *Let G be a connected graph with $n \geq 8$ vertices, $diam(G) = 4$ and $m(\partial_1^L) = n - 4$. Then, H_2 is not an induced subgraph of G.*

Proof Suppose that H_2 is an induced subgraph of G. Then, the principal submatrix of $\mathcal{D}^L(G)$ with respect to H_2 is

$$U_1 = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -3 \\ -1 & t_2 & -1 & -2 & -3 & -2 \\ -2 & -1 & t_3 & -1 & -2 & -1 \\ -3 & -2 & -1 & t_4 & -1 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -3 \\ -3 & -2 & -1 & -2 & -3 & t_6 \end{bmatrix} \text{ or } U_2 = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -2 \\ -1 & t_2 & -1 & -2 & -3 & -2 \\ -2 & -1 & t_3 & -1 & -2 & -1 \\ -3 & -2 & -1 & t_4 & -1 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -3 \\ -2 & -2 & -1 & -2 & -3 & t_6 \end{bmatrix},$$

$$U_3 = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -3 \\ -1 & t_2 & -1 & -2 & -3 & -2 \\ -2 & -1 & t_3 & -1 & -2 & -1 \\ -3 & -2 & -1 & t_4 & -1 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -2 \\ -3 & -2 & -1 & -2 & -2 & t_6 \end{bmatrix} \text{ or } U_4 = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -2 \\ -1 & t_2 & -1 & -2 & -3 & -2 \\ -2 & -1 & t_3 & -1 & -2 & -1 \\ -3 & -2 & -1 & t_4 & -1 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -2 \\ -2 & -2 & -1 & -2 & -2 & t_6 \end{bmatrix}.$$

- Suppose it is U_1 (or U_2). Let $z = (z_1, z_2, z_3, z_4, 0, z_6)$ be a vector satisfying $z \perp \mathbf{1}$ and $U_1 z = \partial_1^L z$ (or $U_1 z = \partial_1^L z$). By Lemma 12, we get a contradiction.
- Suppose it is U_3. Let $z = (0, z_2, z_3, z_4, z_5, z_6)$ be a vector satisfying $z \perp \mathbf{1}$ and $U_3 z = \partial_1^L z$. By Lemma 13, we get a contradiction.

- Suppose it is U_4. Since $d(v_6, v_5) = 2 = d(v_1, v_6)$ and $v_1v_2v_3v_4v_5$ is a path between v_1 and v_5, then there are vertices v_7 and v_8 such that v_8 is adjacent to v_1 and v_6 and v_7 is adjacent to v_5 and v_6. Moreover, v_8 is neither adjacent to v_4 nor v_5, and v_7 is neither adjacent to v_1 nor v_2. So, v_7 can be adjacent to v_3 or v_4 and v_8 can be adjacent to v_3 or v_2.
  If v_3 is adjacent to v_7 and to v_8, then the subgraph of G induced by vertices $v_1, v_3, v_5, v_6, v_7, v_8$ is isomorphic to H_8. If v_3 is adjacent to v_7 or to v_8, but not both, then the subgraph of G induced by vertices $v_1, v_3, v_5, v_6, v_7, v_8$ is isomorphic to H_5. If v_3 is neither adjacent to v_7 nor to v_8, but v_4 is adjacent to v_7, then the subgraph of G induced by vertices $v_1, v_2, v_3, v_4, v_5, v_7$ is isomorphic to H_3. In any of these cases we have a contradiction.
  If v_3 is neither adjacent to v_7 nor to v_8 and v_4 is not adjacent to v_7, then the principal submatrix of $\mathcal{D}^L(G)$ with respect to the subgraph of G, induced by vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7$, is given by W_1 or W_2, respectively defined as:

$$\begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -2 & -3 \\ -1 & t_2 & -1 & -2 & -3 & -2 & -2 \\ -2 & -1 & t_3 & -1 & -2 & -1 & -2 \\ -3 & -2 & -1 & t_4 & -1 & -2 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -2 & -1 \\ -2 & -2 & -1 & -2 & -2 & t_6 & -1 \\ -3 & -2 & -2 & -2 & -1 & -1 & t_7 \end{bmatrix}, \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -2 & -3 \\ -1 & t_2 & -1 & -2 & -3 & -2 & -3 \\ -2 & -1 & t_3 & -1 & -2 & -1 & -2 \\ -3 & -2 & -1 & t_4 & -1 & -2 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -2 & -1 \\ -2 & -2 & -1 & -2 & -2 & t_6 & -1 \\ -3 & -3 & -2 & -2 & -1 & -1 & t_7 \end{bmatrix}.$$

  Suppose it is W_1. Let $z = (0, 0, z_3, z_4, z_5, z_6, z_7)$ be a vector satisfying $z \perp \mathbf{1}$ and $W_1 z = \partial_1^L z$. Considering, respectively, the first, second, sixth and fifth entries of $W_1 z = \partial_1^L z$, we obtain that $z = \mathbf{0}$, contradiction.
  In case it is W_2, we get also get a contradiction from considering $z = (0, 0, z_3, z_4, z_5, z_6, z_7)$ be a vector satisfying $z \perp \mathbf{1}$ and $W_2 z = \partial_1^L z$ and, respectively, first, second, fourth and fifth entries of this eigenequation. □

Proposition 18 *Let G be a connected graph with $n \geq 8$ vertices, $diam(G) = 4$ and $m(\partial_1^L) = n - 4$. If P_6 is not an induced subgraph of G, then H_6 is not an induced subgraph of G.*

Proof Suppose that H_6 is an induced subgraph of G. Then, the principal submatrix of $\mathcal{D}^L(G)$ with respect to H_6 is

$$Y = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -2 \\ -1 & t_2 & -1 & -2 & -3 & -1 \\ -2 & -1 & t_3 & -1 & -2 & -2 \\ -3 & -2 & -1 & t_4 & -1 & -1 \\ -4 & -3 & -2 & -1 & t_5 & -2 \\ -2 & -1 & -2 & -1 & -2 & t_6 \end{bmatrix}.$$

Let $z = (z_1, z_2, z_3, z_4, 0, z_6)$ be a vector satisfying $z \perp \mathbf{1}$ and $Yz = \partial_1^L z$. By Lemma 12, we get $z = (z_1, z_2, z_3, 0, 0, z_6)$ and $-2z_1 - z_2 = 0$. Considering the third entry of $Yz = \partial_1^L z$, it follows that $z_3 = \frac{2}{t_1 - \partial_1^L} z_6$. If $z_6 = 0$, then $z_3 = 0$ and $z = \mathbf{0}$, which is impossible. So, $z_6 \neq 0$. Considering the sixth entry of $Yz = \partial_1^L z$, it follows that $z_3 = \frac{t_6 - \partial_1^L}{2} z_6$ and, then, $\frac{2}{t_1 - \partial_1^L} = \frac{t_6 - \partial_1^L}{2}$.

Since $n \geq 8$, we get $\partial_1^L \geq t_6+2$ and $\partial_1^L \geq t_3+2$. Consequently, $\partial_1^L = t_6+2 = t_3+2$ and v_3 is a vertex of G with maximum transmission. Because $\sum_{i=1, i\neq 3}^{6} Y_{3i} = 8 < \sum_{i=1, i\neq 5}^{6} Y_{5i} = 12$, there is a vertex v_7 such that $d(v_7, v_5) = 1 < d(v_7, v_3)$.

Concluding, as P_6 is not an induced subgraph of G and v_7 is not adjacent to v_3, v_1, v_2, then v_7 is adjacent to v_4 and the subgraph of G induced by vertices $v_1, v_2, v_3, v_4, v_5, v_7$ is isomorphic to H_3. So, we get an impossible situation. □

Proposition 19 *Let G be a connected graph with $n \geq 8$ vertices, $diam(G) = 4$ and $m(\partial_1^L) = n-4$. If P_6 is not an induced subgraph of G, then H_1 is not an induced subgraph of G.*

Proof Suppose that H_1 is an induced subgraph of G.

- If $d(v_6, v_5) = 2$, then there is a vertex v_7 adjacent to v_6 and v_5. Since $d(v_1, v_5) = 4$, then v_7 is neither adjacent to v_1 nor v_2. Since P_6 is not an induced subgraph of G, then v_7 is adjacent to v_3 or v_4.
  If v_7 is adjacent to v_3 and v_4, then the subgraph of G induced by vertices $v_1, v_2, v_3, v_4, v_5, v_7$ is isomorphic to H_7. If v_7 is adjacent to v_3 and it is not adjacent to v_4, then the subgraph of G induced by vertices $v_1, v_2, v_6, v_7, v_5, v_4$ is isomorphic to P_6. If v_7 is adjacent to v_4 and it is not adjacent to v_3, then the subgraph of G induced by vertices $v_1, v_2, v_3, v_4, v_5, v_7$ is isomorphic to H_3. In any of these cases we get a contradiction.
- If $d(v_6, v_5) = 3$, then there are vertices v_7 and v_8 such that v_7 is adjacent to v_6 and v_8 and the vertex v_8 is adjacent to v_7 and v_5. Since $d(v_1, v_5) = 4$, then v_8 is neither adjacent to v_1 nor v_2. Using similar argument, as used when $d(v_6, v_5) = 2$, with v_8 instead of v_7, we get a contradiction.
- If $d(v_6, v_5) = 4$, then the principal submatrix of $\mathcal{D}^L(G)$ with respect to H_1 is

$$M = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -2 \\ -1 & t_2 & -1 & -2 & -3 & -1 \\ -2 & -1 & t_3 & -1 & -2 & -2 \\ -3 & -2 & -1 & t_4 & -1 & -3 \\ -4 & -3 & -2 & -1 & t_5 & -4 \\ -2 & -1 & -2 & -3 & -4 & t_6 \end{bmatrix}.$$

  Let $z = (z_1, z_2, z_3, z_4, 0, z_6)$ be a vector satisfying $z_1+z_2+z_3+z_4+z_6 = 0$ and $Mz = \partial_1^L z$. By Lemma 12, $z = (z_1, 0, 0, 0, 0, z_6)$. Considering the first entry of both sides of $Mz = \partial_1^L z$, we have $t_1 z_1 - 2z_6 = \partial_1^L z_1$. Since $n \geq 8$, then $t_1 + 2 = \partial_1^L$ and v_1 is a vertex with maximum transmission in G. Because $\sum_{i=1, i\neq 1}^{6} M_{3i} = 12 < \sum_{i=1, i\neq 5}^{6} M_{5i} = 14$ there is a vertex v_7 such that $d(v_7, v_5) < d(v_7, v_1)$. Considering all cases we have just seen, the subgraph of G induced by vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ is isomorphic to graphs R_1 or R_2 (Fig. 3).
  Since $d(v_6, v_5) = 4$, then, in both cases, we have v_6 is not adjacent to v_7.
  Let E be the principal submatrix of $\mathcal{D}^L(G)$ with respect to R_1 :

$$E = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -2 & -3 \\ -1 & t_2 & -1 & -2 & -3 & -1 & -2 \\ -2 & -1 & t_3 & -1 & -2 & -2 & -1 \\ -3 & -2 & -1 & t_4 & -1 & -3 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -4 & -1 \\ -2 & -1 & -2 & -3 & -4 & t_6 & -3 \\ -3 & -2 & -1 & -2 & -1 & -3 & t_7 \end{bmatrix}.$$

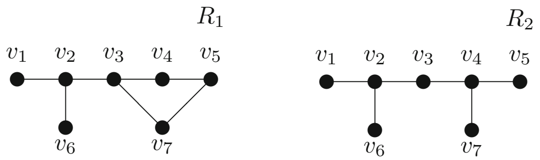

Fig. 3 Graphs R_1 and R_2

and $z = (0, z_2, z_3, 0, z_5, z_6, z_7)$ be a vector satisfying $z \perp \mathbf{1}$ and $Ez = \partial_1^L z$. Considering, respectively, the first, second, fourth and fifth entries of this eigenequation, we conclude that $z = \mathbf{0}$.

If R_2 (Fig. 3) is an induced subgraph of G, then $d(v_6, v_7) = 2$ or $d(v_6, v_7) = 3$ or $d(v_6, v_7) = 4$.

If $d(v_6, v_7) = 2$ and $d(v_6, v_5) = 4$, then there is a vertex v_8 adjacent to v_6 and v_7, and v_8 is neither adjacent to v_4 nor v_5. So, v_8 can be adjacent to v_1 or v_2 or v_3. If v_8 is adjacent to v_3, then P_6 or H_2 or H_5 or H_7 is an induced subgraph of G. If v_8 is not adjacent to v_3, then P_6 or H_3 is an induced subgraph of G. In any case of these cases we get a contradiction.

If $d(v_6, v_7) = 3$ and $d(v_6, v_5) = 4$, then there are vertices v_8 and v_9 such that v_9 is adjacent to v_6 and v_8, and v_8 is adjacent to v_7. Moreover, v_8 is not adjacent to v_5, and v_9 is neither adjacent to v_4 nor v_5. If v_8 is not adjacent to v_4, then the subgraph of G induced by vertices $v_6, v_9, v_8, v_7, v_4, v_5$ is isomorphic to P_6. If v_8 is adjacent to v_4, then P_6 or H_5 or H_6 or H_8 is an induced subgraph of G. All cases are not possible.

If $d(v_6, v_7) = 3$, then $d(v_1, v_7) = 4$, $d(v_2, v_7) = 3$ and the principal submatrix of $\mathcal{D}^L(G)$ with respect to R_2 is

$$J = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -2 & -4 \\ -1 & t_2 & -1 & -2 & -3 & -1 & -3 \\ -2 & -1 & t_3 & -1 & -2 & -2 & -2 \\ -3 & -2 & -1 & t_4 & -1 & -3 & -1 \\ -4 & -3 & -2 & -1 & t_5 & -4 & -2 \\ -2 & -1 & -2 & -3 & -4 & t_6 & -4 \\ -4 & -3 & -2 & -1 & -2 & -4 & t_7 \end{bmatrix}.$$

Let $z = (0, z_2, z_3, z_4, 0, z_6, z_7)$ be a vector satisfying $z \perp \mathbf{1}$ and $Jz = \partial_1^L z$. Considering, respectively, the first, second, fifth and fourth entries of $Jz = \partial_1^L z$, we conclude that $z = \mathbf{0}$. □

For the last results, we need to introduce a definition and a lemma.

Definition 1 Let a, b be two positive integers. We denote by $S(a, b)$ the graph obtained from $K_{a,1} \cup K_1 \cup K_{b,1}$ by joining each pendant vertex of $K_{a,1}$ and $K_{b,1}$ with the vertex of K_1.

Proposition 20 *Let a, b be two positive integers such that $a \leq b$. Let v be a pendant vertex of $K_{a,1}$ and u be its central vertex. Let w be a pendant vertex of $K_{b,1}$. Then, in $S(a, b)$ it follows that $Tr(v) = Tr(w) < Tr(u)$.*

Proof It is easy to check that $Tr(v) = 3 + 2(a + b) = Tr(w)$ and $Tr(u) = 6 + a + 3b$. Also, $Tr(v) < Tr(u)$ since $a \leq b$. □

Proposition 21 *Let G be a connected graph with $n \geq 8$ vertices, $diam(G) = 4$ and $m(\partial_1^L) = n - 4$. If P_6 is not an induced subgraph of G, then H_4 is not an induced subgraph of G.*

Proof Suppose that H_4 is an induced subgraph of G. Then, the principal submatrix of $\mathcal{D}^L(G)$ with respect to H_4 is

$$M = \begin{bmatrix} t_1 & -1 & -2 & -3 & -4 & -1 \\ -1 & t_2 & -1 & -2 & -3 & -2 \\ -2 & -1 & t_3 & -1 & -2 & -1 \\ -3 & -2 & -1 & t_4 & -1 & -2 \\ -4 & -3 & -2 & -1 & t_5 & -3 \\ -1 & -2 & -1 & -2 & -3 & t_6 \end{bmatrix}.$$

Let $z = (z_1, z_2, z_3, z_4, 0, z_6)$ be a vector satisfying $z \perp \mathbf{1}$ and $Mz = \partial_1^L z$. By Lemma 12, $z = (0, z_2, 0, 0, 0, z_6)$. Consider the second entry of $Mz = \partial_1^L z$ and, since $n \geq 8$, then $t_2 + 2 = \partial_1^L$ and v_2 is a vertex with maximum transmission in G. Because $\sum_{i=1, i\neq 2}^{6} M_{3i} = 9 < \sum_{i=1, i\neq 5}^{6} M_{5i} = 15$, there is a vertex v_7 such that $d(v_7, v_5) < d(v_7, v_2)$. Considering all cases we have just seen we conclude v_7 is adjacent to v_4 and v_5. Using a similar argument, as we used before, we get v_4 is a vertex of G with a maximum transmission. Consequently, G has an induced subgraph isomorphic to $S(a, b)$, with $a, b \geq 1$, v_1, v_5 as the central vertices of the stars in $S(a, b)$, v_3 as the vertex of K_1, v_4 as a pendent vertex of the star with central vertex v_5 and $a + 1$ vertices, and v_2 as a pendent vertex of the star with central vertex v_1 and $b + 1$ vertices. By Proposition 20, if $a \leq b$, then $Tr(v_5) > Tr(v_2)$ and if $a \geq b$, then $Tr(v_1) > Tr(v_2)$. So, we get a contradiction. □

As any diameter four graph with P_6 as a forbidden subgraph must have an induced subgraph isomorphic to some of the graphs H_i, $1 \leq i \leq 8$, from Propositions 14, 15, 16, 17, 18, 19, 21, we can state:

Theorem 22 *Let G be a connected graph with $n \geq 8$ vertices such that $m(\partial_1^L) = n - 4$. If P_6 is not an induced subgraph of G, then $diam(G) \leq 3$.*

Concluding, based on results presented in Sect. 3 and a computational search for graphs on n vertices, $6 \leq n \leq 11$, we could not find a graph with $m(\partial_1^L(G)) = n - 4$ and P_6 as an induced subgraph. So, we propose the following conjecture:

Conjecture 23 *Let G be a connected graph with $n \geq 6$ vertices. If $m(\partial_1^L(G)) = n - 4$, then P_6 is a forbidden subgraph.*

In case that P_6 is a forbidden subgraph, Theorem 22 presents some advances in an effort to determine all graphs such that $m(\partial_1^L) = n - 4$.

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Data availability Files used to support Conj. 23 are available from the authors upon request.

Declarations

Conflict of interest No potential conflict of interest was reported by the authors.

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