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(p, q)-Rung linear Diophantine fuzzy sets and their application in decision-making

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Abstract

The q-rung linear Diophantine fuzzy set is one of the effective generalizations of Diophantine fuzzy set for dealing with uncertainties in information. Under this environment, in this study, we define a new type of extensions of Diophantine fuzzy sets called (p, q)-rung linear Diophantine fuzzy sets. The (p, q)-rung linear Diophantine fuzzy sets can supply with more doubtful circumstances than q-rung linear Diophantine fuzzy sets and intuitionistic q-rung linear Diophantine fuzzy sets because of their larger range of depicting the membership grades. The values of this membership grades function and the non-membership grades function are symmetric. Moreover, the novel notion of a (p, q)-rung linear Diophantine fuzzy set through double universes is more flexible when debating the symmetry between two or more objects that are better than the diffusing concept of a p-rung linear Diophantine fuzzy, as well as q-rung linear Diophantine fuzzy set. The main advantage of (p, q)-rung linear Diophantine fuzzy sets is that it can describe more uncertainties than linear Diophantine fuzzy sets, which can be applied in many decision-making problems. Then, we suggest a number of geometric and averaging operators based on defined operating laws for a (p, q)-rung linear Diophantine fuzzy set. To address the emergency situation under (p, q)-rung linear Diophantine fuzzy information, two ranking algorithms based on proposed aggregation operators are presented in the last section of the paper. The goal of this study is to present a (p, q)-rung linear Diophantine fuzzy multi-attribute decision-making ((p, q)RLDFMADM) model for controlling emergency circumstances, because doing so is difficult. Fundamentally, the achievement of suitable and accurate responses to the emergency multi-attribute decision-making circumstance.

Keywords Intuitionistic fuzzy set · Linear Diophantine fuzzy set · (p, q)-Rung linear Diophantine fuzzy set · Decision-making · MADM

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1 Introduction

Multi-attribute decision-making (MADM) is a process that can give the ranking results for the finite alternatives according to the attribute values of different alternatives, and it is an important aspect of decision-making sciences. The problem of MADM has received a lot of attention recently, since it is connected to how businesses develop and how society makes decisions in many spheres. In the decision process, an important problem is how to express the attribute value more efficiently and accurately. Decision-makers (DMs) use several methodologies to address MADM problems, such as fuzzy sets (FSs) Zadeh (1965), bipolar complex fuzzy sets (BCFSs) Mahmood et al. (2023), cubic intuitionistic complex fuzzy soft sets (CICFSSs) Mahmood et al. (2022), linear Diophantine uncertain linguistic sets (LDULSs) Mahmood et al. (2021), linear Diophantine fuzzy sets (LDFSs) Yousafzai et al. (2023), and T-spherical linear Diophantine fuzzy sets (T-SLDFSs) Quran (2023). As a generalization of the FS and intuitionistic fuzzy set (IFS) Atanassov (1968), Riaz and Hashmi (2019) introduced the idea of a linear Diophantine fuzzy set (LDFS) in which the satisfaction of an attribute is represented in terms of membership, non-membership, and reference parameters between zero and one. In LDFS, it is observed that each object has three membership degree, membership $\mu_A(a)$, non-membership $\mu_A(a)$, and the reference parameters α_A, β_A which satisfying the condition $0 \leq \alpha_A \mu_A(a) + \beta_A \eta_A(a) \leq 1$ and $0 \leq \alpha_A + \beta_A \leq 1$ for all $a \in U$. Because linear Diophantine fuzzy numbers (LDFNs) endow the decision-makers with more flexibility, many researchers devoted themselves to studying linear Diophantine fuzzy decision-making. The linear Diophantine fuzzy numbers (LDFNs) has been widely used in many applications Farid et al. (2022) and has been integrated with other set theories such as spherical linear Diophantine fuzzy sets Alshammari et al. (2022). In 2021, Hashmi et al. (2021) proposed the spherical linear Diophantine fuzzy numbers (SLDFNs) and certain properties of SLDFSs and SLDFNs and the spherical linear Diophantine fuzzy soft rough set (SLDFSRS) and spherical linear Diophantine fuzzy soft approximation space to solve multi-criteria decision-making (MCDM) problem. In addition, Parimala et al. (2021) proposed the concepts of the conditions of optimality in Linear Diophantine fuzzy (LDF) networks for the solution algorithm's design. Furthermore, (Ayub et al. 2021) generalized the notion of LDFS and defined a new concept of f linear Diophantine fuzzy relation (LDF-relation). Also, (Iampan et al. 2021) proposed some aggregation operators for LDFSs, i.e., linear Diophantine fuzzy Einstein weighted averaging (LDFEWA), linear Diophantine fuzzy Einstein ordered weighted averaging (LDFEOWA), linear Diophantine fuzzy Einstein weighted geometric (LDFEWG), and linear Diophantine fuzzy Einstein ordered weighted geometric (LDFEOWG) operators and discussed their applications in multi-criteria decision-making (MCDM). In Kamacı (2021), Kamacı introduced some algebraic properties of finite linear Diophantine fuzzy subsets of group, ring, and field. They also proposed the linear Diophantine fuzzy subgroup and normal subgroup of a group, linear Diophantine fuzzy subring and ideal of a ring, and linear Diophantine fuzzy subfield of a field. In 2022, (Petchimuthu et al. 2022) developed a multi-criteria decision-making (MCDM) process based on interval-valued linear Diophantine fuzzy weighted average (IVLDFWA) and interval-valued linear Diophantine fuzzy weighted geometric aggregation (IVLDFWGA) operators for fusing interval-valued linear Diophantine fuzzy information. Another generalization of LDFSs was introduced in 2022, when Kamacı (2022) introduced complex linear Diophantine fuzzy set (CLDFS), gave examples on them, and investigated their properties. Later, Tahan et al. (2022) introduced linear Diophantine fuzzy subpolygroups of a polygroup, linear Diophantine fuzzy normal subpolygroups of a polygroup, and linear Diophantine anti-fuzzy subpolygroups of a polygroup as a generalization of fuzzy subpolygroups of a polygroup, fuzzy normal subpolygroups of a polygroup, and anti-fuzzy subpolygroups of a polygroup and presented some examples and results on them. Besides, Mohammad et al. (2022) developed a multi-criteria group decision-making (MCGDM) process based on similarity measures of a Linear Diophantine fuzzy (LDF) information. In Prakash et al. (2022), Prakash et al. presented the special forms of linear Diophantine fuzzy bridges, linear Diophantine fuzzy cut-vertices, linear Diophantine fuzzy cycles, linear Diophantine fuzzy trees, and linear Diophantine fuzzy forests. Based on the proposed basic operations and comparison method for q-rung linear Diophantine fuzzy number (q-RLDFN), Almagrabi et al. (2022) strengthened the classical arithmetic average and geometric average to the q-rung linear Diophantine fuzzy environment and developed two q-rung linear Diophantine fuzzy operators. The q-rung linear Diophantine fuzzy set (q-RLDFS) contains the membership degree and non-membership degree which assure that the sum of qth power of membership $\mu_A(a)$, non-membership $\mu_A(a)$ and the reference parameters α_A, β_A is not greater than one, i.e., $0 \leq \alpha_A^q \mu_A(a) + \beta_A^q \eta_A(a) \leq 1$ and $0 \leq \alpha_A^q + \beta_A^q \leq 1$ for all $a \in U$. We can conclude that the q-RLDFS is more precise as the required orthopair capacity raises. In addition, the qRLDFS gives membership degrees greater freedom to communicate their linear Diophantine fuzzy information. They also proposed the q-rung linear Diophantine fuzzy weighted geometric aggregation (q-RLDFWGA), q-RLDFOWGA, and q-RLDFHWGA-operators. In 2023, Ashraf et al. (2023) introduced the spherical q-linear Diophantine fuzzy set (Sq-LDFS), an unique generalization of the LDFS, q-LDFS, and spherical linear Diophantine fuzzy set (SLDFS), and also highlighted its significant aspects. Furthermore, aggregation operators contribute significantly to the efficient aggregation of uncertainty in multiple attribute decision-making (MADM) situations. The Sq-LDFS cover the membership α_A, neutral γ_A and non-membership β_A with control factors $\alpha_A, \gamma_A, \beta_A$ having restrictions $\alpha_A, \gamma_A, \beta_A \in [0, 1]$ are control factors also $0 \leq \alpha_A^p \mu_A(a) + \gamma_A^q \nu_A(a) + \beta_A^r \eta_A(a) \leq 1$ and $0 \leq \alpha_A^p + \gamma_A^q + \beta_A^r \leq 1$ for all $a \in U$. It gives us an open choice to select the membership, neutral and non-membership values. Subsequently, Iqbal and Yaqoob (2023) proposed an idea of trapezoidal linear Diophantine fuzzy numbers in general (TrapLDFNs). In (Riaz et al. 2023), developed a multi-criteria decision-making (MCDM) process based on linear Diophantine fuzzy weighted average (LDFWA) and linear Diophantine fuzzy weighted geometric (LDFWG) operators for linear Diophantine fuzzy information. The concepts of LDFSs and q-LDFSs have numerous applications in various fields of real life, but these theories have their own limitations related to the membership, non-membership, and reference parameters. To eradicate these restrictions, we introduce the novel concept of (p, q)-rung linear Diophantine fuzzy set ((p, q)RLDFS). The proposed decision-making model of (p, q)-rung linear Diophantine fuzzy number ((p, q)RLDFN) is more efficient and flexible rather than other approaches due to the use of reference parameters $\alpha_A^p + \beta_A^q \in [0, 1]$. The (p, q)RLDFS also categorizes the data in multi-attribute decision-making (MADM) problems by changing the physical sense of reference parameters. The inspiration for the proposed work is addressed in every part of this article. This paper has multiple objectives as follows:

1. To define a new linear Diophantine fuzzy set.
2. To define some properties of (p, q)-rung linear Diophantine fuzzy sets.
3. To define a new score function useful for (p, q)-rung linear Diophantine fuzzy multi-attribute decision-making ((p, q)RLDFMADM) applications.

4. To define a new aggregation operators, i.e., the (p, q)-rung linear Diophantine fuzzy weighted averaging ((p, q)RLDFWA) and (p, q)-rung linear Diophantine fuzzy weighted geometric ((p, q)RLDFWG) operators and study some of its properties.
5. To develop optimization (p, q)RLDFMADM models to determine the weights of attributes.

The layout of this paper is systematized as follows: Sect. 2 reviews some preliminaries regarding intuitionistic fuzzy sets, Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, n, m-rung orthopair fuzzy sets, linguistic intuitionistic fuzzy variables, linguistic Pythagorean fuzzy numbers, and linguistic q-rung orthopair fuzzy numbers. Section 3 proposes the concepts of a linguistic (p, q)-rung orthopair fuzzy variable and defines some basic operations of linguistic (p, q)-rung orthopair fuzzy variables and the score function of a linguistic (p, q)-rung orthopair fuzzy variable. In Sect. 4, we develop linguistic (p, q)-rung orthopair fuzzy weighted averaging operator for linguistic (p, q)-rung orthopair fuzzy variables and investigate their properties. In Sect. 5, we propose linguistic (p, q)-rung orthopair fuzzy weighted geometric operator and discuss some properties and some special cases of these operators. In Sect. 6, an illustrative example of a company selecting the most suitable green supplier is provided to illustrate the application of the developed method. Finally, the concluding remarks are given in Sect. 7.

2 Notation and background

2.1 Notation

We use the following notation throughout the model:

IFS	Intuitionistic fuzzy set
PFS	Pythagorean fuzzy set
q-ROFS	q-rung orthopair fuzzy set
n, m-ROFS	n, m-rung orthopair fuzzy set
n, m-ROFN	n, m-rung orthopair fuzzy number
LDFS	Linear Diophantine fuzzy set
q-RLDFS	q-rung linear Diophantine fuzzy set
(p, q)RLDFS	(p, q)-rung linear Diophantine fuzzy set
(p, q)RLDFN	(p, q)-rung linear Diophantine fuzzy number
$\mathcal{SC}$	Score function
$\mathcal{AC}$	Accuracy function
(p, q)RLDFWA	(p, q)-rung linear Diophantine fuzzy weighted averaging
$\mathcal{WA}$	(p, q)RLDFWA-operator
(p, q)RLDFWG	(p, q)-rung linear Diophantine fuzzy weighted geometric
$\mathcal{WG}$	(p, q)RLDFWG-operator
MADM	Multi-attribute decision-making
(p, q)RLDFMADM	(p, q)-rung linear Diophantine fuzzy multi-attribute decision-making

2.2 Background

In this section, we recall some basic concepts of intuitionistic fuzzy sets, Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, n, m-rung orthopair fuzzy sets, linear Diophantine fuzzy sets, and q-rung linear Diophantine fuzzy sets. We utilize these basic components for the construction of a hybrid structure called (p, q)-rung linear Diophantine fuzzy sets.

The concept of intuitionistic fuzzy sets was introduced in Atanassov (1968) as a generalization of the notion of fuzzy sets as follows:

Definition 1 Atanassov (1968) An **intuitionistic fuzzy set (IFS)** $\mathcal{A}$ over a fixed set U is an object having the form

$$\mathcal{A} = \{\{a, \mu_A(a), \eta_A(a)\} : a \in U\}, \tag{1}$$

where the functions $\mu_A : U \to [0, 1]$ and $\eta_A : U \to [0, 1]$ denote the degree of **membership** and the degree of **non-membership**, respectively, and $\mu_A(a) + \eta_A(a) \in [0, 1]$ for all $a \in U$.

Remark 1 Let $\mathcal{A} = \{\{a, \mu_A(a), \nu_A(a)\} : a \in U\}$ be an IFS over a fixed set U. For the sake of simplicity, IFS will be denoted by $\mathcal{A} = (\mu_A, \nu_A)$.

Definition of Pythagorean fuzzy sets is given in Yager (2014) as follows:

Definition 2 Yager (2014) A **Pythagorean fuzzy set (PFS)** $\mathcal{A}$ over a fixed set U is an object having the form

$$\mathcal{A} = \{\{a, \mu_A(a), \eta_A(a)\} : a \in U\}, \tag{2}$$

where the functions $\mu_A : U \to [0, 1]$ and $\eta_A : U \to [0, 1]$ denote the degree of **membership** and the degree of **non-membership**, respectively, and $\mu_A^2(a) + \eta_A^2(a) \in [0, 1]$ for all $a \in U$.

In 2017, Yager (2017) presented the concept of q-rung orthopair fuzzy sets which is the generalization of Pythagorean fuzzy sets as follows:

Definition 3 Yager (2017) A **q-rung orthopair fuzzy set (q-ROFS)** $\mathcal{A}$ on a fixed set U is an object having the form

$$\mathcal{A} = \{\{a, \mu_A(a), \eta_A(a)\} : a \in U\}, \tag{3}$$

where the functions $\mu_A : U \to [0, 1]$ and $\eta_A : U \to [0, 1]$ denote the degree of **membership** and the degree of **non-membership**, respectively, and $\mu_A^q(a) + \eta_A^q(a) \in [0, 1]$ for all $a \in U$.

In 2022, Ibrahim and Alshammari (2022) redefined the notion of q-rung orthopair fuzzy set (q-ROFS), as a useful extension of q-ROFSs as follows:

Definition 4 Ibrahim and Alshammari (2022) A **n, m-rung orthopair fuzzy set (n, m-ROFS)** $\mathcal{A}$ on a fixed set U is an object having the form

$$\mathcal{A} = \{\{a, \mu_A(a), \eta_A(a)\} : a \in U\}, \tag{4}$$

where the functions $\mu_A : U \to [0, 1]$ and $\eta_A : U \to [0, 1]$ denote the degree of **membership** and the degree of **non-membership**, respectively, and $\mu_A^n(a) + \eta_A^m(a) \in [0, 1]$ for all $a \in U$.

Remark 2 Let $\mathcal{A} = \{\{a, \mu_A(a), \eta_A(a)\} : a \in U\}$ be a n, m-ROFS on a fixed set U. In particular, if U has only one element, $\mathcal{A}$ is called a **n, m-rung orthopair fuzzy number (n, m-ROFN)**.

The theory of linear Diophantine fuzzy sets, which was initially introduced in 2019 by Riaz and Hashmi in Riaz and Hashmi (2019), has been applied to many mathematical branches.

Definition 5 Riaz and Hashmi (2019) A **linear Diophantine fuzzy set (LDFS)** $\mathcal{A}$ on a fixed set U is an object having the form

$$\mathcal{A} = \{\{a, \mu_A(a), \eta_A(a)\}, \{\alpha_A, \beta_A\} : a \in U\}, \tag{5}$$

where the functions $\mu_A : U \to [0, 1]$ and $\eta_A : U \to [0, 1]$ denote the degree of **membership grades** and the degree of **non-membership grades**, respectively, and $\alpha_A, \beta_A \in [0, 1]$ are reference parameters also $0 \le \alpha_A \mu_A(a) + \beta_A \eta_A(a) \le 1$ and $0 \le \alpha_A + \beta_A \le 1$ for all $a \in U$.

In 2022, Almagrabi et al. (2022) defined the q-rung linear Diophantine fuzzy set as an extension of linear Diophantine fuzzy set and q-rung orthopair fuzzy set as follows.

Definition 6 Almagrabi et al. (2022) A **q-rung linear Diophantine fuzzy set (q-RLDFS)** $\mathcal{A}$ on a fixed set U is an object having the form

$$\mathcal{A} = \{\{a, \mu_A(a), \eta_A(a)\}, \{\alpha_A, \beta_A\} : a \in U\}, \tag{6}$$

where the functions $\mu_A : U \to [0, 1]$ and $\eta_A : U \to [0, 1]$ denote the degree of **membership grades** and the degree of **non-membership grades**, respectively, and $\alpha_A, \beta_A \in [0, 1]$ are reference parameters also $0 \le \alpha_A^q \mu_A(a) + \beta_A^q \eta_A(a) \le 1$ and $0 \le \alpha_A^q + \beta_A^q \le 1$ for all $a \in U$.

Now, we introduce the concepts of t-norms and t-conorms which are defined as follows:

Definition 7 Klir and Yuan (1995) A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a **t-norm (TN)** if it satisfies the following four conditions:

1. $T(1, x) = x$ for every x of $[0, 1]$.
2. $T(x, y) = T(y, x)$ for every x and y of $[0, 1]$.
3. $T(x, T(y, z)) = T(T(x, y), z)$ for every x, y and z of $[0, 1]$.
4. If $w \le x$ and $y \le z$, then $T(w, y) \le T(x, z)$.

A TN T is called **Archimedean t-norm** if it is continuous and $T(x, x) < x$ for every x of $[0, 1]$

Definition 8 Klir and Yuan (1995) A function $T^* : [0, 1] \times [0, 1] \to [0, 1]$ is called a **t-conorm (TC)** if it satisfies the following four conditions:

1. $T^*(0, x) = x$ for all x.
2. $T^*(x, y) = T^*(y, x)$ for all x and y.
3. $T^*(x, T^*(y, z)) = T^*(T^*(x, y), z)$ for all x, y and z.
4. If $w \le x$ and $y \le z$, then $T^*(w, y) \le T^*(x, z)$.

A TC T^* is called **Archimedean t-conorm** if it is continuous and $T^*(x, x) < x$ for every x of $[0, 1]$

3 (p, q)-Rung linear Diophantine fuzzy sets

In this section, we describe the concepts of (p, q)-rung linear Diophantine fuzzy set and (p, q)-rung linear Diophantine fuzzy number, and then investigate their operational laws.

The (p, q)-rung linear Diophantine fuzzy sets are generated with the help of n, m-ROFSs and q-RLDFSs. Therefore, we can say that (p, q)-rung linear Diophantine fuzzy set is the generalization of previously defined concepts related to linear Diophantine fuzzy set theory. The (p, q)-rung linear Diophantine fuzzy set gives more opportunities to deal with uncertainty in data.

Definition 9 A (p, q)**-rung linear Diophantine fuzzy set (**(p, q)**RLDFS)** $\mathcal{A}$ on a fixed set U is an object having the form

$$\mathcal{A} = \{\{a, \mu_A(a), \eta_A(a)\}, \{\alpha_A, \beta_A\} : a \in U\}, \tag{7}$$

where the functions $\mu_A : U \to [0, 1]$ and $\eta_A : U \to [0, 1]$ denote the degree of **membership grades** and the degree of **non-membership grades**, respectively, and $\alpha_A, \beta_A \in [0, 1]$ are reference parameters also $\alpha_A^p \mu_A(a) + \beta_A^q \eta_A(a) \in [0, 1]$ and $\alpha_A^p + \beta_A^q \in [0, 1]$ for all $a \in U$.

Remark 3 By Definition 9, we observe that (Almagrabi et al. 2022) q-RLDFSs are (p, q)RLDFSs with $p = q$.

Let $\mathcal{A} = \{\{a, \mu_A(a), \eta_A(a)\}, \{\alpha_A, \beta_A\} : a \in U\}$ be a (p, q)RLDFS on a fixed set U. In particular, if U has only one element, $\mathcal{A}$ is called a (p, q)**-rung linear Diophantine fuzzy number (**(p, q)**RLDFN)**. For convenience, a (p, q)RLDFN is denoted by $\mathcal{A} = ((\mu_A, \eta_A), (\alpha_A, \beta_A))$. Moreover, we denote by $\mathcal{RL}(U)$ the collection of (p, q)RLDFNs on a on a fixed set U with $\mathbb{U} = ((1, 0), (1, 0))$ and $\emptyset = ((0, 1), (0, 1))$.

Example 1 Suppose that $\mu_A = 0.7, \eta_A = 1, \alpha_A = 0.9, \beta_A = 0.7$ and $U = \{a\}$. Hence, $(0.9)(0.7)+(0.7)(1) = 1.33 > 1, 0.9+0.7 = 1.6 > 1, (0.9)^2(0.7)+(0.7)^2(1) = 1.057 > 1$, $(0.9)^2 + (0.7)^2 = 1.3 > 1$, $(0.9)^3(0.7) + (0.7)^3(1) = 0.8533 < 1$, $(0.9)^3 + (0.7)^3 = 1.072 > 1$, but $(0.9)^3(0.7) + (0.7)^4(1) = 0.7504 < 1$, $(0.9)^3 + (0.7)^4 = 0.9691 < 1$, and $(0.9)^4(0.7) + (0.7)^3(1) = 0.80227 < 1$, $(0.9)^4 + (0.7)^3 = 0.9991 < 1$. Therefore, $\mathcal{A} = ((\mu_A, \eta_A), (\alpha_A, \beta_A))$ is both (3, 4)RLDFN and (4, 3)RLDFN, but $\mathcal{A}$ is neither LDFN nor 2-RLDFN nor 3-RLDFN.

For this relation of less than and equality are defined as follows:

Definition 10 For any two (p, q)RLDFNs $\mathcal{A} = ((\mu_A, \eta_A), (\alpha_A, \beta_A))$ and $\mathcal{B} = ((\mu_B, \eta_B), (\alpha_B, \beta_B))$ on a fixed set U, the corresponding operations are defined as follows:

1. $(\mu_A, \eta_A) \preceq (\mu_B, \eta_B)$ if and only if $\mu_A \leq \mu_B$ and $\eta_A \geq \eta_B$,
2. $(\alpha_A, \beta_A) \preceq (\alpha_B, \beta_B)$ if and only if $\alpha_A \leq \alpha_B$ and $\beta_A \geq \beta_B$,
3. $\mathcal{A} \preceq \mathcal{B}$ if and only if $(\mu_A, \eta_A) \preceq (\mu_B, \eta_B)$ and $(\alpha_A, \beta_A) \preceq (\alpha_B, \beta_B)$,
4. $\mathcal{A} = \mathcal{B}$ if and only if $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{B} \preceq \mathcal{A}$.

Motivated by the operations of the LDFNs and q-RLDFNs, in the following, we shall define some operational laws of (p, q)RLDFNs.

Definition 11 Let $\mathcal{A} = ((\mu_A, \eta_A), (\alpha_A, \beta_A))$ and $\mathcal{B} = ((\mu_B, \eta_B), (\alpha_B, \beta_B))$ be any (p, q)RLDFNs on a fixed set U. For every element $\lambda \in (0, \infty)$, then their operations are defined as follows:

1. $(\mu_A, \eta_A) \oplus (\mu_B, \eta_B) = \left(\sqrt[p]{(\mu_A)^p + (\mu_B)^p - (\mu_A)^p (\mu_B)^p}, \eta_A \eta_B\right)$,
2. $(\alpha_A, \beta_A) \oplus (\alpha_B, \beta_B) = \left(\sqrt[p]{(\alpha_A)^p + (\alpha_B)^p - (\alpha_A)^p (\alpha_B)^p}, \beta_A \beta_B\right)$,

3. $(\mu_A, \eta_A) \otimes (\mu_B, \eta_B) = \left(\mu_A \mu_B, \sqrt[q]{(\eta_A)^q + (\eta_B)^q - (\eta_A)^q (\eta_B)^q}\right)$,
4. $(\alpha_A, \beta_A) \otimes (\alpha_B, \beta_B) = \left(\alpha_A \alpha_B, \sqrt[q]{(\beta_A)^q + (\beta_B)^q - (\beta_A)^q (\beta_B)^q}\right)$,
5. $\lambda (\mu_A, \eta_A) = \left(\sqrt[p]{1 - \left(1 - (\mu_A)^p\right)^\lambda}, (\eta_A)^\lambda\right)$,
6. $\lambda (\alpha_A, \beta_A) = \left(\sqrt[p]{1 - \left(1 - (\alpha_A)^p\right)^\lambda}, (\beta_A)^\lambda\right)$,
7. $(\mu_A, \eta_A)^\lambda = \left((\mu_A)^\lambda, \sqrt[q]{1 - \left(1 - (\eta_A)^q\right)^\lambda}\right)$,
8. $(\alpha_A, \beta_A)^\lambda = \left((\alpha_A)^\lambda, \sqrt[q]{1 - \left(1 - (\beta_A)^q\right)^\lambda}\right)$,
9. $\mathcal{A} \oplus \mathcal{B} = ((\mu_A, \eta_A) \oplus (\mu_B, \eta_B), (\alpha_A, \beta_A) \oplus (\alpha_B, \beta_B))$,
10. $\mathcal{A} \otimes \mathcal{B} = ((\mu_A, \eta_A) \otimes (\mu_B, \eta_B), (\alpha_A, \beta_A) \otimes (\alpha_B, \beta_B))$,
11. $\lambda \mathcal{A} = (\lambda (\mu_A, \eta_A), \lambda (\alpha_A, \beta_A))$,
12. $\mathcal{A}^\lambda = \left((\mu_A, \eta_A)^\lambda, (\alpha_A, \beta_A)^\lambda\right)$.

It can be easily proved that the (p, q)RLDFN has the following properties.

Theorem 1 *Let* $\mathcal{A} = ((\mu_A, \eta_A), (\alpha_A, \beta_A)), \mathcal{B} = ((\mu_B, \eta_B), (\alpha_B, \beta_B))$ *and* $\mathcal{C} = ((\mu_C, \eta_C), (\alpha_C, \beta_C))$ *be any (p, q)RLDFNs on a fixed set U. For every elements $\lambda, \xi \in (0, \infty)$, the following properties hold:*

1. $\mathcal{A} \oplus \mathcal{B} = \mathcal{B} \oplus \mathcal{A}$.
2. $\mathcal{A} \otimes \mathcal{B} = \mathcal{B} \otimes \mathcal{A}$.
3. $(\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{C} = \mathcal{A} \oplus (\mathcal{B} \oplus \mathcal{C})$.
4. $(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$.
5. $\lambda (\mathcal{A} \oplus \mathcal{B}) = \lambda \mathcal{A} \oplus \lambda \mathcal{B}$.
6. $(\mathcal{A} \otimes \mathcal{B})^\lambda = \mathcal{A}^\lambda \otimes \mathcal{B}^\lambda$.
7. $(\lambda + \xi) \mathcal{A} = \lambda \mathcal{A} \oplus \xi \mathcal{A}$.
8. $\mathcal{A}^{\lambda+\xi} = \mathcal{A}^\lambda \otimes \mathcal{A}^\xi$.
9. $\left(\mathcal{A}^\lambda\right)^\xi = \mathcal{A}^{\lambda\xi}$.

For applications of (p, q)RLDFNs in decision-making problems, we need to know how to rank the (p, q)RLDFNs. For ranking a (p, q)RLDFN, we need to calculate the score and accuracy of that (p, q)RLDFN as follows defined in the next definition.

Definition 12 Let $\mathcal{A} = ((\mu_A, \eta_A), (\alpha_A, \beta_A))$ be a (p, q)RLDFN on a fixed set U. Then, the **score function (accuracy function)** of $\mathcal{A}$ can be defined as follows:

$$\mathcal{SC}(\mathcal{A}) = \frac{1}{2} \left(\mu_A - \eta_A + (\alpha_A)^p - (\beta_A)^q\right) \tag{8}$$

$$\mathcal{AC}(\mathcal{A}) = \frac{1}{4} (\mu_A + \eta_A) + \frac{1}{2} \left((\alpha_A)^p + (\beta_A)^q\right). \tag{9}$$

Remark 4 It is clear that if membership grades $\mu_A(\alpha_A)$ are bigger, and the non-membership grades $\eta_A(\beta_A)$ are smaller, then the score value of the (p, q)RLDFN $\mathcal{A} = ((\mu_A, \eta_A), (\alpha_A, \beta_A))$ is greater.

It is noted that the score function (accuracy function) $\mathcal{SC}(\mathcal{A})$ $(\mathcal{AC}(\mathcal{A}))$ has some desirable properties as below.

Theorem 2 *Let $\mathcal{A} = ((\mu_A, \eta_A), (\alpha_A, \beta_A))$ be a (p, q)RLDFN on a fixed set U. Then, the following properties hold.*

1. $-1 \leq \mathcal{SC}(\mathcal{A}) \leq 1$.
2. $0 \leq \mathcal{AC}(\mathcal{A}) \leq 1$.
3. $\mathcal{SC}(\mathbb{U}) = 1$ *(one property).*
4. $\mathcal{SC}(\emptyset) = -1$ *(zero property).*
5. $\mathcal{AC}(\mathbb{U}) = 1$ *(one property).*
6. $\mathcal{AC}(\emptyset) = 1$ *(zero property).*

Proof The proof is obvious.

4 (p, q)-Rung linear Diophantine fuzzy weighted averaging operators

In this section, we shall develop some weighted averaging operators with (p, q)RLDFN, such as (p, q)-rung linear Diophantine fuzzy weighted averaging operator.

We now discuss some aggregation operators based on operational rules of (p, q)RLDFNs. Thus, from the above definition, it is clear that (p, q)-rung linear Diophantine fuzzy weighted averaging operators are a generalization of a q-rung linear Diophantine fuzzy weighted averaging operator.

Definition 13 Let $\Delta = \left\{\mathcal{A}_i = \left(\left(\mu_{A_i}, \eta_{A_i}\right), \left(\alpha_{A_i}, \beta_{A_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ be a collection of (p, q)RLDFNs on a fixed set U and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ be the weight vector of Δ indicates the importance degree of Δ satisfying $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0, 1]$ and $\sum_{i=1}^{n} \lambda_i = 1$, and let (p, q)**-rung linear Diophantine fuzzy weighted averaging (**(p, q)**RLDFWA)** $\mathcal{WA} : (\mathcal{RL}(U))^n \to \mathcal{RL}(U)$ if

$$\mathcal{WA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigoplus_{i=1}^{n} \lambda_i \mathcal{A}_i, \tag{10}$$

then the function (p, q)RLDFWA is called the (p, q)**-rung linear Diophantine fuzzy weighted averaging operator (**(p, q)**RLDFWA-operator)**.

On the basis of the operational rules of the (p, q)RLDFNs on U, we can get the aggregation result described as Theorem 3.

Theorem 3 *Let $\Delta = \left\{\mathcal{A}_i = \left(\left(\mu_{A_i}, \eta_{A_i}\right), \left(\alpha_{A_i}, \beta_{A_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ be a collection of (p, q)RLDFNs on a fixed set U and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ be the weight vector of Δ, where λ_j indicates the importance degree of Δ, satisfying $0 \leq \lambda_1, \lambda_2, \ldots, \lambda_n \leq 1$ and $\sum_{i=1}^{n} \lambda_i = 1$. Then, their accumulated outcome utilizing the (p, q)RLDFWA-operator is again (p, q)RLDFN and*

$$\mathcal{WA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \left(\begin{array}{c} \left(\begin{array}{c} \sqrt[p]{1 - \prod_{i=1}^{n} \left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i}}, \\ \prod_{i=1}^{n} \left(\eta_{A_i}\right)^{\lambda_i} \end{array} \right), \\ \left(\begin{array}{c} \sqrt[p]{1 - \prod_{i=1}^{n} \left(1 - \left(\alpha_{A_i}\right)^p\right)^{\lambda_i}}, \\ \prod_{i=1}^{n} \left(\beta_{A_i}\right)^{\lambda_i} \end{array} \right) \end{array} \right). \tag{11}$$

From Theorem 3, it is observed that our proposed (p, q)RLDFWA-operator satisfies idempotency, boundedness, and monotonicity properties which are as follows:

Theorem 4 *(Idempotency) Let* $\left\{\mathcal{A}_i = \left(\left(\mu_{A_i}, \eta_{A_i}\right), \left(\alpha_{A_i}, \beta_{A_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ *be a collection of* (p, q)*RLDFNs on a fixed set* U. *If* $\mathcal{A}_i = \mathcal{A} = \left(\left(\mu_A, \eta_A\right), \left(\alpha_A, \beta_A\right)\right)$ *for every* $i = 1, 2, \ldots, n$, *then* $\mathcal{WA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \mathcal{A}$.

It can be easily proved that the (p, q)RLDFWA-operator has the following properties.

Corollary 1 *Let* $\Delta = \left\{\mathcal{A}_i = \left(\left(\mu_{A_i}, \eta_{A_i}\right), \left(\alpha_{A_i}, \beta_{A_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ *be a collection of* (p, q)*RLDFNs on a fixed set* U. *Then, the following properties hold.*

1. *If* $\mathcal{A}_i = \emptyset$ *for every* $i = 1, 2, 3, \ldots, n$, *then* $\mathcal{WA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \emptyset$.
2. *If* $\mathcal{A}_i = \mathbb{U}$ *for every* $i = 1, 2, 3, \ldots, n$, *then* $\mathcal{WA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \mathbb{U}$.

Theorem 5 establishes some properties of (p, q)RLDFWA-operator.

Theorem 5 *(Monotonicity) Let* $\left\{\mathcal{A}_i = \left(\left(\mu_{A_i}, \eta_{A_i}\right), \left(\alpha_{A_i}, \beta_{A_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ *and* $\left\{\mathcal{B}_i = \left(\left(\mu_{B_i}, \eta_{B_i}\right), \left(\alpha_{B_i}, \beta_{B_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ *be any collections of* (p, q)*RLDFNs on a fixed set* U. *If* $\mathcal{A}_i \preceq \mathcal{B}_i$ *for every* $i = 1, 2, 3, \ldots, n$, *then* $\mathcal{WA}\left(\mathcal{A}_1, \ldots, \mathcal{A}_n\right) \preceq \mathcal{WA}\left(\mathcal{B}_1, \ldots, \mathcal{B}_n\right)$.

Now, we prove properties of (p, q)RLDFWA-operator.

Theorem 6 *(Boundedness) Let* $\left\{\mathcal{A}_i = \left(\left(\mu_{A_i}, \eta_{A_i}\right), \left(\alpha_{A_i}, \beta_{A_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ *be a collection of* (p, q)*RLDFNs on a fixed set* U. *If*

$$\mathcal{A}^+ = \left(\left(\bigvee_{i=1}^{n} \mu_{A_i}, \bigwedge_{i=1}^{n} \eta_{A_i}\right), \left(\bigvee_{i=1}^{n} \alpha_{A_i}, \bigwedge_{i=1}^{n} \beta_{A_i}\right)\right)$$

and

$$\mathcal{A}^- = \left(\left(\bigwedge_{i=1}^{n} \mu_{A_i}, \bigvee_{i=1}^{n} \eta_{A_i}\right), \left(\bigwedge_{i=1}^{n} \alpha_{A_i}, \bigvee_{i=1}^{n} \beta_{A_i}\right)\right),$$

then $\mathcal{A}^- \preceq \mathcal{WA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) \preceq \mathcal{A}^+$.

5 (p, q)-Rung linear Diophantine fuzzy weighted geometric operators

In the following, we shall propose the concepts and basic operations of the (p, q)-rung linear Diophantine fuzzy weighted geometric operator on the basis of the (p, q)RLDFNs.

The following definition describes the (p, q)-rung linear Diophantine fuzzy weighted geometric operators.

Definition 14 Let $\Delta = \left\{\mathcal{A}_i = \left(\left(\mu_{A_i}, \eta_{A_i}\right), \left(\alpha_{A_i}, \beta_{A_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ be a collection of (p, q)RLDFNs on a fixed set U and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ be the weight vector of Δ, where λ_j indicates the importance degree of Δ, satisfying $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0, 1]$ and $\sum_{i=1}^{n} \lambda_i = 1$, and let **$(p, q)$-rung linear Diophantine fuzzy weighted geometric ((p, q)RLDFWG)** $\mathcal{WG} : (\mathcal{RL}(U))^n \to \mathcal{RL}(U)$ if

$$\mathcal{WG}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \bigotimes_{i=1}^{n} \mathcal{A}_i^{\lambda_i}, \tag{12}$$

then the function (p, q)RLDFWG is called the **(p, q)-rung linear Diophantine fuzzy weighted geometric operator ((p, q)RLDFWG-operator)**.

The general expression of the (p, q)RLDFWG-operator is constructed in the following theorem:

Theorem 7 *Let $\Delta = \left\{\mathcal{A}_i = \left(\left(\mu_{A_i}, \eta_{A_i}\right), \left(\alpha_{A_i}, \beta_{A_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ be a collection of (p, q)RLDFNs on a fixed set U and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ be the weight vector of Δ, where λ_j indicates the importance degree of Δ, satisfying $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0, 1]$ and $\sum_{i=1}^{n} \lambda_i = 1$. Then, their accumulated outcome utilizing the (p, q)RLDFWG-operator is again (p, q)RLDFN, and*

$$\mathcal{WG}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \left(\begin{array}{c} \left(\begin{array}{c} \prod_{i=1}^{n} \left(\mu_{A_i}\right)^{\lambda_i}, \\ \sqrt[q]{1 - \prod_{i=1}^{n} \left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i}} \end{array}\right), \\ \left(\begin{array}{c} \prod_{i=1}^{n} \left(\alpha_{A_i}\right)^{\lambda_i}, \\ \sqrt[q]{1 - \prod_{i=1}^{n} \left(1 - \left(\beta_{A_i}\right)^q\right)^{\lambda_i}} \end{array}\right) \end{array}\right). \tag{13}$$

From Theorem 7, it is observed that our proposed (p, q)RLDFWG-operator satisfies idempotency, boundedness, and monotonicity properties which are as follows:

Theorem 8 (Idempotency) *Let $\left\{\mathcal{A}_i = \left(\left(\mu_{A_i}, \eta_{A_i}\right), \left(\alpha_{A_i}, \beta_{A_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ be a collection of (p, q)RLDFNs on a fixed set U. If $\mathcal{A}_i = \mathcal{A} = \left(\left(\mu_A, \eta_A\right), \left(\alpha_A, \beta_A\right)\right)$ for every $i = 1, 2, \ldots, n$, then $\mathcal{WG}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \mathcal{A}$.*

It can be easily proved that the (p, q)RLDFWG-operator has the following properties.

Corollary 2 *Let $\left\{\mathcal{A}_i = \left(\left(\mu_{A_i}, \eta_{A_i}\right), \left(\alpha_{A_i}, \beta_{A_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ be a collection of (p, q)RLDFNs on a fixed set U. Then, the following properties hold.*

1. *If* $\mathcal{A}_i = \emptyset$ *for every* $i = 1, 2, 3, \ldots, n$, *then* $\mathcal{WG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \emptyset$.
2. *If* $\mathcal{A}_i = \mathbb{U}$ *for every* $i = 1, 2, 3, \ldots, n$, *then* $\mathcal{WG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathbb{U}$.

The properties of the (p, q)RLDFWG-operator are stated in the following theorem:

Theorem 9 *(Monotonicity) Let* $\left\{\mathcal{A}_i = \left(\left(\mu_{A_i}, \eta_{A_i}\right), \left(\alpha_{A_i}, \beta_{A_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ *and* $\left\{\mathcal{B}_i = \left(\left(\mu_{B_i}, \eta_{B_i}\right), \left(\alpha_{B_i}, \beta_{B_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ *be any collections of* (p, q)*RLDFNs on a fixed set* U. *If* $\mathcal{A}_i \preceq \mathcal{B}_i$ *for every* $i = 1, 2, \ldots, n$, *then* $\mathcal{WG}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \preceq \mathcal{WG}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$.

The following Theorem 10 establishes some properties of (p, q)RLDFWG-operator.

Theorem 10 *(Boundedness) Let* $\left\{\mathcal{A}_i = \left(\left(\mu_{A_i}, \eta_{A_i}\right), \left(\alpha_{A_i}, \beta_{A_i}\right)\right) : i = 1, 2, 3, \ldots, n\right\}$ *be a collection of* (p, q)*RLDFNs on a fixed set* U. *If*

$$\mathcal{A}^+ = \left(\left(\bigvee_{i=1}^{n} \mu_{A_i}, \bigwedge_{i=1}^{n} \eta_{A_i}\right), \left(\bigvee_{i=1}^{n} \alpha_{A_i}, \bigwedge_{i=1}^{n} \beta_{A_i}\right)\right)$$

and

$$\mathcal{A}^- = \left(\left(\bigwedge_{i=1}^{n} \mu_{A_i}, \bigvee_{i=1}^{n} \eta_{A_i}\right), \left(\bigwedge_{i=1}^{n} \alpha_{A_i}, \bigvee_{i=1}^{n} \beta_{A_i}\right)\right),$$

then $\mathcal{A}^- \preceq \mathcal{WG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \preceq \mathcal{A}^+$.

6 A new multi-attribute decision-making method for (p, q)RLDFNs based on the proposed operators

In this section, we define a multi-attribute decision-making (MADM) method, the so-called (p, q)-rung linear Diophantine fuzzy multi-attribute decision-making ((p, q)RLDFMADM) method. Its adopted from Almagrabi et al. (2022).

In the DM problems, we propose a (p, q)RLDFMADM approach using (p, q)-rung linear Diophantine fuzzy aggregation operators in which weights of the real numbers under (p, q)-rung linear Diophantine fuzzy environment. Here, (p, q)RLDFMADM method is used to find a favorable alternative for a selecting the most suitable supplier. The set $A = \{A_1, A_2, A_3, \ldots, A_m\}$ contains the numbers of alternatives, while $G = \{G_1, G_2, \ldots, G_n\}$ represents the numbers of attributes. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ be the weight vector met the criteria as $0 \leq \lambda_1, \lambda_2, \ldots, \lambda_n \leq 1$ and $\sum_{i=1}^{n} \lambda_i = 1$. A set of decision-makers (DMs) $D = \{D_1, D_2, D_3, \ldots, D_k\}$ are invited to appraisal all the alternatives. The expert's decisions are stated as (p, q)RLDFNs, i.e., $\left\{\mathcal{A}_{ij} = \left(\left(\mu_{A_{ij}}, \eta_{A_{ij}}\right), \left(\alpha_{A_{ij}}, \beta_{A_{ij}}\right)\right) : i = 1, 2, 3, \ldots, m, j = 1, 2, 3, \ldots, n\right\}$, such that $0 \leq \alpha^p_{A_{ij}} \mu_{A_{ij}} + \beta^q_{A_{ij}} \eta_{A_{ij}} \leq 1$ and $0 \leq \alpha^p_{A_{ij}} + \beta^q_{A_{ij}} \leq 1$. Thus, the (p, q)-rung linear Diophantine fuzzy decision matrix is given as $D = \left[\left(\left(\mu_{A_{ij}}, \eta_{A_{ij}}\right), \left(\alpha_{A_{ij}}, \beta_{A_{ij}}\right)\right)\right]_{m \times n}$ and presented as follows:

$$D = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \ldots & \mathcal{A}_{1n} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \ldots & \mathcal{A}_{2n} \\ \vdots & & & \\ \mathcal{A}_{m1} & \mathcal{A}_{m2} & \ldots & \mathcal{A}_{mn} \end{bmatrix}. \tag{14}$$

The algorithm follows a (p, q)RLDFMADM method to interpret a (p, q)RLDFMCDM problem under (p, q)-rung linear Diophantine fuzzy information using (p, q)RLDFWA and (p, q)RLDFWG-operators.

Algorithm

Input: (p, q)-rung linear Diophantine fuzzy information

Output: Best alternative

Step 1. Calculate the (p, q)-rung linear Diophantine fuzzy decision matrix.

Step 2. To eliminate the influence of attribute type, we consider the following technique and obtain the standardize (p, q)-rung linear Diophantine fuzzy matrix $R = \left[\mathcal{R}_{ij}\right]_{m\times n}$, where $\mathcal{R}_{ij} = \mathcal{A}_{ij} = \left(\left(\mu_{A_{ij}}, \eta_{A_{ij}}\right), \left(\alpha_{A_{ij}}, \beta_{A_{ij}}\right)\right)$ is (p, q)RLDFN. Then, we have

$$\mathcal{R}_{ij} = \begin{cases} \mathcal{A}_{ij} & \text{; for same type input data} \\ \mathcal{A}_{ij}^{c} & \text{; for different type input data.} \end{cases} \tag{15}$$

Step 3. Based on the standardize (p, q)-rung linear Diophantine fuzzy matrix, as obtained from step 2, the overall aggregated value of the alternative A_i for all $i = 1, 2, \ldots, m$, under the different criteria G_j for all $j = 1, 2, \ldots, n$ is obtained using either

$$r_i = \mathcal{WA}\left(\mathcal{R}_{i1}, \mathcal{R}_{i2}, \ldots, \mathcal{R}_{in}\right) \tag{16}$$

or

$$r_i = \mathcal{WG}\left(\mathcal{R}_{i1}, \mathcal{R}_{i2}, \ldots, \mathcal{R}_{in}\right) \tag{17}$$

operator and hence get the collective value r_i for each alternative A_i for all $i = 1, 2, \ldots, m$.

Step 4. Next, the score values of $\mathcal{SC}(r_i)$ $(i = 1, 2, 3, \ldots, m)$ of the overall (p, q)RLDFNs of $r_i (i = 1, 2, 3, \ldots, m)$ is obtained to rank the alternatives $A_i (i = 1, 2, \ldots, m)$. If the score values of $\mathcal{SC}(r_i)$ and $\mathcal{SC}\left(r_j\right)$ are equal for two alternatives A_i and A_j, then it is required to calculate accuracy degrees of $\mathcal{AC}(r_i)$ and $\mathcal{AC}\left(r_j\right)$ with respect to the overall collective (p, q)RLDFNs to rank the alternatives A_i and A_j, respectively, based on the aforementioned accuracy degrees $\mathcal{AC}(r_i)$ and $\mathcal{AC}\left(r_j\right)$.

Step 5. We select the best alternative from the rankings of all alternatives A_i $(i = 1, 2, \ldots, m)$ according to $\mathcal{SC}(r_i)$ $(i = 1, 2, \ldots, m)$.

Step 6.: End.

An illustrative example: We consider a (p, q)RLDFMADM problem given below. To illustrate the proposed (p, q)RLDFMADM strategy, we solve a (p, q)RLDFMADM problem adapted from Almagrabi et al. (2022).

We are suggesting a novel emergency public health solution to determine the best attribute under (p, q)-rung linear Diophantine fuzzy information. We are applying our input data over this algorithm. Three health experts are appointed to evaluate the four alternatives for emergencies $A = \{A_1, A_2, A_3, A_4\}$ with respect to five attributes $G = \{G_1, G_2, G_3, G_4\}$, and the decision matrices $D = \{D_1, D_2, D_3\}$ are constructed, as shown in Table 1. Suppose that the four experts'risk attitudes are $\lambda_1 = 0.2$, $\lambda_2 = 0.3$, $\lambda_3 = 0.4$ and $\lambda_4 = 0.1$. It is clear from final weight vectors that $\sum_{i=1}^{4} \lambda_i = 0.2 + 0.3 + 0.4 + 0.1 = 1$ (Tables 2, 3, 4).

The (p, q)RLDFMADM steps based on (p, q)RLDFWA and (p, q)RLDFWG-operators:

Step 1. Calculate the (p, q)-rung linear Diophantine fuzzy decision matrix. These five alternatives A_1, A_2, A_3 and A_4 are to be evaluated by an expert under the four aspects G_1, G_2, G_3 and G_4 using (p, q)-rung linear Diophantine fuzzy decision matrix

Table 1 Evaluations of decision-makers

	G_1	G_2	G_3	G_4
A_1	$\begin{pmatrix}(0.9, 1),\\(0.8, 0.7)\end{pmatrix}$	$\begin{pmatrix}(0.8, 0.9),\\(0.6, 0.9)\end{pmatrix}$	$\begin{pmatrix}(0.8, 0.9),\\(0.7, 0.8)\end{pmatrix}$	$\begin{pmatrix}(0.8, 0.8),\\(0.7, 0.8)\end{pmatrix}$
A_2	$\begin{pmatrix}(1, 0.9),\\(0.7, 0.8)\end{pmatrix}$	$\begin{pmatrix}(0.9, 0.7),\\(0.8, 0.7)\end{pmatrix}$	$\begin{pmatrix}(0.8, 1),\\(0.7, 0.8)\end{pmatrix}$	$\begin{pmatrix}(1, 0.8),\\(0.7, 0.8)\end{pmatrix}$
A_3	$\begin{pmatrix}(0.8, 1),\\(0.8, 0.7)\end{pmatrix}$	$\begin{pmatrix}(1, 1),\\(0.7, 0.8)\end{pmatrix}$	$\begin{pmatrix}(0.8, 0.9),\\(0.7, 0.8)\end{pmatrix}$	$\begin{pmatrix}(0.8, 0.9),\\(0.8, 0.7)\end{pmatrix}$
A_4	$\begin{pmatrix}(0.9, 0.8),\\(0.6, 0.9)\end{pmatrix}$	$\begin{pmatrix}(1, 0.9),\\(0.6, 0.9)\end{pmatrix}$	$\begin{pmatrix}(1, 1),\\(0.8, 0.7)\end{pmatrix}$	$\begin{pmatrix}(0.9, 0.8),\\(0.7, 0.8)\end{pmatrix}$

Table 2 Aggregated values of the alternatives using (p, q)RLDFWA-operator

	$(p, q) = (3, 3)$	$(p, q) = (3, 4)$	$(p, q) = (4, 3)$	$(p, q) = (10, 15)$
r_1	$\begin{pmatrix}(0.83, 0.91),\\(0.70, 081)\end{pmatrix}$	$\begin{pmatrix}(0.83, 0.91),\\(0.70, 0.81)\end{pmatrix}$	$\begin{pmatrix}(0.83, 0.91),\\(0.71, 0.81)\end{pmatrix}$	$\begin{pmatrix}(0.83, 0.91),\\(0.72, 0.81)\end{pmatrix}$
r_2	$\begin{pmatrix}(1.00, 0.86),\\(0.74, 0.77)\end{pmatrix}$	$\begin{pmatrix}(1.00, 0.86),\\(0.74, 0.77)\end{pmatrix}$	$\begin{pmatrix}(1.00, 0.86),\\(0.74, 0.77)\end{pmatrix}$	$\begin{pmatrix}(1.00, 0.86),\\(0.75, 0.77)\end{pmatrix}$
r_3	$\begin{pmatrix}(1.00, 0.95),\\(0.74, 0.77)\end{pmatrix}$	$\begin{pmatrix}(1.00, 0.95),\\(0.74, 0.77)\end{pmatrix}$	$\begin{pmatrix}(1.00, 0.95),\\(0.74, 0.77)\end{pmatrix}$	$\begin{pmatrix}(1.00, 0.95),\\(0.75, 0.77)\end{pmatrix}$
r_4	$\begin{pmatrix}(1.00, 0.96),\\(0.71, 0.80)\end{pmatrix}$	$\begin{pmatrix}(1.00, 0.96),\\(0.71, 0.80)\end{pmatrix}$	$\begin{pmatrix}(1.00, 0.96),\\(0.72, 0.80)\end{pmatrix}$	$\begin{pmatrix}(1.00, 0.96),\\(0.74, 0.80)\end{pmatrix}$

Table 3 Aggregated values of the alternatives using (p, q)RLDFWG-operator

	$(p, q) = (3, 3)$	$(p, q) = (3, 4)$	$(p, q) = (4, 3)$	$(p, q) = (15, 10)$
r_1	$\begin{pmatrix}(0.82, 1.00),\\(0.69, 0.83)\end{pmatrix}$	$\begin{pmatrix}(0.82, 1.00),\\(0.69, 0.83)\end{pmatrix}$	$\begin{pmatrix}(0.82, 1.00),\\(0.69, 0.83)\end{pmatrix}$	$\begin{pmatrix}(0.82, 1.00),\\(0.69, 0.85)\end{pmatrix}$
r_2	$\begin{pmatrix}(0.89, 1.00),\\(0.73, 0.78)\end{pmatrix}$	$\begin{pmatrix}(0.89, 1.00),\\(0.73, 0.78)\end{pmatrix}$	$\begin{pmatrix}(0.89, 1.00),\\(0.73, 0.78)\end{pmatrix}$	$\begin{pmatrix}(0.89, 1.00),\\(0.73, 0.78)\end{pmatrix}$
r_3	$\begin{pmatrix}(0.86, 1.00),\\(0.73, 0.78)\end{pmatrix}$	$\begin{pmatrix}(0.86, 1.00),\\(0.73, 0.78)\end{pmatrix}$	$\begin{pmatrix}(0.86, 1.00),\\(0.73, 0.78)\end{pmatrix}$	$\begin{pmatrix}(0.86, 1.00),\\(0.73, 0.78)\end{pmatrix}$
r_4	$\begin{pmatrix}(0.97, 1.00),\\(0.68, 0.84)\end{pmatrix}$	$\begin{pmatrix}(0.97, 1.00),\\(0.68, 0.84)\end{pmatrix}$	$\begin{pmatrix}(0.97, 1.00),\\(0.68, 0.84)\end{pmatrix}$	$\begin{pmatrix}(0.97, 1.00),\\(0.68, 0.87)\end{pmatrix}$

$D = \left[\mathcal{A}_{ij}\right]_{4\times 4}$ and their corresponding rating values are shown in Eq. (18)

$$D = \begin{bmatrix} \begin{pmatrix}(0.9, 1),\\(0.8, 0.7)\end{pmatrix} & \begin{pmatrix}(0.8, 0.9),\\(0.6, 0.9)\end{pmatrix} & \begin{pmatrix}(0.8, 0.9),\\(0.7, 0.8)\end{pmatrix} & \begin{pmatrix}(0.8, 0.8),\\(0.7, 0.8)\end{pmatrix} \\ \begin{pmatrix}(1, 0.9),\\(0.7, 0.8)\end{pmatrix} & \begin{pmatrix}(0.9, 0.7),\\(0.8, 0.7)\end{pmatrix} & \begin{pmatrix}(0.8, 1),\\(0.7, 0.8)\end{pmatrix} & \begin{pmatrix}(1, 0.8),\\(0.7, 0.8)\end{pmatrix} \\ \begin{pmatrix}(0.8, 1),\\(0.8, 0.7)\end{pmatrix} & \begin{pmatrix}(1, 1),\\(0.7, 0.8)\end{pmatrix} & \begin{pmatrix}(0.8, 0.9),\\(0.7, 0.8)\end{pmatrix} & \begin{pmatrix}(0.8, 0.9),\\(0.8, 0.7)\end{pmatrix} \\ \begin{pmatrix}(0.9, 0.8),\\(0.6, 0.9)\end{pmatrix} & \begin{pmatrix}(1, 0.9),\\(0.6, 0.9)\end{pmatrix} & \begin{pmatrix}(1, 1),\\(0.8, 0.7)\end{pmatrix} & \begin{pmatrix}(0.9, 0.8),\\(0.7, 0.8)\end{pmatrix} \end{bmatrix}. \tag{18}$$

Table 4 Score values of alternatives using (p, q)RLDFWA-operator

(p, q)	$(3, 3)$	$(3, 4)$	$(4, 3)$	$(10, 15)$
$\mathcal{SC}(r_1)$	−0.1293	−0.0785	−0.1782	−0.0381
$\mathcal{SC}(r_2)$	0.0428	0.0953	−0.0089	0.0867
$\mathcal{SC}(r_3)$	−0.0011	0.0513	−0.0529	0.0427
$\mathcal{SC}(r_4)$	−0.0586	−0.0076	−0.1077	0.0259

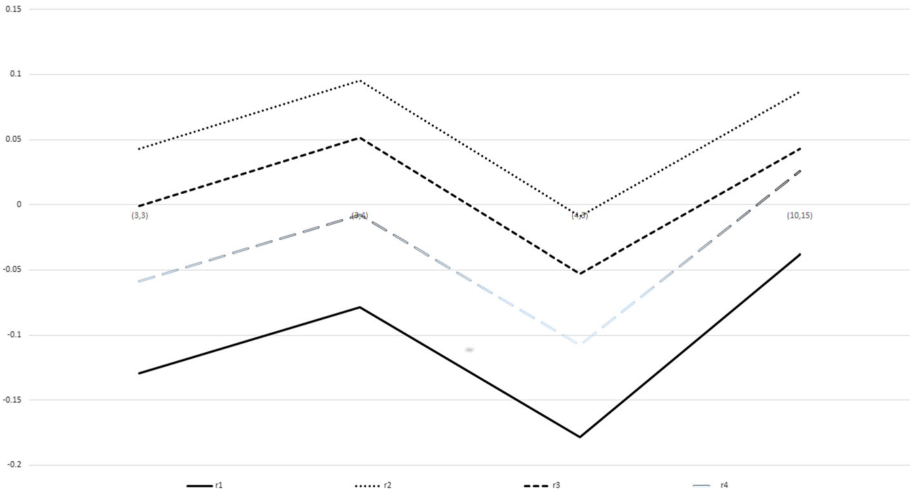

Fig. 1 Scores of alternative based on the (p, q)RLDFWA-operator

Step 2. In this case, the input data for all attributes are the same, then we do not require to normalize the data.

Step 3. By following the (p, q)RLDFWA and (p, q)RLDFWG-operators given in Eqs. (16) and (17), we obtain the overall rating values of each alternative A_i.

Step 4. Calculate the scores function of r_i for all $i = 1, 2, 3, 4$.

Step 5.

To explain the effect of the parameters p and q on (p, q)RLDFMCDM end results, we have utilized different values of p and q to rank the alternatives (Figs. 1, 2).

The results of ranking order of the alternatives based on (p, q)ROFWA and (p, q)ROFWG-operators are presented in Table 5. When $(p, q) = (3, 3)$, $(p, q) = (3, 4)$ and $(p, q) = (4, 3)$, we obtained a rank of alternatives as $A_2 \geq A_3 \geq A_4 \geq A_1$; here, A_2 is the best choice and A_3 is the best second choice, but, when $(p, q) = (10, 15)$, we obtained a rank of alternatives as $A_2 \geq A_4 \geq A_3 \geq A_1$, here, A_2 is the best choice, and A_4 is the best second choice. Thus, the overall best rank is A_2.

This section gives the comparison analysis of the proposed (p, q)RLDFWA and (p, q)RLDFWG-operators under (p, q)RLDFNs with other well-known operator (Table 6).

We compared the results of (p, q)RLDFWA and (p, q)RLDFWG-operators with n, m-ROFWPA-operator Ibrahim and Alshammari (2022), LDFWGA-operator Riaz and Hashmi (2019), LDFEPWA and LDFEPWG-operators Farid et al. (2022), LDFWA and LDFWG-operators Riaz et al. (2023), LDFEWA, LDFEOWA, LDFEWG, and LDFEOWG-operators Iampan et al. (2021), and q-RLDFOWAA, q-RLDFHWAA, q-RLDFOWGA, and q-

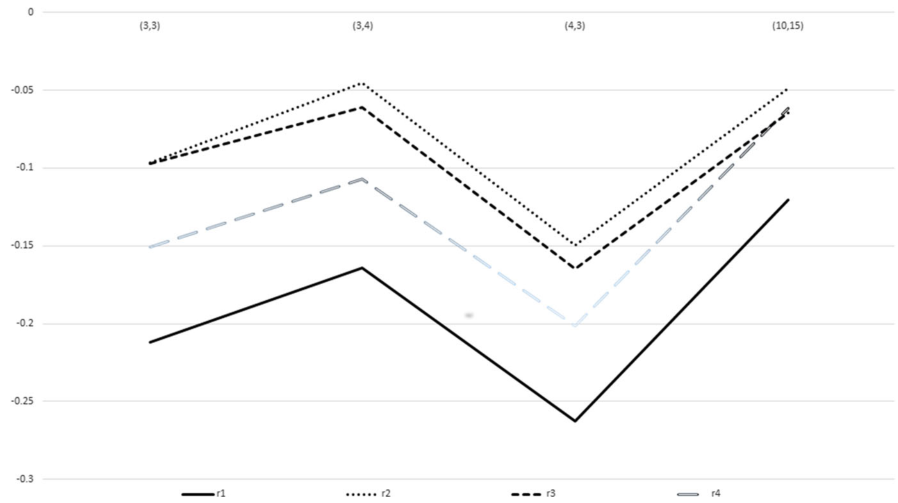

Fig. 2 Scores of alternative based on the (p, q)RLDFWG-operator

Table 5 Score values of alternatives using (p, q)RLDFWG-operator

(p, q)	(3, 3)	(3, 4)	(4, 3)	(10, 15)
$\mathcal{SC}(r_1)$	−0.2118	−0.1640	−0.2625	−0.1203
$\mathcal{SC}(r_2)$	−0.0968	−0.0453	−0.1493	−0.0490
$\mathcal{SC}(r_3)$	−0.0973	−0.0608	−0.1648	−0.0645
$\mathcal{SC}(r_4)$	−0.1509	−0.1068	−0.2014	−0.0617

RLDFHWGA-operators Almagrabi et al. (2022). The results are summed up as follows in Table 7.

According to Table 7, the proposed operators have the following advantages over the current operators:

1. It is clear that our results of (3, 3)RLDFWA, (3, 4)RLDFWA, (4, 3)RLDFWA, (10, 15) RLDFWA, (3, 3)RLDFWG, (3, 4)RLDFWG and (4, 3)RLDFWG-operators are similar to 3-RLDFOWAA, 3-RLDFHWAA, 3-RLDFOWGA and 3-RLDFHWGA-operators, but results of (3, 3)RLDFWA, (3, 4)RLDFWA, (4, 3)RLDFWA, (10, 15)RLDFWA, (3, 3)RLDFWG, (3, 4)RLDFWG and (4, 3)RLDFWG-operators give A_2 is the best choice and A_3 is the best second choice, but results of (10, 15)RLDFGA operator give A_2 is the best choice and A_4 is the best second choice. Therefore, our proposed method is more flexible than other existing methods.
2. The L(p, q)ROFMCDM, being the more flexible, accomplished and general model, covers more decision-making problems using different values and befitting the needs of MCDM problems.
3. (p, q)RLDFN is used for the representation of information in (p, q)RLDFWA and (p, q)RLDFWG-operators. The (p, q)RLDFN is the combination of LDFN with n,m-rung orthopair fuzzy set (n,m-ROFS) which gives full details of the evaluation. The (p, q)RLDFS handles the quantitative as well as qualitative information, so the proposed operator is more general.

Table 6 Ranking order of the alternatives

Aggregation operator	Ranking ordered
(3, 3)RLDFWA-operator	$A_2 \geq A_3 \geq A_4 \geq A_1$
(3, 4)RLDFWA-operator	$A_2 \geq A_3 \geq A_4 \geq A_1$
(4, 3)RLDFWA-operator	$A_2 \geq A_3 \geq A_4 \geq A_1$
(10, 15)RLDFWA-operator	$A_2 \geq A_3 \geq A_4 \geq A_1$
(3, 3)RLDFWG-operator	$A_2 \geq A_3 \geq A_4 \geq A_1$
(3, 4)RLDFWG-operator	$A_2 \geq A_3 \geq A_4 \geq A_1$
(4, 3)RLDFWG-operator	$A_2 \geq A_3 \geq A_4 \geq A_1$
(10, 15)RLDFWG-operator	$A_2 \geq A_4 \geq A_3 \geq A_1$

Table 7 Comparison between existing work with the proposed work

Operator	Parameter	Score function
n, m-ROFWPA-operator Ibrahim and Alshammari (2022)	$p = n, m = q$	Cannot be calculated
LDFWGA-operator Riaz and Hashmi (2019)	$(p, q) = (1, 1)$	Cannot be calculated
LDFEPWA operator Farid et al. (2022)	$(p, q) = (1, 1)$	Cannot be calculated
LDFEPWG operator Farid et al. (2022)	$(p, q) = (1, 1)$	Cannot be calculated
LDFWA operator Riaz et al. (2023)	$(p, q) = (1, 1)$	Cannot be calculated
LDFWG operator Riaz et al. (2023)	$(p, q) = (1, 1)$	Cannot be calculated
LDFEWA operator Iampan et al. (2021)	$(p, q) = (1, 1)$	Cannot be calculated
LDFEOWA operator Iampan et al. (2021)	$(p, q) = (1, 1)$	Cannot be calculated
LDFEWG operator Iampan et al. (2021)	$(p, q) = (1, 1)$	Cannot be calculated
LDFEOWG operator Iampan et al. (2021)	$(p, q) = (1, 1)$	Cannot be calculated
q-RLDFOWAA operator Almagrabi et al. (2022)	$(p, q) = (3, 3)$	$A_2 \geq A_3 \geq A_4 \geq A_1$
q-RLDFHWAA operator Almagrabi et al. (2022)	$(p, q) = (3, 3)$	$A_2 \geq A_3 \geq A_4 \geq A_1$
q-RLDFOWGA operator Almagrabi et al. (2022)	$(p, q) = (3, 3)$	$A_2 \geq A_3 \geq A_4 \geq A_1$
q-RLDFHWGA-operator Almagrabi et al. (2022)	$(p, q) = (3, 3)$	$A_2 \geq A_3 \geq A_4 \geq A_1$
The propounded methodology		
(p, q)RLDFWA	$(p, q) = (10, 15)$	$A_2 \geq A_3 \geq A_4 \geq A_1$
(p, q)RLDFGA	$(p, q) = (10, 15)$	$A_2 \geq A_4 \geq A_3 \geq A_1$

4. The existing operators given in (Ibrahim and Alshammari 2022; Riaz and Hashmi 2019; Farid et al. 2022; Riaz et al. 2023; Iampan et al. 2021; Almagrabi et al. 2022) do not handle the problems when data are in the (p, q)-rung linear Diophantine fuzzy information form. The results obtained from the proposed approach are more accurate and precise.
5. The LIFPWA, LIFPWG, LPFPWA, LPFPWG, Lq-ROFWA, and Lq-ROFWG-operators are the special cases of our developed operators, but these operators have some limitations as the uncertainty of decision data does not reduce. Therefore, our developed operators are much more efficient.

To compare the new method with previous methods more clearly, please see Table 8.

Table 8 The characteristic comparisons of different methods

Methods	Information correlation	Monotonicity	Flexibility	Deal with (p, q)RLDF
n, m-ROFWPA operator Ibrahim and Alshammari (2022)	×	×	×	×
LDFWGA-operator Riaz and Hashmi (2019)	×	×	×	×
LDFEPWA operator Farid et al. (2022)	×	×	×	×
LDFEPWG operator Farid et al. (2022)	×	×	×	×
LDFWA operator Riaz et al. (2023)	×	×	×	×
LDFWG operator Riaz et al. (2023)	×	×	×	×
LDFEWA operator Iampan et al. (2021)	×	×	×	×
LDFEOWA operator Iampan et al. (2021)	×	×	×	×
LDFEWG operator Iampan et al. (2021)	×	×	×	×
LDFEOWG operator Iampan et al. (2021)	×	×	×	×
q-RLDFOWAA operator Almagrabi et al. (2022)	×	×	×	×
q-RLDFHWAA operator Almagrabi et al. (2022)	×	×	×	×
q-RLDFOWGA operator Almagrabi et al. (2022)	×	×	×	×
q-RLDFHWGA-operator Almagrabi et al. (2022)	×	×	×	×
(p, q)RLDFWA	✓	✓	✓	✓
(p, q)RLDFGA	✓	✓	✓	✓

7 Conclusion

The publication provided a brief explanation of how the q-linear Diophantine fuzzy set framework expands all current ideas and offers a solid foundation without constraints. The formal definition of (p, q)-RLDFS was stated which is generalization of q-linear Diophantine fuzzy set by merging it with n, m-rung orthopair fuzzy set to enhance the memberships space. Set theoretical operations were developed and a number of aggregation operators were built under the (p, q)-RLDFN. The suggested aggregating operators' intriguing characteristics were investigated. A (p, q)RLDFMADM approach was also built using recommended aggregating operators and scoring functions. To illustrate how the proposed technique should be applied, a case study was provided. We only consider four options as a restriction on our analysis to show the validity of the proposed technique. The suggested technique works where the n, m-ROFSs, linear Diophantine fuzzy number did not work. The q-linear Diophantine fuzzy set only works for $p = q \geq 1$, but (p, q)-RLDFS works for $p \geq 1$ and $q \geq 1$. The suggested methodology has the following notable successes when compared to some of the current MAGDM methods: (1) defining a huge information space with flexible membership grades and non-membership grades constraints, (2) covering a range of diverse operations, and (3) decreasing the negative effects of decision-makers' excessive evaluation values and dynamically adjusting the weights allocated to input arguments. The approach of the provided strategy may be turned into a computer program, which enables us to carry out our investigation for a limited number of characteristics and options while using enormous data and taking into consideration extra elements. Future research aims include extending the recommended operators to the Archimedean norm and examining other aggregating operators, such as Hamacher and Bonferroni.

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Appendix: Proof of the Theorems

Proof of Theorem 1 Here, we prove part 1 and 5. Proofs of 2, 3, 4, and 6 are easy to reader can get by replacing first vector in (p, q)-RLDFN as in 1 and 5, whereas 7 can get by rule 8 in the Definition 11. The rest follows on the similar lines.

1. Let $\mathcal{A} = ((\mu_A, \eta_A), (\alpha_A, \beta_A))$ and $\mathcal{B} = ((\mu_B, \eta_B), (\alpha_B, \beta_B))$ be any (p, q)RLDFNs on U. Where from it can be easily proved that

$$\begin{aligned}(\mu_A, \eta_A) \oplus (\mu_B, \eta_B) &= \left(\sqrt[p]{(\mu_A)^p + (\mu_B)^p - (\mu_A)^p (\mu_B)^p}, \eta_A \eta_B\right) \\ &= \left(\sqrt[p]{(\mu_B)^p + (\mu_A)^p - (\mu_B)^p (\mu_A)^p}, \eta_B \eta_A\right) \\ &= (\mu_B, \eta_B) \oplus (\mu_A, \eta_A).\end{aligned}$$

Similarly, we can prove that $(\alpha_A, \beta_A) \oplus (\alpha_B, \beta_B) = (\mu_B, \eta_B) \oplus (\alpha_A, \beta_A)$. Therefore, we obtain that $\mathcal{A} \oplus \mathcal{B} = \mathcal{B} \oplus \mathcal{A}$.

5. Let $\mathcal{A} = ((\mu_A, \eta_A), (\alpha_A, \beta_A))$ and $\mathcal{B} = ((\mu_B, \eta_B), (\alpha_B, \beta_B))$ be any (p, q)RLDFNs on U. Where from it can be easily proved that

$$\begin{aligned}\lambda (\mu_A, \eta_A) \oplus \lambda (\mu_B, \eta_B) &= \begin{array}{c}\left(\sqrt[p]{1 - \left(1 - (\mu_A)^p\right)^\lambda}, (\eta_A)^\lambda\right) \oplus \\ \left(\sqrt[p]{1 - \left(1 - (\mu_B)^p\right)^\lambda}, (\eta_B)^\lambda\right)\end{array} \\ &= \left(\begin{array}{c}\sqrt[p]{\begin{array}{c}\left(\sqrt[p]{1 - \left(1 - (\mu_A)^p\right)^\lambda}\right)^p + \\ \left(\sqrt[p]{1 - \left(1 - (\mu_B)^p\right)^\lambda}\right)^p + \\ -\left(\begin{array}{c}\left(\sqrt[p]{1 - \left(1 - (\mu_A)^p\right)^\lambda}\right)^p \\ \left(\sqrt[p]{1 - \left(1 - (\mu_B)^p\right)^\lambda}\right)^p\end{array}\right)\end{array}}, \\ (\eta_A)^\lambda (\eta_B)^\lambda\end{array}\right)\end{aligned}$$

and

$$\begin{aligned}\lambda ((\mu_A, \eta_A) \oplus (\mu_B, \eta_B)) &= \lambda \left(\sqrt[p]{(\mu_A)^p + (\mu_B)^p - (\mu_A)^p (\mu_B)^p}, \eta_A \eta_B\right) \\ &= \left(\begin{array}{c}\sqrt[p]{1 - \left(1 - \left(\sqrt[p]{(\mu_A)^p + (\mu_B)^p - (\mu_A)^p (\mu_B)^p}\right)^p\right)^\lambda}, \\ (\eta_A \eta_B)^\lambda\end{array}\right) \\ &= \left(\begin{array}{c}\sqrt[p]{1 - \left(1 - \left((\mu_A)^p + (\mu_B)^p - (\mu_A)^p (\mu_B)^p\right)\right)^\lambda}, \\ (\eta_A \eta_B)^\lambda\end{array}\right)\end{aligned}$$

$$= \begin{pmatrix} \sqrt[p]{1-\left(\left(1-(\mu_A)^p\right)\left(1-(\mu_B)^p\right)\right)^{\lambda}}, \\ (\eta_A\eta_B)^{\lambda} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[p]{1-\left(\left(1-(\mu_A)^p\right)\left(1-(\mu_B)^p\right)\right)^{\lambda}}, \\ (\eta_A\eta_B)^{\lambda} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[p]{1-\left(1-(\mu_A)^p\right)^{\lambda}\left(1-(\mu_B)^p\right)^{\lambda}}, \\ (\eta_A\eta_B)^{\lambda} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[p]{\begin{array}{c} 1-\left(1-(\mu_A)^p\right)^{\lambda}+1-\left(1-(\mu_B)^p\right)^{\lambda}+ \\ -\left(1-\left(1-(\mu_A)^p\right)^{\lambda}\right)\left(1-\left(1-(\mu_B)^p\right)^{\lambda}\right) \end{array}}, \\ (\eta_A\eta_B)^{\lambda} \end{pmatrix},$$

and so, $\lambda(\mu_A,\eta_A)\oplus\lambda(\mu_B,\eta_B) = \lambda((\mu_A,\eta_A)\oplus(\mu_B,\eta_B))$. Similarly, it follows that $\lambda(\alpha_A,\beta_A)\oplus\lambda(\alpha_B,\beta_B) = \lambda((\alpha_A,\beta_A)\oplus(\alpha_B,\beta_B))$. Therefore, we obtain that $\lambda(\mathcal{A}\oplus\mathcal{B}) = \lambda\mathcal{A}\oplus\lambda\mathcal{B}$.

7. Let $\mathcal{A} = ((\mu_A,\eta_A),(\alpha_A,\beta_A))$ be a (p,q)RLDFN on U. Where from it can be easily proved that

$$\begin{aligned}(\lambda+\xi)(\mu_A,\eta_A) &= \left(\sqrt[p]{1-\left(1-(\mu_A)^p\right)^{\lambda+\xi}}, (\eta_A)^{\lambda+\xi}\right) \\ &= \begin{pmatrix} \sqrt[p]{1-\left(1-(\mu_A)^p\right)^{\lambda}\left(1-(\mu_A)^p\right)^{\xi}}, \\ (\eta_A)^{\lambda+\xi} \end{pmatrix}\end{aligned}$$

and

$$\lambda(\mu_A,\eta_A)\oplus\xi(\mu_A,\eta_A) = \begin{array}{c} \left(\sqrt[p]{1-\left(1-(\mu_A)^p\right)^{\lambda}}, (\eta_A)^{\lambda}\right)\oplus \\ \left(\sqrt[p]{1-\left(1-(\mu_A)^p\right)^{\xi}}, (\eta_A)^{\xi}\right) \end{array}$$

$$= \begin{pmatrix} \sqrt[p]{\begin{array}{c} \left(\sqrt[p]{1-\left(1-(\mu_A)^p\right)^{\lambda}}\right)^p + \\ \left(\sqrt[p]{1-\left(1-(\mu_A)^p\right)^{\xi}}\right)^p + \\ -\begin{pmatrix} \left(\sqrt[p]{1-\left(1-(\mu_A)^p\right)^{\lambda}}\right)^p \\ \left(\sqrt[p]{1-\left(1-(\mu_A)^p\right)^{\xi}}\right)^p \end{pmatrix} \end{array}}, \\ (\eta_A)^{\lambda}(\eta_A)^{\xi} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[p]{\begin{array}{c} 1-\left(1-(\mu_A)^p\right)^{\lambda}+1-\left(1-(\mu_A)^p\right)^{\xi}+ \\ -\left(1-\left(1-(\mu_A)^p\right)^{\lambda}\right)\left(1-\left(1-(\mu_A)^p\right)^{\xi}\right) \end{array}}, \\ (\eta_A)^{\lambda+\xi} \end{pmatrix},$$

and so, $(\lambda + \xi)(\mu_A, \eta_A) = \lambda(\mu_A, \eta_A) \oplus \xi(\mu_A, \eta_A)$. It can be similarly proved that $(\lambda + \xi)(\alpha_A, \beta_A) = \lambda(\alpha_A, \beta_A) \oplus \xi(\alpha_A, \beta_A)$. Therefore, we obtain that $(\lambda + \xi)\mathcal{A} = \lambda\mathcal{A} \oplus \xi\mathcal{A}$. □

Proof of Theorem 3 To prove the theorem, we use mathematical induction on n. Therefore, we have the following.

Step 1. Now, for $n = 2$, we get

$$
\begin{aligned}
\bigoplus_{i=1}^{2} \lambda_i \left(\mu_{A_i}, \eta_{A_i}\right) &= \lambda_1\left(\mu_{A_1}, \eta_{A_1}\right) \oplus \lambda_2\left(\mu_{A_2}, \eta_{A_2}\right) \\
&= \begin{array}{c} \left(\sqrt[p]{1-\left(1-\left(\mu_{A_1}\right)^p\right)^{\lambda_1}}, \left(\eta_{A_1}\right)^{\lambda_1}\right) \oplus \\ \left(\sqrt[p]{1-\left(1-\left(\mu_{A_2}\right)^p\right)^{\lambda_2}}, \left(\eta_{A_2}\right)^{\lambda_2}\right) \end{array} \\
&= \left(\begin{array}{c} \sqrt[p]{\begin{array}{c} \left(\sqrt[p]{1-\left(1-\left(\mu_{A_1}\right)^p\right)^{\lambda_1}}\right)^p + \\ \left(\sqrt[p]{1-\left(1-\left(\mu_{A_2}\right)^p\right)^{\lambda_2}}\right)^p + \\ -\left(\begin{array}{c} \left(\sqrt[p]{1-\left(1-\left(\mu_{A_1}\right)^p\right)^{\lambda_1}}\right)^p \\ \left(\sqrt[p]{1-\left(1-\left(\mu_{A_2}\right)^p\right)^{\lambda_2}}\right)^p \end{array}\right) \end{array}}, \\ \left(\beta_{A_1}\right)^{\lambda_1}\left(\eta_{A_2}\right)^{\lambda_2} \end{array}\right) \\
&= \left(\begin{array}{c} \sqrt[p]{\begin{array}{c} 1-\left(1-\left(\mu_{A_1}\right)^p\right)^{\lambda_1} + 1-\left(1-\left(\mu_{A_2}\right)^p\right)^{\lambda_2} + \\ -\left(1-\left(1-\left(\mu_{A_1}\right)^p\right)^{\lambda_1}\right)\left(1-\left(1-\left(\mu_{A_2}\right)^p\right)^{\lambda_2}\right) \end{array}}, \\ \left(\eta_{A_1}\right)^{\lambda_1}\left(\eta_{A_2}\right)^{\lambda_2} \end{array}\right) \\
&= \left(\begin{array}{c} \sqrt[p]{1-\left(1-\left(\mu_{A_1}\right)^p\right)^{\lambda_1}\left(1-\left(\mu_{A_2}\right)^p\right)^{\lambda_2}}, \\ \prod_{i=1}^{2}\left(\eta_{A_i}\right)^{\lambda_i} \end{array}\right) \\
&= \left(\sqrt[p]{1-\prod_{i=1}^{2}\left(1-\left(\mu_{A_i}\right)^p\right)^{\lambda_i}}, \prod_{i=1}^{2}\left(\eta_{A_i}\right)^{\lambda_i}\right).
\end{aligned}
$$

Similarly, $\bigoplus_{i=1}^{2} \lambda_i \left(\alpha_{A_i}, \beta_{A_i}\right) = \left(\sqrt[p]{1-\prod_{i=1}^{2}\left(1-\left(\alpha_{A_i}\right)^p\right)^{\lambda_i}}, \prod_{i=1}^{2}\left(\beta_{A_i}\right)^{\lambda_i}\right)$. Thus, Eq. 11 holds.

Step 2. Suppose that Eq. 11 holds for $n = k$, that is

$$\mathcal{WA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k) = \left(\begin{pmatrix} \sqrt[p]{1 - \prod_{i=1}^{k} \left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i}}, \\ \prod_{i=1}^{k} \left(\eta_{A_i}\right)^{\lambda_i} \end{pmatrix}, \begin{pmatrix} \sqrt[p]{1 - \prod_{i=1}^{k} \left(1 - \left(\alpha_{A_i}\right)^p\right)^{\lambda_i}}, \\ \prod_{i=1}^{k} \left(\beta_{A_i}\right)^{\lambda_i} \end{pmatrix} \right).$$

Step 3. Now, we have to prove that Eq. 11 holds for $n = k + 1$, based on the operational laws of the (p, q)RLDFNs, we can get

$$\begin{aligned} \bigoplus_{i=1}^{k+1} \lambda_i \left(\mu_{A_i}, \eta_{A_i}\right) &= \bigoplus_{i=1}^{k} \lambda_i \left(\mu_{A_i}, \eta_{A_i}\right) \oplus \lambda_{k+1} \left(\mu_{A_{k+1}}, \eta_{A_{k+1}}\right) \\ &= \begin{matrix} \left(\sqrt[p]{1 - \prod_{i=1}^{k} \left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i}}, \prod_{i=1}^{k} \left(\eta_{A_i}\right)^{\lambda_i} \right) \\ \oplus \left(\sqrt[p]{1 - \left(1 - \left(\mu_{A_{k+1}}\right)^p\right)^{\lambda_{k+1}}}, \left(\eta_{A_{k+1}}\right)^{\lambda_{k+1}} \right) \end{matrix} \\ &= \left(\begin{matrix} \sqrt[p]{ \begin{matrix} \left(\sqrt[p]{1 - \prod_{i=1}^{k} \left(1 - (\mu_i)^p\right)^{\lambda_i}} \right)^p + \\ \left(\sqrt[p]{1 - \left(1 - (\mu_{k+1})^q\right)^{\lambda_{k+1}}} \right)^p + \\ - \left(\begin{matrix} \left(\sqrt[p]{1 - \prod_{i=1}^{k} \left(1 - (\mu_i)^p\right)^{\lambda_i}} \right)^p \\ \left(\sqrt[p]{1 - \left(1 - (\mu_{k+1})^p\right)^{\lambda_{k+1}}} \right)^p \end{matrix} \right) \end{matrix} }, \\ \prod_{i=1}^{k} (\eta_i)^{\lambda_1} (\eta_{k+1})^{\lambda_{k+1}} \end{matrix} \right) \\ &= \left(\begin{matrix} \sqrt[p]{ \begin{matrix} 1 - \prod_{i=1}^{k} \left(1 - (\mu_i)^p\right)^{\lambda_i} + 1 - \left(1 - (\mu_{k+1})^p\right)^{\lambda_{k+1}} + \\ - \left(1 - \prod_{i=1}^{k} \left(1 - (\mu_i)^p\right)^{\lambda_i}\right) \left(1 - \left(1 - (\mu_{k+1})^p\right)^{\lambda_{k+1}}\right) \end{matrix} }, \\ \prod_{i=1}^{k} \left(\eta_{A_i}\right)^{\lambda_i} \left(\eta_{A_{k+1}}\right)^{\lambda_{k+1}} \end{matrix} \right) \end{aligned}$$

$$= \left(\begin{array}{c} \sqrt[p]{1 - \prod_{i=1}^{k} \left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i} \left(1 - \left(\mu_{A_{k+1}}\right)^p\right)^{\lambda_{k+1}}}, \\ \prod_{i=1}^{k+1} \left(\eta_{A_i}\right)^{\lambda_i} \end{array} \right)$$

$$= \left(\sqrt[p]{1 - \prod_{i=1}^{k+1} \left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i}}, \prod_{i=1}^{k+1} \left(\eta_{A_i}\right)^{\lambda_i} \right).$$

Similarly, $\bigoplus_{i=1}^{k+1} \lambda_i \left(\alpha_{A_i}, \beta_{A_i}\right) = \left(\sqrt[p]{1 - \prod_{i=1}^{k+1} \left(1 - \left(\alpha_{A_i}\right)^p\right)^{\lambda_i}}, \prod_{i=1}^{k+1} \left(\beta_{A_i}\right)^{\lambda_i} \right)$. Therefore, Eq. 11 holds for $n = k + 1$, and hence, Eq. 11 holds for any i.

In the following, we will prove that $\mathcal{WA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right)$ is also a (p, q)RLDFN. Then, since $0 \leq \left(\alpha_{A_i}\right)^p + \left(\beta_{A_i}\right)^q \leq 1$, we have

$$\begin{aligned} &\left(\beta_{A_i}\right)^q \leq 1 - \left(\alpha_{A_i}\right)^p \\ &\left(\left(\beta_{A_i}\right)^q\right)^{\lambda_i} \leq \left(1 - \left(\alpha_{A_i}\right)^p\right)^{\lambda_i} \\ &\prod_{i=1}^{n} \left(\left(\beta_{A_i}\right)^q\right)^{\lambda_i} \leq \prod_{i=1}^{n} \left(1 - \left(\alpha_{A_i}\right)^p\right)^{\lambda_i} \\ &-\prod_{i=1}^{n} \left(1 - \left(\alpha_{A_i}\right)^p\right)^{\lambda_i} + \prod_{i=1}^{n} \left(\left(\beta_{A_i}\right)^q\right)^{\lambda_i} \leq 0 \\ &0 \leq 1 - \prod_{i=1}^{n} \left(1 - \left(\alpha_{A_i}\right)^p\right)^{\lambda_i} + \prod_{i=1}^{n} \left(\left(\beta_{A_i}\right)^{\lambda_i}\right)^q \leq 1. \end{aligned}$$

Thus, we obtain that

$$0 \leq \left(1 - \prod_{i=1}^{n} \left(1 - \left(\alpha_{A_i}\right)^p\right)^{\lambda_i}\right) \sqrt[p]{1 - \prod_{i=1}^{n} \left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i}} + \prod_{i=1}^{n} \left(\left(\beta_{A_i}\right)^{\lambda_i}\right)^q \prod_{i=1}^{n} \left(\eta_{A_i}\right)^{\lambda_i} \leq 1$$

since

$$0 \leq \sqrt[p]{1 - \prod_{i=1}^{n} \left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i}}, \prod_{i=1}^{n} \left(\eta_{A_i}\right)^{\lambda_i} \leq 1$$

and

$$0 \leq 1 - \prod_{i=1}^{n} \left(1 - \left(\alpha_{A_i}\right)^p\right)^{\lambda_i} + \prod_{i=1}^{n} \left(\left(\beta_{A_i}\right)^{\lambda_i}\right)^q \leq 1.$$

□

Proof of Theorem 4 Then, since $\mathcal{A}_1 = \mathcal{A}_2 = \ldots = \mathcal{A}_n = \mathcal{A}$ by Theorem 3, we have

$$
\begin{aligned}
\mathcal{WA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \left(\begin{array}{c} \left(\begin{array}{c} \sqrt[p]{1 - \prod_{i=1}^{n} \left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i}}, \\ \prod_{i=1}^{n} \left(\eta_{A_i}\right)^{\lambda_i} \end{array} \right), \\ \left(\begin{array}{c} \sqrt[p]{1 - \prod_{i=1}^{n} \left(1 - \left(\alpha_{A_i}\right)^p\right)^{\lambda_i}}, \\ \prod_{i=1}^{n} \left(\beta_{A_i}\right)^{\lambda_i} \end{array} \right) \end{array} \right) \\
&= \left(\begin{array}{c} \left(\begin{array}{c} \sqrt[p]{1 - \left(1 - (\mu_A)^p\right)^{\sum_{i=1}^{n} \lambda_i}}, \\ (\eta_A)^{\sum_{i=1}^{n} \lambda_i} \end{array} \right), \\ \left(\begin{array}{c} \sqrt[p]{1 - \left(1 - (\alpha_A)^p\right)^{\sum_{i=1}^{n} \lambda_i}}, \\ (\beta_A)^{\sum_{i=1}^{n} \lambda_i} \end{array} \right) \end{array} \right) \\
&= \left(\begin{array}{c} \left(\sqrt[p]{1 - \left(1 - (\mu_A)^p\right)^1}, (\eta_A)^1 \right), \\ \left(\sqrt[p]{1 - \left(1 - (\alpha_A)^p\right)^1}, (\beta_A)^1 \right) \end{array} \right) \\
&= \left((\mu_A, \eta_A), (\alpha_A, \beta_A) \right) \\
&= \mathcal{A}.
\end{aligned}
$$

Therefore, we obtain that $\mathcal{WA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}$. □

Proof of Theorem 5 Then, since $\mathcal{A}_i \preceq \mathcal{B}_i$ for every $i = 1, 2, 3, \ldots, n$, we have $\mu_{A_i} \leq \mu_{B_i}$ and $\eta_{A_i} \geq \eta_{B_i}$, that is

$$
\begin{aligned}
&\left(\mu_{A_i}\right)^p \leq \left(\mu_{B_i}\right)^p \\
&1 - \left(\mu_{A_i}\right)^p \geq 1 - \left(\mu_{B_i}\right)^p \\
&\left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i} \geq \left(1 - \left(\mu_{B_i}\right)^p\right)^{\lambda_i} \\
&\prod_{i=1}^{n} \left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i} \geq \prod_{i=1}^{n} \left(1 - \left(\mu_{B_i}\right)^p\right)^{\lambda_i}
\end{aligned}
$$

$$1-\prod_{i=1}^{n}\left(1-\left(\mu_{A_i}\right)^p\right)^{\lambda_i} \leq 1-\prod_{i=1}^{n}\left(1-\left(\mu_{B_i}\right)^p\right)^{\lambda_i}$$
$$\sqrt[p]{1-\prod_{i=1}^{n}\left(1-\left(\mu_{A_i}\right)^p\right)^{\lambda_i}} \leq \sqrt[p]{1-\prod_{i=1}^{n}\left(1-\left(\mu_{B_i}\right)^p\right)^{\lambda_i}}. \tag{19}$$

From these calculations, we obtain

$$\left(\eta_{A_i}\right)^{\lambda_i} \geq \left(\eta_{B_i}\right)^{\lambda_i}$$
$$\prod_{i=1}^{n}\left(\eta_{A_i}\right)^{\lambda_i} \geq \prod_{i=1}^{n}\left(\eta_{B_i}\right)^{\lambda_i}. \tag{20}$$

Thus, by Eqs. (19) and (20), we have

$$\begin{pmatrix} \sqrt[p]{1-\prod_{i=1}^{n}\left(1-\left(\mu_{A_i}\right)^p\right)^{\lambda_i}}, \\ \prod_{i=1}^{n}\left(\eta_{A_i}\right)^{\lambda_i} \end{pmatrix} \preceq \begin{pmatrix} \sqrt[p]{1-\prod_{i=1}^{n}\left(1-\left(\mu_{B_i}\right)^p\right)^{\lambda_i}}, \\ \prod_{i=1}^{n}\left(\eta_{B_i}\right)^{\lambda_i} \end{pmatrix}. \tag{21}$$

Similarly, we can show that

$$\begin{pmatrix} \sqrt[p]{1-\prod_{i=1}^{n}\left(1-\left(\alpha_{A_i}\right)^p\right)^{\lambda_i}}, \\ \prod_{i=1}^{n}\left(\beta_{A_i}\right)^{\lambda_i} \end{pmatrix} \preceq \begin{pmatrix} \sqrt[p]{1-\prod_{i=1}^{n}\left(1-\left(\alpha_{B_i}\right)^p\right)^{\lambda_i}}, \\ \prod_{i=1}^{n}\left(\beta_{B_i}\right)^{\lambda_i} \end{pmatrix}. \tag{22}$$

Therefore, by Eqs. (21) and (22), we get $\mathcal{WA}\left(\mathcal{A}_1,\ldots,\mathcal{A}_n\right) \preceq \mathcal{WA}\left(\mathcal{B}_1,\ldots,\mathcal{B}_n\right)$. □

Proof of Theorem 6 For the membership grades of $\mathcal{WA}\left(\mathcal{A}_1,\ldots,\mathcal{A}_n\right)$, we have

$$\bigwedge_{i=1}^{n}\left(\mu_{A_i}\right)^p \leq \left(\mu_{A_i}\right)^p$$
$$1-\bigwedge_{i=1}^{n}\left(\mu_{A_i}\right)^p \geq 1-\left(\mu_{A_i}\right)^p$$
$$\left(1-\bigwedge_{i=1}^{n}\left(\mu_{A_i}\right)^p\right)^{\lambda_i} \geq \left(1-\left(\mu_{A_i}\right)^p\right)^{\lambda_i}$$
$$\prod_{i=1}^{n}\left(1-\bigwedge_{i=1}^{n}\left(\mu_{A_i}\right)^p\right)^{\lambda_i} \geq \prod_{i=1}^{n}\left(1-\left(\mu_{A_i}\right)^p\right)^{\lambda_i}$$
$$1-\prod_{i=1}^{n}\left(1-\bigwedge_{i=1}^{n}\left(\mu_{A_i}\right)^p\right)^{\lambda_i} \leq 1-\prod_{i=1}^{n}\left(1-\left(\mu_{A_i}\right)^p\right)^{\lambda_i}$$
$$\sqrt[p]{1-\prod_{i=1}^{n}\left(1-\bigwedge_{i=1}^{n}\left(\mu_{A_i}\right)^p\right)^{\lambda_i}} \leq \sqrt[p]{1-\prod_{i=1}^{n}\left(1-\left(\mu_{A_i}\right)^p\right)^{\lambda_i}}$$

$$\begin{aligned}
\sqrt[p]{1-\left(1-\bigwedge_{i=1}^{n}(\mu_{A_i})^p\right)^{\sum_{i=1}^{n}\lambda_i}} &\leq \sqrt[p]{1-\prod_{i=1}^{n}\left(1-(\mu_{A_i})^p\right)^{\lambda_i}}\\
\sqrt[p]{1-\left(1-\bigwedge_{i=1}^{n}(\mu_{A_i})^p\right)} &\leq \sqrt[p]{1-\prod_{i=1}^{n}\left(1-(\mu_{A_i})^p\right)^{\lambda_i}}\\
\bigwedge_{i=1}^{n}\mu_{A_i} &\leq \sqrt[p]{1-\prod_{i=1}^{n}\left(1-(\mu_{A_i})^p\right)^{\lambda_i}}
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\bigvee_{i=1}^{n}(\mu_{A_i})^p &\geq (\mu_{A_i})^p\\
1-\bigvee_{i=1}^{n}(\mu_{A_i})^p &\leq 1-(\mu_{A_i})^p\\
\left(1-\bigvee_{i=1}^{n}(\mu_{A_i})^p\right)^{\lambda_i} &\leq \left(1-(\mu_{A_i})^p\right)^{\lambda_i}\\
\prod_{i=1}^{n}\left(1-\bigvee_{i=1}^{n}(\mu_{A_i})^p\right)^{\lambda_i} &\leq \prod_{i=1}^{n}\left(1-(\mu_{A_i})^p\right)^{\lambda_i}\\
1-\prod_{i=1}^{n}\left(1-\bigvee_{i=1}^{n}(\mu_{A_i})^p\right)^{\lambda_i} &\geq 1-\prod_{i=1}^{n}\left(1-(\mu_{A_i})^p\right)^{\lambda_i}\\
\sqrt[p]{1-\prod_{i=1}^{n}\left(1-\bigvee_{i=1}^{n}(\mu_{A_i})^p\right)^{\lambda_i}} &\geq \sqrt[p]{1-\prod_{i=1}^{n}\left(1-(\mu_{A_i})^p\right)^{\lambda_i}}\\
\sqrt[p]{1-\left(1-\bigvee_{i=1}^{n}(\mu_{A_i})^p\right)^{\sum_{i=1}^{n}\lambda_i}} &\geq \sqrt[p]{1-\prod_{i=1}^{n}\left(1-(\mu_{A_i})^p\right)^{\lambda_i}}\\
\sqrt[p]{1-\left(1-\bigvee_{i=1}^{n}(\mu_{A_i})^p\right)^{1}} &\geq \sqrt[p]{1-\prod_{i=1}^{n}\left(1-(\mu_{A_i})^p\right)^{\lambda_i}}\\
\bigvee_{i=1}^{n}\mu_{A_i} &\geq \sqrt[p]{1-\prod_{i=1}^{n}\left(1-(\mu_{A_i})^p\right)^{\lambda_i}}.
\end{aligned} \tag{24}$$

Then, by Eqs. (23) and (24), we get

$$\bigwedge_{i=1}^{n}\mu_{A_i} \leq \sqrt[p]{1-\prod_{i=1}^{n}\left(1-(\mu_{A_i})^p\right)^{\lambda_i}} \leq \bigvee_{i=1}^{n}\mu_{A_i}. \tag{25}$$

Similarly we can show that

$$\bigwedge_{i=1}^{n} \alpha_{A_i} \leq \sqrt[p]{1-\prod_{i=1}^{n}\left(1-\left(\alpha_{A_i}\right)^{p}\right)^{\lambda_i}} \leq \bigvee_{i=1}^{n} \alpha_{A_i}. \tag{26}$$

For the non-membership grades of $\mathcal{WA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right)$, we have

$$\begin{aligned}
&\bigwedge_{i=1}^{n}\left(\eta_{A_i}\right)^{\lambda_i} \leq \left(\eta_{A_i}\right)^{\lambda_i} \leq \bigvee_{i=1}^{n}\left(\eta_{A_i}\right)^{\lambda_i}\\
&\prod_{i=1}^{n}\bigwedge_{i=1}^{n}\left(\eta_{A_i}\right)^{\lambda_i} \leq \prod_{i=1}^{n}\left(\eta_{A_i}\right)^{\lambda_i} \leq \prod_{i=1}^{n}\bigvee_{i=1}^{n}\left(\eta_{A_i}\right)^{\lambda_i}\\
&\bigwedge_{i=1}^{n}\left(\eta_{A_i}\right)^{\sum_{i=1}^{n}\lambda_i} \leq \prod_{i=1}^{n}\left(\eta_{A_i}\right)^{\lambda_i} \leq \bigvee_{i=1}^{n}\left(\eta_{A_i}\right)^{\sum_{i=1}^{n}\lambda_i}\\
&\bigwedge_{i=1}^{n}\left(\eta_{A_i}\right) \leq \prod_{i=1}^{n}\left(\eta_{A_i}\right)^{\lambda_i} \leq \bigvee_{i=1}^{n}\left(\eta_{A_i}\right)\\
&\bigwedge_{i=1}^{n}\left(\eta_{A_i}\right) \leq \prod_{i=1}^{n}\left(\eta_{A_i}\right)^{\lambda_i} \leq \bigvee_{i=1}^{n}\left(\eta_{A_i}\right).
\end{aligned} \tag{27}$$

It can be similarly proved that

$$\bigwedge_{i=1}^{n}\left(\beta_{A_i}\right) \leq \prod_{i=1}^{n}\left(\beta_{A_i}\right)^{\lambda_i} \leq \bigvee_{i=1}^{n}\left(\beta_{A_i}\right). \tag{28}$$

Therefore, by Eqs. (25), (26), (27), and (28), we have $\mathcal{A}^{-} \preceq \mathcal{WA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) \preceq \mathcal{A}^{+}$. □

Proof of Theorem 7 To prove the Theorem 7, we use mathematical induction on n. For this, we proceed as follows.

Step 1. Now, for $n = 2$, we have

$$\begin{aligned}
\bigotimes_{i=1}^{2}\left(\mu_{A_i}, \eta_{A_i}\right)^{\lambda_i} &= \left(\mu_{A_1}, \eta_{A_1}\right)^{\lambda_1} \otimes \left(\mu_{A_2}, \eta_{A_2}\right)^{\lambda_2}\\
&= \begin{array}{c}\left(\left(\mu_1\right)^{\lambda_{A_1}}, \sqrt[q]{1-\left(1-\left(\eta_{A_1}\right)^{q}\right)^{\lambda_1}}\right) \otimes \\ \left(\left(\mu_2\right)^{\lambda_{A_2}}, \sqrt[q]{1-\left(1-\left(\eta_{A_2}\right)^{q}\right)^{\lambda_2}}\right)\end{array}
\end{aligned}$$

$$= \left(\begin{array}{c} \left(\mu_{A_1}\right)^{\lambda_1} \left(\mu_{A_2}\right)^{\lambda_2}, \\ \sqrt[q]{\begin{array}{c} \left(\sqrt[q]{1-\left(1-\left(\eta_{A_1}\right)^q\right)^{\lambda_1}}\right)^q + \\ \left(\sqrt[q]{1-\left(1-\left(\eta_{A_2}\right)^q\right)^{\lambda_2}}\right)^q + \\ -\left(\begin{array}{c} \left(\sqrt[q]{1-\left(1-\left(\eta_{A_1}\right)^q\right)^{\lambda_1}}\right)^q \\ \left(\sqrt[q]{1-\left(1-\left(\eta_{A_2}\right)^q\right)^{\lambda_2}}\right)^q \end{array} \right) \end{array}} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\mu_{A_1}\right)^{\lambda_1} \left(\mu_{A_2}\right)^{\lambda_2}, \\ \sqrt[q]{\begin{array}{c} 1-\left(1-\left(\eta_{A_1}\right)^q\right)^{\lambda_1} + 1-\left(1-\left(\eta_{A_2}\right)^q\right)^{\lambda_2} + \\ -\left(1-\left(1-\left(\eta_{A_1}\right)^q\right)^{\lambda_1}\right)\left(1-\left(1-\left(\eta_{A_2}\right)^q\right)^{\lambda_2}\right) \end{array}} \end{array} \right)$$

$$= \left(\begin{array}{c} \prod_{i=1}^{2} \left(\mu_{A_i}\right)^{\lambda_i}, \\ \sqrt[q]{1-\left(1-\left(\eta_{A_1}\right)^q\right)^{\lambda_1}\left(1-\left(\eta_{A_2}\right)^q\right)^{\lambda_2}} \end{array} \right)$$

$$= \left(\prod_{i=1}^{2} \left(\mu_{A_i}\right)^{\lambda_i}, \sqrt[q]{1-\prod_{i=1}^{2}\left(1-\left(\eta_{A_i}\right)^q\right)^{\lambda_i}} \right).$$

Similarly, $\bigotimes_{i=1}^{2} \left(\alpha_{A_i}, \beta_{A_i}\right)^{\lambda_i} = \left(\prod_{i=1}^{2} \left(\alpha_{A_i}\right)^{\lambda_i}, \sqrt[q]{1-\prod_{i=1}^{2}\left(1-\left(\beta_{A_i}\right)^q\right)^{\lambda_i}} \right)$. Thus, Eq. (13) holds.

Step 2. Suppose that Eq. (13) holds for $n = k$, that is

$$\mathcal{WG}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k\right) = \left(\begin{array}{c} \left(\begin{array}{c} \prod_{i=1}^{k} \left(\mu_{A_i}\right)^{\lambda_i}, \\ \sqrt[q]{1-\prod_{i=1}^{k}\left(1-\left(\eta_{A_i}\right)^q\right)^{\lambda_i}} \end{array} \right), \\ \left(\begin{array}{c} \prod_{i=1}^{k} \left(\alpha_{A_i}\right)^{\lambda_i}, \\ \sqrt[q]{1-\prod_{i=1}^{k}\left(1-\left(\beta_{A_i}\right)^q\right)^{\lambda_i}} \end{array} \right) \end{array} \right).$$

Step 3. Now, we have to prove that Eq. 13 holds for $n = k + 1$, based on the operational laws of the (p, q)RLDFNs, we can get

$$\begin{aligned} \bigotimes_{i=1}^{k+1} \left(\mu_{A_i}, \eta_{A_i}\right)_i^{\lambda_i} &= \bigotimes_{i=1}^{k} \left(\mu_{A_i}, \eta_{A_i}\right)^{\lambda_i} \otimes \left(\mu_{A_{k+1}}, \eta_{A_{k+1}}\right)^{\lambda_{k+1}} \\ &= \mathcal{WG}\left(\mathcal{A}_1, \ldots, \mathcal{A}_k\right) \otimes \left(\mu_{A_{k+1}}, \eta_{A_{k+1}}\right)^{\lambda_{k+1}} \end{aligned}$$

$$= \left(\prod_{i=1}^{k} \left(\mu_{A_i}\right)^{\lambda_i}, \sqrt[q]{1 - \prod_{i=1}^{k} \left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i}} \right) \otimes \left(\left(\mu_{A_{k+1}}\right)^{\lambda_{k+1}}, \sqrt[q]{1 - \left(1 - \left(\eta_{A_{k+1}}\right)^q\right)^{\lambda_{k+1}}} \right)$$

$$= \left(\begin{array}{c} \prod_{i=1}^{k} \left(\mu_{A_i}\right)^{\lambda_1} \left(\mu_{A_{k+1}}\right)^{\lambda_{k+1}}, \\ \sqrt[q]{\begin{array}{c} \left(\sqrt[q]{1 - \prod_{i=1}^{k} \left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i}}\right)^q + \\ \left(\sqrt[q]{1 - \left(1 - \left(\eta_{A_{k+1}}\right)^q\right)^{\lambda_{k+1}}}\right)^q + \\ - \left(\begin{array}{c} \left(\sqrt[q]{1 - \prod_{i=1}^{k} \left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i}}\right)^q \\ \left(\sqrt[q]{1 - \left(1 - \left(\eta_{A_{k+1}}\right)^q\right)^{\lambda_{k+1}}}\right)^q \end{array} \right) \end{array}} \end{array} \right)$$

$$= \left(\begin{array}{c} \prod_{i=1}^{k+1} \left(\mu_{A_i}\right)^{\lambda_i}, \\ \sqrt[q]{1 - \prod_{i=1}^{k} \left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i} \left(1 - \left(\eta_{A_{k+1}}\right)^q\right)^{\lambda_{k+1}}} \end{array} \right)$$

$$= \left(\prod_{i=1}^{k+1} \left(\mu_{A_i}\right)^{\lambda_i}, \sqrt[q]{1 - \prod_{i=1}^{k+1} \left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i}} \right).$$

Similarly, $\bigotimes_{i=1}^{k+1} \left(\alpha_{A_i}, \beta_{A_i}\right)_i^{\lambda_i} = \left(\prod_{i=1}^{k+1} \left(\alpha_{A_i}\right)^{\lambda_i}, \sqrt[q]{1 - \prod_{i=1}^{k+1} \left(1 - \left(\beta_{A_i}\right)^q\right)^{\lambda_i}} \right)$. Therefore, Eq. (13) holds for $n = k + 1$, and hence, Eq. (13) holds for any i.

In the following, we will prove that $\mathcal{WG}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right)$ is also a (p, q)RLDFN. Then, since $\left(\alpha_{A_i}\right)^p + \left(\beta_{A_i}\right)^q \leq 1$, we have

$$\begin{aligned} &\left(\alpha_{A_i}\right)^p \leq 1 - \left(\beta_{A_i}\right)^q \\ &\left(\left(\alpha_{A_i}\right)^p\right)^{\lambda_i} \leq \left(1 - \left(\beta_{A_i}\right)^q\right)^{\lambda_i} \\ &\prod_{i=1}^{n} \left(\left(\alpha_{A_i}\right)^p\right)^{\lambda_i} \leq \prod_{i=1}^{n} \left(1 - \left(\beta_{A_i}\right)^q\right)^{\lambda_i} \\ &\prod_{i=1}^{n} \left(\left(\alpha_{A_i}\right)^p\right)^{\lambda_i} - \prod_{i=1}^{n} \left(1 - \left(\beta_{A_i}\right)^q\right)^{\lambda_i} \leq 0 \\ &0 \leq \prod_{i=1}^{n} \left(\left(\alpha_{A_i}\right)^{\lambda_i}\right)^p + 1 - \prod_{i=1}^{n} \left(1 - \left(\beta_{A_i}\right)^q\right)^{\lambda_i} \leq 1. \end{aligned}$$

Thus, we obtain that

$$0 \leq \left(1-\prod_{i=1}^{n}\left(1-\left(\beta_{A_i}\right)^{q}\right)^{\lambda_i}\right)\sqrt[q]{1-\prod_{i=1}^{n}\left(1-\left(\eta_{A_i}\right)^{q}\right)^{\lambda_i}}+\prod_{i=1}^{n}\left(\left(\alpha_{A_i}\right)^{\lambda_i}\right)^{p}\prod_{i=1}^{n}\left(\mu_{A_i}\right)^{\lambda_i} \leq 1$$

since

$$0 \leq \prod_{i=1}^{n}\left(\mu_{A_i}\right)^{\lambda_i}, \sqrt[q]{1-\prod_{i=1}^{n}\left(1-\left(\eta_{A_i}\right)^{q}\right)^{\lambda_i}} \leq 1$$

and

$$0 \leq \prod_{i=1}^{n}\left(\left(\alpha_{A_i}\right)^{\lambda_i}\right)^{p}+1-\prod_{i=1}^{n}\left(1-\left(\beta_{A_i}\right)^{q}\right)^{\lambda_i} \leq 1.$$

□

Proof of Theorem 8 Then, since $\mathcal{A}_1 = \mathcal{A}_2 = \ldots = \mathcal{A}_n = \mathcal{A}$ by Theorem 7, we have

$$\begin{aligned}
\mathcal{WG}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) &= \left(\begin{array}{c}\left(\begin{array}{c}\prod_{i=1}^{n}\left(\mu_{A_i}\right)^{\lambda_i}, \\ \sqrt[q]{1-\prod_{i=1}^{n}\left(1-\left(\eta_{A_i}\right)^{q}\right)^{\lambda_i}}\end{array}\right), \\ \left(\begin{array}{c}\prod_{i=1}^{n}\left(\alpha_{A_i}\right)^{\lambda_i}, \\ \sqrt[q]{1-\prod_{i=1}^{n}\left(1-\left(\beta_{A_i}\right)^{q}\right)^{\lambda_i}}\end{array}\right)\end{array}\right) \\
&= \left(\begin{array}{c}\left(\begin{array}{c}\left(\mu_A\right)^{\sum_{i=1}^{n}\lambda_i}, \\ \sqrt[q]{1-\left(1-\left(\eta_A\right)^{q}\right)^{\sum_{i=1}^{n}\lambda_i}}\end{array}\right), \\ \left(\begin{array}{c}\left(\alpha_A\right)^{\sum_{i=1}^{n}\lambda_i}, \\ \sqrt[q]{1-\left(1-\left(\beta_A\right)^{q}\right)^{\sum_{i=1}^{n}\lambda_i}}\end{array}\right)\end{array}\right) \\
&= \left(\begin{array}{c}\left(\left(\mu_A\right)^{1}, \sqrt[q]{1-\left(1-\left(\eta_A\right)^{q}\right)^{1}}\right), \\ \left(\left(\alpha_A\right)^{1}, \sqrt[q]{1-\left(1-\left(\beta_A\right)^{q}\right)^{1}}\right)\end{array}\right) \\
&= \left(\left(\mu_A, \eta_A\right), \left(\alpha_A, \beta_A\right)\right) \\
&= \mathcal{A}.
\end{aligned}$$

Therefore, we obtain that $\mathcal{WG}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\right) = \mathcal{A}$. □

Proof of Theorem 9 Then, since $\mathcal{A}_i \preceq \mathcal{B}_i$ for every $i = 1, 2, 3, \ldots, n$, we have $\mu_{A_i} \leq \mu_{B_i}$, $\eta_{A_i} \geq \eta_{B_i}$, $\alpha_{A_i} \leq \alpha_{B_i}$ and $\beta_{A_i} \geq \beta_{B_i}$, that is

$$\begin{aligned} \mu_{A_i} &\geq \mu_{B_i} \\ \left(\mu_{A_i}\right)^{\lambda_i} &\geq \left(\mu_{B_i}\right)^{\lambda_i} \\ \prod_{i=1}^{n}\left(\mu_{A_i}\right)^{\lambda_i} &\geq \prod_{i=1}^{n}\left(\mu_{B_i}\right)^{\lambda_i}. \end{aligned} \tag{29}$$

It can be similarly proved that

$$\prod_{i=1}^{n}\left(\alpha_{A_i}\right)^{\lambda_i} \geq \prod_{i=1}^{n}\left(\alpha_{B_i}\right)^{\lambda_i}. \tag{30}$$

From these calculations, we obtain

$$\begin{aligned} \left(\eta_{A_i}\right)^q &\leq \left(\eta_{B_i}\right)^q \\ 1 - \left(\eta_{A_i}\right)^q &\geq 1 - \left(\eta_{B_i}\right)^q \\ \left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i} &\geq \left(1 - \left(\eta_{B_i}\right)^q\right)^{\lambda_i} \\ \prod_{i=1}^{n}\left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i} &\geq \prod_{i=1}^{n}\left(1 - \left(\eta_{B_i}\right)^q\right)^{\lambda_i} \\ 1 - \prod_{i=1}^{n}\left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i} &\leq 1 - \prod_{i=1}^{n}\left(1 - \left(\eta_{B_i}\right)^q\right)^{\lambda_i} \\ \sqrt[q]{1 - \prod_{i=1}^{n}\left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i}} &\leq \sqrt[q]{1 - \prod_{i=1}^{n}\left(1 - \left(\eta_{B_i}\right)^q\right)^{\lambda_i}}. \end{aligned} \tag{31}$$

Similarly, we can prove

$$\sqrt[q]{1 - \prod_{i=1}^{n}\left(1 - \left(\beta_{A_i}\right)^q\right)^{\lambda_i}} \leq \sqrt[q]{1 - \prod_{i=1}^{n}\left(1 - \left(\beta_{B_i}\right)^q\right)^{\lambda_i}}. \tag{32}$$

Therefore, we obtain that, by Eqs. (29), (30), (31), and (32), we have

$$\left(\begin{array}{c} \left(\begin{array}{c} \prod_{i=1}^{n}\left(\mu_{A_i}\right)^{\lambda_i}, \\ \sqrt[q]{1 - \prod_{i=1}^{n}\left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i}} \end{array}\right), \\ \left(\begin{array}{c} \prod_{i=1}^{n}\left(\alpha_{A_i}\right)^{\lambda_i}, \\ \sqrt[q]{1 - \prod_{i=1}^{n}\left(1 - \left(\beta_{A_i}\right)^q\right)^{\lambda_i}} \end{array}\right) \end{array}\right) \preceq \left(\begin{array}{c} \left(\begin{array}{c} \prod_{i=1}^{n}\left(\mu_{B_i}\right)^{\lambda_i}, \\ \sqrt[q]{1 - \prod_{i=1}^{n}\left(1 - \left(\eta_{B_i}\right)^q\right)^{\lambda_i}} \end{array}\right), \\ \left(\begin{array}{c} \prod_{i=1}^{n}\left(\alpha_{B_i}\right)^{\lambda_i}, \\ \sqrt[q]{1 - \prod_{i=1}^{n}\left(1 - \left(\beta_{B_i}\right)^q\right)^{\lambda_i}} \end{array}\right) \end{array}\right),$$

and hence, $\mathcal{WG}\left(\mathcal{A}_1, \ldots, \mathcal{A}_n\right) \preceq \mathcal{WG}\left(\mathcal{B}_1, \ldots, \mathcal{B}_n\right)$. □

Proof of Theorem 10 For the membership grades of $\mathcal{WG}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, we have

$$
\begin{aligned}
\bigwedge_{i=1}^{n} (\mu_i)^{\lambda_i} &\leq (\mu_i)^{\lambda_i} \leq \bigvee_{i=1}^{n} (\mu_i)^{\lambda_i} \\
\prod_{i=1}^{n} \bigwedge_{i=1}^{n} \left(\mu_{A_i}\right)^{\lambda_i} &\leq \prod_{i=1}^{n} \left(\mu_{A_i}\right)^{\lambda_i} \leq \prod_{i=1}^{n} \bigvee_{i=1}^{n} \left(\mu_{A_i}\right)^{\lambda_i} \\
\bigwedge_{i=1}^{n} \left(\mu_{A_i}\right)^{\sum_{i=1}^{n} \lambda_i} &\leq \prod_{i=1}^{n} \left(\mu_{A_i}\right)^{\lambda_i} \leq \bigvee_{i=1}^{n} \left(\mu_{A_i}\right)^{\sum_{i=1}^{n} \lambda_i} \\
\bigwedge_{i=1}^{n} \left(\mu_{A_i}\right) &\leq \prod_{i=1}^{n} (\mu_i)^{\lambda_i} \leq \bigvee_{i=1}^{n} \left(\mu_{A_i}\right).
\end{aligned} \tag{33}
$$

Similarly we can prove

$$
\bigwedge_{i=1}^{n} \left(\alpha_{A_i}\right) \leq \prod_{i=1}^{n} (\alpha_i)^{\lambda_i} \leq \bigvee_{i=1}^{n} \left(\alpha_{A_i}\right). \tag{34}
$$

For the non-membership grades of $\mathcal{WG}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, we have

$$
\begin{aligned}
\bigwedge_{i=1}^{n} \left(\eta_{A_i}\right)^q &\leq \left(\eta_{A_i}\right)^q \\
1 - \bigwedge_{i=1}^{n} \left(\eta_{A_i}\right)^q &\geq 1 - \left(\eta_{A_i}\right)^q \\
\left(1 - \bigwedge_{i=1}^{n} \left(\eta_{A_i}\right)^q\right)^{\lambda_i} &\geq \left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i} \\
\prod_{i=1}^{n} \left(1 - \bigwedge_{i=1}^{n} \left(\eta_{A_i}\right)^q\right)^{\lambda_i} &\geq \prod_{i=1}^{n} \left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i} \\
1 - \prod_{i=1}^{n} \left(1 - \bigwedge_{i=1}^{n} \left(\eta_{A_i}\right)^q\right)^{\lambda_i} &\leq 1 - \prod_{i=1}^{n} \left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i} \\
\sqrt[q]{1 - \prod_{i=1}^{n} \left(1 - \bigwedge_{i=1}^{n} \left(\eta_{A_i}\right)^q\right)^{\lambda_i}} &\leq \sqrt[q]{1 - \prod_{i=1}^{n} \left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i}} \\
\sqrt[q]{1 - \left(1 - \bigwedge_{i=1}^{n} \left(\eta_{A_i}\right)^q\right)^{\sum_{i=1}^{n} \lambda_i}} &\leq \sqrt[q]{1 - \prod_{i=1}^{n} \left(1 - \left(\eta_{A_i}\right)^q\right)^{\lambda_i}}
\end{aligned}
$$

$$\begin{aligned}
\sqrt[q]{1-\left(1-\bigwedge_{i=1}^{n}(\eta_{A_i})^q\right)} &\leq \sqrt[q]{1-\prod_{i=1}^{n}\left(1-(\eta_{A_i})^q\right)^{\lambda_i}} \\
\bigwedge_{i=1}^{n}\eta_{A_i} &\leq \sqrt[q]{1-\prod_{i=1}^{n}\left(1-(\eta_{A_i})^q\right)^{\lambda_i}} \qquad (35)
\end{aligned}$$

and

$$\begin{aligned}
\bigvee_{i=1}^{n}(\eta_{A_i})^q &\geq (\eta_{A_i})^q \\
1-\bigvee_{i=1}^{n}(\eta_{A_i})^q &\leq 1-(\eta_{A_i})^q \\
\left(1-\bigvee_{i=1}^{n}(\eta_{A_i})^q\right)^{\lambda_i} &\leq \left(1-(\eta_{A_i})^q\right)^{\lambda_i} \\
\prod_{i=1}^{n}\left(1-\bigvee_{i=1}^{n}(\eta_{A_i})^q\right)^{\lambda_i} &\leq \prod_{i=1}^{n}\left(1-(\eta_{A_i})^q\right)^{\lambda_i} \\
1-\prod_{i=1}^{n}\left(1-\bigvee_{i=1}^{n}(\eta_{A_i})^q\right)^{\lambda_i} &\geq 1-\prod_{i=1}^{n}\left(1-(\eta_{A_i})^q\right)^{\lambda_i} \\
\sqrt[q]{1-\prod_{i=1}^{n}\left(1-\bigvee_{i=1}^{n}(\eta_{A_i})^q\right)^{\lambda_i}} &\geq \sqrt[q]{1-\prod_{i=1}^{n}\left(1-(\eta_{A_i})^q\right)^{\lambda_i}} \\
\sqrt[q]{1-\left(1-\bigvee_{i=1}^{n}(\eta_{A_i})^q\right)^{\sum_{i=1}^{n}\lambda_i}} &\geq \sqrt[q]{1-\prod_{i=1}^{n}\left(1-(\eta_{A_i})^q\right)^{\lambda_i}} \\
\sqrt[q]{1-\left(1-\bigvee_{i=1}^{n}(\eta_{A_i})^q\right)^{1}} &\geq \sqrt[q]{1-\prod_{i=1}^{n}\left(1-(\eta_{A_i})^q\right)^{\lambda_i}} \\
\bigvee_{i=1}^{n}\eta_{A_i} &\geq \sqrt[q]{1-\prod_{i=1}^{n}\left(1-(\eta_{A_i})^q\right)^{\lambda_i}}. \qquad (36)
\end{aligned}$$

Then, by Eqs. (35) and (36), we get

$$\bigwedge_{i=1}^{n}\eta_{A_i} \leq \sqrt[q]{1-\prod_{i=1}^{n}\left(1-(\eta_{A_i})^q\right)^{\lambda_i}} \leq \bigvee_{i=1}^{n}\eta_{A_i}. \qquad (37)$$

Similarly, we can show that

$$\bigwedge_{i=1}^{n}\beta_{A_i} \leq \sqrt[q]{1-\prod_{i=1}^{n}\left(1-(\beta_{A_i})^q\right)^{\lambda_i}} \leq \bigvee_{i=1}^{n}\beta_{A_i}. \qquad (38)$$

Therefore, by Eqs. (33), (34), (37), and (38), we have $\mathcal{A}^{-} \preceq \mathcal{WG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \preceq \mathcal{A}^{+}$. □

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