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The Schoenberg kernel and more flexible multivariate covariance models in Euclidean spaces

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Abstract
The J-Bessel univariate kernel Ω_d introduced by Schoenberg plays a central role in the characterization of stationary isotropic covariance models defined in a d-dimensional Euclidean space. In the multivariate setting, a matrix-valued isotropic covariance is a scale mixture of the kernel Ω_d against a matrix-valued measure that is nondecreasing with respect to matrix inequality. We prove that constructions based on a p-variate kernel $[\Omega_{d_{ij}}]_{i,j=1}^{p}$ are feasible for different dimensions d_{ij}, at the expense of some parametric restrictions. We illustrate how multivariate covariance models inherit such restrictions and provide new classes of hypergeometric, Matérn, Cauchy and compactly-supported models to illustrate our findings.

Keywords Matrix-valued covariance · Schoenberg measure · Multivariate hypergeometric covariance · Multivariate Matérn covariance · Multivariate Cauchy covariance

Mathematics Subject Classification 42A82 · 60G10 · 86A32

1 Introduction

The last 30 years have seen a plethora of approaches to multivariate modeling, estimation and prediction in spatial statistics. Chilès and Delfiner (2012) and Genton and Kleiber (2015) provide an overview of modeling approaches that are centered on the second-order properties of a zero-mean p-variate random field $\boldsymbol{Z}$ in $\mathbb{R}^d$ with real-valued components. The mapping $\boldsymbol{C}$, defined through

$$\boldsymbol{C}(\boldsymbol{s}, \boldsymbol{s}') = \mathbb{E}(\boldsymbol{Z}(\boldsymbol{s}) \cdot \boldsymbol{Z}(\boldsymbol{s}')^\top), \quad \boldsymbol{s}, \boldsymbol{s}' \in \mathbb{R}^d, \tag{1}$$

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with $\mathbb{E}$ denoting the mathematical expectation, is the matrix-valued (or multivariate) covariance function of $\boldsymbol{Z}$.

A necessary and sufficient condition for a function $\boldsymbol{C} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{p\times p}$ to be the covariance of a p-variate random field in $\mathbb{R}^d$ is that $\boldsymbol{C}$ is positive semidefinite, i.e., the matrix $[[C_{ij}(\boldsymbol{s}_k, \boldsymbol{s}_\ell)]_{i,j=1}^p]_{k,\ell=1}^n$, where C_{ij} denotes the (i, j)-th entry of $\boldsymbol{C}$, is symmetric and positive semidefinite for any choice of the positive integer n and of the set of points $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$ in $\mathbb{R}^d$.

Under assumptions of second-order stationarity and isotropy, one has

$$\boldsymbol{C}(\boldsymbol{s}, \boldsymbol{s}') = \boldsymbol{\varphi}(\|\boldsymbol{s} - \boldsymbol{s}'\|), \quad \boldsymbol{s}, \boldsymbol{s}' \in \mathbb{R}^d, \tag{2}$$

where the matrix-valued function $\boldsymbol{\varphi} : [0, \infty) \to \mathbb{R}^{p\times p}$ is known as the isotropic or radial part of $\boldsymbol{C}$, and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$.

Hereinafter, we denote Φ_d^p the class of continuous matrix-valued mappings $\boldsymbol{\varphi} : [0, \infty) \to \mathbb{R}^{p\times p}$ such that (2) holds for a covariance function $\boldsymbol{C}$ defined in $\mathbb{R}^d \times \mathbb{R}^d$. We abuse of notation by writing Φ_d for Φ_d^1.

1.1 Context and problem

Let d be a positive integer. Following Schoenberg (1938), we define the mapping $\Omega_d : [0, \infty) \to \mathbb{R}$ through

$$\Omega_d(x) = \Gamma\left(\frac{d}{2}\right)\left(\frac{2}{x}\right)^{\frac{d}{2}-1} J_{\frac{d}{2}-1}(x), \quad x \geq 0, \tag{3}$$

with J_ν being the Bessel function of the first kind of order $\nu > 0$ (Olver et al. 2010, formula 10.2.2). Ω_d plays a crucial role to characterize the class Φ_d^p: in fact, Alonso-Malaver et al. (2015) proved that a continuous mapping $\boldsymbol{\varphi} : [0, \infty) \to \mathbb{R}^{p\times p}$ belongs to Φ_d^p if and only if it can be uniquely written as

$$\boldsymbol{\varphi}(x) = \int_0^\infty \Omega_d(rx)\boldsymbol{F}_{d,p}(\mathrm{d}r), \quad x \geq 0, \tag{4}$$

where the integral is understood as componentwise and where the measure $\boldsymbol{F}_{d,p} = \left[F_{ij;d,p}\right]_{i,j=1}^p$ is finite and nondecreasing with respect to matrix inequality, i.e., the matrix

$$\left[F_{ij;d,p}(r + \Delta) - F_{ij;d,p}(r)\right]_{i,j=1}^p$$

is positive semidefinite for all positive r and Δ.
The case $p = 1$ is due to Schoenberg (1938) and is especially useful as it offers the dual view of the members $\varphi(\cdot)/\varphi(0)$ in Φ_d as being both the isotropic parts of correlation functions in $\mathbb{R}^d$ and the characteristic functions of random vectors that are equal in distribution to the product between a nonnegative random variable with probability distribution F_d (the scalar-valued version of $\boldsymbol{F}_{d,p}$) with a random vector that is uniformly distributed over the unit sphere embedded in $\mathbb{R}^d$, with Ω_d being its characteristic function. Following Daley and Porcu (2014), we term $\boldsymbol{F}_{d,p}$ a (d, p)-Schoenberg measure and, for the scalar-valued case, we term F_d a d-Schoenberg measure.

There has been considerable criticism about the flexibility of current multivariate covariance models. Most of them are based on the adaptation principle (Porcu et al. 2018). Let $\{\varphi(\cdot; \boldsymbol{\theta})\ ; \boldsymbol{\theta} \in \mathbb{R}^m\}$ be a parametric family of members of Φ_d, with $\boldsymbol{\theta}$ being a parameter vector, such that $\varphi(0; \boldsymbol{\theta}) = 1$. Let $F_d(\cdot; \boldsymbol{\theta})$ be the d-Schoenberg measure of $\varphi(\cdot; \boldsymbol{\theta})$. Then, most of the proposals in the literature provide members $\boldsymbol{\varphi} \in \Phi_d^p$ having elements φ_{ij} that are identically equal to

$$\begin{aligned}\varphi_{ij}(x) &= \sigma_i \sigma_j \rho_{ij} \varphi(x; \boldsymbol{\theta}_{ij}) \\ &= \sigma_i \sigma_j \rho_{ij} \int_0^\infty \Omega_d(rx) F_d(\mathrm{d}r; \boldsymbol{\theta}_{ij}), \quad x \geq 0,\ i, j = 1, \ldots, p,\end{aligned} \tag{5}$$

where σ_i is the standard deviation of the i-th component Z_i of $\boldsymbol{Z}$, ρ_{ij} is the collocated correlation coefficient between Z_i and Z_j, and $\boldsymbol{\theta}_{ij}$ belongs to $\mathbb{R}^m$. This principle is the core of the celebrated multivariate Matérn model (Gneiting et al. 2010). Similar constructions have been proposed by Daley et al. (2015) for a multivariate model with compact support, by Emery and Alegría (2022) for a general formulation that encompasses both Matérn and compactly-supported models, and by Bourotte et al. (2016) and Allard et al. (2022) for the nonseparable Gneiting space-time model.

There are certainly issues with the formulation (5), which appears as a particular case of (4) where the components of $\boldsymbol{F}_{d,p}$ are determined from a single parametric family $F_d(\cdot, \boldsymbol{\theta})$ of univariate Schoenberg measures. Bevilacqua et al. (2015) note how the constraints on the vectors $\boldsymbol{\theta}_{ij}$ imply restrictions on the collocated correlations coefficient ρ_{ij}, which are no longer free to vary between -1 and 1. Further limits can be found by studying measures of discrepancies between the elements on the diagonal and those on the anti-diagonal of the mapping $\boldsymbol{\varphi}$. Yet, the adaptation principle remains a valid instrument to provide parametric families of matrix-valued covariance functions and to determine sufficient validity conditions on their parameters. The present paper digs into this principle and introduces more flexibility by allowing the generating kernel Ω_d to have multiple indices d_{ij} associated with the components φ_{ij} in Eq. (5).

1.2 Our contribution

Schoenberg's representation for the case $p = 1$ in (4) implies that Ω_d is a member of the class Φ_d. Hence, all members $\boldsymbol{\varphi}$ in Φ_d^p are written as a scale mixture of a univariate member of Φ_d against a (d, p)-Schoenberg measure. A tempting choice for flexible multivariate modeling would be to consider a matrix of integers $\boldsymbol{d} = \left[d_{ij}\right]_{i,j=1}^p$ and representations of the type

$$\boldsymbol{\varphi}(x) = \int_0^\infty \boldsymbol{\Omega_d}(rx) \boldsymbol{F}_{d,p}(\mathrm{d}r), \qquad x \geq 0, \tag{6}$$

with $\boldsymbol{\Omega_d}(x) = \left[\Omega_{d_{ij}}(x)\right]_{i,j=1}^p$. Again, the integration is taken componentwise and $\boldsymbol{\Omega_d}(\cdot x)\boldsymbol{F}_{d,p}(\mathrm{d}\cdot)$ is the matrix-valued function having elements $\Omega_{d_{ij}}(\cdot x) F_{ij;d,p}(\mathrm{d}\cdot)$. We will show that such a construction is possible under suitable parametric restrictions. As a consequence, we will prove that there is room for improving the classical adaptation construction (5) that has been the gold standard for many years in multivariate spatial statistics modeling. An important by-product of the representation (6) will be the possibility to devise parametric families of members of Φ_d^p and to derive sufficient validity conditions on their parameters, thus to extend the current state of knowledge on multivariate covariance modeling. Some examples will illustrate the versatility of our approach. We will also provide an operator viewpoint for the $\boldsymbol{\Omega_d}$-based construction and prove that this entirely maps Φ_d into Φ_d^p.

Table 1 Ordinary and special functions used in the paper; see Olver et al. (2010) for mathematical definitions

Notation	Function name
$\mathbb{I}_{(a,b)}(\cdot)$	Indicator function of the open interval (a, b)
$(\cdot)_+$	Positive part function
$\lceil\cdot\rceil$	Ceil function
$J_\nu(\cdot)$	Bessel function of the first kind of order ν
$K_\nu(\cdot)$	Modified Bessel function of the second kind of order ν
$M_{\lambda,\mu}(\cdot)$, $W_{\lambda,\mu}(\cdot)$	Whittaker functions
$\Gamma(\cdot)$	Gamma function
$B(\cdot,\cdot)$	Beta function
$(\cdot)^{\underline{k}}$	Falling factorial
${}_0F_1(;\beta;\cdot)$	Confluent hypergeometric function
${}_1F_1(\alpha;\beta;\cdot)$	Kummer confluent hypergeometric function
${}_2F_1(\alpha_1,\alpha_2;\beta;\cdot)$	Gauss hypergeometric function
${}_qF_{q'}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_{q'};\cdot)$	Generalized hypergeometric function

1.3 Notation

Throughout, the functions listed in Table 1will be used. Bold letters will refer to matrices and vectors (one-column matrices), p and d will denote positive integers, $\mathbf{0}$ and $\mathbf{1}$ the zero and all-ones matrices of size $p \times p$, and $\top$ the transposition operator. Continuity, differentiation and integration involving matrix-valued functions are understood as componentwise. So will be any mathematical operation (e.g., product, ratio, square root, power, exponentiation, composition, and indicator function) involving matrices or matrix-valued functions.

The next section provides a technical result that supports the main findings contained in Sect. 3. Technical definitions, lemmas and proofs are deferred to appendices for a neater exposition.

2 An auxiliary result

Proposition 1 *Let* $\boldsymbol{\rho} = \left[\rho_{ij}\right]_{i,j=1}^p$, $\boldsymbol{b} = \left[b_{ij}\right]_{i,j=1}^p$ *and* $\boldsymbol{\nu} = \left[\nu_{ij}\right]_{i,j=1}^p$ *be real symmetric matrices, with* $b_{ij} > 0$ *and* $\nu_{ij} > d$ *for* $i, j = 1, \ldots, p$. *Define* γ_{ij} *and* $\kappa_{ij,d}$ *as*

$$\gamma_{ij} = \frac{\nu_{ij} - d}{2} - 1 \tag{7}$$

and

$$\kappa_{ij,d} = \frac{2\Gamma\left(\frac{\nu_{ij}}{2}\right)}{\Gamma(\gamma_{ij}+1)\Gamma\left(\frac{d}{2}\right)}. \tag{8}$$

Let $B_{ij}(r) = \left(1 - b_{ij}^2 r^2\right)_+^{\gamma_{ij}}$ *for* $r > 0$ *and* $i, j = 1, \ldots, p$. *Let the matrix* $\boldsymbol{A}(r) = \left[A_{ij}(r)\right]_{i,j=1}^p$ *have entries*

$$A_{ij}(r) = \rho_{ij}\,\kappa_{ij,d}\,B_{ij}(r), \quad i, j = 1, \ldots, p. \tag{9}$$

Then, $\boldsymbol{A}(r)$ is positive semidefinite for all $r > 0$ under any of the three following sets of conditions:

1. (a) $[\rho_{ij}]_{i,j=1}^p$ *is positive semidefinite;*
   (b) $b_{ij} = \max\{b_i, b_j\}$ *for* $i \neq j$ *and* $b_{ii} = b_i - \beta_i$, *with* $b_1, \ldots, b_p > 0$ *and* $\beta_1, \ldots, \beta_p \geq 0$;
   (c) $\nu_{ij} = \nu$ *for* $i, j = 1, \ldots, p$;
   or
2. (a) $[-\gamma_{ij}]_{i,j=1}^p$ *is positive semidefinite;*
   (b) $[b_{ij}^2]_{i,j=1}^p$ *is positive semidefinite;*
   (c) $[\rho_{ij}\, \kappa_{ij,d}\, \mathbb{I}_{(b_{ij},\infty)}(z)]_{i,j=1}^p$ *is positive semidefinite for any* $z > 0$;
   or
3. (a) $\rho_{ii} \geq 0$ *for* $i = 1, \ldots, p$;
   (b) $b_{ii} < b_{ij}$ *and* $\gamma_{ij} \geq 0$, *or* $b_{ii} > b_{ij}$ *and* $\gamma_{ii} < 0 \leq \gamma_{ij}$, *or* $b_{ii} = b_{ij}$ *and* $\gamma_{ii} \leq \gamma_{ij}$, *for* $i, j = 1, \ldots, p$ *with* $i \neq j$;
   (c) *for* $i = 1, \ldots, p$,

$$\rho_{ii}\, \kappa_{ii,d} \geq \sum_{i \neq j} \rho_{ij}\, \kappa_{ij,d} \left\{ \left(1 - \mathbb{I}_{(0,1)} \left(\frac{\gamma_{ij} b_{ij}^2 - \gamma_{ii} b_{ii}^2}{b_{ii}^2 (\gamma_{ij} - \gamma_{ii})} \right) \right) \right.$$
$$\left. + \mathbb{I}_{(0,1)} \left(\frac{\gamma_{ij} b_{ij}^2 - \gamma_{ii} b_{ii}^2}{b_{ii}^2 (\gamma_{ij} - \gamma_{ii})} \right) \left(\frac{\gamma_{ij} (b_{ij}^2 - b_{ii}^2)}{b_{ii}^2 (\gamma_{ii} - \gamma_{ij})} \right)^{\gamma_{ij}} \left(\frac{\gamma_{ii} (b_{ij}^2 - b_{ii}^2)}{b_{ij}^2 (\gamma_{ii} - \gamma_{ij})} \right)^{-\gamma_{ii}} \right\},$$

with the convention $0^0 = 1$.

3 Main results

3.1 The Schoenberg kernel $\boldsymbol{\Omega_\nu}$

Let ν be a positive real number. We generalize the exposition in Sect. 1 by considering the mapping $\Omega_\nu : [0, \infty) \to \mathbb{R}$ through the identity

$$\Omega_\nu(x) = \Gamma\left(\frac{\nu}{2}\right) \left(\frac{2}{x}\right)^{\frac{\nu}{2}-1} J_{\frac{\nu}{2}-1}(x), \quad x \geq 0. \tag{10}$$

Arguments in Schoenberg (1938) prove that, for $d \leq \nu < d + 1$ with d a positive integer, Ω_ν belongs to $\Phi_d \setminus \Phi_{d+1}$.

Proposition 2 *Let* $\boldsymbol{\rho} = [\rho_{ij}]_{i,j=1}^p$, $\boldsymbol{b} = [b_{ij}]_{i,j=1}^p$ *and* $\boldsymbol{\nu} = [\nu_{ij}]_{i,j=1}^p$ *be real symmetric matrices with the restriction that* $b_{ij} > 0$ *and* $\nu_{ij} > d$ *for* $i, j = 1, \ldots, p$. *Then, the mapping* $\boldsymbol{\lambda} : [0, \infty) \to \mathbb{R}^{p \times p}$ *defined through*

$$\lambda_{ij}(x) = \frac{\rho_{ij}}{b_{ij}^d} \Omega_{\nu_{ij}} \left(\frac{x}{b_{ij}} \right), \qquad x \geq 0, \quad i, j = 1, \ldots, p, \tag{11}$$

belongs to Φ_d^p *provided that the matrix* $\boldsymbol{A}(r)$ *defined at* (9) *is positive semidefinite for any* $r > 0$.

Some comments are in order. Proposition 2 is related to the kernel $\boldsymbol{\Omega_d}$ as in (6) when the matrix $\boldsymbol{\nu}$ is restricted to coefficients being integers and representing spatial dimensions.

Clearly, Proposition 1 shows that the choice of these dimensions cannot be arbitrary. On the other hand, the proof of Proposition 2 (Appendix B) shows that $\boldsymbol{\lambda}$ is actually the scale mixture of Ω_d against a (d, p)-Schoenberg measure that is absolutely continuous with respect to the Lebesgue measure, with a (d, p)-Schoenberg density equal to the mapping $v \mapsto v^{d-1}\boldsymbol{A}(v)$, with $\boldsymbol{A}$ defined at (9). This fact suggests to take the following operator perspective.

Proposition 3 *Let $\varphi : [0, \infty) \to \mathbb{R}$ be a member of Φ_d. Let Υ_d be the operator from Φ_d into $\mathbb{R}^{p\times p}$ defined through*

$$\Upsilon_d(\varphi)(x) = \boldsymbol{\psi}(x) := \left[\int_0^\infty \varphi(vx)v^{d-1}\boldsymbol{A}(v)\mathrm{d}v \right]_{i,j=1}^p, \quad x \geq 0, \tag{12}$$

with $\boldsymbol{A}(\cdot)$ defined at (9), such that $\boldsymbol{b}$ and $\boldsymbol{\nu} - d$ have positive entries and $\boldsymbol{\rho}$, $\boldsymbol{b}$ and $\boldsymbol{\nu}$ satisfy one of the three sets of conditions in Proposition 1. Then, Υ_d maps Φ_d into Φ_d^p.

We finally prove that the operator Υ_d can provide walks through dimensions as much as in Matheron (1965) and in Daley and Porcu (2014). Let $\varphi \in \Phi_d$ and $\boldsymbol{\psi}$ be as defined at (12). We now define the operator $\mathcal{I} : \Phi_d^p \to \mathbb{R}^{p\times p}$ through $\mathcal{I}(\boldsymbol{\psi})$ having components

$$\mathcal{I}(\psi_{ij})(x) = \int_x^\infty u\psi_{ij}(u)\mathrm{d}u, \quad x \geq 0, i, j = 1, \ldots, p, \tag{13}$$

provided that the integral is convergent. The following result shows that a suitable combination of $\mathcal{I}$ with Υ_d allows mapping Φ_d into Φ_{d-2}^p.

Proposition 4 *Let $d \geq 3$. Let $\varphi \in \Phi_d$ with d-Schoenberg measure F_d such that $\int_0^\infty u^{-2}F_d(\mathrm{d}u)$ is well-defined. Let Υ_d be the operator defined at (12), and let $\mathcal{I}$ be the operator defined at (13). Then, $\mathcal{I}(\Upsilon_d(\varphi))$ is well-defined. Furthermore, $\boldsymbol{\varphi}(\cdot) := \mathcal{I}(\Upsilon_d(\varphi))(\cdot)$ belongs to Φ_{d-2}^p.*

3.2 Application: multivariate hypergeometric covariances

Table 2 provides some examples of absolutely continuous d-Schoenberg measures F_d, associated with a probability density function f_d, for which an analytical expression of $\Upsilon_d(\varphi)$ can be obtained. In this table, κ is a normalization constant to ensure that f_d has a unit integral, while κ' is a positive constant depending on κ and on the parameters of f_d. The first four entries of the table have been established by using formulae 6.621.1, 6.631.1, 6.569 and 7.661.3 of Gradshteyn and Ryzhik (2007), respectively, and the last three entries by using formulae 8.5.4, 8.5.24 and 8.13.4 of Erdélyi (1954).

More hypergeometric models than those reported in Table 2 can be designed, as explained next. First, note that Ω_ν can be written as (Olver et al. 2010, formula 10.16.9)

$$\Omega_\nu(x) = {}_0F_1\left(; \frac{\nu}{2}; -\frac{x^2}{4}\right), \quad x \geq 0. \tag{14}$$

Second, a beta or a gamma mixture of hypergeometric functions is another hypergeometric function (Olver et al. 2010, formulae 16.5.2 and 16.5.3):

$$\begin{aligned} &{}_{q+1}F_{q'+1}(\alpha_0, \ldots, \alpha_q; \beta_0 + \alpha_0, \beta_1, \ldots, \beta_{q'}; -x) \\ &= \frac{1}{B(\alpha_0, \beta_0)} \int_0^1 t^{\alpha_0-1}(1-t)^{\beta_0-1} {}_qF_{q'}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_{q'}; -xt)\mathrm{d}t, \quad x \geq 0, \end{aligned} \tag{15}$$

Table 2 Analytical expressions of the (i, j)-th entry ψ_{ij} of $\boldsymbol{\psi} = \Upsilon_d(\varphi)$, where φ is the member of Φ_d with Schoenberg density f_d, for some specific choices of f_d

$f_d(r)$	$\psi_{ij}(x)$	Restrictions
$\frac{1}{\Gamma(\mu)} e^{-r} r^{\mu-1}$	$\frac{\rho_{ij}}{b_{ij}^d}\, {}_2F_1\left(\frac{\mu}{2}, \frac{\mu+1}{2}; \frac{\nu_{ij}}{2}; -\frac{x^2}{b_{ij}^2}\right)$	$\mu > 0$
$\kappa e^{-r^2} r^{\mu-1}$	$\frac{\rho_{ij}\,\kappa'}{b_{ij}^d}\, {}_1F_1\left(\frac{\mu}{2}; \frac{\nu_{ij}}{2}; -\frac{x^2}{4b_{ij}^2}\right)$	$\mu > 0$
$\kappa\, r^{\delta}(1-r)^{\mu-1}\mathbb{I}_{(0,1)}(r)$	$\frac{\rho_{ij}\,\kappa'}{b_{ij}^d}\, {}_2F_3\left(\frac{\delta+1}{2}, \frac{\delta}{2}+1; \frac{\nu_{ij}}{2}, \frac{\delta+\mu+1}{2}, \frac{\delta+\mu}{2}+1; -\frac{x^2}{4\,b_{ij}^2}\right)$	$\mu > 0,\ \delta > -1$
$\kappa\, r^{2\mu} W_{\frac{1}{2}-\mu,\mu}(r) M_{\mu-\frac{1}{2},\mu}(r)$	$\frac{\rho_{ij}\,\kappa'}{b_{ij}^d}\left(\frac{\nu_{ij}}{2}-1\right)\frac{b_{ij}^2}{x^2}\, {}_2F_1\left(\mu, \mu+\frac{1}{2}; \frac{\nu_{ij}}{2}-1; -\frac{x^2}{b_{ij}^2}\right)$	$\mu > -\frac{1}{2},\ \nu_{ij} > 3$
$\frac{3}{2} r\, \mathbb{I}_{(0,1)}(r)$	$\frac{\rho_{ij}\,\kappa'}{b_{ij}^d}\left(\frac{\nu_{ij}}{2}-1\right)\frac{b_{ij}^2}{x^2}\left[1 - {}_0F_1\left(; \frac{\nu_{ij}}{2}-1; -\frac{x^2}{4b_{ij}^2}\right)\right]$	
$\kappa \frac{r}{\sqrt{1-r^2}}\, \mathbb{I}_{(0,1)}(r)$	$\frac{\rho_{ij}\,\kappa'}{b_{ij}^d}\, {}_1F_2\left(1; \frac{3}{2}, \frac{\nu_{ij}}{2}; -\frac{x^2}{4b_{ij}^2}\right)$	
$\kappa\, K_{\mu}(r)$	$\frac{\rho_{ij}\,\kappa'}{b_{ij}^d}\, {}_2F_1\left(\frac{1-\mu}{2}, \frac{1+\mu}{2}; \frac{\nu_{ij}}{2}; -\frac{x^2}{b_{ij}^2}\right)$	$1 > \mu \geq 0$

and

$$
\begin{aligned}
&{}_{q+1}F_{q'}(\alpha_0, \ldots, \alpha_q; \beta_1, \ldots, \beta_{q'}; -x) \\
&= \frac{1}{\Gamma(\alpha_0)} \int_0^\infty e^{-t} t^{\alpha_0 - 1} {}_qF_{q'}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_{q'}; -xt) \mathrm{d}t, \quad x \geq 0, \qquad (16)
\end{aligned}
$$

where $q, q' \in \mathbb{N}$, $\beta_0 > 0$, $\alpha_0 > 0$, and the α's and β's are such that the above hypergeometric functions are well-defined.

Combining the previous facts, one obtains the following result.

Proposition 5 *Let $q, q' \in \mathbb{N}$ with $q \geq q'$. Let $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{q'}$, $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{q'}$ be conditionally negative semidefinite matrices of size $p \times p$ with positive entries. Let $\boldsymbol{\alpha}_{q'+1}, \ldots, \boldsymbol{\alpha}_q$ be conditionally null semidefinite matrices of size $p \times p$ with positive entries. Let $\boldsymbol{\rho}'$ be a real symmetric matrix of size $p \times p$. Then, the matrix-valued function $\boldsymbol{C}$ defined by*

$$
\boldsymbol{C}(x) = \frac{\boldsymbol{\rho}'}{\boldsymbol{b}^d} {}_qF_{q'+1}\left(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_q; \frac{\boldsymbol{\nu}}{2}, \boldsymbol{\alpha}_1 + \boldsymbol{\beta}_1, \ldots, \boldsymbol{\alpha}_{q'} + \boldsymbol{\beta}_{q'}; -\frac{x^2}{4\boldsymbol{b}^2}\right), \quad x \geq 0, \qquad (17)
$$

belongs to Φ_d^p if $\boldsymbol{\rho}$, $\boldsymbol{b}$ and $\boldsymbol{\nu}$ are real symmetric matrices of size $p \times p$ satisfying one of the three sets of conditions in Proposition 1, with

$$
\boldsymbol{\rho} = \frac{\boldsymbol{\rho}'}{\prod_{k=1}^{q'} B(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k) \prod_{k=q'+1}^{q} \Gamma(\boldsymbol{\alpha}_k)}.
$$

Remark 1 The first three entries and the last entry of Table 2 are particular cases of (17).

3.3 Matérn, compactly-supported and Cauchy multivariate covariances

The following proposition provides an integral representation of members of Φ_d^p in a slightly more general form than (6). Applications to the determination of validity conditions for three parametric families of isotropic covariances (Matérn, compactly-supported hypergeometric and Cauchy) follow in Propositions 7–9.

Proposition 6 *Let $\boldsymbol{F}_{d,p}$ be a (d, p)-Schoenberg measure. Let $\boldsymbol{\psi}$ be a matrix-valued function defined through*

$$
\boldsymbol{\psi}(x) := \int_0^\infty \boldsymbol{\lambda}(rx) \boldsymbol{F}_{d,p}(\mathrm{d}r), \quad x \geq 0, \qquad (18)
$$

with $\boldsymbol{\lambda}$ as in (11), $\boldsymbol{\rho}$, $\boldsymbol{b}$ and $\boldsymbol{\nu}$ satisfying one of the three sets of conditions in Proposition 1, and $\boldsymbol{b}$ and $\boldsymbol{\nu} - d$ having positive entries. Then, $\boldsymbol{\psi}$ belongs to Φ_d^p.

Proposition 7 (Multivariate Matérn covariance) *Define the isotropic part of the univariate Matérn covariance with range $b > 0$ and shape parameter $\mu > 0$ as*

$$
\mathcal{M}(x; b, \mu) = \frac{2^{1-\mu}}{\Gamma(\mu)} \left(\frac{x}{b}\right)^\mu K_\mu\left(\frac{x}{b}\right), \quad x \geq 0.
$$

Let $\boldsymbol{\sigma}$, $\boldsymbol{b}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu} - d$ be symmetric matrices of size $p \times p$, the latter three with positive entries. Then, the matrix-valued function $\mathcal{M}(\cdot; \boldsymbol{b}, \boldsymbol{\mu}, \boldsymbol{\sigma}) = [\sigma_{ij}\, \mathcal{M}(\cdot; b_{ij}, \mu_{ij})]_{i,j=1}^p$ belongs to Φ_d^p if the following sufficient conditions hold:

1. *$\boldsymbol{\mu}$ is conditionally negative semidefinite;*

2. $\boldsymbol{\nu}$ *is conditionally negative semidefinite*;
3. $\boldsymbol{b}$, $\boldsymbol{\nu}$ *and* $\boldsymbol{\rho} = \frac{\boldsymbol{b}^d \boldsymbol{\sigma}}{B(\boldsymbol{\mu}, \frac{\boldsymbol{\nu}}{2})}$ *fulfill the conditions of Proposition* 1.

Proposition 8 (Multivariate compactly-supported hypergeometric covariance) *For* $b > 0$, α, β, γ *and* ν *such that* $\frac{\nu}{2} < \alpha < \min\{\beta, \gamma\}$, *define the function* $\mathcal{H}(\cdot; b, \alpha, \beta, \gamma, \nu)$ *on* $[0, \infty)$ *by*

$$\mathcal{H}(x; b, \alpha, \beta, \gamma, \nu) = \frac{\Gamma(\beta - \frac{\nu}{2})\Gamma(\gamma - \frac{\nu}{2})}{\Gamma(\beta - \alpha + \gamma - \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})} \left(1 - \frac{x^2}{b^2}\right)_+^{\beta-\alpha+\gamma-\frac{\nu}{2}-1} \times {}_2F_1\left(\beta - \alpha, \gamma - \alpha; \beta - \alpha + \gamma - \frac{\nu}{2}; \left(1 - \frac{x^2}{b^2}\right)_+\right), \quad x \geq 0.$$

Let $\alpha > \frac{d}{2}$, *and let* $\boldsymbol{b}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\sigma}$ *and* $\boldsymbol{\nu}$ *be symmetric matrices of size* $p \times p$, *the former* ($\boldsymbol{b}$) *with positive entries and the latter* ($\boldsymbol{\nu}$) *with entries in* $(d, 2\alpha)$. *Then, the matrix-valued function* $\mathcal{H}(\cdot; \boldsymbol{b}, \alpha, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\nu}, \boldsymbol{\sigma}) = [\sigma_{ij}\, \mathcal{H}(\cdot; b_{ij}, \alpha, \beta_{ij}, \gamma_{ij}, \nu_{ij})]_{i,j=1}^p$ *belongs to* Φ_d^p *if one of the two sets of sufficient conditions holds*:

1. (a) $\boldsymbol{\beta} = \beta \mathbf{1}$ *with* $\beta > 0$;
   (b) $\boldsymbol{\gamma} = \gamma \mathbf{1}$ *with* $\gamma > 0$;
   (c) $2(\beta - \alpha)(\gamma - \alpha) \geq \alpha$ *and* $2(\beta + \gamma) \geq 6\alpha + 1$;
   (d) $\boldsymbol{\nu}$ *is conditionally null semidefinite*;
   (e) $\boldsymbol{\rho}$, $\boldsymbol{b}$ *and* $\boldsymbol{\nu}$ *fulfill the conditions of Proposition* 1, *where*

$$\boldsymbol{\rho} = \frac{\boldsymbol{b}^d\, \Gamma\left(\beta - \frac{\boldsymbol{\nu}}{2}\right) \Gamma\left(\gamma - \frac{\boldsymbol{\nu}}{2}\right) \boldsymbol{\sigma}}{2^{\boldsymbol{\nu}} \Gamma\left(\frac{\boldsymbol{\nu}}{2}\right) \Gamma\left(\alpha - \frac{\boldsymbol{\nu}}{2}\right)}; \tag{19}$$

   or

2. (a) $\boldsymbol{\beta}$ *is conditionally negative semidefinite, with entries greater than* β;
   (b) $\boldsymbol{\gamma}$ *is conditionally negative semidefinite, with entries greater than* γ;
   (c) $2(\beta - \alpha)(\gamma - \alpha) \geq \alpha$ *and* $2(\beta + \gamma) \geq 6\alpha + 1$;
   (d) $\boldsymbol{\nu}$ *is conditionally null semidefinite*;
   (e) $\boldsymbol{\rho}$, $\boldsymbol{b}$ *and* $\boldsymbol{\nu}$ *fulfill the conditions of Proposition* 1, *where*

$$\boldsymbol{\rho} = \frac{\boldsymbol{b}^d\, \Gamma\left(\boldsymbol{\beta} - \frac{\boldsymbol{\nu}}{2}\right) \Gamma\left(\boldsymbol{\gamma} - \frac{\boldsymbol{\nu}}{2}\right) \boldsymbol{\sigma}}{2^{\boldsymbol{\nu}} \Gamma(\boldsymbol{\beta} - \beta)\Gamma(\boldsymbol{\gamma} - \gamma) \Gamma\left(\frac{\boldsymbol{\nu}}{2}\right) \Gamma\left(\alpha - \frac{\boldsymbol{\nu}}{2}\right)}. \tag{20}$$

Proposition 9 (Multivariate Cauchy covariance) *For* $b > 0$ *and* $\mu > 0$, *define the function* $\mathcal{C}(\cdot; b, \mu)$ *on* $[0, \infty)$ *by*

$$\mathcal{C}(x; b, \mu) = \left(1 + \frac{x^2}{b^2}\right)^{-\mu}, \quad x \geq 0.$$

Let $\boldsymbol{b}$, $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ *and* $\boldsymbol{\sigma}$ *be symmetric matrices of size* $p \times p$, *the former two with positive entries. Then, the matrix-valued function* $\mathcal{C}(\cdot; \boldsymbol{b}, \boldsymbol{\mu}, \boldsymbol{\sigma}) = [\sigma_{ij}\, \mathcal{C}(\cdot; b_{ij}, \mu_{ij})]_{i,j=1}^p$ *belongs to* Φ_d^p *if the following conditions hold*:

1. $\boldsymbol{\nu} - d$ *and* $\boldsymbol{\nu} - \boldsymbol{\mu}$ *are conditionally null semidefinite, with positive entries*;
2. $\boldsymbol{\rho}$, $\boldsymbol{b}$ *and* $\boldsymbol{\nu}$ *fulfill the conditions of Proposition* 1, *where*

$$\boldsymbol{\rho} = \frac{\boldsymbol{b}^d\, \boldsymbol{\sigma}}{\Gamma(\boldsymbol{\mu}) \Gamma\left(\frac{\boldsymbol{\nu}}{2}\right)}. \tag{21}$$

3.4 Comparison with previously proposed models

The validity conditions found in Propositions 7 and 8 differ from those reported in the literature, see Gneiting et al. (2010), Apanasovich et al. (2012), Du et al. (2012) and Emery et al. (2022) for the multivariate Matérn covariance and Emery and Alegría (2022) for the multivariate Gauss hypergeometric covariance.

For instance, for the multivariate Matérn model $\mathcal{M}(\cdot; \boldsymbol{b}, \boldsymbol{\mu}, \boldsymbol{\sigma})$, combining the conditions given in Proposition 7 with the second set of conditions given in Proposition 1, one finds the following sufficient validity conditions:

1. $\boldsymbol{\mu}$ is conditionally negative semidefinite;
2. $(d+2) - \boldsymbol{\nu}$ is positive semidefinite, with $\boldsymbol{\nu}$ having entries greater than d;
3. $\boldsymbol{b}^2$ is positive semidefinite;
4. $\mathbb{I}_{(\boldsymbol{b},\infty)}(z) \frac{\Gamma(\boldsymbol{\mu}+\frac{\boldsymbol{\nu}}{2})}{\Gamma(\frac{\boldsymbol{\nu}-d}{2})\Gamma(\boldsymbol{\mu})} \boldsymbol{b}^d \, \boldsymbol{\sigma}$ is positive semidefinite for any $z > 0$,

where $\mathbb{I}_{(\boldsymbol{b},\infty)}(z) = [\mathbb{I}_{(b_{ij},\infty)}(z)]_{i,j=1}^p$. Note that the last condition only requires checking the positive semidefiniteness of finitely many (at most $\frac{p(p+1)}{2}$) matrices. In particular, choosing $\boldsymbol{\nu} = (d+\varepsilon)\mathbf{1}$ with $\varepsilon \in (0, 2]$ and letting ε tend to zero stills yields valid conditions, since positive semidefiniteness is preserved under limits, namely:

1. $\boldsymbol{\mu}$ is conditionally negative semidefinite;
2. $\boldsymbol{b}^2$ is positive semidefinite;
3. $\mathbb{I}_{(\boldsymbol{b},\infty)}(z) \frac{\Gamma(\boldsymbol{\mu}+\frac{d}{2})}{\Gamma(\boldsymbol{\mu})} \boldsymbol{b}^d \, \boldsymbol{\sigma}$ is positive semidefinite for any $z > 0$.

Clearly, these conditions evade from any of the conditions provided in the cited literature.

Concerning the Gauss hypergeometric model $\mathcal{H}(\cdot; \boldsymbol{b}, \alpha, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\nu}, \boldsymbol{\sigma})$, conditions (2) in Proposition 8 bear resemblance to conditions (1) of Theorem 17 in Emery and Alegría (2022). Yet, any set of parameters $(\boldsymbol{b}, \alpha, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\sigma})$ satisfying the latter conditions is the limit of a set of parameters satisfying the former conditions (take $\boldsymbol{b} = b\mathbf{1}$ with $b > 0$ and $\boldsymbol{\nu} = (d+\varepsilon)\mathbf{1}$ with $\varepsilon > 0$, and then let ε tend to zero), which means that the conditions in Proposition 8 are more general. In particular, they are not limited to matrices $\boldsymbol{b}$ and $\boldsymbol{\nu}$ that are proportional to the all-ones matrix, and therefore allow more varied shapes for the direct and cross-covariance functions, which can be associated with different dimension parameters ν_{ij} and correlation ranges b_{ij}. For instance, with the same reasoning as above, by combining results of Propositions 1 and 8, one finds the following simplified set of validity conditions for $\mathcal{H}(\cdot; \boldsymbol{b}, \alpha, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\nu}, \boldsymbol{\sigma})$:

1. $\boldsymbol{\beta}$ is conditionally negative semidefinite, with entries greater than β;
2. $\boldsymbol{\gamma}$ is conditionally negative semidefinite, with entries greater than γ;
3. $2(\beta - \alpha)(\gamma - \alpha) \geq \alpha > \frac{d}{2}$ and $2(\beta + \gamma) \geq 6\alpha + 1$;
4. $\boldsymbol{b}^2$ is positive semidefinite;
5. $\mathbb{I}_{(\boldsymbol{b},\infty)}(z) \frac{\Gamma(\boldsymbol{\beta}-\frac{d}{2})\Gamma(\boldsymbol{\gamma}-\frac{d}{2})}{\Gamma(\boldsymbol{\beta}-\beta)\Gamma(\boldsymbol{\gamma}-\gamma)} \boldsymbol{b}^d \, \boldsymbol{\sigma}$ is positive semidefinite for any $z > 0$.

4 Conclusions

The findings of this paper contribute to the construction of parametric families of multivariate covariance models in Euclidean spaces and to the determination of sufficient validity conditions on their parameters. We have proven that the parametric adaptation modeling strategy based on the representation (6) may be more versatile and allow identifying wider validity conditions than the traditional strategy based on the representation (5). In particular, the multivariate hypergeometric models given in Table 2 and Proposition 5 are, to the best of our knowledge, novel and provide a wealth of matrix-valued covariances in Euclidean spaces. Also, the conditions given in Propositions 7–9 extend currently known validity conditions for the multivariate Matérn, compactly-supported hypergeometric, and Cauchy covariances, respectively.

Convolution-based approaches have been successful in multivariate covariance modeling (Gaspari and Cohn 1999). It would therefore be extremely useful to construct covariance models from kernels that are closed under convolution, instead of the Schoenberg kernel Ω_ν, so as to be able to build new models based on the convolution principle. Also, the results of this paper could be the starting point for future research to provide more general covariance structures that are not stationary and isotropic. This represents a major challenge.

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Appendix A: Definitions and lemmas

Definition 1 A real symmetric matrix $\boldsymbol{a} = [a_{ij}]_{i,j=1}^p$ is conditionally negative semidefinite if $\sum_{i,j=1}^p \lambda_i \lambda_j a_{ij} \le 0$ for all $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$ such that $\sum_{i=1}^p \lambda_i = 0$.

Definition 2 A real symmetric matrix $\boldsymbol{a} = [a_{ij}]_{i,j=1}^p$ is conditionally null semidefinite if both $\boldsymbol{a}$ and $-\boldsymbol{a}$ are conditionally negative semidefinite.

Lemma 1 *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be symmetric conditionally negative semidefinite matrices of size $p \times p$. Then, for any $t \in (0, 1)$, the matrix $t^{\boldsymbol{\alpha}-1}(1-t)^{\boldsymbol{\beta}-1}$ is positive semidefinite.*

Proof The claim follows from the Schur product theorem and the fact that, under the mentioned conditions, both $e^{(\boldsymbol{\alpha}-1)\ln(t)}$ and $e^{(\boldsymbol{\beta}-1)\ln(1-t)}$ are positive semidefinite (Berg et al. 1984, Chapter 3, Theorem 2.2). □

Lemma 2 *Let $\boldsymbol{\alpha}$ be a conditionally null semidefinite real matrix of size $p \times p$. Then, for any $t \in (0, \infty)$, the matrix $t^{\boldsymbol{\alpha}-1}$ is positive semidefinite.*

Proof The claim follows from the fact that $\boldsymbol{\alpha}$ is addition-separable, i.e., the (i, j)-th entry is the arithmetic average of the (i, i)-th and (j, j)-th entries (Allard et al. 2022). Accordingly, for any $t \in (0, \infty)$, $t^{\boldsymbol{\alpha}-1}$ has positive entries and is product-separable, i.e., the (i, j)-th entry is the geometric average of the (i, i)-th and (j, j)-th entries, which entails that it is positive semidefinite (Berg et al. 1984, Chapter 3, Property 1.9). □

Lemma 3 *Let $\boldsymbol{a} = [a_{ij}]_{i,j=1}^p$ be a symmetric conditionally negative semidefinite matrix with nonnegative entries. Then*:

(1) *for any $t \le 1$, $t^{\boldsymbol{a}}$ is positive semidefinite*;
(2) *for any $t \ge 1$, $t^{-\boldsymbol{a}}$ is positive semidefinite*;
(3) *for any $t \ge 0$, $\left(\frac{1}{1+\boldsymbol{a}}\right)^t$ is positive semidefinite.*

Proof Assertions (1) and (2) are a consequence of Theorem 2.2 in Chapter 3 of Berg et al. (1984). Assertion (3) holds for $t = 0$; for positive t, one has

$$\left(\frac{1}{1+\boldsymbol{a}}\right)^t = \left(\frac{1}{(1+\boldsymbol{a})^{\frac{t}{\lceil t\rceil}}}\right)^{\lceil t\rceil}.$$

The result follows from the application of Corollary 2.10 and Exercise 2.21 in Chapter 3 of Berg et al. (1984) and of the Schur product theorem. □

Appendix B: Proofs

Proof of Proposition 1 We start by proving the first set of conditions. Under conditions (*1.b*) and (*1.c*), for any $r > 0$, $[B_{ij}(r)]_{i,j=1}^p$ is the sum of a nonnegative diagonal matrix and a *min* matrix with nonnegative entries, hence it is positive semidefinite (Horn and Johnson 2013, problem 7.1.P18). So is $[\rho_{ij}\,\kappa_{ij,d}]_{i,j=1}^p$ due to conditions (*1.a*) and (*1.c*). The positive semidefiniteness of $\boldsymbol{A}(r)$ then stems from the Schur product theorem.

To prove the sufficiency of the second set of conditions, let us use Newton's generalized binomial theorem (Olver et al. 2010, formula 4.6.7) to rewrite $A_{ij}(r)$ as

$$\begin{aligned}
A_{ij}(r) &= \sum_{k=0}^{\infty} \rho_{ij}\,\kappa_{ij,d}\,\mathbb{I}_{(b_{ij},\infty)}(r^{-1})\,(\gamma_{ij})_{\underline{k}}\,\frac{(-b_{ij}^2\,r^2)^k}{k!}\\
&= \sum_{k=0}^{\infty} \rho_{ij}\,\kappa_{ij,d}\,\mathbb{I}_{(b_{ij},\infty)}(r^{-1})(\gamma_{ij})_{\underline{2k}}\,(b_{ij}^2\,r^2)^{2k}\left[\frac{1}{(2k)!}+\frac{2k-\gamma_{ij}}{(2k+1)!}b_{ij}^2\,r^2\right]\\
&= \sum_{k=0}^{\infty} \rho_{ij}\,\kappa_{ij,d}\,\mathbb{I}_{(b_{ij},\infty)}(r^{-1})\prod_{q=0}^{2k-1}\left(q-\gamma_{ij}\right)(b_{ij}^2\,r^2)^{2k}\left[\frac{1}{(2k)!}+\frac{2k-\gamma_{ij}}{(2k+1)!}b_{ij}^2\,r^2\right].
\end{aligned}$$

Under conditions (*2.a*), (*2.b*) and (*2.c*), the claim follows from the fact that the set of symmetric positive semidefinite matrices is closed under Schur products, sums and limits.

The third set of conditions is obtained by following the reasoning of Daley et al. (2015, Lemma 2); it ensures that, for all $r > 0$, $\boldsymbol{A}(r)$ is a diagonally dominant matrix with nonnegative diagonal entries, hence positive semidefinite (Horn and Johnson 2013, Theorem 6.1.10). □

Proof of Proposition 2 We invoke formula 6.567.1 in Gradshteyn and Ryzhik (2007) to write $\Omega_{\nu_{ij}}$ as

$$\Omega_{\nu_{ij}}(x) = \kappa_{ij,d}\int_0^1 \Omega_d(rx)r^{d-1}\big(1-r^2\big)^{\gamma_{ij}}\,\mathrm{d}r,$$

with $\kappa_{ij,d}$ defined through (8). Hence, we can write λ_{ij} in (11) as

$$\begin{aligned}
\lambda_{ij}(x) &= \rho_{ij}\frac{\kappa_{ij,d}}{b_{ij}^d}\int_0^1 \Omega_d\Big(r\frac{x}{b_{ij}}\Big)r^{d-1}\big(1-r^2\big)^{\gamma_{ij}}\,\mathrm{d}r\\
&= \rho_{ij}\,\kappa_{ij,d}\int_0^{b_{ij}^{-1}} \Omega_d\big(vx\big)v^{d-1}\big(1-b_{ij}^2v^2\big)^{\gamma_{ij}}\,\mathrm{d}v\\
&= \rho_{ij}\,\kappa_{ij,d}\int_0^{\infty} \Omega_d\big(vx\big)v^{d-1}\big(1-b_{ij}^2v^2\big)_+^{\gamma_{ij}}\,\mathrm{d}v\\
&= \int_0^{\infty} \Omega_d\big(vx\big)v^{d-1}A_{ij}(v)\mathrm{d}v. \qquad (22)
\end{aligned}$$

We note that $x \mapsto \Omega_d(vx)$ belongs to Φ_d for any $v > 0$. Also, by assumption, $\boldsymbol{A}(v)$ is symmetric positive semidefinite for each $v > 0$. Hence, we can invoke Theorem 1 in Porcu and Zastavnyi (2011) to claim that the matrix-valued function

$$x \mapsto \Omega_d(vx)\boldsymbol{A}(v), \quad x \geq 0,$$

belongs to Φ_d^p for all $v > 0$. The proof is completed by invoking again Theorem 1 in Porcu and Zastavnyi (2011) in concert with the fact that the integral above is well-defined because $0 \leq x \mapsto |\Omega_d(x)|$ is uniformly bounded by 1 and the function A_{ij} is compactly supported, strictly decreasing, and bounded at zero, which ensures integrability. □

Proof of Proposition 3 We provide a constructive proof. First, note that the integral in (12) is well-defined because $v \mapsto A_{ij}(v)$ is compactly supported and the integrand is a continuous function of v. By assumption φ is a member of the class Φ_d. Hence, we can invoke Schoenberg's theorem (Schoenberg 1938) to claim that φ admits a uniquely determined expansion of the type

$$\varphi(x) = \int_0^\infty \Omega_d(rx)F_d(\mathrm{d}r), \quad x \geq 0, \tag{23}$$

where F_d is a d-Schoenberg measure on $[0, \infty)$. Hence, we have $\boldsymbol{\psi} = [\psi_{ij}]_{i,j=1}^p$ with

$$\begin{aligned}
\psi_{ij}(x) &= \int_0^\infty \varphi(vx)v^{d-1}A_{ij}(v)\mathrm{d}v \\
&= \int_0^\infty \int_0^\infty \Omega_d(rvx)F_d(\mathrm{d}r)v^{d-1}A_{ij}(v)\mathrm{d}v \\
&= \int_0^\infty \left(\int_0^\infty \Omega_d(rvx)v^{d-1}A_{ij}(v)\mathrm{d}v\right) F_d(\mathrm{d}r) \\
&= \int_0^\infty \lambda_{ij}(rx)F_d(\mathrm{d}r),
\end{aligned} \tag{24}$$

which completes the proof because the class Φ_d^p is closed in the topology of finite measures, so that scale mixtures provide elements within the same class. The interchange of the integrals in (24) is justified by Fubini's theorem, insofar as $|\Omega_d|$ is uniformly bounded by 1 and A_{ij} is continuous and compactly supported, so that

$$\int_0^\infty \int_0^\infty |\Omega_d(rvx)v^{d-1}A_{ij}(v)|F_d(\mathrm{d}r)\mathrm{d}v \leq \int_0^\infty F_d(\mathrm{d}r) \int_0^{b_{ij}^{-1}} v^{d-1}A_{ij}(v)\mathrm{d}v < \infty.$$

□

Proof of Proposition 4 We start by evaluating the integral

$$\begin{aligned}
\int_0^x u\Upsilon_d(\varphi)(u)\mathrm{d}u &= \int_0^x u \int_0^\infty \left(\int_0^\infty \Omega_d(vur)v^{d-1}\boldsymbol{A}(v)\mathrm{d}v\right) F_d(\mathrm{d}r)\mathrm{d}u \\
&= \int_0^\infty v^{d-1}\boldsymbol{A}(v) \int_0^\infty F_d(\mathrm{d}r) \int_0^x u\,\Omega_d(vur)\mathrm{d}u\,\mathrm{d}v \\
&= \int_0^\infty v^{d-3}\boldsymbol{A}(v) \int_0^\infty \frac{d-2}{r^2}\Big(\Omega_{d-2}(0) - \Omega_{d-2}(vxr)\Big)F_d(\mathrm{d}r)\mathrm{d}v,
\end{aligned}$$

where the last identity comes from Equation (2.3) in Daley and Porcu (2014). The last inner integrand is everywhere positive for $x > 0$. When $F_{d-2}(\mathrm{d}r) = r^{-2}F_d(\mathrm{d}r)$ is a finite measure on $\mathbb{R}_+$, we can bound the absolute value of the difference in the last inner integral by 2 and use dominated convergence to justify taking the limit for $x \to \infty$ there. This proves that $\int_0^\infty u \Upsilon_d(\varphi)(u) \mathrm{d}u$ is well-defined. Also, we notice that F_{d-2} is the $(d-2)$-Schoenberg measure associated with the montée of order 2 (sensu Matheron 1965) of φ, say $\hat{\varphi}$. We can now use the previous chain of equalities to write

$$\begin{aligned} \mathcal{I}(\Upsilon_d(\varphi))(x) &= \int_0^\infty v^{d-3} \boldsymbol{A}(v) \int_0^\infty \Omega_{d-2}(vxr) F_{d-2}(\mathrm{d}r) \mathrm{d}v \\ &= \Upsilon_{d-2}(\hat{\varphi}(x)), \end{aligned} \tag{25}$$

which provides an element of Φ_{d-2}^p thanks to Proposition 3. □

Proof of Proposition 5 The proof can be made by recursivity on account of (14), (15), (16) and Lemmas 1 and 2 in Appendix A, based on the fact that a mixture of functions belonging to Φ_d^p weighted by positive semidefinite matrices still belongs to Φ_d^p, insofar as Φ_d^p is closed under Schur products, sums and limits. □

Proof of Proposition 6 The proposition results from Proposition 2 and the fact that Φ_d^p is closed under Schur products, sums and limits. □

Proof of Proposition 7 Let ν be a positive integer. From the spectral representation of the Matérn covariance in $\mathbb{R}^\nu$ (Lantuéjoul 2002; Arroyo and Emery 2021), one has

$$\mathcal{M}(x; b, \mu) = \frac{2}{B(\mu, \frac{\nu}{2})} \int_0^\infty \Omega_\nu \left(\frac{rx}{b}\right) \frac{r^{\nu-1}}{(1+r^2)^{\mu+\frac{\nu}{2}}} \mathrm{d}r, \quad x \geq 0,$$

a formula that is actually valid for any $\nu > 0$ (not necessarily an integer) and $\mu > 0$ (Erdélyi 1954, formula 8.5.20). Accordingly, for $\boldsymbol{\mu}$ and $\boldsymbol{\nu} - d$ with positive entries:

$$\mathcal{M}(x; \boldsymbol{b}, \boldsymbol{\mu}, \boldsymbol{\sigma}) = 2 \frac{\boldsymbol{\sigma}}{B(\boldsymbol{\mu}, \frac{\boldsymbol{\nu}}{2})} \int_0^\infty \Omega_{\boldsymbol{\nu}} \left(\frac{rx}{\boldsymbol{b}}\right) \frac{r^{\boldsymbol{\nu}-1}}{(1+r^2)^{\boldsymbol{\mu}+\frac{\boldsymbol{\nu}}{2}}} \mathrm{d}r, \quad x \geq 0.$$

Under condition (*3*), $x \mapsto \frac{\boldsymbol{\rho}}{\boldsymbol{b}^d} \Omega_{\boldsymbol{\nu}} \left(\frac{rx}{\boldsymbol{b}}\right)$ belongs to Φ_d^p (Proposition 2). Furthermore, owing to Lemma 3 in Appendix A, $(r^2/(1+r^2))^{\frac{\boldsymbol{\nu}}{2}}$ and $(1+r^2)^{-\boldsymbol{\mu}}$ are positive semidefinite for all $r \geq 0$ under conditions (*1*) and (*2*). The claim follows from Proposition 6. □

Proof of Proposition 8 Let ν be a positive integer less than 2α. From the spectral representation of the Gauss hypergeometric covariance in $\mathbb{R}^\nu$ (Emery and Alegría 2022), one has

$$\begin{aligned} \mathcal{H}(x; b, \alpha, \beta, \gamma, \nu) = &\frac{2^{1-\nu}\Gamma(\alpha)\Gamma(\beta - \frac{\nu}{2})\Gamma(\gamma - \frac{\nu}{2})}{\Gamma(\frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})\Gamma(\beta)\Gamma(\gamma)} \\ &\times \int_0^\infty \Omega_\nu \left(\frac{rx}{b}\right) r^{\nu-1} {}_1F_2 \left(\alpha; \beta, \gamma; -\frac{r^2}{4}\right) \mathrm{d}r, \quad x \geq 0, \end{aligned}$$

a formula that is actually valid for any real value (not necessarily an integer) $\nu \in (0, 2\alpha)$ owing to formulae 3–10 in Emery and Alegría (2022).

We now prove (*1*). Under the assumption that $\boldsymbol{\nu}$ has entries in $(0, 2\alpha)$, one can write:

$$\mathcal{H}(x; \boldsymbol{b}, \alpha, \beta, \gamma, \boldsymbol{\nu}, \boldsymbol{\sigma}) = \frac{2\Gamma(\alpha)\boldsymbol{\rho}}{\Gamma(\beta)\,\Gamma(\gamma)\boldsymbol{b}^d} \int_0^\infty \Omega_{\boldsymbol{\nu}} \left(\frac{rx}{\boldsymbol{b}}\right) r^{\boldsymbol{\nu}-1} {}_1F_2 \left(\alpha; \beta, \gamma; -\frac{r^2}{4}\right) \mathrm{d}r,$$

with $\boldsymbol{\rho}$ defined as in (19). The claim then follows from Proposition 6, Lemma 2 and from the fact that, under the specified conditions on (α, β, γ), the mapping $r \mapsto {}_1F_2(\alpha; \beta, \gamma; -\frac{r^2}{4})$ is nonnegative on $[0, \infty)$ (Cho et al. 2020).

Concerning (*2*), one has (Emery and Alegría 2022, Equation 25)

$$
{}_1F_2\left(\alpha; \boldsymbol{\beta}, \boldsymbol{\gamma}; -\frac{r^2}{4}\right) = \frac{\Gamma(\boldsymbol{\beta})\Gamma(\boldsymbol{\gamma})}{\Gamma(\beta)\Gamma(\boldsymbol{\beta}-\beta)\Gamma(\gamma)\Gamma(\boldsymbol{\gamma}-\gamma)}
$$
$$
\times \int_0^1 \int_0^1 {}_1F_2\left(\alpha; \beta, \gamma; -t_1 t_2 \frac{r^2}{4}\right) t_1^{\beta-1}(1-t_1)^{\boldsymbol{\beta}-\beta-1} t_2^{\gamma-1}(1-t_2)^{\boldsymbol{\gamma}-\gamma-1} \mathrm{d}t_1 \mathrm{d}t_2,
$$

with the integrand being a positive semidefinite matrix for any $t_1, t_2 \in [0, 1]$ and $r \in [0, \infty)$ under conditions (a), (b) and (c) (Lemma 3). The claim follows from Proposition 6 and Lemma 2, which apply under conditions (d) and (e). □

Proof of Proposition 9 For $b > 0$ and $\nu > \mu > 0$, one has (Gradshteyn and Ryzhik 2007, formula 6.576.7)

$$
\mathcal{C}(x; b, \mu) = \frac{2}{\Gamma(\mu)\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty \Omega_\nu\left(\frac{rx}{b}\right)\left(\frac{r}{2}\right)^{\mu+\frac{\nu}{2}-1} K_{\mu-\frac{\nu}{2}}(r)\mathrm{d}r, \quad x \geq 0.
$$

Accordingly, under the assumption that $\boldsymbol{b}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}-\boldsymbol{\mu}$ have positive entries, one has:

$$
\mathcal{C}(x; \boldsymbol{b}, \boldsymbol{\mu}, \boldsymbol{\sigma}) = \frac{2\boldsymbol{\sigma}}{\Gamma(\boldsymbol{\mu})\Gamma\left(\frac{\boldsymbol{\nu}}{2}\right)} \int_0^\infty \Omega_{\boldsymbol{\nu}}\left(\frac{rx}{\boldsymbol{b}}\right)\left(\frac{r}{2}\right)^{\boldsymbol{\mu}+\frac{\boldsymbol{\nu}}{2}-1} K_{\boldsymbol{\mu}-\frac{\boldsymbol{\nu}}{2}}(r)\mathrm{d}r, \quad x \geq 0,
$$

where (Gradshteyn and Ryzhik 2007, formula 3.471.9)

$$
2\left(\frac{r}{2}\right)^{\boldsymbol{\mu}+\frac{\boldsymbol{\nu}}{2}-1} K_{\boldsymbol{\mu}-\frac{\boldsymbol{\nu}}{2}}(r) = \left(\frac{r}{2}\right)^{\boldsymbol{\nu}-1} \int_0^\infty \exp\left(-\frac{r^2}{4t}\right)\exp(-t)t^{\boldsymbol{\mu}-\frac{\boldsymbol{\nu}}{2}-1}\mathrm{d}t
$$

is, under condition (*1*), positive semidefinite for all $r \in [0, \infty)$ owing to Lemma 2 and the fact that positive semidefinite matrices are closed under Schur products, sums and limits. The claim follows from Proposition 6, considering condition (*2*) and the fact that, from condition (*1*), the entries of $\boldsymbol{\nu}$ are greater than d. □

References

Allard D, Clarotto L, Emery X (2022) Fully nonseparable Gneiting covariance functions for multivariate space-time data. Spat Stat 52:100706

Alonso-Malaver C, Porcu E, Giraldo R (2015) Multivariate and multiradial Schoenberg measures with their dimension walks. J Multivar Anal 133:251–265

Apanasovich TV, Genton MG, Sun Y (2012) A valid Matérn class of cross-covariance functions for multivariate random fields with any number of components. J Am Stat Assoc 107(497):180–193

Arroyo D, Emery X (2021) Algorithm 1013: an R implementation of a continuous spectral algorithm for simulating vector Gaussian random fields in Euclidean spaces. ACM Trans Math Softw 47(1):8

Berg C, Christensen JPR, Ressel P (1984) Harmonic analysis on semigroups: theory of positive definite and related functions. Springer, New York

Bevilacqua M, Hering AS, Porcu E (2015) On the flexibility of multivariate covariance models: comment on the paper by Genton and Kleiber. Stat Sci 30(2):167–169

Bourotte M, Allard D, Porcu E (2016) A flexible class of non-separable cross-covariance functions for multivariate space-time data. Spat Stat 18:125–146

Chilès J-P, Delfiner P (2012) Geostatistics: modeling spatial uncertainty, 2nd edn. Wiley, New York

Cho Y-K, Chung S-Y, Yun H (2020) Rational extension of the Newton diagram for the positivity of ${}_1F_2$ hypergeometric functions and Askey–Szegö problem. Constr Approx 51(1):49–72

Daley D, Porcu E (2014) Dimension walks and Schoenberg spectral measures. Proc Am Math Soc 142(5):1813–1824

Daley DJ, Porcu E, Bevilacqua M (2015) Classes of compactly supported covariance functions for multivariate random fields. Stoch Environ Res Risk Assess 29(4):1249–1263

Du J, Leonenko N, Ma C, Shu H (2012) Hyperbolic vector random fields with hyperbolic direct and cross covariance functions. Stoch Anal Appl 30(4):662–674

Emery X, Alegría A (2022) The Gauss hypergeometric covariance kernel for modeling second-order stationary random fields in Euclidean spaces: its compact support, properties and spectral representation. Stoch Environ Res Risk Assess 36:2819–2834

Emery X, Porcu E, White P (2022) New validity conditions for the multivariate Matérn coregionalization model, with an application to exploration geochemistry. Math Geosci 54(6):1043–1068

Erdélyi A (1954) Tables of integral transforms, vol II. McGraw-Hill, New York

Gaspari G, Cohn SE (1999) Construction of correlation functions in two and three dimensions. Q J R Meteorol Soc 125(554):723–757

Genton MG, Kleiber W (2015) Cross-covariance functions for multivariate geostatistics. Stat Sci 30(2):147–163

Gneiting T, Kleiber W, Schlather M (2010) Matérn cross-covariance functions for multivariate random fields. J Am Stat Assoc 105:1167–1177

Gradshteyn I, Ryzhik I (2007) Table of integrals, series, and products, 7th edn. Academic Press, Amsterdam

Horn RA, Johnson CR (2013) Matrix analysis, 2nd edn. Cambridge University Press, Cambridge

Lantuéjoul C (2002) Geostatistical simulation: models and algorithms. Springer, Berlin

Matheron G (1965) Les Variables Régionalisées et Leur Estimation. Masson, Paris

Olver FW, Lozier DM, Boisvert RF, Clark CW (2010) NIST handbook of mathematical functions. Cambridge University Press, Cambridge

Porcu E, Zastavnyi V (2011) Characterization theorems for some classes of covariance functions associated to vector valued random fields. J Multivar Anal 102(9):1293–1301

Porcu E, Bevilacqua M, Hering AS (2018) The Shkarofsky-Gneiting class of covariance models for bivariate Gaussian random fields. Stat 7(1):e207

Schoenberg I (1938) Metric spaces and completely monotone functions. Ann Math 39(4):811–841