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Cubical fuzzy Hamacher aggregation operators in multi-attribute decision-making problems

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Abstract

Cubical fuzzy sets, a novel structure, deal with the fuzziness of information more effectively than picture fuzzy sets and spherical fuzzy sets. Each member of a CFS is an ordered quadruple consisting of an element of the universe of discourse and three numbers in the unit interval called, respectively, membership grade, neutral membership grade, and non-membership grade, such that their cubic sum is bounded by one. CFSs, being an extension of PFSs and CFSs by enlarging the space of membership grades, give decision-makers more leeway in assigning values. In this paper, we first integrate the notion of Hamacher T-norm and T-conorm with the structure of CFSs to devise the novel cubical fuzzy Hamacher operations and discuss their essential properties. Then, by employing the idea of cubical fuzzy Hamacher operations, we introduce novel aggregation operators called cubical fuzzy Hamacher aggregation operators. The concepts of the cubical fuzzy Hamacher weighted average operator, the cubical fuzzy Hamacher ordered weighted average operator, and the cubical fuzzy Hamacher hybrid average weighted operator are presented and discussed thoroughly in the first section of this work. The cubical fuzzy Hamacher weighted geometric operator, the cubical fuzzy Hamacher ordered weighted geometric operator, and the cubical fuzzy Hamacher hybrid geometric oper-

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ator are introduced in the second portion, and its essential features are explored. The proposed operators are then used to create some strategies for solving cubical fuzzy information-based multiple attribute decision-making issues. Finally, to demonstrate the applicability and efficiency of the proposed methodology, a real-world example of cyclone disaster appraisal is provided.

Keywords Multiple attribute decision-making (MADM) · Cubical fuzzy Hamacher weighted average (CFHWA) operator · Cubical fuzzy Hamacher ordered weighted average (CFHOWA) operator · Cubical fuzzy Hamacher hybrid weighted average (CFHHWA) operator · Cubical fuzzy Hamacher weighted geometric (CFHWG) operator · Cubical fuzzy Hamacher ordered weighted geometric (CFHOWG) operator · Cubical fuzzy Hamacher hybrid weighted geometric (CFHHWG) operator

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1 Introduction

Over the decades, decision-making problems have grown more complex. Consequently, there has been a lot of focus on creating and putting into practice effective mathematical models to support these type of problems. In recent years, scholars have paid increasing attention to the science of decision-making. Decision-making is the process in which an individual, team, or organization determines what possible steps to take in light of a set of goals and resource constraints. This iterative approach will include problem formulation, information gathering, drawing conclusions, and gaining understanding. Multiple attribute decision-making (MADM) is the branch of decision science in which the decision-makers evaluate a finite number of alternatives subject to a finite number of attributes and thus find the best alternative(s). MADM approaches are widely used in a variety of domains, including operations research, economics, and engineering. In the study of MADM problems, the aggregation operators provide a broad range of analysis. Many researchers have contributed to MADM by devising noval techniques based on aggregation operators (AOs) defined on various extensions of the fuzzy set (FS). The field of AOs is highly applicable and is gaining more and more attention from researchers. The concept of T-norms (TNs) and T-conorms (TCNs), as well as their generalizations, underpins these AOs. Algebraic, Einstein, Hamacher, Archimedean, and Dombi, to name a few, have been suggested as generalizations of the ordinary TNs and TCNs. Hamacher TN and TCN are comprehensive and powerful generalizations of the algebraic and Einstein TN and TCN. The study of Hamacher operations-based aggregation operators and their implementation to MADM scenarios is crucially significant. Wu and Wei (2017) developed various Pythagorean fuzzy (PyF) arithmetic and geometric AOs and offered techniques to tackle PyF MADM issues using Hamacher operations. By the consolidation of the conceptions of Hamacher operations and attribute prioritization, Gao (2018) suggested some PyF Hamacher Prioritized AOs. Hadi et al. (2021) constructed Fermatean fuzzy Hamacher arithmetic and geometric AOs by applying Hamacher operations to Fermatean fuzzy numbers. By developing bipolar fuzzy Hamacher arithmetic and geometric AOs, Wei et al. (2018) probed the MADM problems. For the MADM problems, Zhou et al. and Wei (2018) studied the picture fuzzy set (PFS) under Hamacher operations and developed various Hamacher AOs for picture fuzzy environment. Jana and Pal (2019) developed MADM techniques for enterprise performance evaluation based on Hamacher operations defined on picture fuzzy numbers. In Jana et al. (2019a) Jana et al. developed noval approaches to address decision-making issues using bipolar fuzzy soft weighted AOs. Jana and Pal (2021) also proposed various dynamic intuitionistic fuzzy weighted aggregation operators for interval uncertainty and applied them to the dynamic hybrid multi-attribute decision-making process. Based on various intuitionistic fuzzy Dombi hybrid aggregation operators, Jana et al. (2021) proposed the MADM approach for the assessment of the enterprise financial performance. A multiple attribute border approximation area comparison approach was used by Jana (2021) to address multiple attribute group decision-making using bipolar fuzzy numbers. Khan et al. (2022a) developed a novel multi-attribute group decision-making approach with linguistic Pythagorean fuzzy numbers. Guner et al. (2022) presented a new approach based on Hamacher AOs for the spherical fuzzy data. To establish some novel Pythagorean fuzzy power Dombi operators, Jana et al. (2022) integrated Dombi operations with PA operations to investigate MADM problems.

Recently, another structure called cubical fuzzy set (CFS) has been developed in Khan et al. (2022b), which is a generalization of picture fuzzy set (PFS) and spherical fuzzy set (SFS). In some situations where neither PFS nor SFS frameworks can be employed, a cubical fuzzy model can be used. In CFS, three real values, namely the membership degree, the neutral membership degree, and the negative membership degree, is used to represent an object. Similar to SFS, the range of membership grades of the cubical fuzzy set is also [0, 1]. But in CFSs, the cubic sum of membership grades is bounded by 0 and 1. Hence a cubical fuzzy model has more potential for application in situations where ambiguous information is involved. CFSs are helpful in circumstances where a person's opinion is not just yes or no, but additionally includes abstinence or refusal. A good example of an CFS could be in decision-making when four decision-makers have four distinct kinds of judgments about a candidate. Another case is of voting, where there are four categories of voters: those who vote in favor, those who vote against, those who avoid to vote or abstain. In SFSs and PFSs, due to the constraints on the membership grades, the decision-makers are still restricted to a specific domain in assigning values to the membership grades. For instance, if $\mu(x) = 0.8$, $\eta(x) = 0.5$, and $\nu(x) = 0.6$, then $\mu(x)+\eta(x)+\nu(x) = 1.9 \nleq 1$, which clearly does not fulfill the PFS requirement. Furthermore, we have $(0.8)^2 + (0.5)^2 + (0.6)^2 = 1.25 \nleq 1$, indicating that the SFS criterion is not met. However, if we take $(0.8)^3 + (0.5)^3 + (0.6)^3 = 0.853 < 1$, which is an adequate rationale to build another type of fuzzy set with greater capacity to capture uncertainty, so we defined a cubical fuzzy set.

The flexibility parameter in Hamacher t-norms allows for more exact decision-making outcomes. On the other hand, CFSs provide a broader range of membership grades. The purpose of this article is to combine the essence of these two ideas in order to construct novel aggregation operators and hence develop decision-making algorithms that will contribute to the field of decision-making science. The rest of the article is organized as follows.

In Sect. 2, a brief summary of the essential ideas behind IFSs, PFSs, SFSs, and CFSs is given, along with an overview of the CFSs' core operations. In Sect. 3, we have proposed cubical fuzzy Hamacher weighted average (CFHWA) operator, cubical fuzzy Hamacher ordered weighted average (CFHOWA) operator, cubical fuzzy Hamacher hybrid average (CFHHA) operator, cubical fuzzy Hamacher weighted geometric (CFHWG) operator, cubical fuzzy Hamacher ordered weighted geometric (CFHOWG) operator, and cubical fuzzy Hamacher hybrid geometric (CFHHG) operator. We also analyze the features and special cases of these operators. We use these operators in Sect. 4 to create some methods for resolving the cubical fuzzy multiple attribute decision-making issues. A cyclone disaster evaluation example is provided in Sect. 5. Section 6 wraps up the article.

2 Preliminaries

In this section, we go over some of the key ideas around PFS and SFS, as well as their operations and properties. In addition, we offer ideas that are employed later on in this article. In this section, $\mathfrak{D}$ and $\mathbb{I}$ denote a universe of discourse and the interval [0, 1] respectively.

Definition 1 (Atanassov 1986) A set of ordered triplets

$$\mathfrak{F} = \{\langle \mathrm{t}, \xi(\mathrm{t}), \partial(\mathrm{t})\rangle \,|\mathrm{t} \in \mathfrak{D}\}$$

is defined to be an intuitionistic fuzzy set (IFS) on $\mathfrak{D}$ where ξ and ∂ are functions from $\mathfrak{D}$ to $\mathbb{I}$ such that for every $\mathrm{t} \in \mathfrak{D}$, the images $\xi(\mathrm{t})$ and $\partial(\mathrm{t})$ denote the membership, and the non-membership degrees of t to $\mathfrak{F}$, respectively, and meet the condition: $\xi(\mathrm{t}) + \partial(\mathrm{t}) \leq 1$ for all $\mathrm{t} \in \mathfrak{D}$. The degree of refusal $\pi(\mathrm{t})$ for each $\mathrm{t} \in \mathfrak{D}$ to $\mathfrak{F}$ is defined as

$$\pi(\mathrm{t}) = 1 - (\xi(\mathrm{t}) + \partial(\mathrm{t}))$$

Definition 2 A set of ordered quadruples

$$\mathfrak{F} = \{\langle \mathrm{t}, \xi(\mathrm{t}), \theta(\mathrm{t}), \partial(\mathrm{t})\rangle \,|\mathrm{t} \in \mathfrak{D}\}$$

is called a picture fuzzy set (PFS) on $\mathfrak{D}$ where ξ, θ and ∂ are functions from $\mathfrak{D}$ to $\mathbb{I}$ such that for every $\mathrm{t} \in \mathfrak{D}$, the images $\xi(\mathrm{t})$, $\theta(\mathrm{t})$ and $\partial(\mathrm{t})$ denote the positive membership, the neutral membership and the negative membership degrees of t to $\mathfrak{F}$, respectively, and meet the condition: $\xi(\mathrm{t}) + \theta(\mathrm{t}) + \partial(\mathrm{t}) \leq 1$. The degree of refusal $\pi(\mathrm{t})$ for each $\mathrm{t} \in \mathfrak{D}$ to $\mathfrak{F}$ is defined as

$$\pi(\mathrm{t}) = 1 - (\xi(\mathrm{t}) + \theta(\mathrm{t}) + \partial(\mathrm{t}))$$

Definition 3 (Ashraf et al. 2019) A set of ordered quadruples

$$\mathfrak{F} = \{\langle \mathrm{t}, \xi(\mathrm{t}), \theta(\mathrm{t}), \partial(\mathrm{t})\rangle \,|\mathrm{t} \in \mathfrak{D}\}$$

is called a spherical fuzzy set (SFS) on $\mathfrak{D}$ where ξ, θ and ∂ are functions from $\mathfrak{D}$ to $\mathbb{I}$ such that for every $\mathrm{t} \in \mathfrak{D}$, the images $\xi(\mathrm{t})$, $\theta(\mathrm{t})$ and $\partial(\mathrm{t})$ denote the positive membership, the neutral membership and the negative membership degrees of t to $\mathfrak{F}$, respectively, and meet the condition: $(\xi(\mathrm{t}))^2 + (\theta(\mathrm{t}))^2 + (\partial(\mathrm{t}))^2 \leq 1$. The degree of refusal $\pi(\mathrm{t})$ for each $\mathrm{t} \in \mathfrak{D}$ to $\mathfrak{F}$ is defined as

$$\pi(\mathrm{t}) = \sqrt{1 - \left((\xi(\mathrm{t}))^2 + (\theta(\mathrm{t}))^2 + (\partial(\mathrm{t}))^2\right)}$$

Ashraf et al. (2019), also proposed the following operations of SFSs.

Definition 4 (Ashraf et al. 2019) The intersection, union, complement, inclusion, and equality of spherical fuzzy sets is given as

(i) $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ if $\xi_1(\mathrm{t}) \leq \xi_2(\mathrm{t})$, $\theta_1(\mathrm{t}) \leq \theta_2(\mathrm{t})$ and $\partial_1(\mathrm{t}) \geq \partial_2(\mathrm{t})$, $\forall$ $\mathrm{t} \in \mathfrak{D}$;
(ii) $\mathfrak{F}_1 = \mathfrak{F}_2$ if $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ and $\mathfrak{F}_2 \subseteq \mathfrak{F}_1$;
(iii) $\mathfrak{F}_1^c = \{(\mathrm{t}, (\partial_1(\mathrm{t}), \theta_1(\mathrm{t}), \xi_1(\mathrm{t})))\,|\mathrm{t} \in \mathfrak{D}\}$;
(iv) $\mathfrak{F}_1 \cap \mathfrak{F}_2 = \{(\mathrm{t}, \min\{\xi_1(\mathrm{t}), \xi_2(\mathrm{t})\}, \min\{\theta_1(\mathrm{t}), \theta_2(\mathrm{t})\}, \max\{\partial_1(\mathrm{t}), \partial_2(\mathrm{t})\})\,|\mathrm{t} \in \mathfrak{D}\}$
(v) $\mathfrak{F}_1 \cup \mathfrak{F}_2 = \{(\mathrm{t}, \max\{\xi_1(\mathrm{t}), \xi_2(\mathrm{t})\}, \min\{\theta_1(\mathrm{t}), \theta_2(\mathrm{t})\}, \min\{\partial_1(\mathrm{t}), \partial_2(\mathrm{t})\})\,|\mathrm{t} \in \mathfrak{D}\}$

3 Cubical fuzzy set

This section comprehensively explores the notion of a cubical fuzzy set (CFS) and its properties. For the purpose of comparing CFEs, the scoring function and accuracy degree are also provided.

Definition 5 (Khan et al. 2022b) A set of ordered quadruples

$$\mathfrak{F} = \{\langle \mathrm{t}, \xi(\mathrm{t}), \theta(\mathrm{t}), \partial(\mathrm{t})\rangle \,|\mathrm{t} \in \mathfrak{D}\}$$

is called a cubical fuzzy set (CFS) on $\mathfrak{D}$ where ξ, θ and ∂ are functions from $\mathfrak{D}$ to $\mathbb{I}$ such that for every $\mathrm{t} \in \mathfrak{D}$, the images $\xi(\mathrm{t})$, $\theta(\mathrm{t})$ and $\partial(\mathrm{t})$ denote the membership, the neutral membership and the non-membership degrees of t to $\mathfrak{F}$, respectively, and meet the condition: $(\xi(\mathrm{t}))^3 + (\theta(\mathrm{t}))^3 + (\partial(\mathrm{t}))^3 \leq 1$. The refusal membership degree $\pi(\mathrm{t})$ for each $\mathrm{t} \in \mathfrak{D}$ to $\mathfrak{F}$ is defined as

$$\pi(\mathrm{t}) = \sqrt[3]{1 - \left((\xi(\mathrm{t}))^3 + (\theta(\mathrm{t}))^3 + (\partial(\mathrm{t}))^3\right)}$$

For a fixed $\mathrm{t} \in \mathfrak{D}$, the ordered triplet $(\xi\,(\mathrm{t})\,, \theta\,(\mathrm{t})\,, \partial\,(\mathrm{t}))$ is called a cubical fuzzy element abbreviated as CFE. For simplicity, we write $(\xi\,(\mathrm{t})\,, \theta\,(\mathrm{t})\,, \partial\,(\mathrm{t}))$ as $\breve{e} = (\xi, \theta, \partial)$.

Definition 6 (Khan et al. 2022b) Let $\breve{e} = (\xi, \theta, \partial)$, $\breve{e}_1 = \left(\xi_1, \theta_1, \partial_1\right)$, and $\breve{e}_2 = \left(\xi_2, \theta_2, \partial_2\right)$, be any three CFEs, then we have the following set operations for CFEs:

(i) $\breve{e}_1 \subseteq \breve{e}_2$ iff $\xi_1 \leq \xi_2$, $\theta_1 \leq \theta_2$ and $\partial_1 \geq \partial_2$;
(i) $\breve{e}_1 = \breve{e}_2$ iff $\breve{e}_1 \subseteq \breve{e}_2$ and $\breve{e}_2 \subseteq \breve{e}_1$;
(ii) $\breve{e}_1 \cup \breve{e}_2 = \left(\max\left\{\xi_1, \xi_2\right\}, \min\left\{\theta_1, \theta_2\right\}, \min\left\{\partial_1, \partial_2\right\}\right)$;
(iii) $\breve{e}_1 \cap \breve{e}_2 = \left(\min\left\{\xi_1, \xi_2\right\}, \min\left\{\theta_1, \theta_2\right\}, \max\left\{\partial_1, \partial_2\right\}\right)$;
(iv) $\breve{e}^c = (\partial, \theta, \xi)$.

Definition 7 Khan et al. (2022b) Let $\breve{e} = (\xi, \theta, \partial)$, $\breve{e}_1 = \left(\xi_1, \theta_1, \partial_1\right)$, and $\breve{e}_2 = \left(\xi_2, \theta_2, \partial_2\right)$, be any three CFEs, and $\kappa > 0$, then the algebraic operations are defined as follows:

(i) $\breve{e}_1 \oplus \breve{e}_2 = \left(\sqrt[3]{\xi_1^3 + \xi_2^3 - \xi_1^3\xi_2^3}, \theta_1.\theta_2, \partial_1.\partial_2\right)$;

(ii) $\breve{e}_1 \otimes \breve{e}_2 = \left(\xi_1.\xi_2, \theta_1.\theta_2, \sqrt[3]{\partial_1^3 + \partial_2^3 - \partial_1^3.\partial_2^3}\right)$;

(iii) $\kappa\,(\breve{e}) = \left(\sqrt[3]{1 - \left(1 - \xi^3\right)^\kappa}, \theta^\kappa, \partial^\kappa\right)$;

(iv) $(\breve{e})^\kappa = \left(\xi^\kappa, \theta^\kappa, \sqrt[3]{1 - \left(1 - \partial^3\right)^\kappa}\right)$.

Theorem 8 (Khan et al. 2022b) *For three CFEs* $\breve{e}$, $\breve{e}_1$, *and* $\breve{e}_2$, *we have*

(i) $\breve{e}_1 \otimes \breve{e}_2 = \breve{e}_2 \otimes \breve{e}_1$;
(ii) $\breve{e}_1 \oplus \breve{e}_2 = \breve{e}_2 \oplus \breve{e}_1$;
(iii) $(\breve{e}_1 \otimes \breve{e}_2)^\kappa = \breve{e}_1^\kappa \otimes \breve{e}_2^\kappa$, $\kappa > 0$;
(iv) $\breve{e}^{\kappa_1} \otimes \breve{e}^{\kappa_2} = \breve{e}^{\kappa_1+\kappa_2}$, $\kappa_1, \kappa_2 > 0$;
(v) $\kappa\breve{e}_1 \oplus \kappa\breve{e}_2 = \kappa\,(\breve{e}_1 \oplus \breve{e}_2)$, $\kappa > 0$;
(vi) $\kappa_1\breve{e} \oplus \kappa_2\breve{e} = (\kappa_1 \oplus \kappa_2)\,\breve{e}$, $\kappa_1, \kappa_2 > 0$.

In the following, we define the score function of the CFEs in order to rank them.

Definition 9 (Khan et al. 2022b) For any given CFE $\breve{e} = (\xi, \theta, \partial)$ the score function is given as

$$\mathfrak{sc}(\breve{e}) = \xi^3 - \partial^3.$$

In particular $\mathfrak{sc}(\breve{e}) = \begin{cases} 1, \text{ if } \breve{e} = (1, 0, 0) \\ -1 \text{ if } \breve{e} = (0, 0, 1) \end{cases}$.

For the comparison of CFEs, we have the following procedure:

Definition 10 For two CFEs $\breve{e}_1$ and $\breve{e}_2$, we have

(i) If $\mathfrak{sc}(\breve{e}_1) < \mathfrak{sc}(\breve{e}_2)$, then $\breve{e}_1 < \breve{e}_2$;
(ii) If $\mathfrak{sc}(\breve{e}_1) > \mathfrak{sc}(\breve{e}_2)$, then $\breve{e}_1 > \breve{e}_2$.

Two CFEs cannot be ranked if their score values are identical. So the accuracy degree of CFEs is introduced.

Definition 11 Let $\breve{e} = (\xi, \theta, \partial)$, be a CFE, then the accuracy degree of $\breve{e}$ is given as

$$\mathfrak{ac}(\breve{e}) = \xi^3 + \theta^3 + \partial^3.$$

Now, we provide a comprehensive criteria for rating CFEs.

Definition 12 Let $\breve{e}_1$, and $\breve{e}_2$ be any two CFEs then

(I) If $\mathfrak{sc}(\breve{e}_1) < \mathfrak{sc}(\breve{e}_2)$, then $\breve{e}_1 < \breve{e}_2$;
(II) If $\mathfrak{sc}(\breve{e}_1) > \mathfrak{sc}(\breve{e}_2)$, then $\breve{e}_1 > \breve{e}_2$;
(III) If $\mathfrak{sc}(\breve{e}_1) = \mathfrak{sc}(\breve{e}_2)$, then

 (i) If $\mathfrak{ac}(\breve{e}_1) < \mathfrak{ac}(\breve{e}_2)$, then $\breve{e}_1 < \breve{e}_2$;
 (ii) If $\mathfrak{ac}(\breve{e}_1) > \mathfrak{ac}(\breve{e}_2)$, then $\breve{e}_1 > \breve{e}_2$;
 (iii) If $\mathfrak{ac}(\breve{e}_1) = \mathfrak{ac}(\breve{e}_2)$, then $\breve{e}_1 \sim \breve{e}_2$.

3.1 Hamacher operations

As a more general form of the usual triangular norms, Hamacher introduced Hamacher triangular norms. The Hamacher generalization of triangular norms is as follows.

Definition 13 (Hamacher 1975) For any $\varepsilon, \varepsilon^{'} \in R$ then, the Hamacher t-norm $\otimes$ and t-conorm $\oplus$ are given as

$$\begin{aligned} \varepsilon \otimes \varepsilon^{'} &= \frac{\varepsilon\varepsilon^{'}}{\sigma + (1-\sigma)\left(\varepsilon + \varepsilon^{'} - \varepsilon\varepsilon^{'}\right)}, \sigma > 0 \\ \varepsilon \oplus \varepsilon^{'} &= \frac{\varepsilon + \varepsilon^{'} - \varepsilon\varepsilon^{'} - (1-\sigma)\varepsilon\varepsilon^{'}}{1-(1-\sigma)\,\varepsilon\varepsilon^{'}}, \sigma > 0 \end{aligned} \tag{1}$$

For $\sigma = 1$, Eqs. (1) reduces to the algebraic TN and TCN given as

$$\begin{aligned} t\left(\varepsilon, \varepsilon^{'}\right) &= \varepsilon \otimes \varepsilon^{'} = \varepsilon\varepsilon^{'} \\ t^*\left(\varepsilon, \varepsilon^{'}\right) &= \varepsilon \oplus \varepsilon^{'} = \varepsilon + \varepsilon^{'} - \varepsilon\varepsilon^{'} \end{aligned} \tag{2}$$

For $\sigma = 2$, Eqs. (1) reduces to the Einstein TN and TCN given as

$$t\left(\varepsilon, \varepsilon^{'}\right) = \varepsilon \otimes \varepsilon^{'} = \frac{\varepsilon\varepsilon^{'}}{1 + (1-\varepsilon)\left(1-\varepsilon^{'}\right)}$$

$$t^{*}\left(\varepsilon,\varepsilon^{'}\right)=\varepsilon\oplus\varepsilon^{'}=\frac{\varepsilon+\varepsilon^{'}}{1+\varepsilon\varepsilon^{'}} \tag{3}$$

4 Cubical fuzzy Hamacher operations

Now, we provide the Hamacher operations for CFEs, referred to as the CF Hamacher operations, using the Hamacher triangular norms given in Definition 13.

Definition 14 Let $\breve{e}_r = (\xi_r, \theta_r, \partial_r)$ $(r = 1, 2)$ be two CFEs, $\sigma > 0$ and $\kappa > 0$, then, the cubical fuzzy Hamacher operations are given as follows:

$$\text{(i)}\ \breve{e}_1 \oplus \breve{e}_2 = \left(\begin{array}{l}\sqrt[3]{\frac{(\xi_1)^3+(\xi_2)^3-(\xi_1)^3(\xi_2)^3-(1-\sigma)(\xi_1)^3(\xi_2)^3}{1-(1-\sigma)(\xi_1)^3(\xi_2)^3}},\ \frac{\theta_1\theta_2}{\sqrt[3]{\sigma+(1-\sigma)\left((\theta_1)^3+(\theta_2)^3-(\theta_1)^3(\theta_2)^3\right)}},\\ \frac{\partial_1\partial_2}{\sqrt[3]{\sigma+(1-\sigma)\left((\partial_1)^3+(\partial_2)^3-(\partial_1)^3(\partial_2)^3\right)}}\end{array}\right);$$

$$\text{(ii)}\ \breve{e}_1 \otimes \breve{e}_2 = \left(\begin{array}{l}\frac{\xi_1\xi_2}{\sqrt[3]{\sigma+(1-\sigma)\left((\xi_1)^3+(\xi_2)^3-(\xi_1)^3(\xi_2)^3\right)}},\ \sqrt[3]{\frac{(\theta_1)^3+(\theta_2)^3-(\theta_1)^3(\theta_2)^3-(1-\sigma)(\theta_1)^3(\theta_2)^3}{1-(1-\sigma)(\theta_1)^3(\theta_2)^3}},\\ \sqrt[3]{\frac{(\partial_1)^3+(\partial_2)^3-(\partial_1)^3(\partial_2)^3-(1-\sigma)(\partial_1)^3(\partial_2)^3}{1-(1-\sigma)(\partial_1)^3(\partial_2)^3}}\end{array}\right);$$

$$\text{(iii)}\ \kappa\left(\breve{e}_1\right) = \left(\begin{array}{l}\sqrt[3]{\frac{\left(1+(\sigma-1)(\xi_1)^3\right)^{\kappa}-\left(1-(\xi_1)^3\right)^{\kappa}}{\left(1+(\sigma-1)(\xi_1)^3\right)^{\kappa}+(\sigma-1)\left(1-(\xi_1)^3\right)^{\kappa}}},\ \frac{\sqrt[3]{\sigma}\theta_1^{\kappa}}{\sqrt[3]{\left(1+(\sigma-1)\left(1-(\theta_1)^3\right)\right)^{\kappa}+(\sigma-1)(\theta_1)^{3\kappa}}},\\ \frac{\sqrt[3]{\sigma}\partial_1^{\kappa}}{\sqrt[3]{\left(1+(\sigma-1)\left(1-(\partial_1)^3\right)\right)^{\kappa}+(\sigma-1)(\partial_1)^{3\kappa}}}\end{array}\right);$$

$$\text{(iv)}\ \left(\breve{e}_1\right)^{\kappa} = \left(\begin{array}{l}\frac{\sqrt[3]{\sigma}(\xi_1)^{\kappa}}{\sqrt[3]{\left(1+(\sigma-1)\left(1-(\xi_1)^3\right)\right)^{\kappa}+(\sigma-1)(\xi_1)^{3\kappa}}},\ \sqrt[3]{\frac{\left(1+(\sigma-1)(\theta_1)^3\right)^{\kappa}-\left(1-(\theta_1)^3\right)^{\kappa}}{\left(1+(\sigma-1)(\theta_1)^3\right)^{\kappa}+(\sigma-1)\left(1-(\theta_1)^3\right)^{\kappa}}},\\ \sqrt[3]{\frac{\left(1+(\sigma-1)(\partial_1)^3\right)^{\kappa}-\left(1-(\partial_1)^3\right)^{\kappa}}{\left(1+(\sigma-1)(\partial_1)^3\right)^{\kappa}+(\sigma-1)\left(1-(\partial_1)^3\right)^{\kappa}}}\end{array}\right).$$

The following theorem can easily be proved.

Theorem 15 *Let $\breve{e}$, $\breve{e}_1$ and $\breve{e}_2$ be three CFEs and $\kappa, \kappa_1, \kappa_2 > 0$, then*

(i) $\breve{e}_1 \oplus \breve{e}_2, \breve{e}_1 \otimes \breve{e}_2, \kappa\left(\breve{e}\right)$, *and* $\left(\breve{e}\right)^{\kappa}$ *are also CFEs*;
(ii) $\breve{e}_1 \oplus \breve{e}_2 = \breve{e}_2 \oplus \breve{e}_1$;
(iii) $\breve{e}_1 \otimes \breve{e}_2 = \breve{e}_2 \otimes \breve{e}_1$;
(iv) $\kappa\left(\breve{e}_1 \oplus \breve{e}_2\right) = \kappa\breve{e}_1 \oplus \kappa\breve{e}_2$;
(v) $\left(\breve{e}_1 \otimes \breve{e}_2\right)^{\kappa} = \breve{e}_1^{\kappa} \otimes \breve{e}_2^{\kappa}$;
(vi) $(\kappa_1 + \kappa_2)\,\breve{e} = \kappa_1\breve{e} \oplus \kappa_2\breve{e}$;
(vii) $\breve{e}^{\kappa_1+\kappa_2} = \breve{e}^{\kappa_1} \otimes \breve{e}^{\kappa_2}$;
(viii) $\left(\breve{e}^{\kappa_1}\right)^{\kappa_2} = \breve{e}^{\kappa_1\kappa_2}$.

5 Cubical fuzzy Hamacher aggregation operators

We employ cubical fuzzy Hamacher operations in this section and offer new aggregation operators based on CFEs. First we suggest the cubical fuzzy arithmetic aggregation operators such as cubical fuzzy Hamacher weighted averaging operator, cubical fuzzy Hamacher ordered weighted averaging operator and cubical fuzzy Hamacher hybrid weighted average operator. Then we propose cubical fuzzy geometric aggregation operators such as cubical fuzzy Hamacher weighted geometric operator, cubical fuzzy Hamacher ordered weighted geometric operator and cubical fuzzy Hamacher hybrid weighted geometric operator. We discuss the key features of both types of the operators. In this section, $\mathfrak{L}$ denotes the collection of all CFEs on some universe of discourse $\mathfrak{D}$.

5.1 Cubical fuzzy Hamacher arithmetic aggregation operators

Definition 16 A function $\mathrm{CFHWA}_{\acute{\mathfrak{z}}} : \mathfrak{L}^p \longrightarrow \mathfrak{L}$ given as

$$\mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \oplus_{r=1}^{p}\left(\acute{\mathfrak{z}}_r \breve{e}_r\right)$$

is called a cubical fuzzy Hamacher weighted averaging (CFHWA) operator of dimension P, where the weight vector of the vector $\breve{e} = \left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right)$ is $\acute{\mathfrak{z}} = \left(\acute{\mathfrak{z}}_1, \acute{\mathfrak{z}}_2, ..., \acute{z}_p\right)^{\mathrm{T}}$ with the conditions $\acute{\mathfrak{z}}_r > 0$ and $\Sigma_{r=1}^{p}\acute{\mathfrak{z}}_r = 1$.

The Hamacher operations on CFEs are used in the following theorem to prove that the aggregate value of a group of CFEs subject to the CFHWA operator is again a CFE. In terms of CFE membership grades, we also propose a formula for the CFHWA operator.

Theorem 17 *For a collection of CFEs $\breve{e}_r = (\xi_r, \theta_r, \partial_r)$ $(r = 1, ..., p)$, $CFHWA_{\acute{\mathfrak{z}}}$ $(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p)$ is again a CFE and is given as*

$$\mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \left(\begin{array}{c} \sqrt[3]{\dfrac{\Pi_{r=1}^{p}\left(1+(\sigma-1)(\xi_r)^3\right)^{\acute{\mathfrak{z}}_r} - \Pi_{r=1}^{p}\left(1-(\xi_r)^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1+(\sigma-1)(\xi_r)^3\right)^{\acute{\mathfrak{z}}_r} + (\sigma-1)\Pi_{r=1}^{p}\left(1-(\xi_r)^3\right)^{\acute{\mathfrak{z}}_r}}}, \\ \dfrac{\sqrt[3]{\sigma}\,\Pi_{r=1}^{p}(\theta_r)^{\acute{\mathfrak{z}}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(1+(\sigma-1)\left(1-(\theta_r)^3\right)\right)^{\acute{\mathfrak{z}}_r} + (\sigma-1)\Pi_{r=1}^{p}(\theta_r)^{3\acute{\mathfrak{z}}_r}}}, \\ \dfrac{\sqrt[3]{\sigma}\,\Pi_{r=1}^{p}(\partial_r)^{\acute{\mathfrak{z}}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(1+(\sigma-1)\left(1-(\partial_r)^3\right)\right)^{\acute{\mathfrak{z}}_r} + (\sigma-1)\Pi_{r=1}^{p}(\partial_r)^{3\acute{\mathfrak{z}}_r}}} \end{array}\right) \tag{4}$$

such that the weight vector of vector $\breve{e} = \left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right)$ is $\acute{\mathfrak{z}} = \left(\acute{z}_1, \acute{\mathfrak{z}}_2, ..., \acute{\mathfrak{z}}_p\right)^{\mathrm{T}}$ with $\acute{\mathfrak{z}}_r > 0$ and $\Sigma_{r=1}^{p}\acute{z}_r = 1$.

Proof We utilize the technique of mathematical induction on p to establish this result. Thus (i) For $p = 1$, we get $\acute{\mathfrak{z}}_1 = 1$ and the left hand side of Eq. (4) yields

$$\mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = (\xi_1, \theta_1, \partial_1)$$

and the right side of Eq. (4) gives

$$\left(\begin{array}{c} \sqrt[3]{\dfrac{\left(1+(\sigma-1)(\xi_1)^3\right) - \left(1-(\xi_1)^3\right)}{\left(1+(\sigma-1)(\xi_1)^3\right) + (\sigma-1)\left(1-(\xi_1)^3\right)}}, \\ \dfrac{\sqrt[3]{\sigma}\theta_1}{\sqrt[3]{\left(1+(\sigma-1)\left(1-(\theta_1)^3\right)\right) + (\sigma-1)(\theta_1)^3}}, \\ \dfrac{\sqrt[3]{\sigma}\partial_1}{\sqrt[3]{\left(1+(\sigma-1)\left(1-(\partial_1)^3\right)\right) + (\sigma-1)(\partial_1)^3}} \end{array}\right) = (\xi_1, \theta_1, \partial_1).$$

Thus, Eq. (4) holds for $p = 1$. Also since $\mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \breve{e}_1$, so for $p = 1$, $\mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right)$ is a CFE.
(ii) Assume that the theorem holds for $p = k$, where $k \in \mathbb{N}$, then $\mathrm{CFHWA}_{\acute{\mathfrak{z}}}$ $(\breve{e}_1, \breve{e}_2, ..., \breve{e}_k)$ is a CFE and Eq. (4) becomes

$$\mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_k\right) = \prod_{r=1}^{k}\left(\acute{\mathfrak{z}}_r \breve{e}_r\right)$$

$$= \left(\begin{array}{c} \sqrt[3]{\frac{\Pi_{r=1}^{k}\left(1+(\sigma-1)(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}-\Pi_{r=1}^{k}\left(1-(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}}{\Pi_{r=1}^{k}\left(1+(\sigma-1)(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}+(\sigma-1)\Pi_{r=1}^{k}\left(1-(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}}}, \\ \frac{\sqrt[3]{\sigma}\Pi_{r=1}^{k}(\theta_r)^{\acute{\mathfrak{z}}r}}{\sqrt[3]{\Pi_{r=1}^{k}\left(1+(\sigma-1)\left(1-(\theta_r)^3\right)\right)^{\acute{\mathfrak{z}}r}+(\sigma-1)\Pi_{r=1}^{k}(\theta_r)^{3\acute{\mathfrak{z}}r}}}, \\ \frac{\sqrt[3]{\sigma}\Pi_{r=1}^{k}(\partial_r)^{\acute{\mathfrak{z}}r}}{\sqrt[3]{\Pi_{r=1}^{k}\left(1+(\sigma-1)\left(1-(\partial_r)^3\right)\right)^{\acute{\mathfrak{z}}r}+(\sigma-1)\Pi_{r=1}^{k}(\partial_r)^{3\acute{\mathfrak{z}}r}}} \end{array} \right)$$

(iii) Now, for $p = k + 1$, we employ the Definition 14 as follows:

$$\begin{aligned} &\mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, \ldots, \breve{e}_k, \breve{e}_{k+1}\right) \\ &= \left(\bigoplus_{r=1}^{k}\left(\acute{z}_r\breve{e}_r\right)\right) \oplus \left(\acute{\mathfrak{z}}_{k+1}\breve{e}_{k+1}\right) \\ &= \left(\begin{array}{c} \sqrt[3]{\frac{\Pi_{r=1}^{k}\left(1+(\sigma-1)(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}-\Pi_{r=1}^{k}\left(1-(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}}{\Pi_{r=1}^{k}\left(1+(\sigma-1)(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}+(\sigma-1)\Pi_{r=1}^{k}\left(1-(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}}}, \\ \frac{\sqrt[3]{\sigma}\Pi_{r=1}^{k}(\theta_r)^{\acute{\mathfrak{z}}r}}{\sqrt[3]{\Pi_{r=1}^{k}\left(1+(\sigma-1)\left(1-(\theta_r)^3\right)\right)^{\acute{\mathfrak{z}}r}+(\sigma-1)\Pi_{r=1}^{k}(\theta_r)^{3\acute{\mathfrak{z}}r}}}, \\ \frac{\sqrt[3]{\sigma}\Pi_{r=1}^{k}(\partial_r)^{\acute{\mathfrak{z}}r}}{\sqrt[3]{\Pi_{r=1}^{k}\left(1+(\sigma-1)\left(1-(\partial_r)^3\right)\right)^{\acute{\mathfrak{z}}r}+(\sigma-1)\Pi_{r=1}^{k}(\partial_r)^{3\acute{\mathfrak{z}}r}}} \end{array} \right) \\ &\oplus \left(\begin{array}{c} \sqrt[3]{\frac{\left(1+(\sigma-1)(\xi_{k+1})^3\right)^{\acute{\mathfrak{z}}k+1}-\left(1-(\xi_{k+1})^3\right)^{\acute{\mathfrak{z}}k+1}}{\left(1+(\sigma-1)(\xi_{k+1})^3\right)^{\acute{\mathfrak{z}}k+1}+(\sigma-1)\left(1-(\xi_{k+1})^3\right)^{\acute{\mathfrak{z}}k+1}}}, \\ \frac{\sqrt[3]{\sigma}(\theta_{k+1})^{\acute{z}k+1}}{\sqrt[3]{\left(1+(\sigma-1)\left(1-(\theta_{k+1})^3\right)\right)^{\acute{\mathfrak{z}}k+1}+(\sigma-1)(\theta_{k+1})^{3\acute{\mathfrak{z}}k+1}}}, \\ \frac{\sqrt[3]{\sigma}(\partial_{k+1})^{\acute{\mathfrak{z}}k+1}}{\sqrt[3]{\left(1+(\sigma-1)\left(1-(\partial_{k+1})^3\right)\right)^{\acute{\mathfrak{z}}k+1}+(\sigma-1)(\partial_{k+1})^{3\acute{\mathfrak{z}}k+1}}} \end{array} \right) \end{aligned}$$

$$= \left(\begin{array}{c} \sqrt[3]{\frac{\begin{array}{c}\Pi_{r=1}^{k}\left(1+(\sigma-1)(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}\left(1+(\sigma-1)(\xi_{k+1})^3\right)^{\acute{\mathfrak{z}}k+1} \\ -\Pi_{r=1}^{k}\left(1-(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}\left(1-(\xi_{k+1})^3\right)^{\acute{\mathfrak{z}}k+1}\end{array}}{\begin{array}{c}\Pi_{r=1}^{k}\left(1+(\sigma-1)(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}\left(1+(\sigma-1)(\xi_{k+1})^3\right)^{\acute{\mathfrak{z}}k+1} \\ +(\sigma-1)\Pi_{r=1}^{k}\left(1-(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}\left(1-(\xi_{k+1})^3\right)^{\acute{\mathfrak{z}}k+1}\end{array}}}, \\ \frac{\sqrt[3]{\sigma}\Pi_{r=1}^{k}(\theta_r)^{\acute{\mathfrak{z}}r}(\theta_{k+1})^{\acute{\mathfrak{z}}k+1}}{\sqrt[3]{\begin{array}{l}\Pi_{r=1}^{k}\left(1+(\sigma-1)\left(1-(\theta_r)^3\right)\right)^{\acute{\mathfrak{z}}r} \\ \left(1+(\sigma-1)\left(1-(\theta_{k+1})^3\right)\right)^{\acute{\mathfrak{z}}k+1} \\ +(\sigma-1)\Pi_{r=1}^{k}(\theta_r)^{3\acute{\mathfrak{z}}r}(\theta_{k+1})^{3\acute{\mathfrak{z}}k+1}\end{array}}}, \\ \frac{\sqrt[3]{\sigma}\Pi_{r=1}^{k}(\partial_r)^{\acute{\mathfrak{z}}r}(\partial_{k+1})^{\acute{\mathfrak{z}}k+1}}{\sqrt[3]{\begin{array}{l}\Pi_{r=1}^{k}\left(1+(\sigma-1)\left(1-(\partial_r)^3\right)\right)^{\acute{\mathfrak{z}}r} \\ \left(1+(\sigma-1)\left(1-(\partial_{k+1})^3\right)\right)^{\acute{\mathfrak{z}}k+1} \\ +(\sigma-1)\Pi_{r=1}^{k}(\partial_r)^{3\acute{\mathfrak{z}}r}(\partial_{k+1})^{3\acute{\mathfrak{z}}k+1}\end{array}}} \end{array} \right).$$

$$= \left(\begin{array}{c} \sqrt[3]{\frac{\Pi_{r=1}^{k+1}\left(1+(\sigma-1)(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}-\Pi_{r=1}^{k+1}\left(1-(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}}{\Pi_{r=1}^{k+1}\left(1+(\sigma-1)(\xi_r)^3\right)^{\acute{z}r}+(\sigma-1)\Pi_{r=1}^{k+1}\left(1-(\xi_r)^3\right)^{\acute{\mathfrak{z}}r}}}, \\ \frac{\sqrt[3]{\sigma}\Pi_{r=1}^{k+1}(\theta_r)^{\acute{\mathfrak{z}}r}}{\sqrt[3]{\Pi_{r=1}^{k+1}\left(1+(\sigma-1)\left(1-(\theta_r)^3\right)\right)^{\acute{\mathfrak{z}}r}+(\sigma-1)\Pi_{r=1}^{k+1}(\theta_r)^{3\acute{\mathfrak{z}}r}}}, \\ \frac{\sqrt[3]{\sigma}\Pi_{r=1}^{k+1}(\partial_r)^{\acute{\mathfrak{z}}r}}{\sqrt[3]{\Pi_{r=1}^{k+1}\left(1+(\sigma-1)\left(1-(\partial_r)^3\right)\right)^{\acute{\mathfrak{z}}r}+(\sigma-1)\Pi_{r=1}^{k+1}(\partial_r)^{3\acute{\mathfrak{z}}r}}} \end{array} \right)$$

Therefore, Eq. (4) holds for $p = k + 1$. Also since $\oplus_{r=1}^{k}\left(\acute{\mathfrak{z}}_r \breve{e}_r\right)$ and $\breve{e}_{k+1}$ are CFEs so by Theorem 15 $\mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_k, \breve{e}_{k+1}\right) = \left(\oplus_{r=1}^{k}\left(\acute{\mathfrak{z}}_r \breve{e}_r\right)\right) \oplus \left(\acute{\mathfrak{z}}_{k+1} \breve{e}_{k+1}\right)$ is also a CFE. Therefore, It is established that the theorem is true for any $p \in \mathbb{N}$. This completes the proof. □

The CFHWA operator may conveniently be shown to meet the following features.

Theorem 18 (Idempotency Property) *If* $\breve{e}_1, \breve{e}_2, ..., \breve{e}_p$ *are CFEs such that* $\breve{e}_r = \breve{e}\ \forall\ r$, *then*

$$\mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_r\right) = \breve{e}.$$

Theorem 19 (Boundedness property) *Let* $\breve{e}_1, \breve{e}_2, ..., \breve{e}_p$ *be CFEs, such that* $\min\left\{\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right\} = \breve{e}^-$ *and* $\max\left\{\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right\} = \breve{e}^+$. *Then*

$$\breve{e}^- \leq \mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) \leq \breve{e}^+.$$

Theorem 20 (Monotonicity property) *If* $\breve{e}_1, \breve{e}_2, ..., \breve{e}_p$ *and* $\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p$ *be two families of CFEs, such that* $\breve{e}_r \leq \breve{e}'_r$ *for all* r, *then*

$$\mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) \leq \mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p\right).$$

Theorem 21 (Commutativity property) *Let* $\breve{e}_1, \breve{e}_2, ..., \breve{e}_p$ *be CFEs, then*

$$\mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) \leq \mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p\right).$$

where $\left(\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p\right)$ is a permutation of $\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right)$.

The CFHWA operator's two special cases are as follows:

(1) In case when $\sigma = 1$, we obtain the cubical fuzzy weighted average (CFWA) operator from the CFHWA operator.

$$\begin{aligned}&\mathrm{CFWA}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right)\\&= \left(\sqrt[3]{1 - \Pi_{r=1}^{p}\left(1 - (\xi_r)^3\right)^{\acute{\mathfrak{z}}_r}}, \Pi_{r=1}^{p}\theta_r^{\acute{\mathfrak{z}}_r}, \Pi_{r=1}^{p}\partial_r^{\acute{\mathfrak{z}}_r}\right).\end{aligned}$$

(2) We get the cubical fuzzy Einstein weighted average (CFEWA) operator from the CFHWA operator when $\sigma = 2$.

$$\mathrm{CFHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \left(\begin{array}{c}\sqrt[3]{\frac{\Pi_{r=1}^{p}\left(1+(\xi_r)^3\right)^{\acute{\mathfrak{z}}_r} - \Pi_{r=1}^{p}\left(1-(\xi_r)^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1+(\xi_r)^3\right)^{\acute{z}_r} + \Pi_{r=1}^{p}\left(1-(\xi_r)^3\right)^{\acute{\mathfrak{z}}_r}}},\\ \frac{\sqrt[3]{2}\,\Pi_{r=1}^{p}(\theta_r)^{\acute{\mathfrak{z}}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(2-(\theta_r)^3\right)^{\acute{\mathfrak{z}}_r} + \Pi_{r=1}^{p}(\theta_r)^{3\acute{\mathfrak{z}}_r}}},\\ \frac{\sqrt[3]{2}\,\Pi_{r=1}^{p}(\partial_r)^{\acute{\mathfrak{z}}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(2-(\partial_r)^3\right)^{\acute{\mathfrak{z}}_r} + \Pi_{r=1}^{p}(\partial_r)^{3\acute{\mathfrak{z}}_r}}}\end{array}\right).$$

The concept of a cubical fuzzy Hamacher ordered weighted averaging (CFHOWA) operator is now offered, along with its fundamental properties.

Definition 22 A function $\mathrm{CFHOWA}_{\acute{\mathfrak{z}}} : \mathfrak{L}^p \longrightarrow \mathfrak{L}$ given as

$$\mathrm{CFHOWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \oplus_{r=1}^{p}\left(\acute{\mathfrak{z}}_r \breve{e}_{\mathrm{j}_r}\right)$$

is called a CFHOWA operator of dimension P, where $(\mathrm{j}_1, \mathrm{j}_2, ..., \mathrm{j}_p)$ is a rearrangement of $(1, 2, ..., p)$ such that $\breve{e}_{\mathrm{j}_{r-1}} \geq \breve{e}_{\mathrm{j}_r}$ for all $r = 2, 3, ..., p$ and the weight vector of $\breve{e} = (\breve{e}_{\mathrm{j}_1}, \breve{e}_{\mathrm{j}_2}, ..., \breve{e}_{\mathrm{j}_p})$ is $\acute{z} = (\acute{z}_1, \acute{z}_2, ..., \acute{z}_p)^{\mathrm{T}}$ with the conditions $\acute{z}_r > 0$ and $\Sigma_{r=1}^{p}\acute{z}_r = 1$.

Theorem 23 *For a collection of CFEs* $\breve{e}_r = (\xi_r, \theta_r, \partial_r)$ $(r = 1, ..., p)$*, CFHOWA*$_{\acute{z}}$ $(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p)$ *is again a CFE and is given as*

$$\mathrm{CFHOWA}_{\acute{z}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \overset{p}{\underset{r=1}{\oplus}} \left(\acute{z}_r \breve{e}_{\mathrm{j}_r}\right)$$

$$= \left(\begin{array}{c} \sqrt[3]{\dfrac{\Pi_{r=1}^{p}\left(1+(\sigma-1)(\xi_{\mathrm{j}_r})^3\right)^{\acute{z}_r} - \Pi_{r=1}^{p}\left(1-(\xi_{\mathrm{j}_r})^3\right)^{\acute{z}_r}}{\Pi_{r=1}^{p}\left(1+(\sigma-1)(\xi_{\mathrm{j}_r})^3\right)^{\acute{z}_r} + (\sigma-1)\Pi_{r=1}^{p}\left(1-(\xi_{\mathrm{j}_r})^3\right)^{\acute{z}_r}}}, \\ \dfrac{\sqrt[3]{\sigma}\,\Pi_{r=1}^{p}(\theta_{\mathrm{j}_r})^{\acute{z}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(1+(\sigma-1)\left(1-(\theta_{\mathrm{j}_r})^3\right)\right)^{\acute{z}_r} + (\sigma-1)\Pi_{r=1}^{p}(\theta_{\mathrm{j}_r})^{3\acute{z}_r}}}, \\ \dfrac{\sqrt[3]{\sigma}\,\Pi_{r=1}^{p}(\partial_{\mathrm{j}_r})^{\acute{z}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(1+(\sigma-1)\left(1-(\partial_{\mathrm{j}_r})^3\right)\right)^{\acute{z}_r} + (\sigma-1)\Pi_{r=1}^{p}(\partial_{\mathrm{j}_r})^{3\acute{z}_r}}} \end{array} \right)$$

where $(\mathrm{j}_1, \mathrm{j}_2, ..., \mathrm{j}_p)$ is a rearrangement of $(1, 2, ..., p)$ with $\breve{e}_{\mathrm{j}_{r-1}} \geq \breve{e}_{\mathrm{j}_r}$ $(r = 2, 3, ..., p)$ and *the weight vector of* $\breve{e} = (\breve{e}_{\mathrm{j}_1}, \breve{e}_{\mathrm{j}_2}, ..., \breve{e}_{\mathrm{j}_p})$ is $\acute{z} = (\acute{z}_1, \acute{z}_2, ..., \acute{z}_p)^{\mathrm{T}}$ with $\acute{z}_r > 0$ and $\Sigma_{r=1}^{p}\acute{z}_r = 1$.

The following characteristics of a CFHOWA operator are simple to prove:

Theorem 24 (Idempotency property) *If* $\breve{e}_1, \breve{e}_2, ..., \breve{e}_p$ $(r = 1, ..., p)$ *are CFEs such that* $\breve{e}_r = \breve{e}$ *for all* r*, then*

$$\mathrm{CFHOWA}_{\acute{z}}\,(\breve{e}_1, \breve{e}_2, ..., \breve{e}_r) = \breve{e}.$$

Theorem 25 (Boundedness property) *Let* $\breve{e}_r$ $(r = 1, 2, ..., p)$ *be CFEs, and* $\breve{e}^- = min\{\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\}$ *and* $\breve{e}^+ = max\{\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\}$. *Then*

$$\breve{e}^- \leq \mathrm{CFHOWA}_{\acute{z}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) \leq \breve{e}^+.$$

Theorem 26 (Monotonicity property) *For two groups of CFEs* $\breve{e}_r$ $(r = 1, 2, ..., p)$ *and* $\breve{e}'_r$ $(r = 1, 2, ..., p)$ *if* $\breve{e}_r \leq \breve{e}'_r$ *for all* r*, then*

$$\mathrm{CFHOWA}_{\acute{z}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) \leq \mathrm{CFHOWA}_{\acute{z}}\left(\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p\right).$$

Theorem 27 (Commutativity property) *Let* $\breve{e}_1, \breve{e}_2, ..., \breve{e}_p$ *be a family of CFEs, then*

$$\mathrm{CFHOWA}_{\acute{z}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) \leq \mathrm{CFHOWA}_{\acute{z}}\left(\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p\right).$$

where $\left(\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p\right)$ is a permutation of $\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right)$.

The CFHOWA operator's two special cases are as follows:

(1) In case when $\sigma = 1$, we obtain the cubical fuzzy ordered weighted average (CFOWA) operator from the CFHOWA operator.

$$\mathrm{CFOWA}_{\acute{z}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \left(\sqrt[3]{1 - \Pi_{r=1}^{p}\left(1 - (\xi_{\mathrm{j}_r})^3\right)^{\acute{z}_r}}, \Pi_{r=1}^{p}\left(\theta_{\mathrm{j}_r}\right)^{\acute{z}_r}, \Pi_{r=1}^{p}\left(\partial_{\mathrm{j}_r}\right)^{\acute{z}_r}\right).$$

(2) We get the cubical fuzzy Einstein ordered weighted average (CFEOWA) operator from the CFHOWA operator when $\sigma = 2$.

$$\mathrm{CFEOWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \left(\begin{array}{c} \sqrt[3]{\dfrac{\Pi_{r=1}^{p}\left(1+\left(\xi_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r} - \Pi_{r=1}^{p}\left(1-\left(\xi_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1+\left(\xi_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r} + \Pi_{r=1}^{p}\left(1-\left(\xi_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}}}, \\ \dfrac{\sqrt[3]{2}\Pi_{r=1}^{p}\left(\theta_{\mathrm{j}_r}\right)^{\acute{\mathfrak{z}}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(2-\left(\theta_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r} + \Pi_{r=1}^{p}\left(\theta_{\mathrm{j}_r}\right)^{3\acute{\mathfrak{z}}_r}}}, \\ \dfrac{\sqrt[3]{2}\Pi_{r=1}^{p}\left(\partial_{\mathrm{j}_r}\right)^{\acute{\mathfrak{z}}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(2-\left(\partial_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r} + \Pi_{r=1}^{p}\left(\partial_{\mathrm{j}_r}\right)^{3\acute{\mathfrak{z}}_r}}} \end{array}\right).$$

The CFHWA operator and the CFHOWA operator, respectively, weigh the CFEs and the ordered position of the CFEs, as we can see in Definitions 16 and 22. In order to combine the qualities of both operators, we define the concept of a cubical fuzzy Hamacher hybrid weighted average (CFHHWA) operator in the next section.

Definition 28 A function $\mathrm{CFHHWA}_{(\acute{\mathfrak{z}},\Omega)} : \mathfrak{L}^p \longrightarrow \mathfrak{L}$ *defined by*

$$\mathrm{CFHHWA}_{(\acute{\mathfrak{z}},\Omega)}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \overset{p}{\underset{r=1}{\oplus}}\left(\acute{\mathfrak{z}}_r \dot{\breve{e}}_{\mathrm{j}_r}\right) \tag{5}$$

is called a CFHHWA operator of dimension P, where the weight vector of $\breve{e} = \left(\dot{\breve{e}}_{\mathrm{j}_1}, \dot{\breve{e}}_{\mathrm{j}_2}, ..., \dot{\breve{e}}_{\mathrm{j}_p}\right)$ *is* $\mathfrak{z} = \left(\acute{\mathfrak{z}}_1, \acute{\mathfrak{z}}_2, ..., \acute{\mathfrak{z}}_p\right)^{\mathrm{T}}$ such that $\acute{\mathfrak{z}}_r > 0$ and $\Sigma_{r=1}^{p}\acute{\mathfrak{z}}_r = 1$ and $\dot{\breve{e}}_{\mathrm{j}_r}$ is the rth maximum of the CFEs $\dot{\breve{e}}_r$ $\left(\dot{\breve{e}}_r = p\Omega_r\breve{e}_r,\ r = 1, 2, ..., p\right)$ and the weight vector of $\dot{\breve{e}} = \left(\dot{\breve{e}}_1, \dot{\breve{e}}_2, ..., \dot{\breve{e}}_p\right)$ is $\Omega = \left(\Omega_1, \Omega_2, ..., \Omega_p\right)^{\mathrm{T}}$ with the condition $\Omega_r > 0$ and $\Sigma_{r=1}^{p}\Omega_r = 1$, and p works as the balancing coefficient.

Theorem 29 *For a collection of CFEs* $\breve{e}_r = (\xi_r, \theta_r, \partial_r)$ $(r = 1, ..., p)$, *CFHHWA*$_{(\acute{\mathfrak{z}},\Omega)}$ $(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p)$ *is again a CFE and is given as*

$$\mathrm{CFHHWA}_{(\acute{\mathfrak{z}},\Omega)}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \left(\begin{array}{c} \sqrt[3]{\dfrac{\Pi_{r=1}^{p}\left(1+(\sigma-1)\left(\dot{\tilde{\xi}}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r} - \Pi_{r=1}^{p}\left(1-\left(\dot{\tilde{\xi}}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1+(\sigma-1)\left(\dot{\tilde{\xi}}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r} + (\sigma-1)\Pi_{r=1}^{p}\left(1-\left(\dot{\tilde{\xi}}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}}}, \\ \dfrac{\sqrt[3]{\sigma}\Pi_{r=1}^{p}\left(\dot{\tilde{\theta}}_{\mathrm{j}_r}\right)^{\acute{\mathfrak{z}}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(1+(\sigma-1)\left(1-\left(\dot{\tilde{\theta}}_{\mathrm{j}_r}\right)^3\right)\right)^{\acute{\mathfrak{z}}_r} + (\sigma-1)\Pi_{r=1}^{p}\left(\dot{\tilde{\theta}}_{\mathrm{j}_r}\right)^{3\acute{\mathfrak{z}}_r}}}, \\ \dfrac{\sqrt[3]{\sigma}\Pi_{r=1}^{p}\left(\dot{\tilde{\partial}}_{\mathrm{j}_r}\right)^{\acute{z}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(1+(\sigma-1)-\left(\dot{\tilde{\partial}}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r} + (\sigma-1)\Pi_{r=1}^{p}\left(\dot{\tilde{\partial}}_{\mathrm{j}_r}\right)^{3\acute{\mathfrak{z}}_r}}} \end{array}\right)$$

where the weight vector of $\dot{\breve{e}} = \left(\dot{\breve{e}}_{\mathrm{j}_1}, \dot{\breve{e}}_{\mathrm{j}_2}, ..., \dot{\breve{e}}_{\mathrm{j}_p}\right)$ *is* $\acute{\mathfrak{z}} = \left(\acute{\mathfrak{z}}_1, \acute{\mathfrak{z}}_2, ..., \acute{\mathfrak{z}}_p\right)^{\mathrm{T}}$ such that $\acute{\mathfrak{z}}_r > 0$ and $\Sigma_{r=1}^{p}\acute{\mathfrak{z}}_r = 1$ and $\dot{\breve{e}}_{\mathrm{j}_r}$ is the rth maximum of the CFEs $\dot{\breve{e}}_r$ $\left(\dot{\breve{e}}_r = p\Omega_r\breve{e}_r,\ r = 1, 2, ..., p\right)$ and the weight vector of $\dot{\breve{e}}^{*} = \left(\dot{\breve{e}}_1, \dot{\breve{e}}_2, ..., \dot{\breve{e}}_p\right)$ is $\Omega = \left(\Omega_1, \Omega_2, ..., \Omega_p\right)^{\mathrm{T}}$ with the condition $\Omega_r > 0$ and $\Sigma_{r=1}^{p}\Omega_r = 1$, and p works as the balancing coefficient.

In the CFHHWA operator

(i) When we take $\acute{\mathfrak{z}} = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^{\mathrm{T}}$, then we obtain CFHWA operator.
(ii) When we take $\Omega = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$, then we get CFHOWA operator.

5.2 Cubical fuzzy Hamacher geometric aggregation operators

Definition 30 A function $CFHWG_{\acute{\mathfrak{z}}} : \mathfrak{L}^p \longrightarrow \mathfrak{L}$ given as

$$\mathrm{CFHWG}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \oplus_{r=1}^{p}\left(\breve{e}_r\right)^{\acute{\mathfrak{z}}_r}$$

is called a cubical fuzzy Hamacher weighted geometric (CFHWG) operator of dimension P, where the weight vector of $\breve{e} = \left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right)$ is $\acute{\mathfrak{z}} = \left(\acute{\mathfrak{z}}_1, \acute{\mathfrak{z}}_2, ..., \acute{\mathfrak{z}}_p\right)^{\mathrm{T}}$ with the conditions $\acute{\mathfrak{z}}_r > 0$ and $\Sigma_{r=1}^{p}\acute{\mathfrak{z}}_r = 1$.

In the following theorem, we establish a relationship between CFHWG operator of a family of CFEs and the membership grades of the CFEs.

Theorem 31 *For a collection of CFEs* $\breve{e}_r = (\xi_r, \theta_r, \partial_r)$ $(r = 1, ..., p)$*, CFHWG*$_{\acute{\mathfrak{z}}}$ $(\breve{e}_1, \breve{e}_2, ,$ $..., \breve{e}_p)$ *is again a CFE and is given as*

$$\mathrm{CFHWG}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \left(\begin{array}{c} \frac{\sqrt[3]{\sigma}\Pi_{r=1}^{p}\xi_r^{\acute{\mathfrak{z}}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(1+(\sigma-1)\left(1-(\xi_r)^3\right)\right)^{\acute{\mathfrak{z}}_r}+(\sigma-1)\Pi_{r=1}^{p}\xi_r^{3\acute{\mathfrak{z}}_r}}}, \\ \sqrt[3]{\frac{\Pi_{r=1}^{p}\left(1+(\sigma-1)(\theta_r)^3\right)^{\acute{\mathfrak{z}}_r}-\Pi_{r=1}^{p}\left(1-(\theta_r)^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1+(\sigma-1)(\theta_r)^3\right)^{\acute{\mathfrak{z}}_r}+(\sigma-1)\Pi_{r=1}^{p}\left(1-(\theta_r)^3\right)^{\acute{\mathfrak{z}}_r}}}, \\ \sqrt[3]{\frac{\Pi_{r=1}^{p}\left(1+(\sigma-1)(\partial_r)^3\right)^{\acute{\mathfrak{z}}_r}-\Pi_{r=1}^{p}\left(1-(\partial_r)^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1+(\sigma-1)(\partial_r)^3\right)^{\acute{\mathfrak{z}}_r}+(\sigma-1)\Pi_{r=1}^{p}\left(1-(\partial_r)^3\right)^{\acute{\mathfrak{z}}_r}}} \end{array}\right)$$

where the weight vector of $\breve{e} = \left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right)$ is $\acute{\mathfrak{z}} = \left(\acute{\mathfrak{z}}_1, \acute{\mathfrak{z}}_2, ..., \acute{\mathfrak{z}}_p\right)^{\mathrm{T}}$ with the conditions $\acute{\mathfrak{z}}_r > 0$ and $\Sigma_{r=1}^{p}\acute{z}_r = 1$.

Proof Same as the proof of Theorem 17. □

The CFHWG operator satisfy the following properties:

Theorem 32 (Idempotency property) *If* $\breve{e}_r$ $(r = 1, ..., p)$ *be CFEs such that* $\breve{e}_r = \breve{e}$ *for all* r*, then*

$$\mathrm{CFHWG}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_r\right) = \breve{e}.$$

Theorem 33 (Boundedness property) *Let* $\breve{e}_1, \breve{e}_2, ..., \breve{e}_p$ *be a family of CFEs, and* $\breve{e}^- = min\{\breve{e}_1, , \breve{e}_2, ..., \breve{e}_p\}$ *and* $\breve{e}^+ = max\{\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\}$ *. Then*

$$\breve{e}^- \leq \mathrm{CFHWG}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) \leq \breve{e}^+.$$

Theorem 34 (Monotonicity property) *For two groups of CFEs* $\breve{e}_1, \breve{e}_2, ..., \breve{e}_p$ *and* $\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p$ *if* $\breve{e}_r \leq \breve{e}'_r$ *for all r, then*

$$\mathrm{CFHWG}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) \leq \mathrm{CFHWG}_{\acute{\mathfrak{z}}}\left(\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p\right).$$

Theorem 35 (Commutativity property) *Let* $\breve{e}_1, \breve{e}_2, ..., \breve{e}_p$ *be a family of CFEs, then*

$$\mathrm{CFHWG}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) \leq \mathrm{CFHWG}_{\acute{\mathfrak{z}}}\left(\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p\right).$$

where $\left(\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p\right)$ is a permutation of $\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right)$.

The CFHWG operator's two special cases are as follows:

(1) In case when $\sigma = 1$, we obtain the cubical fuzzy weighted geometric (CFWG) operator from the CFHWG operator.

$$\mathrm{CFWG}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \left(\prod_{r=1}^{p}(\xi_r)^{\acute{\mathfrak{z}}_r}, \sqrt[3]{1-\prod_{r=1}^{p}\left(1-(\theta_r)^3\right)^{\acute{\mathfrak{z}}_r}}, \sqrt[3]{1-\prod_{r=1}^{p}\left(1-(\partial_r)^3\right)^{\acute{\mathfrak{z}}_r}}\right).$$

(2) We get the cubical fuzzy Einstein weighted geometric (CFEWG) operator from the CFHWG operator when $\sigma = 2$.

$$\mathrm{CFEWG}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \left(\begin{array}{c} \frac{\sqrt[3]{2}\Pi_{r=1}^{p}\xi_r^{\acute{\mathfrak{z}}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(2-(\xi_r)^3\right)^{\acute{\mathfrak{z}}_r}+\Pi_{r=1}^{p}\xi_r^{3\acute{\mathfrak{z}}_r}}}, \\ \sqrt[3]{\frac{\Pi_{r=1}^{p}\left(1+(\theta_r)^3\right)^{\acute{\mathfrak{z}}_r}-\Pi_{r=1}^{p}\left(1-(\theta_r)^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1+(\theta_r)^3\right)^{\acute{\mathfrak{z}}_r}+\Pi_{r=1}^{p}\left(1-(\theta_r)^3\right)^{\acute{\mathfrak{z}}_r}}}, \\ \sqrt[3]{\frac{\Pi_{r=1}^{p}\left(1+(\partial_r)^3\right)^{\acute{\mathfrak{z}}_r}-\Pi_{r=1}^{p}\left(1-(\partial_r)^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1+(\partial_r)^3\right)^{\acute{\mathfrak{z}}_r}+\Pi_{r=1}^{p}\left(1-(\partial_r)^3\right)^{\acute{\mathfrak{z}}_r}}} \end{array}\right).$$

The idea of a cubical fuzzy Hamacher ordered weighted geometric (CFHOWG) operator is now proposed, along with its basic features.

Definition 36 A function $\mathrm{CFHOWG}_{\acute{\mathfrak{z}}} : \mathfrak{L}^p \longrightarrow \mathfrak{L}$ given as

$$\mathrm{CFHOWG}_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \oplus_{r=1}^{p}\left(\breve{e}_{\mathfrak{j}_r}\right)^{\acute{\mathfrak{z}}_r}$$

is called a CFHOWG operator of dimension P, where $\left(\mathfrak{j}_1, \mathfrak{j}_2, ..., \mathfrak{j}_p\right)$ is a rearrangement of $(1, 2, ..., p)$ such that $\breve{e}_{\mathfrak{j}_{r-1}} \geq \breve{e}_{\mathfrak{j}_r}$ for all $r = 2, 3, ..., p$ and the weight vector of $\breve{e} = \left(\breve{e}_{\mathfrak{j}_1}, \breve{e}_{\mathfrak{j}_2}, ..., \breve{e}_{\mathfrak{j}_p}\right)$ is $\acute{\mathfrak{z}} = \left(\acute{z}_1, \acute{\mathfrak{z}}_2, ..., \acute{\mathfrak{z}}_p\right)^{\mathrm{T}}$ with the conditions $\acute{\mathfrak{z}}_r > 0$ and $\Sigma_{r=1}^{p}\acute{\mathfrak{z}}_r = 1$.

Now, we establish a useful formula for CFHOWG operator.

Theorem 37 *For a collection of CFEs* $\breve{e}_r = (\xi_r, \theta_r, \partial_r)$ $(r = 1, ..., p)$, *CFHOWG*$_{\acute{\mathfrak{z}}}$ $(\breve{e}_1, \breve{e}_2, , ..., \breve{e}_p)$ *is again a CFE and is given as*

$$\mathrm{CFHOWG}_w\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \left(\begin{array}{c} \frac{\sqrt[3]{\sigma}\Pi_{r=1}^{p}\left(\xi_{\mathfrak{j}_r}\right)^{\acute{\mathfrak{z}}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(1-(1-\sigma)\left(1-\left(\xi_{\mathfrak{j}_r}\right)^3\right)\right)^{\acute{z}_r}-(1-\sigma)\Pi_{r=1}^{p}\left(\xi_{\mathfrak{j}_r}\right)^{3\acute{\mathfrak{z}}_r}}}, \\ \sqrt[3]{\frac{\Pi_{r=1}^{p}\left(1-(1-\sigma)\left(\theta_{\mathfrak{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}-\Pi_{r=1}^{p}\left(1-\left(\theta_{\mathfrak{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1-(1-\sigma)\left(\theta_{\mathfrak{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}-(1-\sigma)\Pi_{r=1}^{p}\left(1-\left(\theta_{\mathfrak{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}}}, \\ \sqrt[3]{\frac{\Pi_{r=1}^{p}\left(1-(1-\sigma)\left(\partial_{\mathfrak{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}-\Pi_{r=1}^{p}\left(1-\left(\partial_{\mathfrak{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1-(1-\sigma)\left(\partial_{\mathfrak{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}-(1-\sigma)\Pi_{r=1}^{p}\left(1-\left(\partial_{\mathfrak{j}_r}\right)^3\right)^{\acute{z}_r}}} \end{array}\right)$$

where $(\mathrm{j}_1, \mathrm{j}_2, ..., \mathrm{j}_p)$ is a permutation of $(1, 2, ..., p)$ such that $\breve{e}_{\mathrm{j}_{r-1}} \geq \breve{e}_{\mathrm{j}_r}$ for all $r = 2, 3, ..., p$ and *the weight vector of* $\breve{e} = (\breve{e}_{\mathrm{j}_1}, \breve{e}_{\mathrm{j}_2}, ..., \breve{e}_{\mathrm{j}_p})$ is $\acute{\mathfrak{z}} = (\acute{\mathfrak{z}}_1, \acute{\mathfrak{z}}_2, ..., \acute{\mathfrak{z}}_p)^{\mathrm{T}}$ with the conditions $\acute{\mathfrak{z}}_r > 0$, $\Sigma_{r=1}^{p} \acute{z}_r = 1$.

Now, we give two special cases of CFHOWG operator.

(1) In case when $\sigma = 1$, we obtain the cubical fuzzy ordered weighted geometric (CFOWG) operator from the CFHOWG operator.

$$\mathrm{CFOWG}_{\acute{\mathfrak{z}}}(\breve{e}_1, \breve{e}_2, ..., \breve{e}_r) = \left(\prod_{r=1}^{p} (\xi_{\mathrm{j}_r})^{\acute{\mathfrak{z}}_r}, \sqrt[3]{1 - \prod_{r=1}^{p} \left(1 - (\theta_{\mathrm{j}_r})^3\right)^{\acute{\mathfrak{z}}_r}}, \sqrt[3]{1 - \prod_{r=1}^{p} \left(1 - (\partial_{\mathrm{j}_r})^3\right)^{\acute{\mathfrak{z}}_r}} \right).$$

(2) We get the cubical fuzzy Einstein weighted ordered geometric (CFEOWG) operator from the CFHOWG operator when $\sigma = 2$.

$$\mathrm{CFEOWG}_{\acute{\mathfrak{z}}}(\breve{e}_1, \breve{e}_2, ..., \breve{e}_r) = \left(\begin{array}{c} \frac{\sqrt[3]{2}\Pi_{r=1}^{p}(\xi_{\mathrm{j}_r})^{\acute{\mathfrak{z}}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(2-(\xi_{\mathrm{j}_r})^3\right)^{\acute{z}_r} + \Pi_{r=1}^{p}(\xi_{\mathrm{j}_r})^{3\acute{\mathfrak{z}}_r}}}, \\ \sqrt[3]{\frac{\Pi_{r=1}^{p}\left(1+(\theta_{\mathrm{j}_r})^3\right)^{\acute{\mathfrak{z}}_r} - \Pi_{r=1}^{p}\left(1-(\theta_{\mathrm{j}_r})^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1+(\theta_{\mathrm{j}_r})^3\right)^{\acute{\mathfrak{z}}_r} + \Pi_{r=1}^{p}\left(1-(\theta_{\mathrm{j}_r})^3\right)^{\acute{\mathfrak{z}}_r}}}, \\ \sqrt[3]{\frac{\Pi_{r=1}^{p}\left(1+(\partial_{\mathrm{j}_r})^3\right)^{\acute{\mathfrak{z}}_r} - \Pi_{r=1}^{p}\left(1-(\partial_{\mathrm{j}_r})^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1+(\partial_{\mathrm{j}_r})^3\right)^{\acute{\mathfrak{z}}_r} + \Pi_{r=1}^{p}\left(1-(\partial_{\mathrm{j}_r})^3\right)^{\acute{\mathfrak{z}}_r}}} \end{array} \right).$$

It is easy to establish the following properties of a CFHOWG operator.

Theorem 38 (Idempotency property) *If* $\breve{e}_r$ $(r = 1, ..., p)$ *is are CFEs such that* $\breve{e}_r = \breve{e}$ *for all* r*, then*

$$\mathrm{CFHOWG}_{\acute{\mathfrak{z}}}(\breve{e}_1, \breve{e}_2, ..., \breve{e}_r) = \breve{e}.$$

Theorem 39 (Boundedness property) *Let* $\breve{e}_r$ $(r = 1, 2, ..., p)$ *be CFEs, and min*$\{\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\} = \breve{e}^-$ *and max*$\{\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\} = \breve{e}^+$*. Then*

$$\breve{e}^- \leq \mathrm{CFHOWG}_{\acute{\mathfrak{z}}}(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p) \leq \breve{e}^+.$$

Theorem 40 (Monotonicity property) *For two groups of CFEs* $\breve{e}_r$ $(r = 1, 2, ..., p)$ *and* $\breve{e}'_r$ $(r = 1, 2, ..., p)$*, if* $\breve{e}_r \leq \breve{e}'_r$ *for all* r*, then*

$$\mathrm{CFHOWG}_{\acute{\mathfrak{z}}}(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p) \leq \mathrm{CFHOWG}_{\acute{\mathfrak{z}}}\left(\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p\right).$$

Theorem 41 (Commutativity property) *Let* $\breve{e}_1, \breve{e}_2, ..., \breve{e}_p$ *be a family of CFEs, then*

$$\mathrm{CFHOWG}_{\acute{\mathfrak{z}}}(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p) \leq \mathrm{CFHOWG}_{\acute{\mathfrak{z}}}\left(\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p\right).$$

where $\left(\breve{e}'_1, \breve{e}'_2, ..., \breve{e}'_p\right)$ is any reordering of $(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p)$.

The CFHWG operator and the CFHOWG operator, respectively, weigh the CFEs and the ordered position of the CFEs, as we can see in Definitions 30 and 36. In order to combine the qualities of both operators, we define the concept of a cubical fuzzy Hamacher hybrid weighted geometric (CFHHWG) operator in the next section.

Definition 42 A CFHHWG operator is a function $\mathrm{CFHHWG}_{(\acute{\mathfrak{z}},\Omega)} : \mathfrak{L}^p \longrightarrow \mathfrak{L}$ *defined by*

$$\mathrm{CFHHWG}_{(\acute{\mathfrak{z}},\Omega)}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right) = \underset{r=1}{\overset{p}{\oplus}} \left(\dot{\breve{e}}_{\mathrm{j}_r}\right)^{\acute{\mathfrak{z}}_r} \tag{6}$$

where the weight vector of $\dot{\breve{e}} = \left(\dot{\breve{e}}_{\mathrm{j}_1}, \dot{\breve{e}}_{\mathrm{j}_2}, ..., \dot{\breve{e}}_{\mathrm{j}_p}\right)$ *is* $\acute{z} = \left(\acute{\mathfrak{z}}_1, \acute{\mathfrak{z}}_2, ..., \acute{\mathfrak{z}}_p\right)^{\mathrm{T}}$ such that $\acute{\mathfrak{z}}_r > 0$ and $\Sigma_{r=1}^{p}\acute{\mathfrak{z}}_r = 1$ and $\dot{\breve{e}}_{\mathrm{j}_r}$ is the rth maximum of the CFEs $\dot{\breve{e}}_r$ $\left(\dot{\breve{e}}_r = (\breve{e}_r)^{p\Omega_r},\ r = 1, 2, ..., p\right)$ and the weight vector of $\breve{e} = \left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right)$ is $\Omega = \left(\Omega_1, \Omega_2, ..., \Omega_p\right)^{\mathrm{T}}$ with the condition $\Omega_r > 0$ and $\Sigma_{r=1}^{p}\Omega_r = 1$, and p works as the balancing coefficient.

Theorem 43 *For a collection of CFEs* $\breve{e}_r = (\xi_r, \theta_r, \partial_r)$ $(r = 1, ..., p)$, *CFHHWG*$_{\acute{\mathfrak{z}}}\left(\breve{e}_1, \breve{e}_2, , ..., \breve{e}_p\right)$ *is again a CFE and is given as*

$$\mathrm{CFHHWG}_{(\acute{\mathfrak{z}},\Omega)}\left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right)$$

$$= \left(\begin{array}{c} \dfrac{\sqrt[3]{\sigma}\,\Pi_{r=1}^{p}\left(\dot{\tilde{\xi}}_{\mathrm{j}_r}\right)^{\acute{\mathfrak{z}}_r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(1+(\sigma-1)\left(1-\left(\dot{\tilde{\xi}}_{\mathrm{j}_r}\right)^3\right)\right)^{\acute{\mathfrak{z}}_r} + (\sigma-1)\Pi_{r=1}^{p}\left(\dot{\tilde{\xi}}_{\mathrm{j}_r}\right)^{3\acute{\mathfrak{z}}_r}}}, \\ \sqrt[3]{\dfrac{\Pi_{r=1}^{p}\left(1+(\sigma-1)\left(\dot{\tilde{\theta}}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r} - \Pi_{r=1}^{p}\left(1-\left(\dot{\tilde{\theta}}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1+(\sigma-1)\left(\dot{\tilde{\theta}}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r} + (\sigma-1)\Pi_{r=1}^{p}\left(1-\left(\dot{\tilde{\theta}}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}}}, \\ \sqrt[3]{\dfrac{\Pi_{r=1}^{p}\left(1+(\sigma-1)\left(\dot{\tilde{\partial}}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r} - \Pi_{r=1}^{p}\left(1-\left(\dot{\tilde{\partial}}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}}{\Pi_{r=1}^{p}\left(1+(\sigma-1)\left(\dot{\tilde{\partial}}_{\mathrm{j}_r}\right)^3\right)^{\acute{z}_r} + (\sigma-1)\Pi_{r=1}^{p}\left(1-\left(\dot{\tilde{\partial}}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}}} \end{array} \right)$$

where the weight vector of $\dot{\breve{e}} = \left(\dot{\breve{e}}_{\mathrm{j}_1}, \dot{\breve{e}}_{\mathrm{j}_2}, ..., \dot{\breve{e}}_{\mathrm{j}_p}\right)$ *is* $\acute{\mathfrak{z}} = \left(\acute{\mathfrak{z}}_1, \acute{\mathfrak{z}}_2, ..., \acute{\mathfrak{z}}_p\right)^{\mathrm{T}}$ such that $\acute{\mathfrak{z}}_r > 0$ and $\Sigma_{r=1}^{p}\acute{\mathfrak{z}}_r = 1$ and $\dot{\breve{e}}_{\mathrm{j}_r}$ is the rth maximum of the CFEs $\dot{\breve{e}}_r$ $\left(\dot{\breve{e}}_r = (\breve{e}_r)^{p\Omega_r},\ r = 1, 2, ..., p\right)$ and the weight vector of $\breve{e} = \left(\breve{e}_1, \breve{e}_2, ..., \breve{e}_p\right)$ is $\Omega = \left(\Omega_1, \Omega_2, ..., \Omega_p\right)^{\mathrm{T}}$ with the condition $\Omega_r > 0$ and $\Sigma_{r=1}^{p}\Omega_r = 1$, and p works as the balancing coefficient.

In the CFHHWG operator

(i) When we take $\acute{\mathfrak{z}} = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^{\mathrm{T}}$, then we obtain CFHWG operator.
(ii) When we take $\Omega = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$, then we get CFHOWG operator.

The CFHHWG operator's two special cases are as follows:

(1) If $\sigma = 1$, then CFHHWG operator changes to CFHWG operator:

$$\mathrm{CFHWG}_{\acute{\mathfrak{z}}}\left(\dot{\breve{e}}_1, \dot{\breve{e}}_2, ..., \dot{\breve{e}}_r\right) = \left(\begin{array}{c} \Pi_{r=1}^{p}\left(\dot{\xi}_{\mathrm{j}_r}\right)^{\acute{\mathfrak{z}}_r}, \sqrt[3]{1-\Pi_{r=1}^{p}\left(1-\left(\dot{\theta}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}}, \\ \sqrt[3]{1-\Pi_{r=1}^{p}\left(1-\left(\dot{\partial}_{\mathrm{j}_r}\right)^3\right)^{\acute{\mathfrak{z}}_r}} \end{array} \right).$$

(2) If $\sigma = 2$, then CFHHWG operator converts to CFEHWG operator:

$$\mathrm{CFEHWG}_{\acute{\mathfrak{z}}}\left(\dot{\breve{e}}_1, \dot{\breve{e}}_2, ..., \dot{\breve{e}}_r\right) = \left(\begin{array}{c} \frac{\sqrt[3]{2}\Pi_{r=1}^{p}\left(\dot{\tilde{\xi}}_{\mathrm{j}r}\right)^{\acute{\mathfrak{z}}r}}{\sqrt[3]{\Pi_{r=1}^{p}\left(2-\left(\dot{\tilde{\xi}}_{\mathrm{j}r}\right)^{3}\right)^{\acute{\mathfrak{z}}r}+\Pi_{r=1}^{p}\left(\dot{\tilde{\xi}}_{\mathrm{j}r}\right)^{3\acute{\mathfrak{z}}r}}}, \\ \sqrt[3]{\frac{\Pi_{r=1}^{p}\left(1+\left(\dot{\tilde{\theta}}_{\mathrm{j}r}\right)^{3}\right)^{\acute{\mathfrak{z}}r}-\Pi_{r=1}^{p}\left(1-\left(\dot{\tilde{\theta}}_{\mathrm{j}r}\right)^{3}\right)^{\acute{\mathfrak{z}}r}}{\Pi_{r=1}^{p}\left(1+\left(\dot{\tilde{\theta}}_{\mathrm{j}r}\right)^{3}\right)^{\acute{\mathfrak{z}}r}+(\sigma-1)\Pi_{r=1}^{p}\left(1-\left(\dot{\tilde{\theta}}_{\mathrm{j}r}\right)^{3}\right)^{\acute{\mathfrak{z}}r}}}, \\ \sqrt[3]{\frac{\Pi_{r=1}^{p}\left(1+\left(\dot{\tilde{\partial}}_{\mathrm{j}r}\right)^{3}\right)^{\acute{z}r}-\Pi_{r=1}^{p}\left(1-\left(\dot{\tilde{\partial}}_{\mathrm{j}r}\right)^{3}\right)^{\acute{z}r}}{\Pi_{r=1}^{p}\left(1+\left(\dot{\tilde{\partial}}_{\mathrm{j}r}\right)^{3}\right)^{\acute{z}r}+\Pi_{r=1}^{p}\left(1-\left(\dot{\tilde{\partial}}_{\mathrm{j}r}\right)^{3}\right)^{\acute{z}r}}} \end{array}\right).$$

6 Decision-making algorithm based on CF Hamacher aggregation operators

We employ our proposed AOs to devise a technique for the solution of MADM problems involving cubical fuzzy data. The technique is formulated as follows. Let $\mathfrak{K}_1, \mathfrak{K}_2, ..., \mathfrak{K}_m$ be m alternatives to be assessed by experts for n attributes $\mathfrak{B}_1, \mathfrak{B}_2, ..., \mathfrak{B}_n$. Let the weight vector of the operator under consideration be $w = \left(\acute{\mathfrak{z}}_1, \acute{z}_2, ..., \acute{\mathfrak{z}}_n\right)$ and its components satisfy the conditions $\acute{\mathfrak{z}}_j > 0$ and $\Sigma_{j=1}^{n}\acute{\mathfrak{z}}_j = 1$. Let the team of experts assess each alternative $\mathfrak{K}_i$ for its satisfaction of the attribute $\mathfrak{B}_i$ and assign it the cubical fuzzy element $\left(\xi_{ij}, \theta_{ij}, \partial_{ij}\right)$ and $M = \left(\breve{e}_{ij}\right)_{m\times n} = \left(\xi_{ij}, \theta_{ij}, \partial_{ij}\right)_{m\times n}$. The algorithm to solve MADM problems under the CFHHWA (or CFHHWG) operator is formulated as given as follows.

Algorithm

Step 1. Aggregate the CFEs $\breve{e}_{i1}, \breve{e}_{i2}, ..., \breve{e}_{in}$ in the decision matrix M corresponding to the alternative $\mathfrak{K}_i$, $i = 1, 2, ..., m$ by employing the CFHHWA operator to find the aggregated CFNs $\breve{e}_i$ given as follows:

$$\begin{aligned} \breve{e}_i &= \mathrm{CFHHWA}_{\acute{\mathfrak{z}}}\left(\breve{e}_{i1}, \breve{e}_{i2}, ..., \breve{e}_{in}\right) \\ &= \left(\begin{array}{c} \sqrt[3]{\frac{\Pi_{j=1}^{n}\left(1+(\sigma-1)\left(\dot{\tilde{\xi}}_{i\mathfrak{d}(j)}\right)^{3}\right)^{\acute{\mathfrak{z}}j}-\Pi_{j=1}^{n}\left(1-\left(\dot{\tilde{\xi}}_{i\mathfrak{d}(j)}\right)^{3}\right)^{\acute{\mathfrak{z}}j}}{\Pi_{j=1}^{n}\left(1+(\sigma-1)\left(\dot{\tilde{\xi}}_{i\mathfrak{d}(j)}\right)^{3}\right)^{\acute{\mathfrak{z}}j}+(\sigma-1)\Pi_{j=1}^{n}\left(1-\left(\dot{\tilde{\xi}}_{i\mathfrak{d}(j)}\right)^{3}\right)^{\acute{\mathfrak{z}}j}}}, \\ \frac{\sqrt[3]{\sigma}\Pi_{j=1}^{n}\left(\dot{\tilde{\theta}}_{i\mathfrak{d}(j)}\right)^{\acute{z}j}}{\sqrt[3]{\Pi_{j=1}^{n}\left(1+(\sigma-1)\left(1-\left(\dot{\tilde{\theta}}_{i\mathfrak{d}(j)}\right)^{3}\right)\right)^{\acute{\mathfrak{z}}j}+(\sigma-1)\Pi_{j=1}^{n}\left(\dot{\tilde{\theta}}_{i\mathfrak{d}(j)}\right)^{3\acute{z}j}}}, \\ \frac{\sqrt[3]{\sigma}\Pi_{j=1}^{n}\left(\dot{\tilde{\partial}}_{i\mathfrak{d}(j)}\right)^{\acute{\mathfrak{z}}j}}{\sqrt[3]{\Pi_{j=1}^{n}\left(1+(\sigma-1)\left(1-\left(\dot{\tilde{\partial}}_{i\mathfrak{d}(j)}\right)^{3}\right)\right)^{\acute{\mathfrak{z}}j}+(\sigma-1)\Pi_{j=1}^{n}\left(\dot{\tilde{\partial}}_{i\mathfrak{d}(j)}\right)^{3\acute{\mathfrak{z}}j}}} \end{array}\right) \end{aligned} \quad (7)$$

Or alternatively use the CFHHWG operator, given as under

$$\breve{e}_i = \mathrm{CFHHWG}_{\acute{z}}\left(\breve{e}_{i1}, \breve{e}_{i2}, ..., \breve{e}_{in}\right)$$
$$= \left(\begin{array}{c} \dfrac{\sqrt[3]{\sigma}\Pi_{j=1}^{n}\left(\dot{\tilde{\xi}}_{i\mathfrak{d}(j)}\right)^{\acute{\mathfrak{z}}_j}}{\sqrt[3]{\Pi_{j=1}^{n}\left(1+(\sigma-1)\left(1-\left(\dot{\tilde{\xi}}_{i\mathfrak{d}(j)}\right)^3\right)\right)^{\acute{z}_j}+(\sigma-1)\Pi_{j=1}^{n}\left(\dot{\tilde{\xi}}_{i\mathfrak{d}(j)}\right)^{3\acute{\mathfrak{z}}_j}}}, \\ \sqrt[3]{\dfrac{\Pi_{j=1}^{n}\left(1+(\sigma-1)\left(\dot{\tilde{\theta}}_{i\mathfrak{d}(j)}\right)^3\right)^{\acute{\mathfrak{z}}_j}-\Pi_{j=1}^{n}\left(1-\left(\dot{\tilde{\theta}}_{i\mathfrak{d}(j)}\right)^3\right)^{\acute{\mathfrak{z}}_j}}{\Pi_{j=1}^{n}\left(1+(\sigma-1)\left(\dot{\tilde{\theta}}_{i\mathfrak{d}(j)}\right)^3\right)^{\acute{z}_j}+(\sigma-1)\Pi_{j=1}^{n}\left(1-\left(\dot{\tilde{\theta}}_{i\mathfrak{d}(j)}\right)^3\right)^{\acute{\mathfrak{z}}_j}}}, \\ \sqrt[3]{\dfrac{\Pi_{j=1}^{n}\left(1+(\sigma-1)\left(\dot{\tilde{\eth}}_{i\mathfrak{d}(j)}\right)^3\right)^{\acute{\mathfrak{z}}_j}-\Pi_{j=1}^{n}\left(1-\left(\dot{\tilde{\eth}}_{i\mathfrak{d}(j)}\right)^3\right)^{\acute{\mathfrak{z}}_j}}{\Pi_{j=1}^{n}\left(1+(\sigma-1)\left(\dot{\tilde{\eth}}_{i\mathfrak{d}(j)}\right)^3\right)^{\acute{\mathfrak{z}}_j}+(\sigma-1)\Pi_{j=1}^{n}\left(1-\left(\dot{\tilde{\eth}}_{i\mathfrak{d}(j)}\right)^3\right)^{\acute{\mathfrak{z}}_j}}} \end{array} \right) \tag{8}$$

Step 2. For every alternative $\mathfrak{K}_i$ calculate the value $\mathfrak{sc}\left(\breve{e}_i\right)$ of the score function of the corresponding aggregated CFE $\breve{e}_i$.

Step 3. Arrange in descending order all of the options $\mathfrak{K}_i$ in accordance with the values of the score function $\mathfrak{sc}\left(\breve{e}_i\right)$. Use accuracy degree $\mathfrak{ac}\left(\breve{e}_i\right)$ for the ranking of the CFEs with equal score values.

Step 4. Pick the most suitable option(s).

7 Cyclone disaster appraisal with the proposed algorithm

To demonstrate how to utilize the suggested techniques, we study a real-world case of the MADM problem in this section. Adverse weather conditions like gales, rainstorms, and storm surges frequently accompany cyclones, an extremely destructive weather system, and cause secondary disasters like flash floods, landslides, and mudslides. Disasters caused by cyclones are unpredictable and inevitable. Consequently, cyclone risk study is essential (adopted from Hadi et al. (2021)). One of the regions most susceptible to cyclones is the Pakistani province of Khyber Pakhtunkhwa (KP), and each year, huge economic losses are caused by cyclones. The catastrophic losses caused by various cyclones have been the subject of numerous studies. A severe storm struck the northwest of Pakistan in the night of April 26, 2015. In several of KP's cities, it seriously damaged the infrastructure. Heavy rain, hail, and gusty winds of more than 120 km/h were all part of the storm's effects. The devastating consequences of the storm resulted in numerous deaths or injuries. The storm also killed several animals, destroyed numerous harvests, toppled numerous electricity poles, and tore down numerous walls, roofs, and houses in rural areas of KP. To accurately evaluate cyclone devastation, it is important to note that a variety of indices must be considered at once. Let $\mathfrak{B} = \{\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3\}$, where $\mathfrak{B}_1$, $\mathfrak{B}_2$, and $\mathfrak{B}_3$ respectively indicate the three most important indicators: economic catastrophe, social impact, and environmental destruction. Let the weight vector of the operator be $\acute{\mathfrak{z}} = (0.23, 0.45, 0.32)^{\mathrm{T}}$ and $\sigma = 3$. Nowshehra ($\mathfrak{K}_1$), Sawat ($\mathfrak{K}_2$), Mardan ($\mathfrak{K}_3$), Charsadda ($\mathfrak{K}_4$), and Peshawar ($\mathfrak{K}_5$) are the five cities in KP that are being appraised. The data for this appraisal is in the form of CFNs, and it is performed by three experts. These CFNs are used to generate the CF decision matrix $M = \left(\breve{e}_{ij}\right)_{5\times3}$, as shown in Table 1.

Table 1 Cubical fuzzy decision matrix

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$
$\mathfrak{K}_1$	$(0.93, 0.53, 0.33)$	$(0.97, 0.39, 0.22)$	$(0.87, 0.65, 0.12)$
$\mathfrak{K}_2$	$(0.80, 0.50, 0.30)$	$(0.63, 0.23, 0.43)$	$(0.43, 0.63, 0.33)$
$\mathfrak{K}_3$	$(0.91, 0.21, 0.33)$	$(0.27, 0.17, 0.87)$	$(0.24, 0.22, 0.88)$
$\mathfrak{K}_4$	$(0.85, 0.25, 0.45)$	$(0.84, 0.24, 0.44)$	$(0.20, 0.89, 0.39)$
$\mathfrak{K}_5$	$(0.90, 0.15, 0.32)$	$(0.68, 0.58, 0.28)$	$(0.45, 0.87, 0.35)$

Table 2 Cubical fuzzy Hamacher ordered weighted decision matrix

	$\mathfrak{B}_{\mathfrak{d}(1)}$	$\mathfrak{B}_{\mathfrak{d}(2)}$	$\mathfrak{B}_{\mathfrak{d}(3)}$
$\mathfrak{K}_1$	$(.9890, .2687, .1267)$	$(.9418, .4959, .2960)$	$(.7595, .7950, .2995)$
$\mathfrak{K}_2$	$(.8177, .4652, .2670)$	$(.6885, .1342, .3061)$	$(.3671, .7823, .5505)$
$\mathfrak{K}_3$	$(.9235, .1817, .2960)$	$(.2945, .0906, .8234)$	$(.2055, .4338, .9293)$
$\mathfrak{K}_4$	$(.8972, .1419, .3157)$	$(.8666, .2193, .4146)$	$(.1713, .9349, .6051)$
$\mathfrak{K}_5$	$(.9142, .1265, .2863)$	$(.7416, .4604, .1736)$	$(.3840, .9236, .5693)$

In the following, we use the algorithm developed in the preceding section to identify the city in KP province that is the most affected. We take $\Omega = (0.3585, 0.4316, 0.2099)^{\mathrm{T}}$ as the weight vector for the attributes $\mathfrak{B}_i$.

7.1 By CFHHWA operator

Step 1. In this step, first, we utilize the relation $\dot{\breve{e}}_{ij} = 3\Omega_j \breve{e}_{ij}$ along with the Definition 14 to Table 1 to order the attribute $\mathfrak{B}_i$ for every alternative $\mathfrak{K}_i$ and obtain Table 2.
Applying the CFHHWA operator from Eq. (7) to Table 2 yields the aggregated values of the cities as follows:

$$\breve{e}_1 = (.9353, .5138, .2447), \breve{e}_2 = (.6626, .3263, .3600), \breve{e}_3 = (.6048, .1764, .7172),$$
$$\breve{e}_4 = (.7879, .3405, .4421), \breve{e}_5 = (.7377, .4573, .2877).$$

Step 2. Score function is applied, and the result is

$$\mathfrak{sc}(\breve{e}_1) = .8035, \mathfrak{sc}(\breve{e}_2) = .2443, \mathfrak{sc}(\breve{e}_3) = -.1477, \mathfrak{sc}(\breve{e}_4) = .4027, \mathfrak{sc}(\breve{e}_5) = .3776.$$

Step 3. According to descending order of scores, the following is how the cities are ranked:

$$\mathfrak{K}_1, \mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_3.$$

Step 4. Nowshehra ($\mathfrak{K}_1$) turns out to be the city that has sustained the most damage as a result.

7.2 By CFHHWG operator

Step 1. In this step first we utilize the relation $\dot{\breve{e}}_{ij} = \left(\breve{e}_{ij}\right)^{3\Omega_j}$ along with the Definition 14 to Table 1 to order the attribute $\mathfrak{B}_i$ for every alternative $\mathfrak{K}_i$ and obtain Table 3.

Table 3 Cubical fuzzy Hamacher ordered weighted decision matrix

	$\mathfrak{B}_{\mathfrak{d}(1)}$	$\mathfrak{B}_{\mathfrak{d}(2)}$	$\mathfrak{B}_{\mathfrak{d}(3)}$
$\mathfrak{K}_1$	(.9602, .4261, .2399)	(.9236, .5535, .1028)	(.9238, .5436, .3382)
$\mathfrak{K}_2$	(.7818, .5128, .3075)	(.6387, .5364, .2823)	(.5169, .2508, .4700)
$\mathfrak{K}_3$	(.9020, .2152, .3382)	(.4569, .1884, .7707)	(.1655, .1853, .9223)
$\mathfrak{K}_4$	(.8364, .2562, .4614)	(.7825, .2617, .4810)	(.4096, .7822, .3332)
$\mathfrak{K}_5$	(.8910, .1537, .3280)	(.6548, .7595, .2993)	(.5761, .6345, .3055)

Applying the CFHHWG operator from Eq. (8) to Table 3 yields the aggregated values of the cities as follows:

$$\breve{e}_1 = (.9323, .5256, .2516), \breve{e}_2 = (.6299, .4708, .3673), \breve{e}_3 = (.4044, .1943, .7939),$$
$$\breve{e}_4 = (.6642, .5457, .4382), \breve{e}_5 = (.6843, .6537, .3083).$$

Step 2. Score function is applied, and the result is

$$\mathfrak{sc}(\breve{e}_1) = .7944, \mathfrak{sc}(\breve{e}_2) = .2004, \mathfrak{sc}(\breve{e}_3) = -.4342, \mathfrak{sc}(\breve{e}_4) = .2089, \mathfrak{sc}(\breve{e}_5) = .2911.$$

Step 3. According to descending order of scores, the following is how the cities are ranked:

$$\mathfrak{K}_1, \mathfrak{K}_5, \mathfrak{K}_4, \mathfrak{K}_2, \mathfrak{K}_3.$$

Step 4. Nowshehra ($\mathfrak{K}_1$) turns out to be the city that has sustained the most damage as a result.

According to the discussion above, the most damaged city, which is $\mathfrak{K}_1$, remains the same even though the rating orders of the alternatives are different when utilizing the CFHHWA and CFHHWG operators.

8 Effect of different values of working parameter on the decision-making process

In order to know the different ranking order of alternatives by assigning values to working parameter σ, in the interval [1, 10] using CFHHA and CFHHG operators, we get Tables 4 and 5 respectively. In Table 4, different ranking orders of alternative are discussed using CFHHWA operator. In Table 5, we adopted CFHHWG operator and selected several values of the working parameter σ, in the interval [1, 10].

From Table 4, we observe that when σ takes on integral values from 1 to 10, the ranking of the alternatives remains the same.

Table 5 shows that with the exception of when σ is 1, the ordering of the alternatives remains the same when σ takes on integral values from 1 to 10.

From the discussion above, we note that the highest grading order of alternatives for both operators of CFHHWA & CFHHWG operators occurred at $\mathfrak{K}_1$ and the lowest alternative is at $\mathfrak{K}_3$.

Table 4 Ranks of alternatives for different values of σ using CFHHWA operator

σ	$\mathfrak{sc}(\breve{e}_1)$	$\mathfrak{sc}(\breve{e}_2)$	$\mathfrak{sc}(\breve{e}_3)$	$\mathfrak{sc}(\breve{e}_4)$	$\mathfrak{sc}(\breve{e}_5)$	Ranking order
1	.8088	.2557	−.0160	.4329	.4054	$\mathfrak{K}_1, \mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_3$
2	.8052	.2485	−.1025	.4150	.3872	$\mathfrak{K}_1, \mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_3$
3	.8035	.2443	−.1477	.4027	.3776	$\mathfrak{K}_1, \mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_3$
4	.8024	.2414	−.1767	.3929	.3715	$\mathfrak{K}_1, \mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_3$
5	.8015	.2392	−.1980	.3846	.3673	$\mathfrak{K}_1, \mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_3$
6	.8010	.2377	−.2132	.3778	.3642	$\mathfrak{K}_1, \mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_3$
7	.8007	.2364	−.2255	.3718	.3619	$\mathfrak{K}_1, \mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_3$
8	.8001	.2354	−.2352	.3665	.3601	$\mathfrak{K}_1, \mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_3$
9	.8001	.2347	−.2432	.3619	.3586	$\mathfrak{K}_1, \mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_3$
10	.7995	.2341	−.2494	.3578	.3575	$\mathfrak{K}_1, \mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_3$

Table 5 Ranks of alternatives for different values of σ using CFHHWG operator

σ	$\mathfrak{sc}(\breve{e}_1)$	$\mathfrak{sc}(\breve{e}_2)$	$\mathfrak{sc}(\breve{e}_3)$	$\mathfrak{sc}(\breve{e}_4)$	$\mathfrak{sc}(\breve{e}_5)$	Ranking order
1	.7859	.1827	−.4863	.1630	.2612	$\mathfrak{K}_1, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_4, \mathfrak{K}_3$
2	.7914	.1945	−.4549	.1956	.2809	$\mathfrak{K}_1, \mathfrak{K}_5, \mathfrak{K}_4, \mathfrak{K}_2, \mathfrak{K}_3$
3	.7944	.2004	−.4342	.2089	.2911	$\mathfrak{K}_1, \mathfrak{K}_5, \mathfrak{K}_4, \mathfrak{K}_2, \mathfrak{K}_3$
4	.7971	.2046	−.4190	.2161	.2981	$\mathfrak{K}_1, \mathfrak{K}_5, \mathfrak{K}_4, \mathfrak{K}_2, \mathfrak{K}_3$
5	.8009	.2082	−.4069	.2212	.3038	$\mathfrak{K}_1, \mathfrak{K}_5, \mathfrak{K}_4, \mathfrak{K}_2, \mathfrak{K}_3$
6	.8028	.2107	−.3971	.2239	.3076	$\mathfrak{K}_1, \mathfrak{K}_5, \mathfrak{K}_4, \mathfrak{K}_2, \mathfrak{K}_3$
7	.8050	.2130	−.3886	.2262	.3111	$\mathfrak{K}_1, \mathfrak{K}_5, \mathfrak{K}_4, \mathfrak{K}_2, \mathfrak{K}_3$
8	.8064	.2148	−.3817	.2276	.3138	$\mathfrak{K}_1, \mathfrak{K}_5, \mathfrak{K}_4, \mathfrak{K}_2, \mathfrak{K}_3$
9	.8078	.2163	−.3756	.2286	.3159	$\mathfrak{K}_1, \mathfrak{K}_5, \mathfrak{K}_4, \mathfrak{K}_2, \mathfrak{K}_3$
10	.8082	.2175	−.3702	.2291	.3176	$\mathfrak{K}_1, \mathfrak{K}_5, \mathfrak{K}_4, \mathfrak{K}_2, \mathfrak{K}_3$

9 Comparative analysis

The proposed and preexisting operators are contrasted in this section. The reader will see our attempt to demonstrate why the suggested operators are more dependable and effective. The PF Hamacher aggregation operators listed in Wei et al. (2018) are taken into consideration in order to accomplish this. For the model proposed in Wei et al. (2018) as an illustrative example, the following decision matrix was taken into account. Assume that the operators' weight vector is $\acute{\mathfrak{z}} = (.4, .1, .2, .3)^{\mathrm{T}}$.

To choose the most favorable alternative, we use the PFHHWA and PFHHWG operators with $\sigma = 3$. We start using the PFHHWA operator. Table 7 is obtained by applying the relation $\dot{\breve{e}}_{ij} = 4\Omega_j \breve{e}_{ij}$ under PF Hamacher operations on Table 6 with $\omega = (.2, .1, .3, .4)^{\mathrm{T}}$ as the attributes' weight vector:

Using the PFHHWA operator on Table 7, we calculate the aggregated values $\breve{e}_i (i = 1, 2, ..., 5)$ of attributes for the alternatives $\mathfrak{K}_i$ $(i = 1, 2, ..., 5)$ given as

$$\breve{e}_1 = (.4127, .4346, .0696), \breve{e}_2 = (.5858, .2423, .0777), \breve{e}_3 = (.5707, .2453, .0452),$$
$$\breve{e}_4 = (.4886, .3744, .2394), \breve{e}_5 = (.4544, .4048, .0247).$$

Table 6 The PF decision matrix

	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{B}_4$
$\mathfrak{K}_1$	(.53, .33, .09)	(.89, .08, .03)	(.42, .35, .18)	(.08, .89, .02)
$\mathfrak{K}_2$	(.73, .12, .08)	(.13, .64, .21)	(.03, .82, .13)	(.73, .15, .08)
$\mathfrak{K}_3$	(.91, .03, .02)	(.07, .09, .05)	(.04, .85, .10)	(.68, .26, .06)
$\mathfrak{K}_4$	(.85, .09, .05)	(.74, .16, .10)	(.02, .89, .05)	(.08, .84, .06)
$\mathfrak{K}_5$	(.90, .05, .02)	(.68, .08, .21)	(.05, .87, .06)	(.13, .75, .09)

Table 7 The PF Hamacher ordered weighted decision matrix

	$\mathfrak{B}_{\mathfrak{d}(1)}$	$\mathfrak{B}_{\mathfrak{d}(2)}$	$\mathfrak{B}_{\mathfrak{d}(3)}$	$\mathfrak{B}_{\mathfrak{d}(4)}$
$\mathfrak{K}_1$	(.4997, .2591, .1107)	(.4298, .4417, .1692)	(.1302, .8205, .0010)	(.4680, .4863, .3632)
$\mathfrak{K}_2$	(.9174, .0289, .0099)	(.6182, .2100, .1548)	(.0361, .7822, .0736)	(.0505, .8608, .6346)
$\mathfrak{K}_3$	(.8866, .0771, .0061)	(.8308, .0729, .0531)	(.0482, .8185, .0531)	(.0275, .5028, .4241)
$\mathfrak{K}_4$	(.7520, .1692, .1081)	(.0240, .8670, .0226)	(.3281, .5897, .5180)	(.2897, .4863, .6346)
$\mathfrak{K}_5$	(.8168, .1081, .0531)	(.2123, .5962, .0120)	(.1302, .7387, .0061)	(.0602, .8428, .0283)

Table 8 The PF Hamacher ordered weighted decision matrix

	$\mathfrak{B}_{\mathfrak{d}(1)}$	$\mathfrak{B}_{\mathfrak{d}(2)}$	$\mathfrak{B}_{\mathfrak{d}(3)}$	$\mathfrak{B}_{\mathfrak{d}(4)}$
$\mathfrak{K}_1$	(.9571, .0314, .0119)	(.6219, .2623, .0715)	(.3284, .4197, .2178)	(.0099, .9831, .0322)
$\mathfrak{K}_2$	(.7859, .0952, .0636)	(.5571, .2669, .0810)	(.5659, .2450, .1302)	(.0121, .8893, .1572)
$\mathfrak{K}_3$	(.9285, .0239, .0160)	(.4680, .0352, .0197)	(.4926, .4190, .0973)	(.0172, .9120, .1208)
$\mathfrak{K}_4$	(.8999, .0620, .0391)	(.8811, .0715, .0398)	(.0074, .9403, .0602)	(.0099, .9677, .0973)
$\mathfrak{K}_5$	(.9206, .0398, .0160)	(.8766, .0314, .0810)	(.0226, .9264, .0723)	(.0225, .9282, .1466)

Score function is utilized, and the result is

$$\mathfrak{sc}\left(\breve{e}_1\right) = .6716, \mathfrak{sc}\left(\breve{e}_2\right) = .7541, \mathfrak{sc}\left(\breve{e}_3\right) = .7628\ , \mathfrak{sc}\left(\breve{e}_4\right) = .6246, \mathfrak{sc}\left(\breve{e}_5\right) = .7149.$$

The alternatives are ordered in descending order of scores as follows:

$$\mathfrak{K}_3, \mathfrak{K}_2, \mathfrak{K}_5, \mathfrak{K}_1, \mathfrak{K}_4.$$

Now, in order to select the optimal substitute, we use the PFHHWG operator(s). Applying the relation $\dot{\breve{e}}_{ij} = \left(\breve{e}_{ij}\right)^{4\Omega_j}$ under PF Hamacher operations to Table 6 yields Table 8. Using the PFHHWG operator on Table 8, we calculate the aggregated values $\breve{e}_i (i = 1, 2, ..., 5)$ of attributes for the alternatives $\mathfrak{K}_i$ $(i = 1, 2, ..., 5)$ given as

$$\breve{e}_1 = (.2591, .6522, .0634)\,, \breve{e}_2 = (.2609, .4662, .1063)\,, \breve{e}_3 = (.3051, .4741, .0633)\,,$$
$$\breve{e}_4 = (.1284, .7326, .0606)\,, \breve{e}_5 = (.1999, .6527, .0720)\,.$$

After employing the score function, the output is

$$\mathfrak{sc}\left(\breve{e}_1\right) = .5979, \mathfrak{sc}\left(\breve{e}_2\right) = .5773, \mathfrak{sc}\left(\breve{e}_3\right) = .6209, \mathfrak{sc}\left(\breve{e}_4\right) = .5339, \mathfrak{sc}\left(\breve{e}_5\right) = .5640.$$

Table 9 The CF Hamacher ordered weighted decision matrix

	$\mathfrak{B}_{\mathfrak{d}(1)}$	$\mathfrak{B}_{\mathfrak{d}(2)}$	$\mathfrak{B}_{\mathfrak{d}(3)}$	$\mathfrak{B}_{\mathfrak{d}(4)}$
$\mathfrak{K}_1$	(.6731, .4445, .3044)	(.4905, .4375, .1566)	(.4472, .2656, .1188)	(.0936, .8085, .0015)
$\mathfrak{K}_2$	(.8440, .0386, .0141)	(.6771, .1970, .1425)	(.0319, .7764, .0804)	(.0958, .8764, .6286)
$\mathfrak{K}_3$	(.8630, .0651, .0470)	(.6771, .1970, .1425)	(.0425, .8137, .0586)	(.0516, .4646, .3714)
$\mathfrak{K}_4$	(.7966, .1566, .0979)	(.5401, .5725, .4831)	(.0936, .7213, .0089)	(.0212, .8637, .0255)
$\mathfrak{K}_5$	(.8514, .0979, .0470)	(.1521, .5771, .0170)	(.0531, .8386, .0318)	(.4948, .4445, .6286)

Table 10 The CF Hamacher ordered weighted decision matrix

	$\mathfrak{B}_{\mathfrak{d}(1)}$	$\mathfrak{B}_{\mathfrak{d}(2)}$	$\mathfrak{B}_{\mathfrak{d}(3)}$	$\mathfrak{B}_{\mathfrak{d}(4)}$
$\mathfrak{K}_1$	(.9604, .0589, .0221)	(.6270, .3060, .0835)	(.3321, .3724, .1913)	(.0141, .9664, .0234)
$\mathfrak{K}_2$	(.7922, .1114, .0743)	(.5477, .1755, .0936)	(.5317, .4656, .1546)	(.0138, .8627, .1382)
$\mathfrak{K}_3$	(.9300, .0278, .0186)	(.4786, .3046, .0702)	(.4226, .0663, .0368)	(.0195, .8904, .1063)
$\mathfrak{K}_4$	(.9112, .1178, .0737)	(.8844, .0835, .0464)	(.0085, .9253, .0531)	(.0141, .9361, .0702)
$\mathfrak{K}_5$	(.9224, .0464, .0186)	(.8907, .0589, .1546)	(.0255, .9081, .0638)	(.0307, .8630, .1053)

The alternatives are ordered in descending order of scores as follows:

$$\mathfrak{K}_3, \mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_5, \mathfrak{K}_4.$$

To assess the alternatives, we now use the CFHHWA and CFHHWG operators. We start using the CFHHWA operator. Applying the relation $\dot{\breve{e}}_{ij} = 4\Omega_j \breve{e}_{ij}$ under CF Hamacher operations to Table 6 yields Table 9.

Using the CFHHWA operator on Table 9, we calculate the aggregated values $\breve{e}_i (i = 1, 2, ..., 5)$ of attributes for the alternatives $\mathfrak{K}_i$ $(i = 1, 2, ..., 5)$ given as

$$\breve{e}_1 = (.5301, 0.4919, .0481), \breve{e}_2 = (.6532, .2293, .0801), \breve{e}_3 = (.6698, .2248, .1024),$$
$$\breve{e}_4 = (.6013, .4296, .0476), \breve{e}_5 = (.6649, .2938, .0870).$$

After employing the score function, the output is

$$\mathfrak{sc}(\breve{e}_1) = .1489, \mathfrak{sc}(\breve{e}_2) = .2782, \mathfrak{sc}(\breve{e}_3) = .2994, \mathfrak{sc}(\breve{e}_4) = .2173, \mathfrak{sc}(\breve{e}_5) = .2933.$$

The alternatives are ordered in descending order of scores as follows:

$$\mathfrak{K}_3, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_4, \mathfrak{K}_1.$$

The CFHHWG operator is now used. When the CF Hamacher operations are used, the relation $\dot{\breve{e}}_{ij} = \left(\breve{e}_{ij}\right)^{4\Omega_j}$ and Table 6 results in Table 10.

Using the CFHHWG operator on Table 10, we calculate the aggregated values $\breve{e}_i (i = 1, 2, ..., 5)$ of attributes for the alternatives $\mathfrak{K}_i$ $(i = 1, 2, ..., 5)$ given as

$$\breve{e}_1 = (.2373, .7251, .1135), \breve{e}_2 = (.2225, .6051, .1211), \breve{e}_3 = (.2588, .6127, .0741),$$
$$\breve{e}_4 = (.1144, .7803, .0672), \breve{e}_5 = (.1819, .7180, .0918).$$

The following are the values of the score function employed to $\breve{e}_i$:

$$\mathfrak{sc}(\breve{e}_1) = .0119, \mathfrak{sc}(\breve{e}_2) = .0092, \mathfrak{sc}(\breve{e}_3) = .0169, \mathfrak{sc}(\breve{e}_4) = .0012, \mathfrak{sc}(\breve{e}_5) = .0052.$$

Table 11 Comparison of PFHHWA and CFHHWA operators

PFHHWA-operator	CFHHWA-operator
$\mathfrak{K}_3, \mathfrak{K}_2, \mathfrak{K}_5, \mathfrak{K}_1, \mathfrak{K}_4$	$\mathfrak{K}_3, \mathfrak{K}_5, \mathfrak{K}_2, \mathfrak{K}_4, \mathfrak{K}_1$

Table 12 Comparison of PFHHWG and CFHHWG operators

PFHHWG operator	CFHHWG operator
$\mathfrak{K}_3, \mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_5, \mathfrak{K}_4$	$\mathfrak{K}_3, \mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_5, \mathfrak{K}_4$

The alternatives are ordered in descending order of scores as follows:

$$\mathfrak{K}_3, \mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_5, \mathfrak{K}_4.$$

Tables 11 and 12 compare the PFHHWA/PFHHWG operator's ranking order of alternatives with the suggested operators.

Table 11 reveals that the optimal alternative for the PFHHWA operator and proposed operator (CFHHWA) is $\mathfrak{K}_3$. The ranking order of alternatives when applying both types of operators differs, which is another important point to consider. According to Table 12, the best alternative is once again $\mathfrak{K}_3$, and the suggested operator (PFHHWG) and the PFHHWG operator both grade the alternatives in the same order.

We can infer from the discussion above that $\mathfrak{K}_3$ is the optimal option for both existing and proposed operators. Since the idea of cubical fuzzy sets has a larger area than that of picture fuzzy sets, our recommended operators are more reliable and effective at solving decision-making problems.

10 Concluding remarks

Using cubical fuzzy information based on Hamacher operations, we investigated a multiple attribute decision-making problem in this paper. From the idea of Hamacher operations, we have introduced arithmetic and geometric operations to construct various cubical fuzzy Hamacher aggregation operators like CFHWA operator, CFHOWA operator, CFHHWA operator, CFHWG operator, CFHOWG operator, and CFHHWG operator. The various properties of these suggested operators are discussed. Then, we developed a few ways to address multi-attribute decision-making issues using these operators. Finally, a real-world example of how to assess a cyclone disaster is presented to show how effective the suggested approaches are. A comparative analysis was also give and it was found that the optimal alternative for the PFHHWA operator and proposed operator (CFHHWA) was the same but the ranking order of alternatives was different differs, an important point to consider. In the case of PFHHWG and the suggested (CFHHWG) operator, the best alternative was once again the same, and both of the operators graded the alternatives in the same order. Therefore, the suggested operators are just as efficient as the existing operators, with the added benefit of having a wider range of membership grades.

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Declarations

Conflict of interest There is no competing interest.

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