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Strong convergence rate of the stochastic theta method for nonlinear hybrid stochastic differential equations with piecewise continuous arguments

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Abstract
We consider the strong convergence of the stochastic theta (ST) method for highly nonlinear hybrid stochastic differential equations with piecewise continuous arguments (SDEPCAs). There are three major ingredients. The first is the pth moment boundedness of the ST method. Second, the mean square convergence rate of the ST method for hybrid SDEPCAs is given by means of the forward–backward Euler–Maruyama method. The third ingredient is a numerical simulation, which shows the agreement with the theoretical convergence rate.

Keywords Stochastic differential equations with piecewise continuous arguments (SDEPCAs) · Stochastic theta (ST) method · Forward–backward Euler–Maruyama (FBEM) method · Convergence rate

Mathematics Subject Classification 65C30 · 60H35

1 Introduction

Stochastic differential equations with piecewise continuous arguments (SDEPCAs) play an important role in stochastic theory. Such models are applicable in a variety areas including biology, control science and neural networks (Li 2014; Mao et al. 2014; You et al. 2015;

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Xie and Zhang 2020). Component failures, changes in subsystem interconnections and sudden environmental disturbances can lead to abrupt changes of structures and parameters in many practical systems. To tackle these problems, hybrid systems driven by continuous-time Markov chains have become a powerful tool (Jobert and Rogers 2006; Smith 2002; Song and Mao 2018). Furthermore, it is well known that Markov chain can work as a stabilizing factor (Li and Mao 2020; Hu et al. 2020), that is, the whole system can be stable even though some subsystems are stable and others are unstable, such property is referred to as switching-dominated stability in Zhang et al. (2019).

Since explicit solutions are almost impossible to obtain for such systems, it becomes extremely important to solve them numerically. Finite time convergence analysis of an Euler type method for stochastic differential equations (SDEs) with Markovian switching was given in Mao et al. (2007) and Yuan and Mao (2004). It has been extended to stochastic differential delay equations (SDDEs) with Markovian switching (Li and Hou 2006; Milošević and Jovanović 2011; Zhang and Xie 2019), SDDEs with Markovian switching and Poisson jump (Li and Chang 2007; Wang and Xue 2007) and neutral SDDEs with Markovian switching (Yin and Ma 2011; Zhou and Wu 2009), etc. Numerical invariant measure of the backward Euler–Maruyama method for SDEs with Markovian switching was investigated in Li et al. (2018). It is worth mentioning that most of previous studies for hybrid systems is devoted to those equations driven by a continuous-time and homogeneous Markov chain independent of the Brownian motion, and the switching process $r(t)$ was assumed to have a finite state space. Recently, Yin et al. extended the study to the Markov process $(X(t), r(t))$ by allowing the generator $r(t)$ to depend on the current state $X(t)$ and to have a countable state space (Yin and Zhu 2010; Nguyen and Yin 2016).

To our best knowledge, there are few works on SDEPCAs with Markovian switching. An SDEPCA belongs to the SDDEs, but the delay term is different from $t - \tau$ and may be a discontinuous function. Moreover, although the SDEPCAs are retarded, the solutions of these equations are determined by only a finite set of initial data, rather than a function, as in the case of general SDDEs (Wiener 1993; Mao 2007). Because of these characteristics, we cannot simply generalize the properties of hybrid SDDEs to hybrid SDEPCAs.

In this work, we concentrate on the numerical solutions for highly nonlinear hybrid SDEPCAs, the stochastic theta (ST) scheme, which is an extension of the Euler–Maruyama method and the backward Euler–Maruyama, is adopted. The rest of this paper is organized as follows. Some basic notations and assumptions are introduced in Sect. 2. The ST method for hybrid SDEPCAs is established in Sect. 3. Section 4 is devoted to the pth moment boundedness of the ST method. Then we go further to reveal the strong convergence rate of the numerical method in Sect. 5. Finally, a numerical experiment is given in Sect. 6 to verify our theoretical convergence order.

2 Notations and preliminaries

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). If A is a vector or matrix, its transpose is denoted by A^{T}. Let $B(t) = (B_1(t), \ldots, B_d(t))^{\mathrm{T}}$ be a d-dimensional Brownian motion defined on the probability space. If x is a vector, $\|x\|$ denotes its Euclidean norm. $\langle x, y\rangle$ denotes the inner product of vectors x and y. If A is a matrix, its trace norm is denoted by $\|A\| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$. For two real numbers a and b, we will use $a \vee b$ and $a \wedge b$ for the max $\{a, b\}$

and min $\{a, b\}$, respectively. Let $\mathcal{L}^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ denotes the family of $\mathcal{F}_t$-measurable $\mathbb{R}^n$-valued random variables ξ with $\mathbb{E}\|\xi\|^p < \infty$. Let $\mathcal{L}^p([a, b]; \mathbb{R}^n)$ denotes the family of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\le t\le b}$ such that $\int_a^b \|f(t)\|^p \mathrm{d}t < \infty$, a.s., $\mathcal{L}^p(\mathbb{R}_+; \mathbb{R}^n)$ denotes the family of processes $\{f(t)\}_{t\ge 0}$ such that for every $T > 0, \{f(t)\}_{0\le t\le T} \in \mathcal{L}^p([0, T]; \mathbb{R}^n)$. $\mathcal{M}^p([a, b]; \mathbb{R}^n)$ denotes the family of processes $\{f(t)\}_{a\le t\le b}$ in $\mathcal{L}^p([a, b]; \mathbb{R}^n)$ such that $\mathbb{E}\int_a^b \|f(t)\|^p \mathrm{d}t < \infty$. $[\cdot]$ denotes the greatest integer function.

Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from i to j when $i \neq j$, and

$$\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

Consider the following hybrid SDEPCAs

$$\mathrm{d}x(t) = f(x(t), x([t]), r(t))\mathrm{d}t + g(x(t), x([t]), r(t))\mathrm{d}B(t), \quad t \ge 0, \tag{1}$$

with initial data $x(0) = x_0 \in \mathbb{R}^n$ and $r(0) = i_0 \in S$, where

$$f : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^{n\times d}.$$

Let us give the definition of the solution.

Definition 1 An $\mathbb{R}^n$-valued stochastic process $\{x(t)\}_{t\ge 0}$ is called a solution of (1) if it has the following properties:

(1) $\{x(t)\}$ is continuous and $\mathcal{F}_t$-adapted;
(2) $\{f(x(t), x([t]), r(t))\} \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^n)$, $\{g(x(t), x([t]), r(t))\} \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{n\times d})$;
(3) Equation (1) is satisfied on each interval $[n, n+1) \subset [0, \infty)$ with integral end-points almost surely.

A solution $\{x(t)\}$ is said to be *unique* if any other solution $\{\bar{x}(t)\}$ is indistinguishable from $\{x(t)\}$, that is

$$\mathbb{P}\{x(t) = \bar{x}(t) \text{ for all } t \ge 0\} = 1.$$

We impose some assumptions:

Assumption 2.1 For every integer $R \ge 1$, there exists a constant $L(R) > 0$ such that

$$\|f(x, y, i) - f(\bar{x}, \bar{y}, i)\| \vee \|g(x, y, i) - g(\bar{x}, \bar{y}, i)\| \le L(R)(\|x - \bar{x}\| + \|y - \bar{y}\|),$$

for all $i \in S$ and those $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ with $\|x\| \vee \|y\| \vee \|\bar{x}\| \vee \|\bar{y}\| \le R$.

Assumption 2.2 There exists a constant $\alpha > 0$ such that

$$x^{\mathrm{T}} f(x, y, i) \le \alpha(1 + \|x\|^2 + \|y\|^2), \ \forall x, y \in \mathbb{R}^n, \ \forall i \in S.$$

Assumption 2.3 There exist constants $L_1 > 0$ and $h_1 \ge 1$ such that

$$\|f(x, y, i)\| \le L_1(1 + \|x\|^{h_1} + \|y\|^{h_1}), \ \forall x, y \in \mathbb{R}^n, \ \forall i \in S.$$

Assumption 2.4 There exist constants $L_2 > 0$ and $h_2 \geq 1$ such that

$$\|g(x, y, i)\| \leq L_2(1 + \|x\| + \|y\|^{h_2}),\ \forall x, y \in \mathbb{R}^n,\ \forall i \in S.$$

Theorem 2.5 *Under Assumptions 2.1–2.4, for any $T > 0$, there exists a unique global solution $x(t)$ to Eq. (1) on $t \in [0, T]$ with the initial data x_0. Moreover, the solution has the property that $\mathbb{E}\|x(t)\|^p < \infty$ for all $t \in [0, T]$.*

Proof For any given $i \in S$, we first prove that there exists a unique global solution to the SDEPCA

$$\mathrm{d}x(t) = f(x(t), x([t]), i)\mathrm{d}t + g(x(t), x([t]), i)\mathrm{d}B(t) \tag{2}$$

on $t \in [0, T]$ with the initial data x_0 and the solution has the property that $\mathbb{E}\|x(t)\|^p < \infty$ for all $t \in [0, T]$. To distinguish between the solution of (1) and that of (2), we denote the solution of (2) by $y(t)$.

In a similar way as the proof of Theorem 3.15 in Mao and Yuan (2006), there is a unique maximal local solution $y(t)$ exists on $[0, \eta_e)$ under the local Lipschitz condition, where η_e is the explosion time. Then for each integer $R \geq \|x_0\|$, define the stopping time $\eta_R = \inf\{t \in [0, \eta_e) : \|y(t)\| \geq R\}$. Clearly, η_R is increasing as $R \to \infty$. We denote that $\eta_\infty = \lim_{R\to\infty} \eta_R$ and $\inf \emptyset = \infty$. Hence, $\eta_\infty \leq \eta_e$ almost surely. If we can obtain $\eta_\infty = \infty$ almost surely, then $\eta_e = \infty$ almost surely. In what follows, we will prove $\eta_\infty = \infty$ almost surely and $\mathbb{E}\|y(t)\|^p < \infty$.

Applying Itô's formula to $\|y(t)\|^p$, $p \geq 2$, we have

$$\begin{aligned} d(\|y(t)\|^p) \leq & p\|y(t)\|^{p-2}\left(y^{\mathrm{T}}(t)f(y(t), y([t]), i) + \frac{p-1}{2}\|g(y(t), y([t]), i)\|^2\right)\mathrm{d}t \\ & + p\|y(t)\|^{p-2}y^{\mathrm{T}}(t)g(y(t), y([t]), i)\mathrm{d}B(t). \end{aligned} \tag{3}$$

- Take any $t \in [0, 1)$, integrating both sides of (3) from 0 to $t \wedge \eta_R$, then

$$\begin{aligned} &\|y(t\wedge\eta_R)\|^p \\ &\quad \leq \|y(0)\|^p + p\int_0^{t\wedge\eta_R}\|y(s)\|^{p-2}\left(y^{\mathrm{T}}(s)f(y(s), y(0), i) + \frac{p-1}{2}\|g(y(s), y(0), i)\|^2\right)\mathrm{d}s \\ &\quad + p\int_0^{t\wedge\eta_R}\|y(s)\|^{p-2}y^{\mathrm{T}}(s)g(y(s), y(0), i)\mathrm{d}B(s). \end{aligned}$$

Hence,

$$\begin{aligned} &\mathbb{E}\|y(t\wedge\eta_R)\|^p \\ &\quad \leq \|x_0\|^p + p\mathbb{E}\int_0^{t\wedge\eta_R}\|y(s)\|^{p-2}\left(y^{\mathrm{T}}(s)f(y(s), x_0, i) + \frac{p-1}{2}\|g(y(s), x_0, i)\|^2\right)\mathrm{d}s. \end{aligned}$$

By Assumptions 2.2 and 2.4, together with Young's inequality, one has

$$\begin{aligned}
\mathbb{E}\|y(t\wedge\eta_R)\|^p \leq& \|x_0\|^p + p\mathbb{E}\int_0^{t\wedge\eta_R} \|y(s)\|^{p-2}\Big(\alpha(1+\|y(s)\|^2+\|x_0\|^2) \\
&+ \frac{(p-1)L_2^2}{2}(1+\|y(s)\|+\|x_0\|^{h_2})^2\Big)\mathrm{d}s \\
\leq& \|x_0\|^p + p\mathbb{E}\int_0^{t\wedge\eta_R}\Big\{\Big(\gamma_0+\alpha\|x_0\|^2+3(p-1)L_2^2/2\|x_0\|^{2h_2}\Big)\|y(s)\|^{p-2} \\
&+\gamma_0\|y(s)\|^p\Big\}\mathrm{d}s \\
\leq& \|x_0\|^p + p\mathbb{E}\int_0^{t\wedge\eta_R}\Big\{\gamma_0^{\frac{p}{2}}+\alpha^{\frac{p}{2}}\|x_0\|^p+(3(p-1)L_2^2/2)^{\frac{p}{2}}\|x_0\|^{ph_2} \\
&+(3+\gamma_0)\|y(s)\|^p\Big\}\mathrm{d}s \\
\leq& pt\gamma_0^{\frac{p}{2}}+(1+pt\alpha^{\frac{p}{2}})\|x_0\|^p+pt(3(p-1)/2)^{\frac{p}{2}}L_2^p\|x_0\|^{ph_2} \\
&+(3+\gamma_0)p\int_0^t\mathbb{E}\|y(s\wedge\eta_R)\|^p\mathrm{d}s,
\end{aligned}$$

where $\gamma_0=\alpha+3(p-1)L_2^2/2$. Now it can be obtained from Gronwall's inequality that

$$\begin{aligned}
\mathbb{E}\|y(t\wedge\eta_R)\|^p \leq& \Big(pt\gamma_0^{\frac{p}{2}}+(1+pt\alpha^{\frac{p}{2}})\|x_0\|^p+pt(3(p-1)/2)^{\frac{p}{2}}L_2^p\|x_0\|^{ph_2}\Big)e^{(3+\gamma_0)pt} \\
\leq& \beta_0 e^{(3+\gamma_0)p},
\end{aligned} \tag{4}$$

where

$$\beta_0 = p\gamma_0^{\frac{p}{2}}+(1+p\alpha^{\frac{p}{2}})\|x_0\|^p+p(3(p-1)/2)^{\frac{p}{2}}L_2^p\|x_0\|^{ph_2}<\infty.$$

Thus,

$$\mathbb{E}\|y(1\wedge\eta_R)\|^p=\lim_{t\to 1}\mathbb{E}\|y(t\wedge\eta_R)\|^p\leq\beta_0e^{(3+\gamma_0)p}. \tag{5}$$

Let I_G denote the indicator function of the set G, then

$$\beta_0e^{(3+\gamma_0)p}\geq\mathbb{E}\|y(1\wedge\eta_R)\|^p\geq\mathbb{E}\left(\|y(\eta_R)\|^pI_{\{\eta_R\leq1\}}\right)\geq R^p\mathbb{P}(\eta_R\leq1);$$

hence,

$$\mathbb{P}(\eta_R\leq1)\leq\frac{\beta_0e^{(3+\gamma_0)p}}{R^p}.$$

Let $R\to\infty$, we have $\mathbb{P}(\eta_\infty\leq1)=0$, which gives

$$\mathbb{P}(\eta_\infty>1)=1,$$

combining (4) and (5), it can be obtained that

$$\mathbb{E}\|y(t)\|^p\leq\beta_0e^{(3+\gamma_0)p},\quad t\in[0,1].$$

- For any $t \in [1, 2)$, similar to the process above, integrating both sides of (3) from 1 to $t \wedge \eta_R$, and then taking the expectations, we can arrive at

$$
\begin{aligned}
&\mathbb{E}\|y(t \wedge \eta_R)\|^p \\
&\le \mathbb{E}\|y(1)\|^p + p\mathbb{E}\int_1^{t\wedge\eta_R} \|y(s)\|^{p-2}\left(y^{\mathrm{T}}(s)f(y(s), y(1), i) + \frac{p-1}{2}\|g(y(s), y(1), i)\|^2\right)\mathrm{d}s \\
&\le \mathbb{E}\|y(1)\|^p + p\mathbb{E}\int_1^{t\wedge\eta_R} \|y(s)\|^{p-2}\Big(\alpha(1 + \|y(s)\|^2 + \|y(1)\|^2) \\
&\quad + \frac{(p-1)L_2^2}{2}(1 + \|y(s)\| + \|y(1)\|^{h_2})^2\Big)\mathrm{d}s \\
&\le \mathbb{E}\|y(1)\|^p + p\mathbb{E}\int_1^{t\wedge\eta_R} \Big\{\left(\gamma_0 + \alpha\|y(1)\|^2 + 3(p-1)L_2^2/2\|y(1)\|^{2h_2}\right)\|y(s)\|^{p-2} \\
&\quad + \gamma_0\|y(s)\|^p\Big\}\mathrm{d}s \\
&\le \mathbb{E}\|y(1)\|^p + p\mathbb{E}\int_1^{t\wedge\eta_R} \Big\{\gamma_0^{\frac{p}{2}} + \alpha^{\frac{p}{2}}\|y(1)\|^p + (3(p-1)L_2^2/2)^{\frac{p}{2}}\|y(1)\|^{ph_2} \\
&\quad + (3+\gamma_0)\|y(s)\|^p\Big\}\mathrm{d}s \\
&\le p(t-1)\gamma_0^{\frac{p}{2}} + (1 + p(t-1)\alpha^{\frac{p}{2}})\mathbb{E}\|y(1)\|^p + p(t-1)(3(p-1)/2)^{\frac{p}{2}}L_2^p\mathbb{E}\|y(1)\|^{ph_2} \\
&\quad + (3+\gamma_0)p\int_1^t \mathbb{E}\|y(s\wedge\eta_R)\|^p\mathrm{d}s.
\end{aligned}
$$

By Gronwall's inequality, one has

$$
\mathbb{E}\|y(t \wedge \eta_R)\|^p \le \beta_1 e^{(3+\gamma_0)p}, \tag{6}
$$

where

$$
\beta_1 = \left(p\gamma_0^{\frac{p}{2}} + (1 + p\alpha^{\frac{p}{2}})\mathbb{E}\|y(1)\|^p + p(3(p-1)/2)^{\frac{p}{2}}L_2^p\mathbb{E}\|y(1)\|^{ph_2}\right) < \infty.
$$

Hence,

$$
\mathbb{E}\|y(2 \wedge \eta_R)\|^p = \lim_{t\to 2}\mathbb{E}\|y(t\wedge\eta_R)\|^p \le \beta_1 e^{(3+\gamma_0)p}, \tag{7}
$$

it gives

$$
\beta_1 e^{(3+\gamma_0)p} \ge \mathbb{E}\|y(2\wedge\eta_R)\|^p \ge \mathbb{E}\left(\|y(\eta_R)\|^p I_{\{\eta_R\le 2\}}\right) \ge R^p\mathbb{P}(\eta_R \le 2).
$$

Taking $R \to \infty$, yields

$$
\mathbb{P}(\eta_\infty \le 2) \le \lim_{R\to\infty}\frac{\beta_1 e^{(3+\gamma_0)p}}{R^p} = 0,
$$

which implies

$$
\mathbb{P}(\eta_\infty > 2) = 1,
$$

then combining (6) and (7), it can be obtained that

$$
\mathbb{E}\|y(t)\|^p \le \beta_1 e^{(3+\gamma_0)p}, \quad t \in [1, 2].
$$

- Repeating this procedure we can deduce that, for any integer $j \geq 1$,

$$\mathbb{P}(\eta_\infty > j) = 1,$$

and

$$\mathbb{E}\|y(t)\|^p \leq \beta_j e^{(3+\gamma_0)p}, \quad t \in [j, j+1], \tag{8}$$

where

$$\beta_j = \left(p\gamma_0^{\frac{p}{2}} + (1 + p\alpha^{\frac{p}{2}})\mathbb{E}\|y(j)\|^p + p(3(p-1)/2)^{\frac{p}{2}} L_2^p \mathbb{E}\|y(j)\|^{ph_2} \right) < \infty.$$

Since $j \geq 1$ is an arbitrary integer, we can conclude that $\eta_\infty = \infty$ almost surely and $\mathbb{E}\|y(t)\|^p < \infty, \forall t \geq 0$.

Now we are in a position to prove Eq.(1) has a unique global solution $x(t)$ and the solution has the property that $\mathbb{E}\|x(t)\|^p < \infty$.

It is well known (see Anderson 2012) that almost every sample path of the Markov chain $r(\cdot)$ is a right-continuous step function with a finite number of sample jumps in any finite subinterval of $\mathbb{R}_+$. Hence, there is a sequence of stopping times $0 = \tau_0 < \tau_1 < \cdots < \tau_k < \cdots$ such that

$$r(t) = \sum_{k=0}^{\infty} r(\tau_k) I_{[\tau_k, \tau_{k+1})}(t), \quad t \geq 0.$$

We first consider Eq. (1) on $t \in [\tau_0, \tau_1)$, which becomes

$$\mathrm{d}x(t) = f(x(t), x([t]), i_0)\mathrm{d}t + g(x(t), x([t]), i_0)\mathrm{d}B(t), \tag{9}$$

with initial data x_0. By the existence-and-uniqueness proof for SDEPCA (2), we know that Eq. (9) has a unique continuous solution which belongs to $\mathcal{M}^2([\tau_0, \tau_1); \mathbb{R}^n)$ and has the property that $\mathbb{E}\|x(t)\|^p < \infty$. In particular, $x(\tau_1) = \lim_{t \to \tau_1^-} x(t) \in L^2_{\mathcal{F}_{\tau_1}}(\Omega; \mathbb{R}^n)$. We next consider Eq. (1) on $t \in [\tau_1, \tau_2)$, which becomes

$$\mathrm{d}x(t) = f(x(t), x([t]), r(\tau_1))\mathrm{d}t + g(x(t), x([t]), r(\tau_1))\mathrm{d}B(t), \tag{10}$$

with initial data $x(\tau_1)$ given by the solution of Eq. (9). Again we know that Eq. (10) has a unique continuous solution which belongs to $\mathcal{M}^2([\tau_1, \tau_2); \mathbb{R}^n)$ and has the property that $\mathbb{E}\|x(t)\|^p < \infty$. Repeating this procedure, we see that Eq. (1) has a unique solution $x(t)$ on $[0, T]$ and has the property that

$$\mathbb{E}\|x(t)\|^p < \infty, \quad \forall t \in [0, T].$$

The proof is completed. □

3 Stochastic theta method

To define the ST scheme, let us first explain how to simulate the discrete Markov chain $\left\{r_k^\Delta, k = 0, 1, 2, \ldots\right\}$. Recall the property of the embedded discrete Markov chain:

Given a step size $\Delta > 0$, let $r_k^\Delta = r(k\Delta)$ for $k \geq 0$. Then $\left\{r_k^\Delta, k = 0, 1, 2, \ldots\right\}$ is a discrete Markov chain with the one-step transition probability matrix

$$\mathbb{P}(\Delta) = (\mathbb{P}_{ij}(\Delta))_{N \times N} = e^{\Gamma\Delta}.$$

Hence, the discrete Markov chain $\left\{r_k^{\Delta}, k=0,1,2,\ldots\right\}$ can be simulated as follows: Let $r_0^{\Delta}=i_0$ and compute a pseudo-random number ζ_1 from the uniform $[0,1]$ distribution. Define

$$r_1^{\Delta}=\begin{cases} i_1, & \text{if } i_1\in S-\{N\} \text{ such that } \sum_{j=1}^{i_1-1}\mathbb{P}_{i_0,j}(\Delta)\le\zeta_1<\sum_{j=1}^{i_1}\mathbb{P}_{i_0,j}(\Delta),\\ N, & \text{if } \sum_{j=1}^{N-1}\mathbb{P}_{i_0,j}(\Delta)\le\zeta_1,\end{cases}$$

where we set $\sum_{j=1}^{0}\mathbb{P}_{i_0,j}(\Delta)=0$ as usual. In other words, we ensure that the probability of state s being chosen is given by $\mathbb{P}(r_1^{\Delta}=s)=\mathbb{P}_{i_0,s}(\Delta)$. Generally, having calculated $r_0^{\Delta},r_1^{\Delta},\ldots,r_k^{\Delta}$, we compute r_{k+1}^{Δ} by drawing a uniform $[0,1]$ pseudo-random number ζ_{k+1} and setting

$$r_{k+1}^{\Delta}=\begin{cases} i_{k+1}, & \text{if } i_{k+1}\in S-\{N\} \text{ such that}\\ & \quad\sum_{j=1}^{i_{k+1}-1}\mathbb{P}_{r_k^{\Delta},j}(\Delta)\le\zeta_{k+1}<\sum_{j=1}^{i_{k+1}}\mathbb{P}_{r_k^{\Delta},j}(\Delta),\\ N, & \text{if } \sum_{j=1}^{N-1}\mathbb{P}_{r_k^{\Delta},j}(\Delta)\le\zeta_{k+1}.\end{cases}$$

This procedure can be carried out independently to obtain more trajectories.

After explaining how to simulate the discrete Markov chain, we can now define the ST approximate solution to Eq. (1). Let $\Delta=1/m$ be a given step size with integer $m\ge 1$, and let the gridpoints t_k be defined by $t_k=k\Delta(k\in\mathbb{N})$. Since for arbitrary $k\in\mathbb{N}$, there exist $s\in\mathbb{N}$ and $l=0,1,2,\ldots,m-1$ such that $k=sm+l$, the adaptation of the ST method to (1) leads to a numerical process of the following type by setting $X_0=x(0)=x_0$, $r_0^{\Delta}=r(0)=i_0$,

$$\begin{aligned} X_{sm+l+1}=&X_{sm+l}+(1-\theta)f\left(X_{sm+l},X_{sm},r_{sm+l}^{\Delta}\right)\Delta\\ &+\theta f\left(X_{sm+l+1},X_{sm},r_{sm+l+1}^{\Delta}\right)\Delta+g\left(X_{sm+l},X_{sm},r_{sm+l}^{\Delta}\right)\Delta B_{sm+l}, \end{aligned}\tag{11}$$

for $s\in\mathbb{N}$ and $l=0,1,2,\ldots,m-1$, where $\Delta B_{sm+l}=B(t_{sm+l+1})-B(t_{sm+l})$, $\theta\in[0,1]$ is a free parameter that is specified a priori. X_{sm+l} is an approximation to the exact solution $x(t_{sm+l})$.

Since the ST scheme is semi-implicit when $\theta\ne 0$, the first item that need to be considered is the existence and uniqueness of solutions of these equations. In that sense, we will employ the one-sided Lipschitz condition in the first argument of the function f, which is given in the following.

Assumption 3.1 There exists a positive constant L such that

$$\langle x-\bar{x}, f(x,y,i)-f(\bar{x},y,i)\rangle\le L\|x-\bar{x}\|^2,$$

for all $x,\bar{x},y\in\mathbb{R}^n$.

Remark 1 By Lemma 3.1 in Mao and Szpruch (2013), it is obvious to obtain that the ST method has a unique solution if $\Delta<\frac{1}{L\theta}$. In the rest of this paper, we always assume that $\Delta<\frac{1}{L\theta}$.

To implement numerical scheme (11), we define a map F, let

$$F(X_{\kappa m+j})=X_{\kappa m+j}-\theta f(X_{\kappa m+j},X_{\kappa m},r_{\kappa m+j}^{\Delta})\Delta,\ \kappa\in\mathbb{N}, j=0,1,\ldots,m-1,\tag{12}$$

then we can represent (11) as follows:

$$F(X_{sm+l+1})=F(X_{sm+l})+f(X_{sm+l},X_{sm},r_{sm+l}^{\Delta})\Delta+g(X_{sm+l},X_{sm},r_{sm+l}^{\Delta})\Delta B_{sm+l},\tag{13}$$

for $l = 0, 1, \ldots, m-2$,

$$\begin{aligned} F(X_{(s+1)m}) =& F(X_{(s+1)m-1}) + f(X_{(s+1)m-1}, X_{sm}, r^{\Delta}_{(s+1)m-1})\Delta \\ &+ g(X_{(s+1)m-1}, X_{sm}, r^{\Delta}_{(s+1)m-1})\Delta B_{(s+1)m-1} \\ &+ \theta\left(f(X_{(s+1)m}, X_{sm}, r^{\Delta}_{(s+1)m}) - f(X_{(s+1)m}, X_{(s+1)m}, r^{\Delta}_{(s+1)m})\right)\Delta, \end{aligned} \tag{14}$$

for $l = m-1$.

According to Assumption 3.1, there exists an inverse mapping F^{-1} and the solution to (11) can be represented in the following form:

$$X_{sm+l+1} = F^{-1}\left(X_{sm+l} + (1-\theta)f(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta + g(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta B_{sm+l}\right),$$

for $l = 0, 1, \ldots, m-2$,

$$\begin{aligned} X_{sm+l+1} = F^{-1}\Big(& X_{sm+l} + (1-\theta)f(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta + g(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta B_{sm+l} \\ &+\theta\left(f(X_{(s+1)m}, X_{sm}, r^{\Delta}_{(s+1)m}) - f(X_{(s+1)m}, X_{(s+1)m}, r^{\Delta}_{(s+1)m})\right)\Delta\Big), \end{aligned}$$

for $l = m-1$. Clearly, X_{sm+l+1} is $\mathcal{F}_{t_{sm+l+1}}$-measurable.

4 *p*th moment boundedness of the ST method

Throughout this section, we fix $T > 0$ be arbitrary and show that the pth moment of the ST method is bounded. The following lemma shows that to guarantee the boundedness of moments for X_{sm+l} it is enough to bound the moments of $F(X_{sm+l})$, where $F(X_{sm+l})$ is defined by (12).

Lemma 4.1 *Suppose that Assumption 2.2 holds. Let δ be any given constant with $1-4\alpha\theta\Delta \geq \delta > 0$. Then for any $p \geq 2$,*

$$\|X_{sm+l}\|^p \leq 3^{\frac{p}{2}-1}\delta^{-\frac{p}{2}}\left\{\|F(X_{sm+l})\|^p + (1-\delta)^{\frac{p}{2}}\|X_{sm}\|^p + (1-\delta)^{\frac{p}{2}}\right\}.$$

Moreover,

$$\|X_{sm+l}\|^p \leq 3^{\frac{p}{2}-1}\delta^{-\frac{p}{2}}\left\{\|F(X_{sm+l})\|^p + (1/\delta-1)^{\frac{p}{2}}\|F(X_{sm})\|^p + (1-\delta)^{\frac{p}{2}}\right\}.$$

Proof Using Assumption 2.2, we can arrive at

$$\begin{aligned} \|F(X_{sm+l})\|^2 =& \|X_{sm+l} - \theta f(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta\|^2 \\ \geq& \|X_{sm+l}\|^2 - 2\theta\Delta X^{\mathrm{T}}_{sm+l} f(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l}) \\ \geq& \|X_{sm+l}\|^2 - 2\alpha\theta\Delta(1 + \|X_{sm+l}\|^2 + \|X_{sm}\|^2), \end{aligned} \tag{15}$$

which implies

$$\|X_{sm+l}\|^2 \leq (1-2\alpha\theta\Delta)^{-1}\|F(X_{sm+l})\|^2 + \frac{2\alpha\theta\Delta}{1-2\alpha\theta\Delta}\|X_{sm}\|^2 + \frac{2\alpha\theta\Delta}{1-2\alpha\theta\Delta}, \tag{16}$$

then applying the inequality $(x+y+z)^{\frac{p}{2}} \leq 3^{\frac{p}{2}-1}(x^{\frac{p}{2}} + y^{\frac{p}{2}} + z^{\frac{p}{2}})$, $\forall x, y, z > 0$, and the fact that $1 - 4\alpha\theta\Delta \geq \delta$, we obtain

$$\|X_{sm+l}\|^p \leq 3^{\frac{p}{2}-1}\delta^{-\frac{p}{2}}\left\{\|F(X_{sm+l})\|^p + (1-\delta)^{\frac{p}{2}}\|X_{sm}\|^p + (1-\delta)^{\frac{p}{2}}\right\}. \tag{17}$$

Furthermore, if $l = 0$, one can get that

$$\|X_{sm}\|^2 \le (1-4\alpha\theta\Delta)^{-1}\|F(X_{sm})\|^2 + \frac{2\alpha\theta\Delta}{1-4\alpha\theta\Delta}, \tag{18}$$

from (15) directly, substituting (18) into (16), then

$$\begin{aligned}\|X_{sm+l}\|^2 \ &\le (1-2\alpha\theta\Delta)^{-1}\|F(X_{sm+l})\|^2 \\ &\quad + \frac{2\alpha\theta\Delta}{(1-2\alpha\theta\Delta)(1-4\alpha\theta\Delta)}\|F(X_{sm})\|^2 + \frac{2\alpha\theta\Delta}{1-4\alpha\theta\Delta},\end{aligned} \tag{19}$$

which gives

$$\|X_{sm+l}\|^p \le 3^{\frac{p}{2}-1}\delta^{-\frac{p}{2}}\left\{\|F(X_{sm+l})\|^p + (1/\delta-1)^{\frac{p}{2}}\ \|F(X_{sm})\|^p + (1-\delta)^{\frac{p}{2}}\right\}. \tag{20}$$

In particular, it from (18) that

$$\|X_{sm}\|^p \le 2^{\frac{p}{2}-1}\delta^{-\frac{p}{2}}\left\{\|F(X_{sm})\|^p + (1-\delta)^{\frac{p}{2}}\right\}. \tag{21}$$

□

In what follows, for notational simplicity, we use the convention that C represents a generic positive constant independent of Δ, the value of which may vary with each appearance. For example, $C = C + C$ and $C = C \times C$ are understood in an appropriate sense. Moreover, we may give specific expressions of C when needed. Let us begin to establish the fundamental result of this paper that reveals the boundedness of the pth moment for the ST scheme.

Theorem 4.2 *Let Assumptions 2.2–2.4 and 3.1 hold, and $\theta \ge 0.5$. Then for any $p \ge 2$, the ST scheme (11) has the following property:*

$$\mathbb{E}\left\{\sup_{0\le t_{sm+l}\le T}\|X_{sm+l}\|^p\right\} \le C.$$

Proof For any $s \in \mathbb{N}, l = 0, 1, \ldots, m-2$, using Assumption 2.2 and $\theta \ge 0.5$, we have

$$\begin{aligned}&\|F(X_{sm+l+1})\|^2 \\ &\quad = \|F(X_{sm+l}) + f(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta + g(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta B_{sm+l}\|^2 \\ &\quad = \|F(X_{sm+l})\|^2 + 2\langle X_{sm+l}, f(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\rangle\Delta \\ &\qquad + (1-2\theta)\|f(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\|^2\Delta^2 + \|g(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta B_{sm+l}\|^2 \\ &\qquad + 2\langle F(X_{sm+l}) + f(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta, g(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta B_{sm+l}\rangle \\ &\quad \le \|F(X_{sm+l})\|^2 + 2\alpha\Delta(1 + \|X_{sm+l}\|^2 + \|X_{sm}\|^2) + \|\Delta M_{sm+l}\|^2 + \Delta N_{sm+l},\end{aligned} \tag{22}$$

where

$$\begin{aligned}\Delta M_{sm+l} &= g(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta B_{sm+l}, \\ \Delta N_{sm+l} &= 2\langle F(X_{sm+l}) + f(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta, \Delta M_{sm+l}\rangle.\end{aligned}$$

Then we can infer that

$$\begin{aligned}\|F(X_{sm+l+1})\|^2 \le &\|F(X_{sm})\|^2 + 2\alpha + 2\alpha\|X_{sm}\|^2 + 2\alpha\Delta\sum_{i=sm}^{sm+l}\|X_i\|^2 \\ &+ \sum_{i=sm}^{sm+l}\|\Delta M_i\|^2 + \sum_{i=sm}^{sm+l}\Delta N_i,\end{aligned}$$

substituting (18) and (19) into the last equation and recalling that $1-4\alpha\theta\Delta \geq \delta$, we acquire

$$\begin{aligned}\|F(X_{sm+l+1})\|^2 \leq & \frac{2\alpha\Delta}{\delta}\sum_{i=sm}^{sm+l}\|F(X_i)\|^2 + \left(1+\frac{2\alpha}{\delta^2}\right)\|F(X_{sm})\|^2 + \frac{2\alpha}{\delta} \\ & + \sum_{i=sm}^{sm+l}\|\Delta M_i\|^2 + \sum_{i=sm}^{sm+l}\Delta N_i.\end{aligned}$$

Thus, according to the inequality $\left(\sum_{i=1}^{k}|a_i|\right)^p \leq k^{p-1}\sum_{i=1}^{k}|a_i|^p$, $p \geq 1$, one can arrive at

$$\begin{aligned}&\mathbb{E}\left\{\sup_{j=0,1,\ldots,l}\|F(X_{sm+j+1})\|^p\right\} \\ &\leq C_1\Delta\sum_{i=sm}^{sm+l}\mathbb{E}\|F(X_i)\|^p + C_2\mathbb{E}\|F(X_{sm})\|^p + C_1 \\ &\quad + C_3(l+1)^{\frac{p}{2}-1}\sum_{i=sm}^{sm+l}\mathbb{E}\|\Delta M_i\|^p + C_3\underbrace{\mathbb{E}\left(\sup_{j=0,1,\ldots,l}\sum_{i=sm}^{sm+j}\Delta N_i\right)^{\frac{p}{2}}}_{A},\end{aligned} \tag{23}$$

where $C_1 = 5^{\frac{p}{2}-1}\left(\frac{2\alpha}{\delta}\right)^{\frac{p}{2}}$, $C_2 = 5^{\frac{p}{2}-1}\left(1+\frac{2\alpha}{\delta^2}\right)^{\frac{p}{2}}$, $C_3 = 5^{\frac{p}{2}-1}$, and we use the fact that $l\Delta \leq 1, l = 0, 1, \ldots, m-2$.

Since ΔB_i is $\mathcal{F}_{t_i}$-independent, with the help of Hölder's inequality and Assumption 2.4, we have

$$\begin{aligned}\mathbb{E}\|\Delta M_i\|^p \leq & \mathbb{E}\|g(X_i, X_{sm}, r_i^{\Delta})\|^p(\mathbb{E}\|\Delta B_i\|^{2p})^{\frac{1}{2}} \\ \leq & C\Delta^{\frac{p}{2}}L_2^p 3^{p-1}\mathbb{E}\left(1+\|X_i\|^p+\|X_{sm}\|^{ph_2}\right),\end{aligned} \tag{24}$$

combining (20) and (21), we can obtain that

$$\mathbb{E}\|\Delta M_i\|^p \leq C\Delta^{\frac{p}{2}}\Big(C_4\mathbb{E}\|F(X_i)\|^p + C_5\mathbb{E}\|F(X_{sm})\|^p + C_6\mathbb{E}\|F(X_{sm})\|^{ph_2} + C_7\Big), \tag{25}$$

where

$$\begin{aligned}&C_4 = L_2^p 3^{\frac{3p}{2}-2}\delta^{-\frac{p}{2}}, \quad C_5 = L_2^p 3^{\frac{3p}{2}-2}(1-\delta)^{\frac{p}{2}}\delta^{-p}, \quad C_6 = L_2^p 3^{p-1}2^{\frac{ph_2}{2}-1}\delta^{-\frac{ph_2}{2}}, \\ &C_7 = L_2^p 3^{p-1}\left(1+3^{\frac{p}{2}-1}(1-\delta)^{\frac{p}{2}}\delta^{-\frac{p}{2}} + 2^{\frac{ph_2}{2}-1}(1-\delta)^{\frac{ph_2}{2}}\delta^{-\frac{ph_2}{2}}\right).\end{aligned}$$

According to the definition of ΔN_{sm+l} and $F(X_{sm+l})$, using the time discrete Burkholder–Davis–Gundy type inequality, we yield

$$\begin{aligned}A \leq & \left(\frac{2}{\theta}\right)^{\frac{p}{2}}\mathbb{E}\left(\sup_{j=0,1,\ldots,l}\sum_{i=sm}^{sm+j}(X_i+(\theta-1)F(X_i))^{\mathrm{T}}g(X_i, X_{sm}, r_i^{\Delta})\Delta B_i\right)^{\frac{p}{2}} \\ \leq & \frac{p}{2}\left(\frac{2}{\theta}\right)^{\frac{p}{2}}\Delta^{\frac{p}{4}}\left(\sum_{i=sm}^{sm+l}\left(\mathbb{E}\|(X_i+(\theta-1)F(X_i))^{\mathrm{T}}g(X_i, X_{sm}, r_i^{\Delta})\|^{\frac{p}{2}}\right)^{\frac{4}{p}}\right)^{\frac{p}{4}}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{p}{2}\left(\frac{2}{\theta}\right)^{\frac{p}{2}} \Delta^{\frac{p}{4}}(l+1)^{\frac{p}{4}-1} \sum_{i=sm}^{sm+l} \mathbb{E}\|(X_i+(\theta-1)F(X_i))^{\mathrm{T}} g(X_i, X_{sm}, r_i^{\Delta})\|^{\frac{p}{2}} \\
&\leq \frac{p}{4}\left(\frac{2}{\theta}\right)^{\frac{p}{2}} \Delta \sum_{i=sm}^{sm+l} \left(\mathbb{E}\|X_i+(\theta-1)F(X_i)\|^p + \mathbb{E}\|g(X_i, X_{sm}, r_i^{\Delta})\|^p\right) \\
&\leq 2^{\frac{3p}{2}-3} p\Delta\theta^{-\frac{p}{2}} \sum_{i=sm}^{sm+l} \left\{\mathbb{E}\|X_i\|^p + \mathbb{E}\|F(X_i)\|^p\right\} + 2^{\frac{p}{2}-2} p\theta^{-\frac{p}{2}} \Delta \sum_{i=sm}^{sm+l} \mathbb{E}\|g(X_i, X_{sm}, r_i^{\Delta})\|^p.
\end{aligned} \tag{26}
$$

Similar to (24) and (25), one can get that

$$
\mathbb{E}\|g(X_i, X_{sm}, r_i^{\Delta})\|^p \leq C_4\mathbb{E}\|F(X_i)\|^p + C_5\mathbb{E}\|F(X_{sm})\|^p + C_6\mathbb{E}\|F(X_{sm})\|^{ph_2} + C_7. \tag{27}
$$

By substituting (20) and (27) into (26), we have

$$
A \leq C_8\Delta \sum_{i=sm}^{sm+l} \mathbb{E}\|F(X_i)\|^p + C_9\mathbb{E}\|F(X_{sm})\|^p + C_{10}\mathbb{E}\|F(X_{sm})\|^{ph_2} + C_{11}, \tag{28}
$$

where

$$
\begin{aligned}
C_8 &= 2^{\frac{3p}{2}-3} p\theta^{-\frac{p}{2}}\left(1+3^{\frac{p}{2}-1}\delta^{-\frac{p}{2}}\right) + 2^{\frac{p}{2}-2}\theta^{-\frac{p}{2}} pC_4, \\
C_9 &= 2^{\frac{3p}{2}-3} p\theta^{-\frac{p}{2}} 3^{\frac{p}{2}-1}(1-\delta)^{\frac{p}{2}}\delta^{-p} + 2^{\frac{p}{2}-2}\theta^{-\frac{p}{2}} pC_5, \\
C_{10} &= 2^{\frac{p}{2}-2}\theta^{-\frac{p}{2}} pC_6, \quad C_{11} = 2^{\frac{3p}{2}-3} p\theta^{-\frac{p}{2}} 3^{\frac{p}{2}-1}(1-\delta)^{\frac{p}{2}}\delta^{-\frac{p}{2}} + 2^{\frac{p}{2}-2}\theta^{-\frac{p}{2}} pC_7.
\end{aligned}
$$

It follows from (23), (25) and (28) that

$$
\begin{aligned}
\mathbb{E}\left\{\sup_{j=0,1,\ldots,l} \|F(X_{sm+j+1})\|^p\right\} \leq & C_{12}\Delta \sum_{i=0}^{l-1} \mathbb{E}\left\{\sup_{j=0,1,\ldots,i} \|F(X_{sm+j+1})\|^p\right\} \\
& + (C_{12}\Delta + C_{13})\mathbb{E}\|F(X_{sm})\|^p + C_{14}\mathbb{E}\|F(X_{sm})\|^{ph_2} + C_{15},
\end{aligned}
$$

where we set $\sum_{i=0}^{-1} \mathbb{E}\|F(X_{sm+i+1})\|^p = 0$ as usual. Here $C_{12} = C_1 + C_3C_4C + C_3C_8$, $C_{13} = C_2 + C_3C_5C + C_3C_9$, $C_{14} = C_3C_6C + C_3C_{10}$, $C_{15} = C_1 + C_3C_7C + C_3C_{11}$. Using the discrete Gronwall inequality (Theorem 2.5 in Mao and Yuan 2006), we obtain

$$
\begin{aligned}
&\mathbb{E}\left\{\sup_{j=0,1,\ldots,l} \|F(X_{sm+j+1})\|^p\right\} \\
&\leq \left((C_{12}\Delta + C_{13})\mathbb{E}\|F(X_{sm})\|^p + C_{14}\mathbb{E}\|F(X_{sm})\|^{ph_2} + C_{15}\right) e^{C_{12}}.
\end{aligned} \tag{29}
$$

By induction, we divide the proof into several steps to show

$$
\mathbb{E}\left\{\sup_{0\leq t_{sm+l}\leq T} \|X_{sm+l}\|^p\right\} \leq C.
$$

Step 1. For $s = 0, l = 0, 1, \ldots, m-2$, (29) implies

$$
\mathbb{E}\left\{\sup_{j=0,1,\ldots,l} \|F(X_{j+1})\|^p\right\} \leq \left((C_{12}\Delta + C_{13})\|F(X_0)\|^p + C_{14}\|F(X_0)\|^{ph_2} + C_{15}\right) e^{C_{12}}.
$$

(30)

Noting that $F(X_0) = X_0 - f(X_0, X_0, i_0)\Delta$, $X_0 = x_0$, using Assumption 2.3, we can easily get that

$$\|F(X_0)\|^p \leq C(p, \|x_0\|, L_1, h_1) \quad \text{and} \quad \|F(X_0)\|^{ph_2} \leq C(p, \|x_0\|, L_1, h_1, h_2).$$

Substituting the last equations into (30) we have

$$\mathbb{E}\left\{\sup_{j=0,\dots,m-1} \|F(X_j)\|^p\right\} \leq C, \tag{31}$$

and it follows from Lemma 4.1 that

$$\mathbb{E}\left\{\sup_{j=0,1,\dots,m-1} \|X_j\|^p\right\} \leq C. \tag{32}$$

Repeating the procedures as discussed above, we can also get that

$$\mathbb{E}\left\{\sup_{j=0,\dots,m-1} \|F(X_j)\|^{ph_2}\right\} \leq C \quad \text{and} \quad \mathbb{E}\left\{\sup_{j=0,1,\dots,m-1} \|X_j\|^{ph_1}\right\} \leq C. \tag{33}$$

Next we show that $\mathbb{E}\|X_m\|^p \leq C$ and $\mathbb{E}\|F(X_m)\|^p \leq C$. Applying (12), (13) and (14) again, by Assumptions 2.3 and 2.4, one has

$$\begin{aligned}
&\|X_m - \theta f(X_m, X_0, r_m^\Delta)\Delta\|^2 \\
&\quad = \|F(X_{m-1}) + f(X_{m-1}, X_0, r_{m-1}^\Delta)\Delta + g(X_{m-1}, X_0, r_{m-1}^\Delta)\Delta B_{m-1}\|^2 \\
&\quad \leq 3\|F(X_{m-1})\|^2 + 3\|f(X_{m-1}, X_0, r_{m-1}^\Delta)\Delta\|^2 + 3\|g(X_{m-1}, X_0, r_{m-1}^\Delta)\Delta B_{m-1}\|^2,
\end{aligned}$$

hence by (31)–(33), we infer that

$$\begin{aligned}
&\mathbb{E}\|X_m - \theta f(X_m, X_0, r_m^\Delta)\Delta\|^p \\
&\quad \leq 3^{p-1}\left\{\mathbb{E}\|F(X_{m-1})\|^p + \mathbb{E}\|f(X_{m-1}, X_0, r_{m-1}^\Delta)\Delta\|^p + \mathbb{E}\|g(X_{m-1}, X_0, r_{m-1}^\Delta)\Delta B_{m-1}\|^p\right\} \\
&\quad \leq 3^{p-1}\Big\{\mathbb{E}\|F(X_{m-1})\|^p + 3^{p-1}L_1^p(1 + \mathbb{E}\|X_{m-1}\|^{ph_1} + \|X_0\|^{ph_1})\Delta^p \\
&\qquad + 3^{p-1}L_2^p(1 + \mathbb{E}\|X_{m-1}\|^p + \|X_0\|^{ph_2})(\mathbb{E}\|\Delta B_{m-1}\|^{2p})^{\frac{1}{2}}\Big\} \\
&\quad \leq C.
\end{aligned}$$

Moreover, repeating the process (15)–(17), one can get that

$$\begin{aligned}
\mathbb{E}\|X_m\|^p \leq& 3^{\frac{p}{2}-1}\left\{\delta^{-\frac{p}{2}}\mathbb{E}\|X_m - \theta f(X_m, X_0, r_m^\Delta)\Delta\|^p + \left(\frac{1-\delta}{\delta}\right)^{\frac{p}{2}}\|X_0\|^p + \left(\frac{1-\delta}{\delta}\right)^{\frac{p}{2}}\right\} \\
\leq& C.
\end{aligned}$$

Using $F(X_{sm+l}) = X_{sm+l} - \theta f(X_{sm+l}, X_{sm}, r_{sm+l}^\Delta)\Delta$ again, we obtain

$$\begin{aligned}
\mathbb{E}\|F(X_m)\|^p =& \mathbb{E}\|X_m - \theta f(X_m, X_m, r_m^\Delta)\Delta\|^p \\
\leq& 2^{p-1}\mathbb{E}\|X_m\|^p + 2^{p-1}\theta^p\Delta^p\mathbb{E}\|f(X_m, X_m, r_m^\Delta)\|^p \\
\leq& 2^{p-1}\mathbb{E}\|X_m\|^p + 6^{p-1}\theta^p\Delta^p L_1^p(1 + 2\mathbb{E}\|X_m\|^p) \\
\leq& C.
\end{aligned}$$

Step 2. For $s = 1$, according to (29), we have

$$\begin{aligned}
&\mathbb{E}\left\{\sup_{j=0,1,\ldots,m-2} \|F(X_{m+j+1})\|^p\right\} \\
&\le \Big((C_{12}\Delta + C_{13})\mathbb{E}\|F(X_m)\|^p + C_{14}\mathbb{E}\|F(X_m)\|^{ph_2} + C_{15}\Big)\, e^{C_{12}} \\
&\le ((C_{12}\Delta + C_{13})C + C_{14}C + C_{15})\, e^{C_{12}} := C,
\end{aligned}$$

which gives

$$\mathbb{E}\left\{\sup_{j=1,2\ldots,m-1} \|F(X_{m+j})\|^p\right\} \le C,$$

then it follows from Lemma 4.1 that

$$\mathbb{E}\left\{\sup_{j=1,2\ldots,m-1} \|X_{m+j}\|^p\right\} \le C.$$

Adopting the same procedures as in the Step 1, we can arrive at $\mathbb{E}\|X_{2m}\|^p \le C$, then $\mathbb{E}\|F(X_{2m})\|^p \le C$ follows from (12).

Step 3. For $s \in \{2, 3, \ldots, [T]\}$, the following assertion can be proved in the same way as shown before, for any fixed T, there exists a constant C independent of Δ such that

$$\mathbb{E}\left\{\sup_{j=1,\ldots,m} \|F(X_{sm+j})\|^p\right\} \le C \quad \text{and} \quad \mathbb{E}\left\{\sup_{j=1,\ldots,m} \|X_{sm+j}\|^p\right\} \le C.$$

Combining Steps 1–3, we can get that

$$\mathbb{E}\left\{\sup_{0\le t_{sm+l}\le T} \|X_{sm+l}\|^p\right\} \le C,$$

for any fixed T. The proof is completed. □

5 Rate of strong convergence

It is convenient to work with a continuous extension of a numerical method here, because the continuous extension enables us to use the powerful continuous-time stochastic analysis to formulate theorems on numerical approximations. For this purpose, we introduce a new numerical scheme, which is called the forward–backward Euler–Maruyama (FBEM) scheme, to help us get a well-defined continuous-time numerical approximation.

First we compute the discrete values X_{sm+l} from the ST method, then we define the discrete FBEM scheme on $[s, s+1) \subset [0, \infty)$, $s \in \mathbb{N}$ by

$$\hat{X}_{sm+l+1} = \hat{X}_{sm+l} + f(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta + g(X_{sm+l}, X_{sm}, r^{\Delta}_{sm+l})\Delta B_{sm+l}, \quad (34)$$

where $l = 0, 1, \ldots, m-1$, $\hat{X}_0 = X_0 = x(0)$.

Let $X(t) = X_{sm+l}$, $\bar{X}(t) = \hat{X}_{sm+l}$, $r^{\Delta}(t) = r^{\Delta}_{sm+l}$ for $t \in [t_{sm+l}, t_{sm+l+1})$, and the continuous FBEM scheme is defined by

$$\hat{X}(t) = \hat{X}_0 + \int_0^t f(X(u), X(s), r^{\Delta}(u))\mathrm{d}u + \int_0^t g(X(u), X(s), r^{\Delta}(u))\mathrm{d}B(u), \quad (35)$$

on each interval $[t_{sm+l}, t_{sm+l+1})$. We would like to remark that the continuous and discrete FBEM schemes coincide at the grid points, that is, $\hat{X}(t_{sm+l}) = \hat{X}_{sm+l}$.

Now we impose some stronger versions of Assumptions 2.2–2.4 to get the convergence rate.

Assumption 5.1 For any constant $K > 0$, there exists a positive constant K_1 such that

$$\begin{aligned}&(x-\bar{x})^{\mathrm{T}}(f(x,y,i)-f(\bar{x},\bar{y},i))+K\|g(x,y,i)-g(\bar{x},\bar{y},i)\|^2\\&\quad\le K_1(\|x-\bar{x}\|^2+\|y-\bar{y}\|^2),\end{aligned}$$

for all $i \in S$ and $x, \bar{x}, y, \bar{y} \in \mathbb{R}^n$.

Assumption 5.2 There exist constants $K_2 > 0$ and $\rho_1 \ge 1$ such that

$$\begin{aligned}\|f(x,y,i)-f(\bar{x},\bar{y},i)\| \le K_2(1&+\|x\|^{\rho_1}+\|y\|^{\rho_1}+\|\bar{x}\|^{\rho_1}\\&+\|\bar{y}\|^{\rho_1})\,(\|x-\bar{x}\|+\|y-\bar{y}\|)\,,\end{aligned}$$

for all $i \in S$ and $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

Assumption 5.3 There exist constants $K_3 > 0$, $K_4 > 0$ and $\rho_2 \ge 1$ such that

$$\begin{aligned}&\|g(x,y,i)-g(\bar{x},y,i)\| \le K_3\|x-\bar{x}\|,\\&\|g(x,y,i)-g(x,\bar{y},i)\| \le K_4(1+\|y\|^{\rho_2}+\|\bar{y}\|^{\rho_2})\|y-\bar{y}\|,\end{aligned}$$

for all $i \in S$ and $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

Assumption 5.4 There exists a positive constant K_5 such that

$$\|f(0,0,i)\| \vee \|g(0,0,i)\| \le K_5,$$

for all $i \in S$.

Remark 2 Assumptions 5.1–5.4 imply Assumptions 2.2–2.4. Suppose that Assumptions 5.1–5.4 hold, we can easily get that

$$\begin{aligned}x^{\mathrm{T}}f(x,y,i) =&(x-0)^{\mathrm{T}}(f(x,y,i)-f(0,0,i))+x^{\mathrm{T}}f(0,0,i)\\\le&K_1(\|x\|^2+\|y\|^2)+\frac{1}{2}\|x\|^2+\frac{1}{2}\|f(0,0,i)\|^2\\\le&\left(K_1+\frac{1}{2}+\frac{1}{2}K_5^2\right)(1+\|x\|^2+\|y\|^2),\end{aligned}$$

which is Assumption 2.2, and

$$\begin{aligned}\|f(x,y,i)\| \le&\|f(x,y,i)-f(0,0,i)\|+\|f(0,0,i)\|\\\le&K_2(1+\|x\|^{\rho_1}+\|y\|^{\rho_1})(\|x\|+\|y\|)+K_5\\\le&(6K_2+K_5)\left(1+\|x\|^{2\rho_1}+\|y\|^{2\rho_1}\right),\end{aligned}$$

which is Assumption 2.3, as well as

$$\begin{aligned}\|g(x,y,i)\| \le&\|g(x,y,i)-g(0,y,i)\|+\|g(0,y,i)-g(0,0,i)\|+\|g(0,0,i)\|\\\le&K_3\|x\|+K_4(1+\|y\|^{\rho_2})\|y\|+K_5\\\le&K_6(1+\|x\|+\|y\|^{\rho_2+1}),\end{aligned}$$

which is Assumption 2.4, where $K_6 = K_3 + 2K_4 + K_5$.

Under the assumptions above, we can determine the rate of strong convergence for ST scheme (11) to the solution of (1). First, we need some lemmas.

Lemma 5.5 *Let Assumptions 5.1–5.4 hold, and $\theta \geq 0.5$. Then for any $p \geq 2$ and sufficiently small step size Δ, $\hat{X}_{sm+l}$ and X_{sm+l} obey*

$$\mathbb{E}\left\{\sup_{0\leq t_{sm+l}\leq T}\|\hat{X}_{sm+l}-X_{sm+l}\|^p\right\}\leq C\Delta^p. \tag{36}$$

Proof For any $s \in \mathbb{N}, l = 0, 1, \ldots, m-1$, summing up both schemes of the discrete FBEM (34) and ST (11), respectively, we have

$$\begin{aligned}\hat{X}_{sm+l+1}-X_{sm+l+1} = \hat{X}_{sm}-X_{sm}+\theta\big(&f(X_{sm},X_{sm},r^{\Delta}_{sm})\\ &-f(X_{sm+l+1},X_{sm},r^{\Delta}_{sm+l+1})\big)\Delta,\end{aligned} \tag{37}$$

then we can infer that

$$\begin{aligned}\hat{X}_{sm+l+1}-X_{sm+l+1} =&\theta\Delta\big(f(X_0,X_0,r^{\Delta}_0)-f(X_{sm+l+1},X_{sm},r^{\Delta}_{sm+l+1})\big)\\ &+\theta\Delta\sum_{i=1}^{s}\big(f(X_{im},X_{im},r^{\Delta}_{im})-f(X_{im},X_{(i-1)m},r^{\Delta}_{im})\big).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\|\hat{X}_{sm+l+1}-X_{sm+l+1}\|^p\\ &\leq 4^{p-1}\theta^p\Delta^p\left(\|f(X_0,X_0,r^{\Delta}_0)\|^p+\|f(X_{sm+l+1},X_{sm},r^{\Delta}_{sm+l+1})\|^p\right)\\ &\quad+2^{p-1}\theta^p\Delta^p s^{p-1}\sum_{i=1}^{s}\|f(X_{im},X_{im},r^{\Delta}_{im})-f(X_{im},X_{(i-1)m},r^{\Delta}_{im})\|^p.\end{aligned} \tag{38}$$

According to Assumptions 5.2, 5.4 and the inequality $(|a|+|b|)^p \leq 2^{p-1}(|a|^p+|b|^p)$, $p \geq 1$, we can acquire

$$\begin{aligned}&\|f(X_{sm+l+1},X_{sm},r^{\Delta}_{sm+l+1})\|^p\\ &\leq 2^{p-1}\|f(X_{sm+l+1},X_{sm},r^{\Delta}_{sm+l+1})-f(0,0,r^{\Delta}_{sm+l+1})\|^p+2^{p-1}\|f(0,0,r^{\Delta}_{sm+l+1})\|^p\\ &\leq 2^{p-1}K_2^p(1+\|X_{sm+l+1}\|^{\rho_1}+\|X_{sm}\|^{\rho_1})^p(\|X_{sm+l+1}\|+\|X_{sm}\|)^p+2^{p-1}K_5^p\\ &\leq 2^{p-2}K_2^p\Big(3^{2p-1}\left(1+\|X_{sm+l+1}\|^{2p\rho_1}+\|X_{sm}\|^{2p\rho_1}\right)\\ &\quad+2^{2p-1}\left(\|X_{sm+l+1}\|^{2p}+\|X_{sm}\|^{2p}\right)\Big)+C.\end{aligned} \tag{39}$$

Similarly, one can also get that

$$\begin{aligned}&\|f(X_{im},X_{im},r^{\Delta}_{im})-f(X_{im},X_{(i-1)m},r^{\Delta}_{im})\|^p\\ &\leq \frac{1}{2}K_2^p\Big(3^{2p-1}(1+3^{2p}\|X_{im}\|^{2p\rho_1}+\|X_{(i-1)m}\|^{2p\rho_1})\\ &\quad+2^{2p-1}(\|X_{im}\|^{2p}+\|X_{(i-1)m}\|^{2p})\Big).\end{aligned} \tag{40}$$

Substituting (39) and (40) into (38), with the help of Theorem 4.2, for $s \in [0, [T]]$, we yield

$$\mathbb{E}\left\{\sup_{0\leq t_{sm+l+1}\leq T}\|\hat{X}_{sm+l+1}-X_{sm+l+1}\|^p\right\}\leq C\Delta^p,$$

and the assertion follows. □

Lemma 5.6 *Let Assumptions 5.1–5.4 hold, and $\theta \geq 0.5$. Then for any $p \geq 2$,*

$$\mathbb{E}\left\{\sup_{0\leq t_{sm+l}\leq T}\|\hat{X}_{sm+l}\|^p\right\}\leq C,\qquad \mathbb{E}\left\{\sup_{0\leq t\leq T}\|\hat{X}(t)\|^p\right\}\leq C.$$

Proof It follows from Theorem 4.2 and Lemma 5.5 that

$$\begin{aligned}\mathbb{E}\left\{\sup_{0\leq t_{sm+l}\leq T}\|\hat{X}_{sm+l}\|^p\right\}\leq& 2^{p-1}\mathbb{E}\left\{\sup_{0\leq t_{sm+l}\leq T}\|\hat{X}_{sm+l}-X_{sm+l}\|^p\right\}\\&+2^{p-1}\mathbb{E}\left\{\sup_{0\leq t_{sm+l}\leq T}\|X_{sm+l}\|^p\right\}\leq C.\end{aligned}\tag{41}$$

Moreover, according to (35) and Hölder's inequality,

$$\begin{aligned}\sup_{t_{sm+l}\leq t<t_{sm+l+1}}\|\hat{X}(t)\|^p\leq& 3^{p-1}\Bigg(\|\hat{X}_{sm+l}\|^p+\Delta^p\|f(X_{sm+l},X_{sm},r^{\Delta}_{sm+l})\|^p\\&+\sup_{t_{sm+l}\leq t<t_{sm+l+1}}\left\|\int_{t_{sm+l}}^t g(X_{sm+l},X_{sm},r^{\Delta}_{sm+l})\mathrm{d}B(u)\right\|^p\Bigg),\end{aligned}$$

it follows that

$$\begin{aligned}&\mathbb{E}\left\{\sup_{0\leq t\leq T}\|\hat{X}(t)\|^p\right\}\\&\leq\mathbb{E}\left\{\sup_{0\leq t_{sm+l}\leq T}\sup_{t_{sm+l}\leq t<t_{sm+l+1}}\|\hat{X}(t)\|^p\right\}\\&\leq 3^{p-1}\mathbb{E}\left\{\sup_{0\leq t_{sm+l}\leq T}\|\hat{X}_{sm+l}\|^p\right\}+3^{p-1}\Delta^p\mathbb{E}\left\{\sup_{0\leq t_{sm+l}\leq T}\|f(X_{sm+l},X_{sm},r^{\Delta}_{sm+l})\|^p\right\}\\&\quad+3^{p-1}\underbrace{\mathbb{E}\left\{\sup_{0\leq t_{sm+l}\leq T}\sup_{t_{sm+l}\leq t<t_{sm+l+1}}\left\|\int_{t_{sm+l}}^t g(X_{sm+l},X_{sm},r^{\Delta}_{sm+l})\mathrm{d}B(u)\right\|^p\right\}}_{I}.\end{aligned}\tag{42}$$

Similar to (39), from Theorem 4.2 we can get that

$$\mathbb{E}\left\{\sup_{0\leq t_{sm+l}\leq T}\|f(X_{sm+l},X_{sm},r^{\Delta}_{sm+l})\|^p\right\}\leq C.\tag{43}$$

According to Remark 2 and Theorem 1.7.2 in Mao (2007), together with Theorem 4.2 again, we can infer that

$$\begin{aligned}I\leq&\sum_{s=0}^{[T]}\sum_{l=0}^{m-1}\mathbb{E}\left\{\sup_{t_{sm+l}\leq t<t_{sm+l+1}}\left\|\int_{t_{sm+l}}^t g(X_{sm+l},X_{sm},r^{\Delta}_{sm+l})\mathrm{d}B(u)\right\|^p\right\}\\&\leq\sum_{s=0}^{[T]}\sum_{l=0}^{m-1}\left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}\Delta^{\frac{p}{2}}\mathbb{E}\|g(X_{sm+l},X_{sm},r^{\Delta}_{sm+l})\|^p\\&\leq 4^{p-1}\left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}K_6^p\Delta^{\frac{p}{2}}\sum_{s=0}^{[T]}\sum_{l=0}^{m-1}\left(1+\mathbb{E}\|X_{sm+l}\|^p+\mathbb{E}\|X_{sm}\|^{p(\rho_2+1)}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq 4^{p-1}\left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} K_6^p \Delta^{\frac{p}{2}-1}([T]+1)\Bigg(1+\mathbb{E}\left\{\sup_{0\leq t_{sm+l}\leq T}\|X_{sm+l}\|^p\right\} \\
&\quad +\mathbb{E}\left\{\sup_{0\leq t_{sm+l}\leq T}\|X_{sm}\|^{p(\rho_2+1)}\right\}\Bigg) \\
&\leq C\Delta^{\frac{p}{2}-1}.
\end{aligned} \tag{44}
$$

Substituting (41), (43) and (44) into (42), one has $\mathbb{E}\left\{\sup_{0\leq t\leq T}\|\hat{X}(t)\|^p\right\}\leq C$. □

Lemma 5.7 *Let Assumptions 5.1–5.4 hold, and $\theta\geq 0.5$, then for any $p\geq 2$,*

$$
\sup_{0\leq t\leq T}\mathbb{E}\|\hat{X}(t)-X(t)\|^p\leq C\Delta^{\frac{p}{2}}.
$$

Proof For any $t\in[0,T]$, there always exist $s\in\mathbb{N}$ and $l\in\{0,1,\ldots,m-1\}$ such that $t\in[t_{sm+l},t_{sm+l+1})$; hence,

$$
\begin{aligned}
\mathbb{E}\|\hat{X}(t)-X(t)\|^p &\leq 2^{p-1}\mathbb{E}\|\hat{X}(t)-\bar{X}(t)\|^p+2^{p-1}\mathbb{E}\|\bar{X}(t)-X(t)\|^p \\
&= 2^{p-1}\mathbb{E}\|\hat{X}(t)-\bar{X}(t)\|^p+2^{p-1}\mathbb{E}\|\hat{X}_{sm+1}-X_{sm+l}\|^p.
\end{aligned} \tag{45}
$$

Applying Hölder's inequality and Theorem 1.7.1 in Mao (2007), for $t\in[t_{sm+l},t_{sm+l+1})$,

$$
\begin{aligned}
&\mathbb{E}\|\hat{X}(t)-\bar{X}(t)\|^p \\
&= \mathbb{E}\left\|\int_{t_{sm+l}}^t f(X_{sm+l},X_{sm},r^\Delta_{sm+l})\mathrm{d}u+\int_{t_{sm+l}}^t g(X_{sm+l},X_{sm},r^\Delta_{sm+l})\mathrm{d}B(u)\right\|^p \\
&\leq 2^{p-1}\Delta^p\mathbb{E}\|f(X_{sm+l},X_{sm},r^\Delta_{sm+l})\|^p \\
&\quad +2^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\Delta^{\frac{p}{2}}\mathbb{E}\left\|g(X_{sm+l},X_{sm},r^\Delta_{sm+l})\right\|^p,
\end{aligned}
$$

similar to (39), by Assumptions 5.2–5.4 and Theorem 4.2, we can easily get

$$
\mathbb{E}\left\|f(X_{sm+l},X_{sm},r^\Delta_{sm+l})\right\|^p\leq C \quad\text{and}\quad \mathbb{E}\left\|g(X_{sm+l},X_{sm},r^\Delta_{sm+l})\right\|^p\leq C,
$$

which means

$$
\mathbb{E}\|\hat{X}(t)-\bar{X}(t)\|^p\leq C\Delta^p+C\Delta^{\frac{p}{2}}\leq C\Delta^{\frac{p}{2}}.
$$

By substituting the last equation and (36) into (45), the desired assertion follows. □

Lemma 5.8 *Let Assumptions 5.1–5.4 hold, and $\theta\geq 0.5$, then for any $s\in\mathbb{N}$,*

$$
\begin{aligned}
\mathbb{E}\int_s^{s+1}\|f(X(u),X(s),r^\Delta(u))-f(X(u),X(s),r(u))\|^2\mathrm{d}u\leq C\Delta, \\
\mathbb{E}\int_s^{s+1}\|g(X(u),X(s),r^\Delta(u))-g(X(u),X(s),r(u))\|^2\mathrm{d}u\leq C\Delta.
\end{aligned}
$$

Proof Using Assumption 5.2, we have

$$
\begin{aligned}
&\mathbb{E}\int_{t_{sm+l}}^{t_{sm+l+1}} \|f(X(u),X(s),r^{\Delta}(u))-f(X(u),X(s),r(u))\|^2 \mathrm{d}u\\
&=\mathbb{E}\int_{t_{sm+l}}^{t_{sm+l+1}} \|f(X(u),X(s),r^{\Delta}(u))-f(X(u),X(s),r(u))\|^2 I_{\{r(u)\neq r(t_{sm+l})\}}\mathrm{d}u\\
&\leq 2\mathbb{E}\int_{t_{sm+l}}^{t_{sm+l+1}} \|f(X(u),X(s),r^{\Delta}(u))-f(0,0,r^{\Delta}(u))\|^2 I_{\{r(u)\neq r(t_{sm+l})\}}\mathrm{d}u\\
&\quad+2\mathbb{E}\int_{t_{sm+l}}^{t_{sm+l+1}} \|f(X(u),X(s),r(u))-f(0,0,r(u))\|^2 I_{\{r(u)\neq r(t_{sm+l})\}}\mathrm{d}u\\
&\leq 24K_2^2\int_{t_{sm+l}}^{t_{sm+l+1}} \mathbb{E}\left\{\left(1+\|X(u)\|^{2\rho_1}+\|X(s)\|^{2\rho_1}\right)\left(\|X(u)\|^2+\|X(s)\|^2\right)I_{\{r(u)\neq r(t_{sm+l})\}}\right\}\mathrm{d}u\\
&=24K_2^2\int_{t_{sm+l}}^{t_{sm+l+1}} \mathbb{E}\Big(\mathbb{E}\big\{\left(1+\|X_{sm+l}\|^{2\rho_1}+\|X_{sm}\|^{2\rho_1}\right)\left(\|X_{sm+l}\|^2+\|X_{sm}\|^2\right)|r(t_{sm+l})\big\}\\
&\quad\times\mathbb{E}\big\{I_{\{r(u)\neq r(t_{sm+l})\}}|r(t_{sm+l})\big\}\Big)\mathrm{d}u,
\end{aligned}
$$

where in the last step we use the fact that $X(u)=X_{sm+l}$, $X(s)=X_{sm}$ and $I_{\{r(u)\neq r(t_{sm+l})\}}$ when $t_{sm+l}<u<t_{sm+l+1}$ are conditionally independent with respect to the σ-algebra generated by $r(t_{sm+l})$. By the property of Markov chain, one has

$$
\begin{aligned}
\mathbb{E}\left\{I_{\{r(u)\neq r(t_{sm+l})\}}|r(t_{sm+l})\right\}&=\sum_{i\in S} I_{\{r(t_{sm+l})=i\}}\mathbb{P}(r(u)\neq i|r(t_{sm+l})=i)\\
&=\sum_{i\in S} I_{\{r(t_{sm+l})=i\}}\sum_{j\neq i}(\gamma_{ij}(u-t_{sm+l})+o(u-t_{sm+l}))\\
&\leq\Big(\max_{1\leq i\leq N}(-\gamma_{ii})\Delta+o(\Delta)\Big)\sum_{i\in S} I_{\{r(t_{sm+l})=i\}}\\
&\leq\bar{C}(\Delta+o(\Delta)),
\end{aligned}
$$

where $\bar{C}=\max_{1\leq i\leq N}(-\gamma_{ii})$. Hence,

$$
\begin{aligned}
&\mathbb{E}\int_{t_{sm+l}}^{t_{sm+l+1}} \|f(X(u),X(s),r^{\Delta}(u))-f(X(u),X(s),r(u))\|^2 \mathrm{d}u\\
&\leq C(\Delta+o(\Delta))\int_{t_{sm+l}}^{t_{sm+l+1}} \mathbb{E}\left\{\left(1+\|X_{sm+l}\|^{2\rho_1}+\|X_{sm}\|^{2\rho_1}\right)\left(\|X_{sm+l}\|^2+\|X_{sm}\|^2\right)\right\}\mathrm{d}u\\
&\leq C\Delta(\Delta+o(\Delta))\left(1+\mathbb{E}\|X_{sm+l}\|^{4\rho_1}+\mathbb{E}\|X_{sm}\|^{4\rho_1}+\mathbb{E}\|X_{sm+l}\|^4+\mathbb{E}\|X_{sm}\|^4\right).
\end{aligned}
$$

Then using Theorem 4.2, one can get that

$$
\begin{aligned}
&\mathbb{E}\int_{s}^{s+1} \|f(X(u),X(s),r^{\Delta}(u))-f(X(u),X(s),r(u))\|^2 \mathrm{d}u\\
&=\sum_{l=0}^{m-1}\mathbb{E}\int_{t_{sm+l}}^{t_{sm+l+1}} \|f(X(u),X(s),r^{\Delta}(u))-f(X(u),X(s),r(u))\|^2 \mathrm{d}u\\
&\leq C(\Delta+o(\Delta))\left(1+\mathbb{E}\|X_{sm+l}\|^{4\rho_1}+\mathbb{E}\|X_{sm}\|^{4\rho_1}+\mathbb{E}\|X_{sm+l}\|^4+\mathbb{E}\|X_{sm}\|^4\right)\\
&\leq C(\Delta+o(\Delta)).
\end{aligned}
\tag{46}
$$

The second inequality can also be proved similarly. □

Theorem 5.9 *Under Assumptions 5.1–5.4, and $\theta \geq 0.5$, the continuous FBEM method (35) strongly converges to the solution of hybrid SDEPCAs (1), that is*

$$\mathbb{E}\left\{\sup_{0\leq t\leq T}\|\hat{X}(t)-x(t)\|^2\right\}\leq C\Delta.$$

Proof Let $e(t)=\hat{X}(t)-x(t)$, $e_\Delta(t)=\hat{X}(t)-X(t)$. For any $t\in[0,T]$, there always exists $s\in\mathbb{N}$ such that $t\in[s,s+1)$, hence

$$\mathbb{E}\left\{\sup_{0\leq t\leq T}\|e(t)\|^2\right\}\leq\mathbb{E}\left\{\sup_{0\leq s\leq[T]}\sup_{s\leq t<s+1}\|e(t)\|^2\right\}\leq\sum_{s=0}^{[T]}\mathbb{E}\left\{\sup_{s\leq t<s+1}\|e(t)\|^2\right\}. \tag{47}$$

For any $t\in[s,s+1)$, it follows from (34) and (35) that

$$\begin{aligned}e(t)=&e(s)+\int_s^t(f(X(u),X(s),r^\Delta(u))-f(x(u),x(s),r(u)))\mathrm{d}u\\&+\int_s^t(g(X(u),X(s),r^\Delta(u))-g(x(u),x(s),r(u)))\mathrm{d}B(u),\end{aligned}$$

then according to the generalised Itô formula (Mao and Yuan 2006), one has

$$\begin{aligned}\|e(t)\|^2=\|e(s)\|^2&+\int_s^t2e(u)^{\mathrm{T}}(f(X(u),X(s),r^\Delta(u))-f(x(u),x(s),r(u)))\mathrm{d}u\\&+\int_s^t\|g(X(u),X(s),r^\Delta(u))-g(x(u),x(s),r(u))\|^2\mathrm{d}u\\&+\int_s^t2e(u)^{\mathrm{T}}(g(X(u),X(s),r^\Delta(u))-g(x(u),x(s),r(u)))\mathrm{d}B(u).\end{aligned}$$

Hence, it is easy to see that

$$\begin{aligned}\mathbb{E}\left\{\sup_{s\leq u\leq t}\|e(u)\|^2\right\}\leq{}&\mathbb{E}\|e(s)\|^2+2\mathbb{E}\left\{\sup_{s\leq u\leq t}M(u)\right\}\\&+2\mathbb{E}\int_s^t e(u)^{\mathrm{T}}\left(f(X(u),X(s),r^\Delta(u))-f(x(u),x(s),r(u))\right)\mathrm{d}u\\&+\mathbb{E}\int_s^t\left\|g(X(u),X(s),r^\Delta(u))-g(x(u),x(s),r(u))\right\|^2\mathrm{d}u,\end{aligned}\tag{48}$$

where

$$M(u)=\int_s^u e(v)^{\mathrm{T}}\left(g(X(v),X(s),r^\Delta(v))-g(x(v),x(s),r(v))\right)\mathrm{d}B(v).$$

Applying the Burkholder–Davis–Gundy inequality and $2ab\leq a^2+b^2$, we obtain

$$\begin{aligned}\mathbb{E}\left\{\sup_{s\leq u\leq t}M(u)\right\}&\leq4\sqrt{2}\mathbb{E}\left(\int_s^t\left\|e(u)^{\mathrm{T}}\left(g(X(u),X(s),r^\Delta(u))-g(x(u),x(s),r(u))\right)\right\|^2\mathrm{d}u\right)^{\frac{1}{2}}\\&\leq\frac{1}{4}\mathbb{E}\left(\sup_{s\leq u\leq t}\|e(u)\|^2\right)\\&\quad+32\mathbb{E}\int_s^t\|g(X(u),X(s),r^\Delta(u))-g(x(u),x(s),r(u))\|^2\mathrm{d}u.\end{aligned}$$

Substituting the last equation into (48), we get

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\left\{\sup_{s\le u\le t}\|e(u)\|^2\right\}\\
&\quad\le \mathbb{E}\|e(s)\|^2 + 2\underbrace{\mathbb{E}\int_s^t e(u)^{\mathrm{T}}\left(f(X(u),X(s),r^{\Delta}(u)) - f(x(u),x(s),r(u))\right)\mathrm{d}u}_{J_1}\\
&\qquad + 65\underbrace{\mathbb{E}\int_s^t \left\|g(X(u),X(s),r^{\Delta}(u)) - g(x(u),x(s),r(u))\right\|^2\mathrm{d}u}_{J_2}.
\end{aligned}
\tag{49}
$$

Using Assumption 5.2, Lemma 5.8, Hölder's inequality, Theorem 4.2, Lemmas 5.6, and 5.7, one can acquire that

$$
\begin{aligned}
J_1 &\le 2\mathbb{E}\int_s^t \|e(u)\|^2\mathrm{d}u + \mathbb{E}\int_s^t \|f(X(u),X(s),r^{\Delta}(u)) - f(X(u),X(s),r(u))\|^2\mathrm{d}u\\
&\quad + \mathbb{E}\int_s^t \|f(X(u),X(s),r(u)) - f(\hat{X}(u),\hat{X}(s),r(u))\|^2\mathrm{d}u\\
&\quad + 2\mathbb{E}\int_s^t e(u)^{\mathrm{T}}\left(f(\hat{X}(u),\hat{X}(s),r(u)) - f(x(u),x(s),r(u))\right)\mathrm{d}u\\
&\le 2\mathbb{E}\int_s^t \sup_{s\le v\le u}\|e(v)\|^2\mathrm{d}u + C\Delta\\
&\quad + 2\mathbb{E}\int_s^t e(u)^{\mathrm{T}}\left(f(\hat{X}(u),\hat{X}(s),r(u)) - f(x(u),x(s),r(u))\right)\mathrm{d}u.
\end{aligned}
$$

According to Lemma 5.8, Assumption 5.3, Theorem 4.2, Lemmas 5.6 and 5.7, we yield

$$
\begin{aligned}
J_2 &\le 3\mathbb{E}\int_s^t \|g(X(u),X(s),r^{\Delta}(u)) - g(X(u),X(s),r(u))\|^2\mathrm{d}u\\
&\quad + 3\mathbb{E}\int_s^t \|g(X(u),X(s),r(u)) - g(\hat{X}(u),\hat{X}(s),r(u))\|^2\mathrm{d}u\\
&\quad + 3\mathbb{E}\int_s^t \|g(\hat{X}(u),\hat{X}(s),r(u)) - g(x(u),x(s),r(u))\|^2\mathrm{d}u\\
&\le C\Delta + 6\mathbb{E}\int_s^t \left(K_3^2\|e_{\Delta}(u)\|^2 + 3K_4^2\left(1+\|X(s)\|^{2\rho_2}+\|\hat{X}(s)\|^{2\rho_2}\right)\|e_{\Delta}(s)\|^2\right)\mathrm{d}u\\
&\quad + 3\mathbb{E}\int_s^t \|g(\hat{X}(u),\hat{X}(s),r(u)) - g(x(u),x(s),r(u))\|^2\mathrm{d}u\\
&\le C\Delta + 18K_4^2\int_s^t \left\{\mathbb{E}\left(1+\|X(s)\|^{2\rho_2}+\|\hat{X}(s)\|^{2\rho_2}\right)^2\right\}^{\frac{1}{2}}\left\{\mathbb{E}\|e_{\Delta}(s)\|^4\right\}^{\frac{1}{2}}\mathrm{d}u\\
&\quad + 3\mathbb{E}\int_s^t \|g(\hat{X}(u),\hat{X}(s),r(u)) - g(x(u),x(s),r(u))\|^2\mathrm{d}u\\
&\le C\Delta + 3\mathbb{E}\int_s^t \|g(\hat{X}(u),\hat{X}(s),r(u)) - g(x(u),x(s),r(u))\|^2\mathrm{d}u.
\end{aligned}
$$

Substituting J_1 and J_2 into (49), using Assumption 5.1, then

$$\begin{aligned}\mathbb{E}\left\{\sup_{s\le u\le t}\|e(u)\|^2\right\} \le & 2\mathbb{E}\|e(s)\|^2 + 8\mathbb{E}\int_s^t \sup_{s\le v\le u}\|e(v)\|^2 \mathrm{d}u + C\Delta \\ & + 8K_1\mathbb{E}\int_s^t (\|e(u)\|^2 + \|e(s)\|^2)\mathrm{d}u \\ \le & 2\mathbb{E}\|e(s)\|^2 + 8(2K_1+1)\mathbb{E}\int_s^t \sup_{s\le v\le u}\|e(v)\|^2 \mathrm{d}u + C\Delta.\end{aligned} \tag{50}$$

For $s = 0$, $t \in [0, 1)$, (50) implies

$$\mathbb{E}\left\{\sup_{0\le u\le t}\|e(u)\|^2\right\} \le 8(2K_1+1)\mathbb{E}\int_0^t \sup_{0\le v\le u}\|e(v)\|^2 \mathrm{d}u + C\Delta,$$

then according to the Gronwall inequality and the continuity of $\|e(u)\|^2$, we have

$$\mathbb{E}\left\{\sup_{0\le u\le 1}\|e(u)\|^2\right\} \le C\Delta e^{8(2K_1+1)} = C\Delta.$$

In particular, we know that $\mathbb{E}\|e(1)\|^2 \le C\Delta$.

For $s = 1$, $t \in [1, 2)$, (50) implies

$$\mathbb{E}\left\{\sup_{1\le u\le t}\|e(u)\|^2\right\} \le 2\mathbb{E}\|e(1)\|^2 + 8(2K_1+1)\mathbb{E}\int_1^t \sup_{1\le v\le u}\|e(v)\|^2 \mathrm{d}u + C\Delta,$$

using the Gronwall inequality and the continuity of $\|e(u)\|^2$ once more, we can also get

$$\mathbb{E}\left\{\sup_{1\le u\le 2}\|e(u)\|^2\right\} \le C\Delta,$$

in particular, $\mathbb{E}\|e(2)\|^2 \le C\Delta$.

Repeating the same procedures, for any $s \in [0, [T]]$, we always have

$$\mathbb{E}\left\{\sup_{s\le u\le s+1}\|e(u)\|^2\right\} \le C\Delta,$$

substituting this inequality into (47), which gives

$$\mathbb{E}\left\{\sup_{0\le t\le T}\|e(t)\|^2\right\} \le C([T]+1)\Delta = C\Delta.$$

The proof is completed. □

We are now ready to formulate the main theorem of this paper.

Theorem 5.10 *Let Assumptions 5.1–5.4 hold, and $\theta \ge 0.5$, there exists a positive constant C, independent of Δ, such that the ST method (11) strongly converges to the solution of hybrid SDEPCAs (1), that is*

$$\sup_{0\le t\le T}\mathbb{E}\|X(t) - x(t)\|^2 \le C\Delta.$$

Proof It is apparent from the triangle inequality that

$$\sup_{0\le t\le T} \mathbb{E}\|X(t)-x(t)\|^2 \le 2\sup_{0\le t\le T} \mathbb{E}\|X(t)-\hat{X}(t)\|^2 + 2\sup_{0\le t\le T} \mathbb{E}\|\hat{X}(t)-x(t)\|^2,$$

then the assertion follows from Lemma 5.7 and Theorem 5.9. □

6 Numerical simulation

In this section, we consider the following scalar nonlinear hybrid SDEPCA

$$\mathrm{d}x(t) = f(x(t), x([t]), r(t))\mathrm{d}t + g(x(t), x([t]), r(t))\mathrm{d}B(t), \quad t \ge 0, \tag{51}$$

where $f : \mathbb{R}\times\mathbb{R}\times S \to \mathbb{R}$,

$$f(x(t), x([t]), i) = \begin{cases} -x^3(t) + x(t), & \text{if } i = 1, \\ -x^3(t) + x([t]), & \text{if } i = 2, \end{cases}$$

and $g : \mathbb{R}\times\mathbb{R}\times S \to \mathbb{R}$,

$$g(x(t), x([t]), i) = \begin{cases} x(t) + \sin(x([t])), & \text{if } i = 1, \\ \sin(x(t)) + \cos(x([t])), & \text{if } i = 2, \end{cases}$$

with the initial conditions $x_0 = 1$ and $i_0 = 1 \in S = \{1, 2\}$. Here $B(t)$ is a scalar Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$, and $r(t)$ is a right-continuous Markov chain taking values in S with the generator $\Gamma = (\gamma_{ij})_{2\times 2} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$.

By a straight calculation, one has

$$\begin{aligned}
&(x-\bar{x})(f(x,y,1) - f(\bar{x},\bar{y},1)) + K\|g(x,y,1) - g(\bar{x},\bar{y},1)\|^2 \\
&\quad = (x-\bar{x})(-x^3 + x + \bar{x}^3 - \bar{x}) + K|x + \sin y - \bar{x} - \sin\bar{y}|^2 \\
&\quad \le -(x^2+\bar{x}^2)|x-\bar{x}|^2 - x\bar{x}|x-\bar{x}|^2 + |x-\bar{x}|^2 + 2K|x-\bar{x}|^2 + 2K|\sin y - \sin\bar{y}|^2 \\
&\quad \le (2K+1)(|x-\bar{x}|^2 + |y-\bar{y}|^2),
\end{aligned}$$

and

$$\begin{aligned}
&(x-\bar{x})(f(x,y,2) - f(\bar{x},\bar{y},2)) + K\|g(x,y,2) - g(\bar{x},\bar{y},2)\|^2 \\
&\quad = (x-\bar{x})(-x^3 + y + \bar{x}^3 - \bar{y}) + K|\sin x - \sin\bar{x} + \cos y - \cos\bar{y}|^2 \\
&\quad = -(x^2+\bar{x}^2+x\bar{x})|x-\bar{x}|^2 + (x-\bar{x})(y-\bar{y}) + 2K|x-\bar{x}|^2 + 2K|y-\bar{y}|^2 \\
&\quad \le \left(2K + \frac{1}{2}\right)(|x-\bar{x}|^2 + |y-\bar{y}|^2),
\end{aligned}$$

which means the coefficients satisfy Assumption 5.1. Similarly we can also verify that the coefficients satisfy other conditions of Theorem 5.10. We generate 2000 different discretized Brownian paths and use the numerical solution of the backward EM method with step-size $\Delta = 2^{-15}$ as the "exact solution".

Let ϵ and η denote the errors in mean square,

$$\epsilon(T) = \mathbb{E}|x(T) - X_{Tm}|^2 = \frac{1}{2000}\sum_{i=1}^{2000} |x(T,\omega_i) - X_{Tm}(\omega_i)|^2,$$

$$\eta(T) = \mathbb{E}\left\{\sup_{0\le t_{sm+l}\le T} |x(t_{sm+l}) - X_{sm+l}|^2\right\}$$

$$= \frac{1}{2000}\sum_{i=1}^{2000}\left(\max_{0\le t_{sm+l}\le T} |x(t_{sm+l},\omega_i) - X_{sm+l}(\omega_i)|^2\right).$$

We calculate the errors in mean square $\epsilon(1), \epsilon(2), \epsilon(3)$ and $\eta(2), \eta(3)$ with step sizes $2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}, 2^{-10}$, respectively. The log–log mean square error plots corresponding to those chosen values of Δ and θ are given in Figs. 1 and 2. It is well known that the slope of a line in the log–log error plot implies the order of convergence for the numerical method. Graphically, the mean square error lines' slopes are close to the reference lines' slope. Therefore, it can be seen from Figs. 1 and 2 that the order of convergence in mean square for the ST method is close to 0.5.

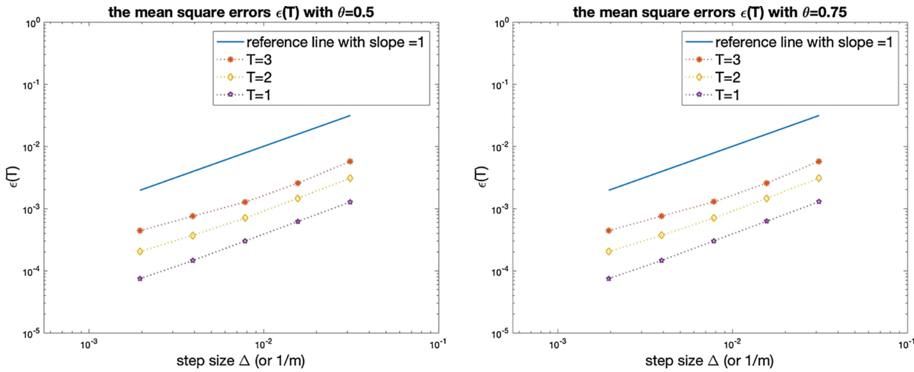

Fig. 1 Convergence rate of the ST method for Eq. (51)

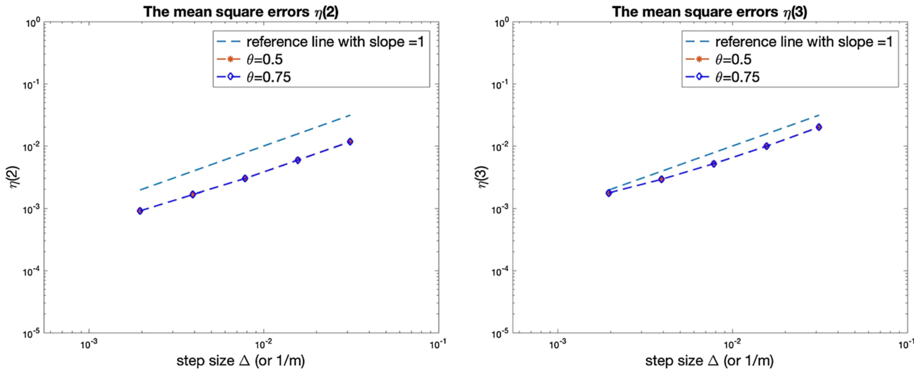

Fig. 2 Convergence rate (uniform) of the ST method for Eq. (51)

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