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Novel distance measures based on complex fuzzy sets with applications in signals

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Abstract

In this paper, we discuss the further development of the theory of complex fuzzy sets (CFSs). The motivation for this extension is the utility of complex-valued function in membership grade which can express the two-dimensional ambiguous information that is prevalent in time-periodic phenomena. We introduce partial order relation on complex fuzzy sets. This partial order relation is then used to define the complex fuzzy maximal, minimal, maximum, and minimum elements. We propose new distance measures such as complex fuzzy distance measures and a complex fuzzy weighted distance measure. We establish some particular examples and basic results of the partial order relations and distance measures. Moreover, we utilize the complex fuzzy sets in signals and systems, because it is the specific form of the Fourier transform by restricting the range of Fourier transform to a complex unit disc. We establish a new algorithm based on the complex fuzzy distance measures and complex fuzzy weighted distance measures for applications in signals and systems by which we determine the degree of high resemblance of signals to the known signal. Further, the comparative study of the proposed distance measures with the Zhang distance measure, Hamming distance measure, and Normalized Hamming distance measure is discussed.

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1 Introduction

Many theories are proposed to cope with uncertainty and imprecision that handle in almost all the real-life problems such as the theory of fuzzy sets (FSs) (Zadeh 1965), theory of rough sets (Pawlak 1982), theory of intuitionistic fuzzy sets (IFSs) (Atanassov 2016), theory of Pythagorean fuzzy sets (Peng and Yang 2015), theory of complex fuzzy sets (CFSs) (Ramot et al. 2002), theory of soft sets (Molodtsov 1999), and the theory of fuzzy soft sets (FSSs) (Maji et al. 2001). All these models have their own limitations, advantages, and characteristics. These models are used in many situations of uncertainties such as engineering, computer science, decision-making problems, networking, pattern recognition, and many other fields of science.

The concept of a fuzzy set was first given by Zadeh (1965). The FSs have desirable applications in economics, engineering, decision-making problems, computer science, pattern recognition, networking, etc. Ibrahim (2021) proposed the notion of (3,2)-fuzzy sets and discussed their applications to topology and optimal choices. Türk et al. (2021) developed a multi-criteria decision-making method based on the interval type-2 fuzzy sets for selecting the best location for electric charging stations. Bulut and Ozcan discussed a new method towards the evaluation of joint technology performances of battery energy storage system under the fuzzy environment (Bulut and Özcan 2021). A novel interval type-2 trapezoid fuzzy multi-attribute group decision-making method was proposed by Meng et al. (2021). They utilized this method to the applications of the evaluation of sponge city construction. Mishra et al. (2021) extended the ARAS technique under the environment of hesitant fuzzy sets to control complex decision-making problems.

Gehrke et al. (1996) proposed the theory of interval-valued fuzzy sets (IVFSs) in which fuzzy values are interval. The interval-valued fuzzy sets have many applications in different fields of science. Dutta (2017) introduced the distance measures for IVFSs and discussed their applications in medical diagnosis. Pękala et al. (2021) defined the inclusion and similarity measures for IVFSs. A multi-criteria decision-making method based on interval-valued Fermatean fuzzy sets was proposed by Jeevaraj in Jeevaraj (2021). Huidobro et al. (2021) defined the concept of convexity of IVFSs and utilized it in decision-making problems.

Atanassov gave the concept of intuitionistic fuzzy sets (IFSs) which is the generalization of fuzzy sets (Atanassov 1986). The intuitionistic fuzzy set model is very useful in various fields of science. Xue and Deng (2021) proposed the decision-making method under the intuitionistic fuzzy environment. Yang and Yao (2021) discussed the two possible solutions to the problem of constructing a shadowed set from an Atanassov IFS. Yang et al. (2022) proposed a method for establishing a three way approximation of an intuitionistic fuzzy set following the trisecting acting outcome framework of three way decision. In Duan and Li (2021), defined four kinds of intuitionistic similarities utilizing the implication operator and corresponding logical metric spaces. They discussed their applications in pattern recognition and robustness analysis. A novel knowledge measure based on intuitionistic fuzzy set was developed by Wu et al. (2021). They utilized the proposed knowledge measure to the multi-criteria decision-making problems. Garg and Rani (2021) defined a new similarity measure based on the transformed right-angled triangles between intuitionistic fuzzy setsand investigated its applications to the decision-making problems.

In Yager (2014), introduced the notion of the Pythagorean fuzzy set (PFS) which is the extension of intuitionistic fuzzy set. The Pythagorean fuzzy sets have been widely used in uncertain problems. Ejegwa and Awolola (2021) introduced some new distance measures for Pythagorean fuzzy sets. They discussed their applications to pattern recognition problems. Farhadinia (2022) defined novel similarity measures based on two notions of t-norm and s-norm together with the distance measure between Pythagorean fuzzy sets. He showed the potential of the proposed similarity measures in medical diagnosis and pattern recognition problems. Boyacı and Şişman (2022) studied the methods used for site selection for a pandemic hospital in Atakum under the Pythagorean fuzzy environment. Ejegwa (2021) generalized the Garg's correlation coefficient for Pythagorean fuzzy sets and applied it to multi-criteria decision-making problems. Some directional correlation coefficient measures for Pythagorean fuzzy sets were introduced by Lin et al. (2021), and discussed their applications in medical diagnosis and cluster analysis. Molla et al. (2021) extended the PROMETHEE method under the environment of Pythagorean fuzzy sets. They solved a medical diagnosis problem utilizing the new proposed Pythagorean fuzzy PROMETHEE method. Ejegwa et al. (2022a, 2021) proposed a three-way approach for the computation of correlation coefficient between PFSs using the concepts of variance and covariance. They discussed decision-making problems based on three-way approach for the computation of correlation coefficient between PFSs. Some methods of calculating the correlation coefficient of PFSs which resolve the setbacks in the existing methods were discussed by Ejegwa et al. (2022b). They studied their applications in decision-making problems. Moreover, a medical diagnostic process based on modified composite relation on Pythagorean fuzzy multi-sets was developed by Ejegwa et al. (2022c).

Fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, and Pythagorean fuzzy sets can not control inconsistent, incomplete, and imprecise information of periodic nature. These models are very useful in different uncertain problems, but these theories can not deal with two-dimensional phenomena. To overcome this deficiency, Ramot et al. (2002) introduced the concept of complex fuzzy sets. The capability of a complex fuzzy set for representing two-dimensional phenomena makes it worthier than the fuzzy set model, intuitionistic fuzzy set, and Pythagorean fuzzy set model. The complex fuzzy sets have desirable applications in advanced control systems and periodic events. Jia et al. (2021) studied a new solution for Z-numbers under the complex fuzzy environment and discussed its applications in decision-making problems. Hu et al. (2017) introduced the orthogonality relation of complex fuzzy sets and discussed its applications in signals and systems. Ma et al. (2019) proposed an algorithm based on complex fuzzy sets for the identification of a high degree of resemblance with the reference signal. Some new types of relations on complex fuzzy sets were proposed by Khan et al. (2021). They developed a decision-making method based on complex fuzzy relations. Khan et al. (2020) defined the notion of complex fuzzy soft matrices and applied it to a decision-making problem in signal processing. Selvachandran et al. (2018) applied the interval-valued complex fuzzy relations in economics problem. Song et al. (2021) proposed the distance measures for interval-valued complex fuzzy sets and utilized them in decision-making problems. Zhang et al. (2009) introduced distance measure between two complex fuzzy sets. They utilized the distance measure to introduce $\delta - equalities$ of CFSs. Dai et al. (2019) developed some series of distance measures between interval-valued CFSs using Hamming and Euclidean metrics. Hu et al. (2018) defined different types of distance measures for CFSs and discussed their applications to continuity problems. The notions of distance measures and cross entropy measures in the environment of CFSs were proposed by Liu et al. (2020); Liu et al. (2022). They discussed the relation between them. Alkouri and

Salleh (2014) introduced several distance measures based on CFSs. They suggested solutions to some problems in different fields through complex fuzzy distance measures. The objectives of this paper are to

(i) define a new distance measure between two CFSs,
(ii) propose a new algorithm based on the complex fuzzy distance measures and complex fuzzy weighted distance measures for applications in signals and systems,
(iii) numerically verify the superiority of the proposed algorithm on CFSs over the existing one.

In this paper, we introduce the partial order relation on complex fuzzy sets. This partial order relation is then used to define the complex fuzzy maximal, minimal, maximum, and minimum elements. We propose new distance measures such as complex fuzzy distance measures and complex fuzzy weighted distance measures. We establish some particular examples and basic results of the partial order relations and distance measures. Moreover, we utilize the complex fuzzy sets in signals and systems. We establish a new algorithm based on the complex fuzzy distance measures and complex fuzzy weighted distance measures for applications in signals and systems by which we determine the degree of high resemblance of signals to the known signal. Further, the comparative study of the proposed distance measures with the Zhang distance measure, Hamming distance measure, and normalized Hamming distance measure is discussed.

2 Complex fuzzy sets

In this section, we will recall the notions of complex fuzzy sets.

Definition 1 (Ramot et al. 2002) A CFS $\Re$, defined on a universe of discourse U, is characterized by a grade value $\mathbb{Z}_{\Re}(\varkappa)$ that assigns any element $\varkappa \in U$ a complex-valued grade of membership in $\Re$. Mathematically, membership function of CFS $\Re$ can be represented by $\mathbb{Z}_{\Re}(\varkappa) = Ⓢ_{\Re}(\varkappa)e^{i\,Arg_{\Re}(\varkappa)}$ where $Ⓢ_{\Re}(\varkappa)$ and $Arg_{\Re}(\varkappa)$ are known as amplitude term and phase term respectively. Both these functions are real-valued and $Ⓢ_{\Re}(\varkappa) \in [0, 1]$. The function $e^{i\,Arg_{\Re}(\varkappa)}$ is a periodic function whose periodic law and principal period are, respectively, 2π and $0 < \arg_{\Re}(\varkappa) \le 2\pi$. Then, $Arg_{\Re}(\varkappa) = \arg_{\Re}(\varkappa) + 2k\pi$, $k = 0, \mp 1, \mp 2, \ldots$. The principle argument $\arg_{\Re}(\varkappa)$ will used on the following text.

Mathematically,
CFS can be expressed as a set of ordered pairs given by

$$\Re = \{(\varkappa; \mathbb{Z}_{\Re}(\varkappa)) : \varkappa \in U\}.$$

Definition 2 (Zhang et al. 2009) Let $\Re_m$, $m = 1, 2, 3, \ldots, M$ be M CFS defined on U and $\mathbb{Z}_{\Re_m}(\varkappa) = Ⓢ_{\Re_m}(\varkappa)e^{i\,\arg_{\Re_m}(\varkappa)}$ their membership functions. The complex fuzzy Cartesian product of $\Re_m$ denoted by $\Re_1 \times \Re_2 \times \Re_3 \times \cdots \times \Re_m$ is specified by a function

$$\begin{aligned}\mathbb{Z}_{\Re_1\times\Re_2\times\Re_3\times\cdots\times\Re_m}(\varkappa) &= Ⓢ_{\Re_1\times\Re_2\times\Re_3\times\cdots\times\Re_m}(\varkappa)e^{i\,\arg_{\Re_1\times\Re_2\times\Re_3\times\cdots\times\Re_m}(\varkappa)}\\ &= \min(Ⓢ_{\Re_1}(\varkappa_1), Ⓢ_{\Re_2}(\varkappa_2), \ldots, Ⓢ_{\Re_m}(\varkappa_m))\\ &\quad e^{i\,\min(\arg_{\Re_1}(\varkappa_1), \arg_{\Re_2}(\varkappa_2), \ldots, \arg_{\Re_m}(\varkappa_m))}.\end{aligned}$$

Definition 3 (Ramot et al. 2002) Let $\Re_1$ and $\Re_2$ be two complex fuzzy sets on U, and $\mathbb{Z}_{\Re_1}(\varkappa) = Ⓢ_{\Re_1}(\varkappa)e^{i\,\arg_{\Re_1}(\varkappa)}$ and $\mathbb{Z}_{\Re_2}(\varkappa) = Ⓢ_{\Re_2}(\varkappa)e^{i\,\arg_{\Re_2}(\varkappa)}$ their grade values, respectively. The intersection of these two complex fuzzy sets $\Re_1$ and $\Re_2$, denoted $\Re_1 \cap \Re_2$, is specified by a function

$$\Re_1 \cap \Re_2 = Ⓢ_{\Re_1}(\varkappa)e^{i \arg_{\Re_1}(\varkappa)} \cap Ⓢ_{\Re_2}(\varkappa)e^{i \arg_{\Re_2}(\varkappa)}$$
$$= \min\left[Ⓢ_{\Re_1}(\varkappa), Ⓢ_{\Re_2}(\varkappa)\right] e^{i \min\left[\arg_{\Re_1}(\varkappa), \arg_{\Re_2}(\varkappa)\right]}.$$

Definition 4 (Ramot et al. 2002) Let $\Re_1$ and $\Re_2$ be two complex fuzzy sets on U, and $\mathbb{Z}_{\Re_1}(\varkappa) = Ⓢ_{\Re_1}(\varkappa)e^{i \arg_{\Re_1}(\varkappa)}$ and $\mathbb{Z}_{\Re_2}(\varkappa) = Ⓢ_{\Re_2}(\varkappa)e^{i \arg_{\Re_2}(\varkappa)}$ their grade values, respectively. The union of these two complex fuzzy sets $\Re_1$ and $\Re_2$, denoted $\Re_1 \cup \Re_2$, is specified by a function

$$\Re_1 \cup \Re_2 = Ⓢ_{\Re_1}(\varkappa)e^{i \arg_{\Re_1}(\varkappa)} \cup Ⓢ_{\Re_2}(\varkappa)e^{i \arg_{\Re_2}(\varkappa)}$$
$$= \max\left[Ⓢ_{\Re_1}(\varkappa), Ⓢ_{\Re_2}(\varkappa)\right] e^{i \max\left[\arg_{\Re_1}(\varkappa), \arg_{\Re_2}(\varkappa)\right]}.$$

Definition 5 Let $\Re_1$ and $\Re_2$ be two complex fuzzy sets on U, and $\mathbb{Z}_{\Re_1}(\varkappa) = Ⓢ_{\Re_1}(\varkappa)e^{i \arg_{\Re_1}(\varkappa)}$ and $\mathbb{Z}_{\Re_2}(\varkappa) = Ⓢ_{\Re_2}(\varkappa)e^{i \arg_{\Re_2}(\varkappa)}$ their grade values, respectively. Then, $\Re_1$ is said to be a subset of $\Re_2$, denoted by $\Re_1 \subseteq \Re_2$ if $Ⓢ_{\Re_1}(\varkappa) \leq Ⓢ_{\Re_2}(\varkappa)$ and $\arg_{\Re_1}(\varkappa) \leq \arg_{\Re_2}(\varkappa)$.

Definition 6 A relation $\leq$ is said to be a partial order on a complex fuzzy set $\Re$ if the following properties hold.

(i) $Ⓢ_{\Re}(\varkappa_i) \leq Ⓢ_{\Re}(\varkappa_i)$ and $\arg_{\Re}(\varkappa_i) \leq \arg_{\Re}(\varkappa_i)$.
(ii) If $Ⓢ_{\Re}(\varkappa_i) \leq Ⓢ_{\Re}(\varkappa_j)$, $\arg_{\Re}(\varkappa_i) \leq \arg_{\Re}(\varkappa_j)$ and $Ⓢ_{\Re}(\varkappa_i) \geq Ⓢ_{\Re}(\varkappa_j)$, $\arg_{\Re}(\varkappa_i) \geq \arg_{\Re}(\varkappa_j)$ then, $Ⓢ_{\Re}(\varkappa_i) = Ⓢ_{\Re}(\varkappa_j)$, $\arg_{\Re}(\varkappa_i) = \arg_{\Re}(\varkappa_j)$.
(iii) If $Ⓢ_{\Re}(\varkappa_i) \leq Ⓢ_{\Re}(\varkappa_j)$, $\arg_{\Re}(\varkappa_i) \leq \arg_{\Re}(\varkappa_j)$ and $Ⓢ_{\Re}(\varkappa_j) \leq Ⓢ_{\Re}(\varkappa_k)$, $\arg_{\Re}(\varkappa_j) \leq \arg_{\Re}(\varkappa_k)$ then, $Ⓢ_{\Re}(\varkappa_i) \leq Ⓢ_{\Re}(\varkappa_k)$, $\arg_{\Re}(\varkappa_i) \leq \arg_{\Re}(\varkappa_k)$.

Definition 7 Let $\Re$ be a complex fuzzy partial order set and $\mathbb{Z}_{\Re}(\varkappa_i) \in \mathbb{Z}_{\Re}(\varkappa)$. We define

(i) $\mathbb{Z}_{\Re}(\varkappa_i)$ is complex fuzzy minimal if $\mathbb{Z}_{\Re}(\varkappa_i) \geq \mathbb{Z}_{\Re}(\varkappa_j)$ then $\mathbb{Z}_{\Re}(\varkappa_i) = \mathbb{Z}_{\Re}(\varkappa_j)$, that is, $Ⓢ_{\Re}(\varkappa_i) \geq Ⓢ_{\Re}(\varkappa_j)$, $\arg_{\Re}(\varkappa_i) \geq \arg_{\Re}(\varkappa_j)$ then, $Ⓢ_{\Re}(\varkappa_i) = Ⓢ_{\Re}(\varkappa_j)$, $\arg_{\Re}(\varkappa_i) = \arg_{\Re}(\varkappa_j)$ for all $\mathbb{Z}_{\Re}(\varkappa_j) \in \mathbb{Z}_{\Re}(\varkappa)$.
(ii) $\mathbb{Z}_{\Re}(\varkappa_i)$ is complex fuzzy maximal if $\mathbb{Z}_{\Re}(\varkappa_i) \leq \mathbb{Z}_{\Re}(\varkappa_j)$ then $\mathbb{Z}_{\Re}(\varkappa_i) = \mathbb{Z}_{\Re}(\varkappa_j)$, that is, $Ⓢ_{\Re}(\varkappa_i) \leq Ⓢ_{\Re}(\varkappa_j)$, $\arg_{\Re}(\varkappa_i) \leq \arg_{\Re}(\varkappa_j)$ then, $Ⓢ_{\Re}(\varkappa_i) = Ⓢ_{\Re}(\varkappa_j)$, $\arg_{\Re}(\varkappa_i) = \arg_{\Re}(\varkappa_j)$ for all $\mathbb{Z}_{\Re}(\varkappa_j) \in \mathbb{Z}_{\Re}(\varkappa)$.
(iii) $\mathbb{Z}_{\Re}(\varkappa_i)$ is a complex fuzzy minimum element if $\mathbb{Z}_{\Re}(\varkappa_i) \leq \mathbb{Z}_{\Re}(\varkappa_j)$, that is, $Ⓢ_{\Re}(\varkappa_i) \leq Ⓢ_{\Re}(\varkappa_j)$, $\arg_{\Re}(\varkappa_i) \leq \arg_{\Re}(\varkappa_j)$ for all $\mathbb{Z}_{\Re}(\varkappa_j) \in \mathbb{Z}_{\Re}(\varkappa)$.
(iv) $\mathbb{Z}_{\Re}(\varkappa_i)$ is a complex fuzzy maximum element if $\mathbb{Z}_{\Re}(\varkappa_i) \geq \mathbb{Z}_{\Re}(\varkappa_j)$, that is, $Ⓢ_{\Re}(\varkappa_i) \geq Ⓢ_{\Re}(\varkappa_j)$, $\arg_{\Re}(\varkappa_i) \geq \arg_{\Re}(\varkappa_j)$ for all $\mathbb{Z}_{\Re}(\varkappa_j) \in \mathbb{Z}_{\Re}(\varkappa)$.

Theorem 1 *Let $\Re$ be a complex fuzzy partial order set and $\mathbb{Z}_{\Re}(\varkappa) = Ⓢ_{\Re}(\varkappa)e^{i \arg_{\Re}(\varkappa)}$ be its membership function. Then,*

(i) *Complex fuzzy maximum elements are complex fuzzy maximal.*
(ii) *Complex fuzzy minimum elements are complex fuzzy minimal.*
(iii) *There can be at most one complex fuzzy maximum element.*
(iv) *There can be at most one complex fuzzy minimum element.*

Proof (i) Let $\mathbb{Z}_{\Re}(\varkappa_i) \in \mathbb{Z}_{\Re}(\varkappa)$ is a complex fuzzy maximum element then,

$$\mathbb{Z}_{\Re}(\varkappa_i) \geq \mathbb{Z}_{\Re}(\varkappa_j) \tag{1}$$

for all $\mathbb{Z}_{\Re}(\varkappa_j) \in \mathbb{Z}_{\Re}(\varkappa)$. We prove that a complex fuzzy maximum element is a complex fuzzy maximal. If

$$\mathbb{Z}_{\Re}(\varkappa_i) \leq \mathbb{Z}_{\Re}(\varkappa_j) \tag{2}$$

From inequalities 1 and 2, we have $\mathbb{Z}_{\Re}(\varkappa_i) = \mathbb{Z}_{\Re}(\varkappa_j)$. Thus, $\mathbb{Z}_{\Re}(\varkappa_i)$ is a complex fuzzy maximal. Since $\mathbb{Z}_{\Re}(\varkappa_i)$ is an arbitrary complex fuzzy maximal elements. Therefore, all the complex fuzzy maximum elements are complex fuzzy maximal.

(ii) It is easy to prove.

(iii) Let $\mathbb{Z}_{\Re}(\varkappa_i)$ and $\mathbb{Z}_{\Re}(\varkappa_j)$ be two complex fuzzy maximum elements of a complex fuzzy set $\Re$. Since $\mathbb{Z}_{\Re}(\varkappa_i)$ is a complex fuzzy maximum element then,

$$\mathbb{Z}_{\Re}(\varkappa_i) \geq \mathbb{Z}_{\Re}(\varkappa_j) \tag{3}$$

for all $\mathbb{Z}_{\Re}(\varkappa_j) \in \mathbb{Z}_{\Re}(\varkappa)$.

Also, $\mathbb{Z}_{\Re}(\varkappa_j)$ is a complex fuzzy maximum element then,

$$\mathbb{Z}_{\Re}(\varkappa_j) \geq \mathbb{Z}_{\Re}(\varkappa_i) \tag{4}$$

for all $\mathbb{Z}_{\Re}(\varkappa_i) \in \mathbb{Z}_{\Re}(\varkappa)$. from inequality 3 and 4 we have $\mathbb{Z}_{\Re}(\varkappa_j) = \mathbb{Z}_{\Re}(\varkappa_i)$. Thus, there exists at most one complex fuzzy maximum element.

(iv). It is easy to prove. □

3 Distance measures of complex fuzzy sets

In this section, we recall some distance measures for complex fuzzy sets such as Zhang distance, Normalized Hamming distance measure, and Hamming distance measure. Moreover, we propose the distance measure and weighted distance measure of CFSs.

(ii) The Zhang distance,

$$\Gamma(\Re_i, \Re_j) = \max\left[\sup_{\varkappa_q \in U} |Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_j}(\varkappa_q)|, \frac{1}{2\pi}\sup_{\varkappa_q \in U} |\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|\right].$$

(iv) The Normalized Hamming distance,

$$\Gamma(\Re_i, \Re_j) = \frac{1}{2n}\left[\sum_{q=1}^{n} |Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_j}(\varkappa_q)| + \frac{1}{2\pi}\sum_{q=1}^{n} |\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|\right].$$

(iii) The Hamming distance,

$$\Gamma(\Re_i, \Re_j) = \frac{1}{2}\left[\sum_{q=1}^{n} |Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_j}(\varkappa_q)| + \frac{1}{2\pi}\sum_{q=1}^{n} |\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|\right].$$

Definition 8 A distance measure of CFSs is a function $\Gamma : \Re^*(U) \times \Re^*(U) \to [0, 1]$ with the properties: for any $\Re_1, \Re_2, \Re_3 \in \Re^*(U)$(collection of CFSs)

(i) $0 \leq \Gamma(\Re_1, \Re_2) \leq 1$, $\Gamma(\Re_1, \Re_2) = 0$ if and only if $\Re_1 = \Re_2$.
(ii) $\Gamma(\Re_1, \Re_2) = \Gamma(\Re_2, \Re_1)$.
(iii) $\Gamma(\Re_1, \Re_3) \leq \Gamma(\Re_1, \Re_2) + \Gamma(\Re_2, \Re_3)$.

We introduce the distance measure Γ as

$$\Gamma(\Re_i, \Re_j) = \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|\circledS_{\Re_i}(\varkappa_q) - \circledS_{\Re_j}(\varkappa_q)|}{1 + |\circledS_{\Re_i}(\varkappa_q) - \circledS_{\Re_j}(\varkappa_q)|} + \frac{|\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|}{2\pi + |\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|}\right]. \tag{5}$$

Note that the distance measure Γ plays a key role in the remainder of this paper.

Example 1 Let

$$\Re_1 = \left\{\frac{0.3e^{1\pi}}{a} + \frac{0.8e^{1.5\pi}}{b} + \frac{0.5e^{2\pi}}{c}\right\},$$
$$\Re_2 = \left\{\frac{0.7e^{2\pi}}{a} + \frac{0.4e^{0.5\pi}}{b} + \frac{0.9e^{1\pi}}{c}\right\},$$

then

$$\begin{aligned}\Gamma(\Re_1, \Re_2) &= \frac{1}{3}\left[\begin{array}{c}\left(\frac{|0.3-0.7|}{1+|0.3-0.7|} + \frac{|1\pi-2\pi|}{2\pi+|1\pi-2\pi|}\right) + \left(\frac{|0.8-0.4|}{1+|0.8-0.4|} + \frac{|1.5\pi-0.5\pi|}{2\pi+|1.5\pi-0.5\pi|}\right)\\ + \left(\frac{|0.5-0.9|}{1+|0.5-0.9|} + \frac{|2\pi-1\pi|}{2\pi+|2\pi-1\pi|}\right)\end{array}\right],\\ &= \frac{1}{3}\,[0.29 + 0.33 + 0.29 + 0.33 + 0.29 + 0.33]\\ &= 0.62.\end{aligned}$$

Theorem 2 *The function Γ defined by the equality 5 is a distance function of CFSs on U.*

Proof The condition $\Gamma(\Re_i, \Re_j) \geq 0$ obviously holds true. Next consider

$$\begin{aligned}\Gamma(\Re_i, \Re_j) &= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|\circledS_{\Re_i}(\varkappa_q) - \circledS_{\Re_j}(\varkappa_q)|}{1 + |\circledS_{\Re_i}(\varkappa_q) - \circledS_{\Re_j}(\varkappa_q)|} + \frac{|\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|}{2\pi + |\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|}\right]\\ &= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{1}{1+1} + \frac{2\pi}{2\pi + 2\pi}\right]\\ &= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{1}{2} + \frac{2\pi}{4\pi}\right] = \frac{1}{n}\left[\frac{1}{2} + \frac{1}{2}\right]\\ &= \frac{1}{n} \leq 1.\end{aligned}$$

Therefore, $0 \leq \Gamma(\Re_i, \Re_j) \leq 1$, and

$$\begin{aligned}\Gamma(\Re_i, \Re_i) &= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|\circledS_{\Re_i}(\varkappa_q) - \circledS_{\Re_i}(\varkappa_q)|}{1 + |\circledS_{\Re_i}(\varkappa_q) - \circledS_{\Re_i}(\varkappa_q)|} + \frac{|\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_i}(\varkappa_q)|}{2\pi + |\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_i}(\varkappa_q)|}\right]\\ &= 0 + 0 = 0.\end{aligned}$$

Condition (ii) is straightforward. To prove (iii), we have

$$\begin{aligned}\Gamma(\Re_i,\Re_k) &= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\right]\\ &= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)+Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)+Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)+\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)+\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\right]\end{aligned}$$

$$\begin{aligned}\Gamma(\Re_i,\Re_k) &\le \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|+|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|+|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\right]\\
&\le \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|+|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}+\frac{|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|+|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\right]\\
&= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|+|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\right]\\
&\quad+\frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|+|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\right]\\
&\le \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\right]\\
&\quad+\frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}{1+|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}{2\pi+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\right]\\
&= \Gamma(\Re_i,\Re_j)+\Gamma(\Re_j,\Re_k).\end{aligned}$$

Thus, $\Gamma(\Re_i,\Re_k)\le\Gamma(\Re_i,\Re_j)+\Gamma(\Re_j,\Re_k)$. □

Note that in the above theorem, if $n=1$ then, $\Gamma(\Re_i,\Re_j)=1$.

Corollary 1 *The distance measure* Γ *of complex fuzzy sets is a fuzzy set.*

Proof The proof is obvious from the definition. □

Proposition 1 *The distance measure* Γ *of complex fuzzy sets is closed with respect to the operations of fuzzy union, fuzzy intersection, and fuzzy complement.*

Proof It is easy to prove. □

Definition 9 Let Γ be a distance measure of complex fuzzy sets. Then, the complement distance measure $\Gamma((\Re_i)^c, (\Re_j)^c)$ of two complex fuzzy sets is defined as

$$\Gamma((\Re_i)^c, (\Re_j)^c) = \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|(1-Ⓢ_{\Re_i}(\varkappa_q))-(1-Ⓢ_{\Re_j}(\varkappa_q))|}{1+|(1-Ⓢ_{\Re_i}(\varkappa_q))-(1-Ⓢ_{\Re_j}(\varkappa_q))|}+\\ \frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\end{array}\right].$$

Example 2 Let

$$\Re_1 = \frac{0.9e^{1\pi}}{a} + \frac{1e^{1.5\pi}}{b} + \frac{0.1e^{2\pi}}{c},$$
$$\Re_2 = \frac{0.2e^{2\pi}}{a} + \frac{0.3e^{0.5\pi}}{b} + \frac{0.8e^{1\pi}}{c},$$

then

$$\Gamma((\Re_1)^c, (\Re_2)^c) = 0.74.$$

Definition 10 Let Γ be a distance measure of complex fuzzy sets. Then, the distance measure $\Gamma(\Re_i \cup \Re_j, \Re_i \cap \Re_j)$ of two complex fuzzy sets is defined as

$$\Gamma(\Re_i \cup \Re_j, \Re_i \cap \Re_j) = \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|(Ⓢ_{\Re_i}(\varkappa_q)\vee Ⓢ_{\Re_j}(\varkappa_q))-(Ⓢ_{\Re_i}(\varkappa_q)\wedge Ⓢ_{\Re_j}(\varkappa_q))|}{1+|(Ⓢ_{\Re_i}(\varkappa_q)\vee Ⓢ_{\Re_j}(\varkappa_q))-(Ⓢ_{\Re_i}(\varkappa_q)\wedge Ⓢ_{\Re_j}(\varkappa_q))|}+\\ \frac{|(\arg_{\Re_i}(\varkappa_q)\vee \arg_{\Re_j}(\varkappa_q))-(\arg_{\Re_i}(\varkappa_q)\wedge \arg_{\Re_j}(\varkappa_q))|}{2\pi+|(\arg_{\Re_i}(\varkappa_q)\vee \arg_{\Re_j}(\varkappa_q))-(\arg_{\Re_i}(\varkappa_q)\wedge \arg_{\Re_j}(\varkappa_q))|}\end{array}\right],$$

where $\vee$ and $\wedge$ denote the max and min operator of complex fuzzy sets.

Example 3 Let

$$\Re_1 = \frac{0.5e^{0.5\pi}}{a} + \frac{0.6e^{2\pi}}{b} + \frac{0.9e^{0.2\pi}}{c},$$
$$\Re_2 = \frac{1e^{1\pi}}{a} + \frac{0.1e^{1.2\pi}}{b} + \frac{0.7e^{1.5\pi}}{c}$$

then

$$\Gamma(\Re_1 \cup \Re_2, \Re_1 \cap \Re_2) = 0.57.$$

Theorem 3 *Let* Γ *be a distance measure of complex fuzzy sets. Then, the following hold.*

(i) $\Gamma((\Re_i)^c, \Re_j) = \Gamma(\Re_i, (\Re_j)^c)$,
(ii) $\Gamma((\Re_i)^c, (\Re_j)^c) = \Gamma(\Re_i, \Re_j)$.

Proof (i) For $\Gamma(\Re_i, \Re_j) = \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\right]$, we have the following:

$$\Gamma((\Re_i)^c, \Re_j) = \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|(1-Ⓢ_{\Re_i}(\varkappa_q))-Ⓢ_{\Re_j}(\varkappa_q)|}{1+|(1-Ⓢ_{\Re_i}(\varkappa_q))-Ⓢ_{\Re_j}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\right]$$
$$= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|(1-Ⓢ_{\Re_j}(\varkappa_q))-Ⓢ_{\Re_i}(\varkappa_q)|}{1+|(1-Ⓢ_{\Re_j}(\varkappa_q))-Ⓢ_{\Re_i}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\right]$$
$$= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_i}(\varkappa_q)-(1-Ⓢ_{\Re_j}(\varkappa_q))|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-(1-Ⓢ_{\Re_j}(\varkappa_q))|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\right]$$
$$= \Gamma(\Re_i, (\Re_j)^c).$$

(ii)

$$\Gamma((\Re_i)^c,(\Re_j)^c) = \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|(1-Ⓢ_{\Re_i}(\varkappa_q))-(1-Ⓢ_{\Re_j}(\varkappa_q))|}{1+|(1-Ⓢ_{\Re_i}(\varkappa_q))-(1-Ⓢ_{\Re_j}(\varkappa_q))|}+\\ \frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\end{array}\right]$$

$$= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_i}(\varkappa_q))|}{1+|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_i}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\right]$$

$$= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q))|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\right]$$

$$= \Gamma(\Re_i,\Re_j).$$

□

Theorem 4 *Let Γ be a distance measure of complex fuzzy sets. Then, the following hold.*

(i) $\Gamma(\Re_i\cup\Re_j,\Re_i\cap\Re_j)=\Gamma(\Re_i,\Re_j)$,
(ii) $\Gamma(\Re_i,\Re_i\cap\Re_j)=\Gamma(\Re_j,\Re_i\cup\Re_j)$,
(iii) $\Gamma(\Re_i,\Re_i\cup\Re_j)=\Gamma(\Re_j,\Re_i\cap\Re_j)$.

Proof (i) To prove 1 there are many cases arise here.
Case 1.

$$Ⓢ_{\Re_i}(\varkappa_q)\le Ⓢ_{\Re_j}(\varkappa_q) \text{ and } \arg_{\Re_i}(\varkappa_q)\le\arg_{\Re_j}(\varkappa_q)$$

$$\Gamma(\Re_i\cup\Re_j,\Re_i\cap\Re_j) = \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|(Ⓢ_{\Re_i}(\varkappa_q)\vee Ⓢ_{\Re_j}(\varkappa_q))-(Ⓢ_{\Re_i}(\varkappa_q)\wedge Ⓢ_{\Re_j}(\varkappa_q))|}{1+|(Ⓢ_{\Re_i}(\varkappa_q)\vee Ⓢ_{\Re_j}(\varkappa_q))-(Ⓢ_{\Re_i}(\varkappa_q)\wedge Ⓢ_{\Re_j}(\varkappa_q))|}+\\ \frac{|(\arg_{\Re_i}(\varkappa_q)\vee\arg_{\Re_j}(\varkappa_q))-(\arg_{\Re_i}(\varkappa_q)\wedge\arg_{\Re_j}(\varkappa_q))|}{2\pi+|(\arg_{\Re_i}(\varkappa_q)\vee\arg_{\Re_j}(\varkappa_q))-(\arg_{\Re_i}(\varkappa_q)\wedge\arg_{\Re_j}(\varkappa_q))|}\end{array}\right]$$

$$= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_j}(\varkappa_q))-Ⓢ_{\Re_i}(\varkappa_q)|}{1+|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_i}(\varkappa_q))|}+\frac{|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_i}(\varkappa_q)|}{2\pi+|(\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_i}(\varkappa_q)|}\right]$$

$$= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_i}(\varkappa_q))-Ⓢ_{\Re_j}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q))|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|(\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\right]$$

$$= \Gamma(\Re_i,\Re_j).$$

Case 2.

$$Ⓢ_{\Re_j}(\varkappa_q)\le Ⓢ_{\Re_i}(\varkappa_q) \text{ and } \arg_{\Re_j}(\varkappa_q)\le\arg_{\Re_i}(\varkappa_q)$$

$$\Gamma(\Re_i\cup\Re_j,\Re_i\cap\Re_j) = \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|(Ⓢ_{\Re_i}(\varkappa_q)\vee Ⓢ_{\Re_j}(\varkappa_q))-(Ⓢ_{\Re_i}(\varkappa_q)\wedge Ⓢ_{\Re_j}(\varkappa_q))|}{1+|(Ⓢ_{\Re_i}(\varkappa_q)\vee Ⓢ_{\Re_j}(\varkappa_q))-(Ⓢ_{\Re_i}(\varkappa_q)\wedge Ⓢ_{\Re_j}(\varkappa_q))|}+\\ \frac{|(\arg_{\Re_i}(\varkappa_q)\vee\arg_{\Re_j}(\varkappa_q))-(\arg_{\Re_i}(\varkappa_q)\wedge\arg_{\Re_j}(\varkappa_q))|}{2\pi+|(\arg_{\Re_i}(\varkappa_q)\vee\arg_{\Re_j}(\varkappa_q))-(\arg_{\Re_i}(\varkappa_q)\wedge\arg_{\Re_j}(\varkappa_q))|}\end{array}\right]$$

$$= \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|Ⓢ_{\Re_i}(\varkappa_q))-Ⓢ_{\Re_j}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q))|}+\\ \frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|(\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\end{array}\right]$$

$$= \Gamma(\Re_i,\Re_j).$$

Case 3.

$$Ⓢ_{\Re_i}(\varkappa_q) \leq Ⓢ_{\Re_j}(\varkappa_q) \text{ and } \arg_{\Re_j}(\varkappa_q) \leq \arg_{\Re_i}(\varkappa_q)$$

$$\begin{aligned}\Gamma(\Re_i \cup \Re_j, \Re_i \cap \Re_j) &= \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|(Ⓢ_{\Re_i}(\varkappa_q) \vee Ⓢ_{\Re_j}(\varkappa_q)) - (Ⓢ_{\Re_i}(\varkappa_q) \wedge Ⓢ_{\Re_j}(\varkappa_q))|}{1+|(Ⓢ_{\Re_i}(\varkappa_q) \vee Ⓢ_{\Re_j}(\varkappa_q)) - (Ⓢ_{\Re_i}(\varkappa_q) \wedge Ⓢ_{\Re_j}(\varkappa_q))|} + \\ \frac{|(\arg_{\Re_i}(\varkappa_q) \vee \arg_{\Re_j}(\varkappa_q)) - (\arg_{\Re_i}(\varkappa_q) \wedge \arg_{\Re_j}(\varkappa_q))|}{2\pi+|(\arg_{\Re_i}(\varkappa_q) \vee \arg_{\Re_j}(\varkappa_q)) - (\arg_{\Re_i}(\varkappa_q) \wedge \arg_{\Re_j}(\varkappa_q))|}\end{array}\right] \\ &= \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|Ⓢ_{\Re_j}(\varkappa_q)) - Ⓢ_{\Re_i}(\varkappa_q)|}{1+|Ⓢ_{\Re_j}(\varkappa_q) - Ⓢ_{\Re_i}(\varkappa_q))|} + \\ \frac{|\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|}{2\pi+|(\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|}\end{array}\right] \\ &= \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|Ⓢ_{\Re_i}(\varkappa_q)) - Ⓢ_{\Re_j}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_j}(\varkappa_q))|} + \\ \frac{|\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|}{2\pi+|(\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|}\end{array}\right] \\ &= \Gamma(\Re_i, \Re_j).\end{aligned}$$

Case 4.

$$Ⓢ_{\Re_j}(\varkappa_q) \leq Ⓢ_{\Re_i}(\varkappa_q) \text{ and } \arg_{\Re_i}(\varkappa_q) \leq \arg_{\Re_j}(\varkappa_q)$$

$$\begin{aligned}\Gamma(\Re_i \cup \Re_j, \Re_i \cap \Re_j) &= \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|(Ⓢ_{\Re_i}(\varkappa_q) \vee Ⓢ_{\Re_j}(\varkappa_q)) - (Ⓢ_{\Re_i}(\varkappa_q) \wedge Ⓢ_{\Re_j}(\varkappa_q))|}{1+|(Ⓢ_{\Re_i}(\varkappa_q) \vee Ⓢ_{\Re_j}(\varkappa_q)) - (Ⓢ_{\Re_i}(\varkappa_q) \wedge Ⓢ_{\Re_j}(\varkappa_q))|} + \\ \frac{|(\arg_{\Re_i}(\varkappa_q) \vee \arg_{\Re_j}(\varkappa_q)) - (\arg_{\Re_i}(\varkappa_q) \wedge \arg_{\Re_j}(\varkappa_q))|}{2\pi+|(\arg_{\Re_i}(\varkappa_q) \vee \arg_{\Re_j}(\varkappa_q)) - (\arg_{\Re_i}(\varkappa_q) \wedge \arg_{\Re_j}(\varkappa_q))|}\end{array}\right] \\ &= \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|Ⓢ_{\Re_i}(\varkappa_q)) - Ⓢ_{\Re_j}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_j}(\varkappa_q))|} + \\ \frac{|\arg_{\Re_j}(\varkappa_q) - \arg_{\Re_i}(\varkappa_q)|}{2\pi+|(\arg_{\Re_j}(\varkappa_q) - \arg_{\Re_i}(\varkappa_q)|}\end{array}\right] \\ &= \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|Ⓢ_{\Re_i}(\varkappa_q)) - Ⓢ_{\Re_j}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_j}(\varkappa_q))|} + \\ \frac{|\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|}{2\pi+|(\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|}\end{array}\right] \\ &= \Gamma(\Re_i, \Re_j).\end{aligned}$$

Thus, in all cases, we have

$$\Gamma(\Re_i \cup \Re_j, \Re_i \cap \Re_j) = \Gamma(\Re_i, \Re_j).$$

(ii) To prove (ii), we use the same cases.
Case 1.

$$Ⓢ_{\Re_i}(\varkappa_q) \leq Ⓢ_{\Re_j}(\varkappa_q) \text{ and } \arg_{\Re_i}(\varkappa_q) \leq \arg_{\Re_j}(\varkappa_q)$$

$$\begin{aligned}\Gamma(\Re_i, \Re_i \cap \Re_j) &= \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|Ⓢ_{\Re_i}(\varkappa_q) - (Ⓢ_{\Re_i}(\varkappa_q) \wedge Ⓢ_{\Re_j}(\varkappa_q))|}{1+|Ⓢ_{\Re_i}(\varkappa_q) - (Ⓢ_{\Re_i}(\varkappa_q) \wedge Ⓢ_{\Re_j}(\varkappa_q))|} + \\ \frac{|\arg_{\Re_i}(\varkappa_q) - (\arg_{\Re_i}(\varkappa_q) \wedge \arg_{\Re_j}(\varkappa_q))|}{2\pi+|\arg_{\Re_i}(\varkappa_q) - (\arg_{\Re_i}(\varkappa_q) \wedge \arg_{\Re_j}(\varkappa_q))|}\end{array}\right] \\ &= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_i}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_i}(\varkappa_q))|} + \frac{|\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_i}(\varkappa_q)|}{2\pi+|(\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_i}(\varkappa_q)|}\right] \\ &= 0.\end{aligned} \tag{6}$$

$$\Gamma(\Re_j, \Re_i \cup \Re_j) = \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|Ⓢ_{\Re_j}(\varkappa_q)-(Ⓢ_{\Re_i}(\varkappa_q)\vee Ⓢ_{\Re_j}(\varkappa_q))|}{1+|Ⓢ_{\Re_j}(\varkappa_q)-(Ⓢ_{\Re_i}(\varkappa_q)\vee Ⓢ_{\Re_j}(\varkappa_q))|}+\\ \frac{|\arg_{\Re_j}(\varkappa_q)-(\arg_{\Re_i}(\varkappa_q)\vee \arg_{\Re_j}(\varkappa_q))|}{2\pi+|\arg_{\Re_j}(\varkappa_q)-(\arg_{\Re_i}(\varkappa_q)\vee \arg_{\Re_j}(\varkappa_q))|}\end{array}\right]$$
$$= \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|}{1+|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q))|}+\frac{|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|(\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\right]$$
$$= 0. \tag{7}$$

From 5 and 7, we have

$$\Gamma(\Re_i, \Re_i \cap \Re_j) = \Gamma(\Re_j, \Re_i \cup \Re_j).$$

Case 2.

$$Ⓢ_{\Re_j}(\varkappa_q) \le Ⓢ_{\Re_i}(\varkappa_q) \text{ and } \arg_{\Re_j}(\varkappa_q) \le \arg_{\Re_i}(\varkappa_q)$$

$$\Gamma(\Re_i, \Re_i \cap \Re_j) = \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|Ⓢ_{\Re_i}(\varkappa_q)-(Ⓢ_{\Re_i}(\varkappa_q)\wedge Ⓢ_{\Re_j}(\varkappa_q))|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-(Ⓢ_{\Re_i}(\varkappa_q)\wedge Ⓢ_{\Re_j}(\varkappa_q))|}+\\ \frac{|\arg_{\Re_i}(\varkappa_q)-(\arg_{\Re_i}(\varkappa_q)\wedge \arg_{\Re_j}(\varkappa_q))|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-(\arg_{\Re_i}(\varkappa_q)\wedge \arg_{\Re_j}(\varkappa_q))|}\end{array}\right]$$
$$= \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q))|}+\\ \frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|(\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\end{array}\right]$$
$$= \Gamma(\Re_i, \Re_j) \tag{8}$$

$$\Gamma(\Re_j, \Re_i \cup \Re_j) = \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|Ⓢ_{\Re_j}(\varkappa_q)-(Ⓢ_{\Re_i}(\varkappa_q)\vee Ⓢ_{\Re_j}(\varkappa_q))|}{1+|Ⓢ_{\Re_j}(\varkappa_q)-(Ⓢ_{\Re_i}(\varkappa_q)\vee Ⓢ_{\Re_j}(\varkappa_q))|}+\\ \frac{|\arg_{\Re_j}(\varkappa_q)-(\arg_{\Re_i}(\varkappa_q)\vee \arg_{\Re_j}(\varkappa_q))|}{2\pi+|\arg_{\Re_j}(\varkappa_q)-(\arg_{\Re_i}(\varkappa_q)\vee \arg_{\Re_j}(\varkappa_q))|}\end{array}\right]$$
$$= \frac{1}{n}\sum_{q=1}^{n}\left[\begin{array}{c}\frac{|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_i}(\varkappa_q)|}{1+|Ⓢ_{\Re_j}(\varkappa_q)-Ⓢ_{\Re_i}(\varkappa_q))|}+\\ \frac{|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_i}(\varkappa_q)|}{2\pi+|(\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_i}(\varkappa_q)|}\end{array}\right]$$
$$= \Gamma(\Re_i, \Re_j). \tag{9}$$

From 8 and 9, we have

$$\Gamma(\Re_i, \Re_i \cap \Re_j) = \Gamma(\Re_j, \Re_i \cup \Re_j).$$

The proof is similar for other cases.
(iii) The Proof of (iii) is similar to the Proof of (ii). □

Corollary 2 *Let* Γ *be a distance measure of complex fuzzy sets. Then,*

(i) $\Gamma(\Re_i \cup \Re_j, \Re_i \cup \Re_j) = 0$,
(ii) $\Gamma(\Re_i \cap \Re_j, \Re_i \cap \Re_j) = 0$.

Proof It is easy to prove. □

Definition 11 A weighted distance measure of CFSs is a function $\Gamma_w : \Re^*(U) \times \Re^*(U) \to [0, 1]$ with the properties: for any $\Re_1, \Re_2, \Re_3 \in \Re^*(U)$(collection of CFSs)

(i) $0 \le \Gamma_w(\Re_1, \Re_2) \le 1$, $\Gamma_w(\Re_1, \Re_2) = 0$ if and only if $\Re_1 = \Re_2$.

(ii) $\Gamma_w(\Re_1, \Re_2) = \Gamma_w(\Re_2, \Re_1)$.
(iii) $\Gamma_w(\Re_1, \Re_3) \le \Gamma_w(\Re_1, \Re_2) + \Gamma_w(\Re_2, \Re_3)$.

We introduce the weighted distance measure Γ_w as

$$\Gamma_w(\Re_i, \Re_j) = \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\begin{array}{c}\frac{|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|}+\\ \frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\end{array}\right]\right]\right] \tag{10}$$

where w_q is a weighted vector.

Theorem 5 *The function Γ_w defined by the equality 10 is a distance function of CFSs on U.*

Proof The condition $\Gamma_w(\Re_i, \Re_j) \ge 0$ obviously holds true. Next consider

$$\begin{aligned}\Gamma_w(\Re_i, \Re_j) &= \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\begin{array}{c}\frac{|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_j}(\varkappa_q)|}+\\ \frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\end{array}\right]\right]\right]\\
&= \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\frac{1}{1+1}+\frac{2\pi}{2\pi+2\pi}\right]\right]\right]\\
&= \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\frac{1}{2}+\frac{1}{2}\right]\right]\right]\\
&= \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n} w_q.1\right] = \frac{1}{n}.\\
&\therefore 0 \le \Gamma_w(\Re_i, \Re_j) \le 1,\ \text{and}\\
\Gamma_w(\Re_i, \Re_i) &= \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\begin{array}{c}\frac{|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_i}(\varkappa_q)|}{1+|Ⓢ_{\Re_i}(\varkappa_q)-Ⓢ_{\Re_i}(\varkappa_q)|}+\\ \frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_i}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_i}(\varkappa_q)|}\end{array}\right]\right]\right]\\
&= \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n} w_q\,[0+0]\right] = 0.\end{aligned}$$

Condition (ii) is straightforward. To prove (iii), we have

$$
\begin{aligned}
\Gamma_w(\Re_i,\Re_k) &= \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\frac{|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}{1+|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\right]\right]\right]\\
&= \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\frac{|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_j}(\varkappa_q)+\circledS_{\Re_j}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}{1+|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_j}(\varkappa_q)+\circledS_{\Re_j}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)+\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)+\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\right]\right]\right]\\
&\leq \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\frac{|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_j}(\varkappa_q)|+|\circledS_{\Re_j}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}{1+|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_j}(\varkappa_q)|+|\circledS_{\Re_j}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\right]\right]\right]\\
&\leq \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\begin{array}{c}\frac{|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_j}(\varkappa_q)|}{1+|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_j}(\varkappa_q)|+|\circledS_{\Re_j}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}+\\ \frac{|\circledS_{\Re_j}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}{1+|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_j}(\varkappa_q)|+|\circledS_{\Re_j}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}+\\ \frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}+\\ \frac{|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\end{array}\right]\right]\right]\\
&= \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\frac{|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_j}(\varkappa_q)|}{1+|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_j}(\varkappa_q)|+|\circledS_{\Re_j}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\right]\right]\right]\\
&\quad+\frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\frac{|\circledS_{\Re_j}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}{1+|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_j}(\varkappa_q)|+|\circledS_{\Re_j}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\right]\right]\right]\\
&\leq \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\frac{|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_j}(\varkappa_q)|}{1+|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_j}(\varkappa_q)|}+\frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_j}(\varkappa_q)|}\right]\right]\right]\\
&\quad+\frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\frac{|\circledS_{\Re_j}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}{1+|\circledS_{\Re_j}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}+\frac{|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}{2\pi+|\arg_{\Re_j}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\right]\right]\right]\\
&= \Gamma_w(\Re_i,\Re_j)+\Gamma_w(\Re_j,\Re_k).
\end{aligned}
$$

Thus $\Gamma_w(\Re_i,\Re_k)\leq\Gamma_w(\Re_i,\Re_j)+\Gamma_w(\Re_j,\Re_k)$. □

4 Application in signal processing

In this section, we will discuss a real-life application of complex fuzzy sets in signals and systems. Especially, the complex fuzzy set explains how to get the highest resemblance of the signal with the known signal R.

Table 1 Received and reference signals

Signals	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	.	.	.	$\varkappa_n$	$\mathbb{R}$
1	$\varkappa_1(1)$	$\varkappa_2(1)$	$\varkappa_3(1)$	.	.	.	$\varkappa_n(1)$	$\mathbb{R}(1)$
2	$\varkappa_1(2)$	$\varkappa_2(2)$	$\varkappa_3(2)$	.	.	.	$\varkappa_n(2)$	$\mathbb{R}(2)$
3	$\varkappa_1(3)$	$\varkappa_2(3)$	$\varkappa_3(3)$	.	.	.	$\varkappa_n(3)$	$\mathbb{R}(3)$
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
m	$\varkappa_1(m)$	$\varkappa_2(m)$	$\varkappa_3(m)$	.	.	.	$\varkappa_n(m)$	$\mathbb{R}(m)$

We propose the following definitions utilized in decision-making algorithm taking the idea of a complex fuzzy set into account:

Definition 12 The Mth inverse discrete Fourier transform (IDFT) coefficient of a length M sequence $\{\varkappa(M)\}$ of the signal $\varkappa_n(m)$ is defined as

$$\varkappa_n(m) = \frac{1}{M}\sum_{q=0}^{M-1} \varkappa'(q)e^{i\frac{2\pi}{M}mq}, m, q \in \{0, 1, 2, \ldots, M-1\},$$

where $\varkappa'(q)$ has different values (Selesnick and Schuller 2001).

If we restrict $\varkappa'(q)$ to a closed interval [0, 1] and tale $\frac{2\pi}{M}mq = \arg_{\mathfrak{R}}(\varkappa)$, then, $\varkappa'(q)e^{i\frac{2\pi}{M}mq}$ is called a complex fuzzy set.

Definition 13 Let $\varkappa_1(m), \varkappa_2(m), \ldots, \varkappa_n(m)$ be different electromagnetic signals and $\mathbb{R}(m)$ be a known signal received by a particular receiver. Then, these signals can be arranged by Table 1.

In this table, take all the signals in columns and each column contains m samples of every signal.

Note that the samples of known signals take in the last column of the table.

To compare the similarity of the received signals with the known signal, we apply the following method.

5 Algorithm

In the following, we develop an algorithm for the identification of receiving signals with the known signal using the proposed concepts of complex fuzzy distance measure and complex fuzzy weighted distance measure.

Step 1. If a digital receiver receives different signals $\varkappa_1(m), \varkappa_2(m), \ldots, \varkappa_n(m)$ from any source. These n signals are sampled M times by the receiver. Let $\varkappa_i(m)$ $(1 = 1, 2, \ldots, n)$ be the nth signal. The inverse discrete Fourier transform of $\varkappa_i(m)$ is

$$\varkappa_n(m) = \frac{1}{M}\sum_{q=0}^{M-1} \varkappa'(q)e^{i\frac{2\pi(q-1)(k-1)+\beta_{\varkappa,q}}{M}}, \quad m, q \in \{0, 1, 2, \ldots, M-1\}. \tag{11}$$

In Eq. 11 $\varkappa'(q)e^{i\frac{2\pi}{M}mq}$ shows the membership function of complex fuzzy sets.

We use the complex fuzzy sets in signals and systems utilizing new kinds of complex fuzzy distance measures to identify a particular signal out of large signals detected by a digital receiver. For this, we have a known signal $\mathbb{R}(m)$. The IDFT of the known signal $\mathbb{R}(m)$ is

$$\mathbb{R}(m) = \frac{1}{M}\sum_{q=0}^{M-1} R[q]e^{i\frac{2\pi(q-1)(k-1)+\beta_{\mathbb{R},q}}{M}}; \quad m, q = 0, 1, 2, \ldots, M-1. \tag{12}$$

The received signals $\varkappa_1(m), \varkappa_2(m), \ldots, \varkappa_n(m)$ can be recognized with respect to the known signal.

Step 2.

Obtain the information about the received signals and known signals in the form of complex fuzzy sets. Then, rearrange by the table defined in (13).

Step 3.

Compute the complex fuzzy distance measure and complex fuzzy weighted distance measure of the received signals and known signal.

Step 4.

Rank the complex fuzzy distance measure and complex fuzzy weighted distance measure to identify the reference signal out of large signals detected by a digital receiver.

Note that the least distance measure of the received signal with the known signal shows a high degree of resemblance.

Example 4 let us assume that the four different electromagnetic signals, $\varkappa_1(m)$, $\varkappa_2(m)$, $\varkappa_3(m)$, and $\varkappa_4(m)$ from four different aircraft S_1, S_2, S_3, and S_4, have been received by a radar system. Each of these time domain signals is sampled four times. Let $\mathbb{R}(m)$ be the known signal. The inverse discrete Fourier transform of the signal $\varkappa_n(m)$; $m, n = 0, 1, 2, 3$ is

$$\varkappa_n(m) = \frac{1}{4}\sum_{q=0}^{3}\left\{U[q]e^{i\frac{2\pi(q-1)(k-1)+\beta_{\varkappa,q}}{M}}\right\} \quad ; m, q = 0, 1, 2, 3, \tag{13}$$

where

$$U[q] \in [0, 1].$$

Also,

$$\mathbb{R}(m) = \frac{1}{4}\sum_{q=0}^{3}\left\{R[q]e^{i\frac{2\pi(q-1)(k-1)+\beta_{\mathbb{R},q}}{M}}\right\} ; m, q = 0, 1, 2, 3, \tag{14}$$

where

$$R[q] \in [0, 1].$$

Now each signal is compared with a known signal to get a high degree of resemblance with the known signal $\mathbb{R}(m)$.

Following steps 1 and 2 in the above algorithm, we take the particular values of amplitude terms and phase terms to explain our proposed method (Table 2).

Step 3. Now the complex fuzzy distance measures and complex fuzzy weighted distance measures of the received signals and known signal are calculated in the following and given in Tables 3 and 4, that is,

Table 2 Particular values of the received signals and reference signal

Signals	$\varkappa_1(m)$	$\varkappa_2(m)$	$\varkappa_3(m)$	$\varkappa_4(m)$	$\mathbb{R}(m)$
0	$0.6e^{i1.3\pi}$	$0.1e^{i0.8\pi}$	$0.9e^{i1\pi}$	$1e^{i2\pi}$	$1e^{i2\pi}$
1	$0.3e^{i1\pi}$	$0.8e^{i2\pi}$	$0.7e^{i0.5\pi}$	$0e^{i0.5\pi}$	$1e^{i2\pi}$
2	$0.9e^{i2\pi}$	$0.4e^{i0.5\pi}$	$0.1e^{i1.2\pi}$	$0.9e^{i1\pi}$	$1e^{i2\pi}$
3	$0.5e^{i0.5\pi}$	$0.1e^{i1\pi}$	$0.2e^{i2\pi}$	$0.3e^{i1.5\pi}$	$1e^{i2\pi}$

Table 3 Values of the proposed distance measures

$\Gamma(\varkappa_1(m), \mathbb{R}(m))$	$\Gamma(\varkappa_2(m), \mathbb{R}(m))$	$\Gamma(\varkappa_3(m), \mathbb{R}(m))$	$\Gamma(\varkappa_4(m), \mathbb{R}(m))$
0.54	0.66	0.57	0.49

$$
\begin{aligned}
\Gamma(\varkappa_1(m), \mathbb{R}(m)) &= \frac{1}{4}\sum_{q=1}^{4}\left[\frac{|U[q]-R[q]|}{1+|U[q]-R[q]|}+\frac{|\arg_{\varkappa_1}-\arg_{\mathbb{R}}|}{2\pi+|\arg_{\varkappa_1}-\arg_{\mathbb{R}}|}\right],\\
&= \frac{1}{4}\left[\begin{array}{c}\left(\frac{|0.6-1|}{1+|0.6-1|}+\frac{|1.3\pi-2\pi|}{2\pi+|1.3\pi-2\pi|}\right)+\\ \left(\frac{|0.3-1|}{1+|0.3-1|}+\frac{|1.\pi-2\pi|}{2\pi+|1.\pi-2\pi|}\right)+\left(\frac{|0.9-1|}{1+|0.9-1|}+\frac{|2\pi-2\pi|}{2\pi+|2\pi-2\pi|}\right)\\ +\left(\frac{|0.5-1|}{1+|0.5-1|}+\frac{|0.5\pi-2\pi|}{2\pi+|0.5\pi-2\pi|}\right)\end{array}\right],\\
&= \frac{1}{4}\left[\begin{array}{c}\left(\frac{0.4}{1.4}+\frac{0.7\pi}{2.7\pi}\right)+\left(\frac{0.7}{1.7}+\frac{1.\pi}{3.\pi}\right)+\left(\frac{0.1}{1.1}+\frac{0.\pi}{2\pi}\right)\\ +\left(\frac{0.5}{1.5}+\frac{1.5\pi}{3.5\pi}\right)\end{array}\right],\\
&= \frac{1}{4}\left[0.29+0.26+0.41+0.33+0.09+0+0.33+0.43\right]\\
&= 0.54.
\end{aligned}
$$

$$
\begin{aligned}
\Gamma(\varkappa_2(m), \mathbb{R}(m)) &= \frac{1}{4}\left[\begin{array}{c}\left(\frac{|0.1-1|}{1+|0.1-1|}+\frac{|0.8\pi-2\pi|}{2\pi+|0.8\pi-2\pi|}\right)+\\ \left(\frac{|0.8-1|}{1+|0.8-1|}+\frac{|2\pi-2\pi|}{2\pi+|2\pi-2\pi|}\right)+\left(\frac{|0.4-1|}{1+|0.4-1|}+\frac{|0.5\pi-2\pi|}{2\pi+|0.5\pi-2\pi|}\right)\\ +\left(\frac{|0.1-1|}{1+|0.1-1|}+\frac{|1\pi-2\pi|}{2\pi+|1\pi-2\pi|}\right)\end{array}\right],\\
&= \frac{1}{4}\left[\begin{array}{c}\left(\frac{0.9}{1.9}+\frac{1.2\pi}{3.2\pi}\right)+\left(\frac{0.2}{1.2}+\frac{0.\pi}{2.\pi}\right)+\left(\frac{0.6}{1.6}+\frac{1.5\pi}{3.5\pi}\right)\\ +\left(\frac{0.9}{1.9}+\frac{1.\pi}{3\pi}\right)\end{array}\right],\\
&= \frac{1}{4}\left[0.47+0.38+0.17+0+0.38+0.43+0.47+0.33\right],\\
&= 0.66.
\end{aligned}
$$

similarly,

$$
\begin{aligned}
\Gamma(\varkappa_3(m), \mathbb{R}(m)) &= 0.57,\\
\Gamma(\varkappa_4(m), \mathbb{R}(m)) &= 0.49.
\end{aligned}
$$

If the weighted vector $w = (0.2, 0.3, 0.2, 0.3)$ is assigned to each sample of the signal, then the complex fuzzy weighted distance measure is

Table 4 Values of the proposed weighted distance measures

$\Gamma_w(\varkappa_1(m), \mathbb{R}(m))$	$\Gamma_w(\varkappa_2(m), \mathbb{R}(m))$	$\Gamma_w(\varkappa_3(m), \mathbb{R}(m))$	$\Gamma_w(\varkappa_4(m), \mathbb{R}(m))$
0.14	0.16	0.15	0.14

$$\begin{aligned}
\Gamma_w(\varkappa_1(m), \mathbb{R}(m)) &= \frac{1}{n\sum_{q=1}^{n} w_q}\left[\sum_{q=1}^{n}\left[w_q\left[\begin{array}{c}\frac{|U[q]-R[q]|}{1+|\circledS_{\Re_i}(\varkappa_q)-\circledS_{\Re_k}(\varkappa_q)|}+\\ \frac{|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}{2\pi+|\arg_{\Re_i}(\varkappa_q)-\arg_{\Re_k}(\varkappa_q)|}\end{array}\right]\right]\right]\\
&= \frac{1}{4}\left[\begin{array}{c}(0.2)\left(\frac{|0.6-1|}{1+|0.6-1|}+\frac{|1.3\pi-2\pi|}{2\pi+|1.3\pi-2\pi|}\right)+\\ (0.3)\left(\frac{|0.3-1|}{1+|0.3-1|}+\frac{|1.\pi-2\pi|}{2\pi+|1.\pi-2\pi|}\right)+(0.2)\left(\frac{|0.9-1|}{1+|0.9-1|}+\frac{|2\pi-2\pi|}{2\pi+|2\pi-2\pi|}\right)\\ +(0.3)\left(\frac{|0.5-1|}{1+|0.5-1|}+\frac{|0.5\pi-2\pi|}{2\pi+|0.5\pi-2\pi|}\right)\end{array}\right],\\
&= \frac{1}{4}\left[\begin{array}{c}(0.2)\left(\frac{0.4}{1.4}+\frac{0.7\pi}{2.7\pi}\right)+(0.3)\left(\frac{0.7}{1.7}+\frac{1.\pi}{3.\pi}\right)+(0.2)\left(\frac{0.1}{1.1}+\frac{0.\pi}{2\pi}\right)\\ +(0.3)\left(\frac{0.5}{1.5}+\frac{1.5\pi}{3.5\pi}\right)\end{array}\right],\\
&= \frac{1}{4}\left[\begin{array}{c}(0.2)(0.47+0.38)+(0.3)(0.17+0)+\\ (0.2)(0.38+0.43)+(0.3)(0.47+0.33)\end{array}\right],
\end{aligned}$$

$$\begin{aligned}
\Gamma_w(\varkappa_1(m), \mathbb{R}(m)) &= \frac{1}{4(1)}\left[\begin{array}{c}(0.2)\left(\frac{|0.6-1|}{1+|0.6-1|}+\frac{|1.3\pi-2\pi|}{2\pi+|1.3\pi-2\pi|}\right)+\\ (0.3)\left(\frac{|0.3-1|}{1+|0.3-1|}+\frac{|1.\pi-2\pi|}{2\pi+|1.\pi-2\pi|}\right)+(0.2)\left(\frac{|0.9-1|}{1+|0.9-1|}+\frac{|2\pi-2\pi|}{2\pi+|2\pi-2\pi|}\right)\\ +(0.3)\left(\frac{|0.5-1|}{1+|0.5-1|}+\frac{|0.5\pi-2\pi|}{2\pi+|0.5\pi-2\pi|}\right)\end{array}\right],\\
&= \frac{1}{4}\left[\begin{array}{c}(0.2)(0.47+0.38)+(0.3)(0.17+0)+\\ (0.2)(0.38+0.43)+(0.3)(0.47+0.33)\end{array}\right],\\
&= \frac{1}{4}(0.623) = 0.16.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Gamma_w(\varkappa_3(m), \mathbb{R}(m)) &= 0.15,\\
\Gamma_w(\varkappa_4(m), \mathbb{R}(m)) &= 0.14,
\end{aligned}$$

Step 4. Using Eq. 5 the rank of the distance measures of the received signals and the known signal is

$$\Gamma(\varkappa_2(m), \mathbb{R}(m)) > \Gamma(\varkappa_3(m), \mathbb{R}(m)) > \Gamma(\varkappa_1(m), \mathbb{R}(m)) > \Gamma(\varkappa_4(m), \mathbb{R}(m)). \quad (15)$$

From the rank of distance measures, we conclude that the signal $\varkappa_4(m)$ has the least distance measure. Thus, the signal $\varkappa_4(m)$ shows a high degree of resemblance with the known signal $\mathbb{R}(m)$.

Also, using Eq. 10, the rank of the distance measures of the received signals and known signal is

$$\Gamma(\varkappa_1(m), \mathbb{R}(m)) > \Gamma(\varkappa_3(m), \mathbb{R}(m)) > \Gamma(\varkappa_4(m), \mathbb{R}(m)) > \Gamma(\varkappa_2(m), \mathbb{R}(m)). \quad (16)$$

From the rank of weighted distance measures, we conclude that the signal $\varkappa_2(m)$ has the least weighted distance measure. Thus, in this case, the signal $\varkappa_2(m)$ shows the high degree of resemblance with the known signal $\mathbb{R}(m)$.

6 Comparison analysis

In this section, we discussed the comparison of the proposed distance measures of complex fuzzy sets with the Zhang distance (Zhang et al. 2009), Hamming distance (Alkouri and Salleh 2014), and Normalized Hamming distance.

Note that our proposed distance measures are different from all the distance measures that exist in the literature.

(i) The proposed distance,

$$\Gamma(\Re_i, \Re_j) = \frac{1}{n}\sum_{q=1}^{n}\left[\frac{|Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_j}(\varkappa_q)|}{1 + |Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_j}(\varkappa_q)|} + \frac{|\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|}{2\pi + |\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|}\right]. \tag{17}$$

(ii) The Zhang distance,

$$\Gamma(\Re_i, \Re_j) = \max\left[\sup_{\varkappa_q \in U}|Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_j}(\varkappa_q)|, \frac{1}{2\pi}\sup_{\varkappa_q \in U}|\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|\right]. \tag{18}$$

(iv) The Normalized Hamming distance,

$$\Gamma(\Re_i, \Re_j) = \frac{1}{2n}\left[\sum_{q=1}^{n}|Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_j}(\varkappa_q)| + \frac{1}{2\pi}\sum_{q=1}^{n}|\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|\right]. \tag{19}$$

(iii) The Hamming distance,

$$\Gamma(\Re_i, \Re_j) = \frac{1}{2}\left[\sum_{q=1}^{n}|Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_j}(\varkappa_q)| + \frac{1}{2\pi}\sum_{q=1}^{n}|\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|\right]. \tag{20}$$

Using Eq. 18 the values of the distance measures of the received signals and known signals are given in Table 5.

$$\begin{aligned}
\Gamma(\varkappa_1(m), \mathbb{R}(m)) &= \max\left[\sup_{\varkappa_q \in U}|Ⓢ_{\Re_i}(\varkappa_q) - Ⓢ_{\Re_j}(\varkappa_q)|, \frac{1}{2\pi}\sup_{\varkappa_q \in U}|\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|\right],\\
&= \max\left[|1 - 0.3|, \frac{1}{2\pi}|0.5\pi - 2\pi|\right],\\
&= \max[0.7, 0.75] = 0.75.\\
\Gamma(\varkappa_2(m), \mathbb{R}(m)) &= \max\left[|0.1 - 1|, \frac{1}{2\pi}|0.5\pi - 2\pi|\right],\\
&= \max[0.9, 0.75] = 0.9.
\end{aligned}$$

Similarly,

$$\Gamma(\varkappa_3(m), \mathbb{R}(m)) = 0.9,$$
$$\Gamma(\varkappa_3(m), \mathbb{R}(m)) = 1,$$

From Table 5, we conclude that $\varkappa_1(m)$ shows a high degree of resemblance with the known signal $\mathbb{R}(m)$.

Table 5 Values of Zhang distance measures

$\Gamma(\varkappa_1(m), \mathbb{R}(m))$	$\Gamma(\varkappa_2(m), \mathbb{R}(m))$	$\Gamma(\varkappa_3(m), \mathbb{R}(m))$	$\Gamma(\varkappa_4(m), \mathbb{R}(m))$
0.75	0.9	0.9	1

Table 6 Values of Normalized Hamming distance measures

$\Gamma(\varkappa_1(m), \mathbb{R}(m))$	$\Gamma(\varkappa_2(m), \mathbb{R}(m))$	$\Gamma(\varkappa_3(m), \mathbb{R}(m))$	$\Gamma(\varkappa_4(m), \mathbb{R}(m))$
0.42	0.56	0.47	0.41

Using Eq. 19, the values of the distance measures of the received signals and known signal are given in Table 6, that is,

$$\begin{aligned}
\Gamma(\varkappa_1(m), \mathbb{R}(m)) &= \frac{1}{2n}\left[\sum_{q=1}^{n}|\circledS_{\Re_i}(\varkappa_q) - \circledS_{\Re_j}(\varkappa_q)| + \frac{1}{2\pi}\sum_{q=1}^{n}|\arg_{\Re_i}(\varkappa_q) - \arg_{\Re_j}(\varkappa_q)|\right],\\
&= \frac{1}{2(4)}\left[\begin{array}{c}\left(|0.6-1| + \frac{1}{2\pi}|1.3\pi - 2\pi|\right) + \left(|0.3-1| + \frac{1}{2\pi}|1\pi - 2\pi|\right)\\ + \left(|0.9-1| + \frac{1}{2\pi}|2\pi - 2\pi|\right) + \left(|0.5-1| + \frac{1}{2\pi}|0.5\pi - 2\pi|\right)\end{array}\right],\\
&= \frac{1}{8}\left[0.4 + 0.35 + 0.7 + 0.5 + 0.1 + 0 + 0.5 + 0.75\right],\\
&= 0.42.\\
\Gamma(\varkappa_2(m), \mathbb{R}(m)) &= \frac{1}{2(4)}\left[\begin{array}{c}\left(|0.1-1| + \frac{1}{2\pi}|0.8\pi - 2\pi|\right) + \left(|0.8-1| + \frac{1}{2\pi}|2\pi - 2\pi|\right)\\ + \left(|0.4-1| + \frac{1}{2\pi}|0.5\pi - 2\pi|\right) + \left(|0.1-1| + \frac{1}{2\pi}|1\pi - 2\pi|\right)\end{array}\right],\\
&= \frac{1}{8}\left[0.9 + 0.6 + 0.2 + 0 + 0.6 + 0.75 + 0.9 + 0.5\right],\\
&= 0.56.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Gamma(\varkappa_3(m), \mathbb{R}(m)) &= 0.47,\\
\Gamma(\varkappa_4(m), \mathbb{R}(m)) &= 0.41.
\end{aligned}$$

From Table 6, we conclude that $\varkappa_4(m)$ shows the high degree of resemblance with the known signal $\mathbb{R}(m)$.

Similarly, the Hamming distance defined in (5.4) can be applied to the problems in signals and systems.

Comparison Table

Table 7 contains the values of the proposed distance measure and the existing distance measures. It is clearly seen that the values of the proposed distance measure are smaller than the values of the existing distance measures. From this, we conclude that our proposed distance measure is more better than the existing distance measures.

Table 7 Comparison Table

Γ	$\Gamma(\varkappa_1(m), \mathbb{R}(m))$	$\Gamma(\varkappa_2(m), \mathbb{R}(m))$	$\Gamma(\varkappa_3(m), \mathbb{R}(m))$	$\Gamma(\varkappa_4(m), \mathbb{R}(m))$
Proposed DM	0.14	0.16	0.15	0.14
Zhang DM	0.75	0.9	0.9	1
Hammin DM	0.42	0.56	0.47	0.41

7 Conclusion

In this paper, we introduced the partial order relation on complex fuzzy sets. We defined the complex fuzzy maximal, minimal, maximum, and minimum elements based on the partial order relations. We proposed new distance measures such as complex fuzzy distance measures and complex fuzzy weighted distance measures. We established some particular examples and basic results of the partial order relations and distance measures. Moreover, we utilized the complex fuzzy sets in signals and systems. We proposed a new decision-making algorithm under the complex fuzzy environment, based on the complex fuzzy distance measures and complex fuzzy weighted distance measures for applications in signals and systems by which we determined the degree of high resemblance of signals to the known signal. Further, we studied the comparative study of the proposed distance measures with the Zhang distance measure, Hamming distance measure, and Normalized Hamming distance measure and proved that our proposed distance measure is more significant than the existing distance measures because the values of the proposed distance measure are smaller than the values of the Zhang distance measure and Hamming distance measure.

In future, we will use the proposed distance measures for interval-valued complex fuzzy sets, complex neutrosophic sets, complex Pythagorean fuzzy sets, complex intuitionistic fuzzy sets, etc., to improve the quality of the research works.

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Author Contributions All authors contributed equally.

Declarations

Conflict of interest The authors declare that they have no conflict of interests.

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