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Efficient alternating direction implicit numerical approaches for multi-dimensional distributed-order fractional integro differential problems

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Abstract

This paper proposes the alternating direction implicit (ADI) numerical approaches for computing the solution of multi-dimensional distributed-order fractional integrodifferential problems. The proposed method discretizes the unknown solution in two stages. First, the Riemann–Liouville fractional integral term and the distributed-order time-fractional derivative are discretized with the help of the second-order convolution quadrature and the weighted and shifted Grünwald formula, respectively. Second, the spatial discretization is obtained by the general centered finite difference (FD) technique. At the same time, the ADI algorithms are devised for reducing the computational burden. Additionally, the convergence analysis of proposed ADI FD schemes is analyzed in detail through the energy method. Finally, two numerical examples highlight the accuracy of the proposed method and verify the theoretical formulations.

Keywords Caputo fractional derivative · Distributed-order integrodifferential equation · Weighted and shifted Grünwald formula · Alternating direction implicit scheme · Second-order convolution quadrature rule · Error estimate

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1 Introduction

This paper considers the distributed-order fractional integrodifferential equation in two/three dimensions

$$\mathbb{D}_t^{\omega} u(\mathbf{x},t) - \mu \Delta u(\mathbf{x},t) - I^{(\beta)} \Delta u(\mathbf{x},t) = f(\mathbf{x},t), \quad 0 < \beta < 1, \quad (\mathbf{x},\ t) \in \Omega \times (0,T]. \tag{1}$$

The initial condition and the boundary condition (IC and BC, respectively) are prescribed as

$$u(\mathbf{x},0) \qquad = \kappa(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{2}$$

$$u(\mathbf{x},t) = 0, \quad (\mathbf{x},\ t) \in \partial\Omega \times [0,T], \tag{3}$$

and the distributed-order integral is defined as

$$\mathbb{D}_t^{\omega} u(\mathbf{x},t) = \int_0^1 \omega(\alpha) D_t^{\alpha} u(\mathbf{x},t) \mathrm{d}\alpha. \tag{4}$$

Following Podlubny (1999), the Caputo fractional derivative (CFD) and the Riemann–Liouville fractional integral (RLFI) are respectively defined in

$$D_t^{\alpha} u(\mathbf{x},t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u}{\partial s}(\mathbf{x},s) \mathrm{d}s, & 0 < \alpha < 1, \\ \frac{\partial u}{\partial t}(\mathbf{x},t), & \alpha = 1, \end{cases} \tag{5}$$

and

$$I^{(\vartheta)} \phi(t) = \int_0^t \beta(t-s)\phi(s) \mathrm{d}s := \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \phi(s) \mathrm{d}s, \\ \vartheta \in (0,1), \qquad t \in (0,\infty), \tag{6}$$

in which $\omega(\alpha) \geq 0$ with $\int_0^1 \omega(\alpha)\mathrm{d}\alpha = c_0 > 0$, $\Omega = \mathbb{R}^2$ or $\mathbb{R}^3$, $\Gamma(\vartheta) = \int_0^{+\infty} \xi^{\vartheta-1} \exp(-\xi)\mathrm{d}\xi$ and $f(\mathbf{x},t)$ represent the weight function, spatial domain, the Euler's Gamma function, and forcing term, respectively. Without loss of generality, we can take a zero initial value $u(\mathbf{x},0)$. If $u(\mathbf{x},0) = \varpi(\mathbf{x})$, then we can consider a transform $w(\mathbf{x},t) = u(\mathbf{x},t) - \varpi(\mathbf{x})$. Theory of fractional calculus (FC) generalizes the integer order derivative to arbitrary order, which can be achieved in space and time with a power law memory kernel of the nonlocal problems (Tarasov 2021a, b; Kumar and Saha Ray 2021; Behera and Ray 2022; Moghaddam et al. 2019; Abdelkawy et al. 2022; Lopes and Machado 2021). With the increasing popularity of FC, fractional differential equations (FDEs) have become an important key for describing and modeling various phenomena phenomena in scopes of engineering and sciences (Podlubny 1999; Hilfer 2000; Akram et al. 2021; Alia et al. 2021). Nakhushev (1998, 2003) discussed the importance of studies on the positivity of continuous and discrete differentiation and integration operators in the theory of mixed type equations and FC and proposed that fractional integrals (FIs) of uniformly distributed order can be expressed in terms of the so-called continual FIs. Then, Pskhu (2004, 2005) suggested the fractional operators which are the opposite of the continual FIs and presented the theory about the continual integro-differentiation operator. Their research has had a significant impact on the study of FC. Furthermore, many scholars have proposed different numerical methods for solving FDEs, including finite difference (FD) (Qiu et al. 2019; Yn et al. 2011), finite element (FE)

(Liu et al. 2015), two-grid methods (Liu et al. 2015; Qiu et al. 2020, mehless method (Nikan et al. 2021b, a; Nikan and Avazzadeh 2021) and etc. In recent decades, distributed-order partial differential equations (DOPDEs) have a wide range of applications in mathematical physics and engineering (Bagley and Torvik 2000; Caputo 2001), and can be used to describe the dynamics of anomalous diffusion and relaxation phenomena. Distributed order derivatives are fractional derivatives that have integrated the order of the derivative over a certain range. On the one hand, the distributed order fractional problem can be extended to a general integer order problem. On the other hand, the distributed order problem can be discretized into a multi-term time fractional order problem. In the past few years, more and more researchers have studied distributed-order differential equations. Naber (2004) obtained the solution for the fractional subdiffusion equations with the distributed-order by means of Laplace transform and variable separation. Kochubei (2008) studied the distributed order derivative and integral. Atanackovic et al. (2009) investigated the Cauchy problem of the time distributed-order diffusion wave problem. Meerschaert et al. (2011) analyzed explicit strong and random analogs solutions. Katsikadelis (2014) adopted a new numerical approach to approximate distributed order FDEs of a general formulation in an integration domain. Morgado and Rebelo (2015) explored an implicit approach for solution of the distributed-order time-fractional nonlinear reaction-diffusion problem. Chen et al. (2016) studied the spectral scheme and pseudo-spectral scheme in a domain of semi-infinite space. Du et al. (2016) analyzed and proposed the higher-order FD techniques having smooth solutions in 1D and 2D spaces. Jin et al. (2016) developed two fully discrete approaches including error analysis to discretize the distributed-order time fractional diffusion problem including nonsmooth initial of data. Abbaszadeh and Dehghan (2017) subsequently presented an improved meshfree technique with error estimation. Gao et al. (2017) developed an interpolation-based approximation for the temporal second order difference scheme to approximate multi-term distributed order time FDEs. Yang et al. (2018) formulated an orthogonal spline collocation (OSC) technique. Qiu et al. (2020) advanced the Galerkin FE technique for the time fractional mobile-immobile model with the distributed-order. Gao et al. (2020) investigated the nonhomogeneous 2D distributed-order time-fractional cable equations by unstructured grids of Galerkin FE. Zhang et al. (2022) presented an ADI Legendre–Laguerre spectral scheme for the 2D time distributed-order diffusion-wave problem on a semi-infinite domain. Jian et al. (2021) established fast numerical algorithms to solve the Riesz space fractional diffusion-wave problem with time distributed-order.

It is well known that the ADI methods have the advantage of reducing the computational burden using decomposing a multidimensional problem into several independent one-dimensional problems Huang et al. (2021). Some fractional order problems have been studied so far by the ADI methods. Chen et al. (2016) and Qiao et al. (2021) proposed the ADI FD technique for fractional order Volterra equation and the 3D nonlocal evolution equation. Pani et al. (2010) implemented ADI OSC method for the single-order time FDEs. Gao and Zz (2016b, a) adopted the ADI FD approach to the distributed-order time-diffusion equations, while Li et al. (2013) used the ADI FE scheme for the investigation of single-order temporal/spatial FDEs. However, the problem (1)–(3) in two/three dimensions has not been studied. In the following, we will discuss and analyze this issue.

For large problems, the ADI method can reduce the storage requirements and computational complexities. In addition, although the implicit method has good stability, it requires a large amount of CPU run time if the number of unknowns is large. Therefore, we construct an ADI FD scheme, which deals with two- and three-dimensional problems by solving a series of smaller, independent one-dimensional problems. The main objective of current work is to develop the efficient ADI numerical approaches for distributed-order integrodifferential equations for the case of two/three dimensions. The time discretization is obtained based on the second-order CQ rule and the weighted and shifted Grünwald formula for the RLFI and the distributed-order time-fractional derivative, respectively. Then, we adopt the central FD technique to establish the fully discrete scheme. Meanwhile, the fully-discrete ADI difference approaches in two/three dimensions are obtained with corresponding ADI algorithms. The numerical results show that our schemes in two/three dimensional cases are convergent, with time convergence of order 2, spatial convergence of order 2, and distributed-order convergence of order 2, respectively.

This paper includes five sections as follows. Section 2 gives the necessary notations, some useful lemmas and derivation of ADI difference approaches and performs the convergence analysis of the two-dimensional distributed order problem. Section 3 constructs the ADI approach of the three-dimensional problem by adding a tiny term, and studies the convergence analysis of the ADI approach by means of the energy method. Section 4 presents two test problems to confirm the theoretical prediction and show effectiveness of the method. Finally, Section 5 summarizes the main concluding remarks.

2 Numerical description and theoretical analysis for the two-dimensional case

2.1 Preliminary

In the following numerical method analysis, we assume that two-dimensional problems (1)–(3) have a sufficiently smooth and unique solution on the domain $\Omega = (0, L_1) \times (0, L_2)$ and its boundary $\partial\Omega$. We will give some useful symbols and significant lemmas, which can help us in the subsequent discussions. First of all, let us define the necessary notations of time and distributed order. For convenience, we consider a temporal step size which is selected as the nodes $\tau = \frac{T}{N}$ and $t_n = n\tau$, $0 \le n \le N$, where N and T are the total number of time steps and a finite time, respectively. For positive integers N and J, we separate $[0, 1]$ into $2J$-subintervals $\alpha_l = l\triangle\alpha,\ \ 0 \le l \le 2J$, so that distributed-order step size $\triangle\alpha = \frac{1}{2J}\cdot$ For $n = 1, 2, \ldots, N$, let us introduce $\delta_t v^{n-\frac{1}{2}} = \frac{1}{\tau}(v^n - v^{n-1})\cdot$ In what follows, we mention the composite trapezoid formulation for discretizing the distributed-order integral.

Lemma 1 (Gao and Zz 2016b) *For* $\sigma(\alpha) \in C^2[0, 1]$, *we have*

$$\int_0^1 \sigma(\alpha)\mathrm{d}\alpha = \triangle\alpha \sum_{l=0}^{2J} c_l \sigma(\alpha_l) - \frac{\sigma''(\xi)}{12}(\triangle\alpha)^2, \quad \xi \in (0, 1),$$

in which

$$c_l = \begin{cases} \frac{1}{2}, & l = 0,\ 2J, \\ 1, & l = 1, 2, \ldots, 2J-1. \end{cases}$$

Next, we describe the process of discretization for the distributed-order CFD. Now, let us introduce

$$\Im^{\alpha+s}(\mathbb{R}) = \left\{\varphi\middle|\varphi \in L^1(\mathbb{R}); \int_{-\infty}^{+\infty} (1 + |\xi|)^{\alpha+s}|\hat{\varphi}(\xi)|\mathrm{d}\xi < \infty\right\}, \qquad s \ge 1,$$

where $\hat{\varphi}(\xi) = \mathcal{F}[\varphi](\xi) = \int_{-\infty}^{+\infty} e^{i\xi t}\varphi(t)\mathrm{d}t$ illustrates the Fourier transformation for the function $\varphi(t)$.

Lemma 2 (Pskhu 2004; Meerschaert and Tadjeran 2004) *For $\varphi \in \Im^{\alpha+1}(\mathbb{R})$, the RL fractional derivative can be stated as*

$$ {}_{-\infty}D_t^{\alpha}\varphi(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t}(t-\theta)^{-\alpha}\varphi(\theta)\mathrm{d}\theta \tag{7}$$

and

$$ B_{\tau,m}^{\alpha}\varphi(t) = \tau^{-\alpha}\sum_{k=0}^{\infty} g_k^{(\alpha)}\varphi\big(t-(k-m)\tau\big), \tag{8}$$

in which $g_k^{(\alpha)} = (-1)^k\binom{\alpha}{k}$ are the coefficients for $\alpha \in (0, 1]$ and m is an integer. Then, we have

$$ B_{\tau,m}^{\alpha}\varphi(t) = {}_{-\infty}D_t^{\alpha}\varphi(t) + \mathcal{O}(\tau),$$

uniformly satisfies in $t \in \mathbb{R}$ when $\tau \to 0$.

Furthermore, in the case of $0 < \alpha \le 1$, the coefficients $g_k^{(\alpha)}$ introduced in (8) satisfy the following properties

$$\begin{aligned} &g_0^{(\alpha)} = 1, \quad g_1^{(\alpha)} = -\alpha < 0,\\ &g_2^{(\alpha)} \le g_3^{(\alpha)} \le g_4^{(\alpha)} \le \cdots \le 0,\\ &\sum_{k=0}^{\infty} g_k^{(\alpha)} = 0, \quad \sum_{k=0}^{n} g_k^{(\alpha)} \ge 0, \quad n \ge 1. \end{aligned} \tag{9}$$

For carrying out a theoretical analysis, we require the following lemma.

Lemma 3 (Tian et al. 2015) *Assume that $\varphi \in \Im^{\alpha+2}(\mathbb{R})$. Then, we have*

$$\begin{aligned} &(1+\tfrac{\alpha}{2})B_{\tau,0}^{\alpha}\varphi(t) - \tfrac{\alpha}{2}B_{\tau,-1}^{\alpha}\varphi(t) = \tau^{-\alpha}\sum_{k=0}^{\infty}\lambda_k^{(\alpha)}\varphi(t-k\tau)\\ &= {}_{-\infty}D_t^{\alpha}\varphi(t) + \mathcal{O}(\tau^2), \end{aligned}$$

uniformly holds for $t \in \mathbb{R}$ when $\tau \to 0$, and the coefficients $\lambda_k^{(\alpha)}$ can be evaluated as follows

$$ \lambda_0^{(\alpha)} = (1+\tfrac{\alpha}{2})g_0^{(\alpha)}, \qquad \lambda_k^{(\alpha)} = (1+\tfrac{\alpha}{2})g_k^{(\alpha)} - \tfrac{\alpha}{2}g_{k-1}^{(\alpha)}, \qquad k \ge 1. \tag{10}$$

Actually, it can be checked for $0 \le \alpha \le 1$ that

$$\begin{aligned} &\lambda_0^{(\alpha)} = 1 + \tfrac{\alpha}{2} > 0, \qquad \lambda_1^{(\alpha)} = -\tfrac{\alpha(3+\alpha)}{2} \le 0,\\ &\lambda_2^{(\alpha)} = \frac{1}{4}\alpha(\alpha^2+3\alpha-2) = \begin{cases} \le 0, & \alpha \in [0, \frac{\sqrt{17}-3}{2}],\\ > 0, & \alpha \in (\frac{\sqrt{17}-3}{2}, 1], \end{cases}\\ &\lambda_k^{(\alpha)} = \big[(1+\tfrac{\alpha}{2})(\tfrac{k-\alpha-1}{k}) - \tfrac{\alpha}{2}\big]g_{k-1}^{(\alpha)} \le 0, \qquad k \ge 3. \end{aligned}$$

From Wang and Vong (2014a), we can obtain the following non-negative properties.

Lemma 4 (Wang and Vong 2014a) *Let the coefficients $\{\lambda_k^{(\alpha)}\}_{k=0}^{\infty}$ are introduced in (10). For any mesh series $(\mathcal{W}^1, \ldots, \mathcal{W}^m)^{\mathrm{T}} \in \mathbb{R}^m$, we have*

$$ \sum_{n=1}^{m}\Big(\sum_{k=0}^{n-1}\lambda_k^{(\alpha)}\mathcal{W}^{n-k}\Big)\mathcal{W}^n \ge 0.$$

According to the aforesaid lemma, we can conclude the lemma as follows.

Lemma 5 (Gao and Zz 2016a, b) *Assume that the coefficients* $\{\lambda_k^{(\alpha)}\}_{k=0}^{\infty}$ *are introduced in Eq. (10). Then, for any mesh series* $\left(\mathcal{W}^0, \ldots, \mathcal{W}^m\right)^{\mathrm{T}} \in \mathbb{R}^{m+1}$, *it follows that*

$$\sum_{n=0}^{m}\left(\sum_{k=0}^{n}\lambda_k^{(\alpha)}\mathcal{W}^{n-k}\right)\mathcal{W}^n \geq 0.$$

Afterwards, we define the following notation

$$\mu_1 := \left[\triangle\alpha\sum_{l=0}^{2J} c_l\omega(\alpha_l)\tau^{-\alpha_l}\lambda_0^{(\alpha_l)}\right]. \tag{11}$$

Then, we get the estimate as follows.

Lemma 6 (Gao and Zz 2016b) *Let* μ_1 *be defined in the relation* (11). *Then, we get* $\mu_1 = \mathcal{O}\Big((\tau|\ln\tau|)^{-1}\Big)$.

Secondly, we take two positive integers M_1 and M_2. Let $h_1 = L_1/M_1$, $h_2 = L_2/M_2$, $h = \max\{h_1, h_2\}$. Define the nodal points $x_i = ih_1$, $0 \leq i \leq M_1$, $y_j = jh_2$, $0 \leq j \leq M_2$, $\chi = \{1 \leq i \leq M_1 - 1, 1 \leq j \leq M_2 - 1\}$, $\gamma = \{(i, j)|(x_i, y_j) \in \partial\Omega\}$. Also, we introduce $\overline{\Omega}_h = \{(x_i, y_j)|\ 0 \leq i \leq M_1, 0 \leq j \leq M_2\}$, $\mathring{\Omega}_h = \{w|w \in \Omega_h;\ w_{ij} = 0,\ \text{when}\ (i, j) \in \gamma\}$, $\Omega_h = \overline{\Omega}_h \cap \Omega$ and $\partial\Omega_h = \Omega_h \cap \partial\Omega$.

In order to facilitate the analysis, suppose that the symbols u_{ij}^n and f_{ij}^n represent the values of functions $u(x, y, t)$ and $f(x, y, t)$ at nodal point (x_i, y_j, t_n), respectively. We define some necessary notations for any grid function $w = \{w_{ij}|0 \leq i \leq M_1, 0 \leq j \leq M_2\}$ over $\overline{\Omega}_h$,

$$\begin{aligned}
\delta_x w_{ij}^n &:= \frac{1}{h_1}(w_{ij}^n - w_{i-1,j}^n), \qquad \delta_y w_{ij}^n := \frac{1}{h_2}(w_{ij}^n - w_{i,j-1}^n),\\
\delta_x^2 w_{ij}^n &:= \frac{1}{h_1}(\delta_x w_{ij}^n - \delta_x w_{i-1,j}^n), \qquad \delta_y^2 w_{ij}^n := \frac{1}{h_2}(\delta_y w_{ij}^n - \delta_y w_{i,j-1}^n),\\
\Delta_h w_{ij}^n &:= \delta_x^2 w_{ij}^n + \delta_y^2 w_{ij}^n, \quad 1 \leq i \leq M_1 - 1, \quad 1 \leq j \leq M_2 - 1, \quad 1 \leq n \leq N.
\end{aligned}$$

Let us define the discrete inner product and the associated norms for $w, v \in \mathring{\Omega}_h$ by

$$(w, v) := h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} w_{ij}v_{ij}, \qquad \|v\| = \sqrt{(v, v)}, \qquad \|v\|_\infty = \max_{\substack{1\leq i\leq M_1-1\\ 1\leq j\leq M_2-1}}|v_{ij}|.$$

Here, we present the associated discrete method and some lemmas to construct the ADI difference approach. First, we introduce the second-order CQ strategy (cf. Lubich 1986, 1988) for discretizing the RLFI $I^{(\beta)}\phi(t_n)$ as

$$Q_n^{(\beta)}(\phi) = \tau^\beta\sum_{p=1}^{n}\omega_{n-p}^{(\beta)}\phi^p + \tau^\beta\tilde{\omega}_n^{(\beta)}\phi^0, \qquad 0 \leq \beta \leq 1, \tag{12}$$

where the CQ weights $\omega_s^{(\beta)}$ can be derived by

$$(\delta(\nu))^{-\beta} = \sum_{s=0}^{\infty}\omega_s^{(\beta)}\nu^s,$$

in which $\delta(\nu)$ denotes the generating function Lubich (1988). For the CQ with second-order accuracy, we get

$$\delta(\nu) = \sum_{s=1}^{2}\tfrac{1}{s}(1-\nu)^s.$$

Therefore, we can get the quadrature weights $\omega_s^{(\beta)}$ via

$$c_s^{(\beta)} = (-1)^s \binom{-\beta}{s} = (-1)^n \frac{\Gamma(1-\beta)}{\Gamma(1+n)\Gamma(1-\beta-n)} > 0,$$
$$\omega_s^{(\beta)} = (\tfrac{2}{3})^\alpha \sum_{i=0}^{s} \frac{c_{s-i}^{(\beta)} c_i^{(\beta)}}{3^i}, \qquad s \geq 0.$$

We can present the correction weights $\tilde{\omega}_n^{(\beta)}$ introduced in (12) for discretizing the integral term with second-order accuracy in the time dimension. When $\phi = 1$, we have

$$Q_n^{(\beta)}(1) = \frac{1}{\Gamma(\beta)} \int_0^{t_n} (t_n - s)^{\beta-1} \mathrm{d}s = \frac{(t_n)^\beta}{\Gamma(1+\beta)}. \tag{13}$$

Thus, we arrive at

$$\tilde{\omega}_n^{(\beta)} = \frac{n^\beta}{\Gamma(1+\beta)} - \sum_{p=1}^{n} \omega_{n-p}^{(\beta)}.$$

Now, we analyze the quadrature error.

Lemma 7 (Chen et al. 2016; Xu 1997) *Suppose that the function $\psi(t)$ is continuously differentiable over $(0, T]$ and real, $\psi_{tt}(t)$ is integrable and continuous on $(0, T]$. Then, the quadrature error can be obtained using*

$$|I^{(\beta)}\psi(t_n) - Q_n^{(\beta)}(\psi)| \leq C\Big[\tau^2 t_n^{\beta-1}|\psi_t(0)| + \tau^2 \int_0^{t_{n-1}} (t_n - \xi)^{\beta-1}|\psi_{tt}(\xi)|\mathrm{d}\xi$$
$$+\tau^{1+\beta} \int_{t_{n-1}}^{t_n} |\psi_{tt}(\xi)|\mathrm{d}\xi\Big], \qquad 0 < \beta < 1,$$

in which $Q_n^{(\beta)}(\psi)$ is presented by (12), and $0 < t_n < T < \infty$.

Remark 1 Throughout the article, C represents a generic positive constant which is independent of the space and time step sizes. In addition, it is not necessarily same in different occurrences.

Remark 2 Through the above-mentioned Lemma, the quadrature error of the CQ is $\mathcal{O}(\tau^{1+\beta})$. However, the quadrature error of second-order CQ is $\mathcal{O}(\tau^2)$ when $|\psi_{tt}(t)| \leq C, t \in [0, T]$.

In the following, we list some useful lemmas based on the Taylor formula with integral remainder.

Lemma 8 (Yn et al. 2011) *Supposing that $u(x, y, \cdot) \in C_{x,y}^{4,4}\big([0, L_1] \times [0, L_2]\big)$. Then, we get*

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_n) = \delta_x^2 U_{ij}^n - \frac{h_1^2}{6} \int_0^1 \Big[\frac{\partial^4 u}{\partial x^4}(x_i + \xi h_1, y_j, t_n) + \frac{\partial^4 u}{\partial x^4}(x_i - \xi h_1, y_j, t_n)\Big](1-\xi)^3 \mathrm{d}\xi, \tag{14}$$

$$\frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_n) = \delta_y^2 U_{ij}^n - \frac{h_2^2}{6} \int_0^1 \Big[\frac{\partial^4 u}{\partial y^4}(x_i, y_j + \xi h_2, t_n) + \frac{\partial^4 u}{\partial y^4}(x_i, y_j - \xi h_2, t_n)\Big](1-\xi)^3 \mathrm{d}\xi. \tag{15}$$

Now, we can obtain the bound of $I^{(\beta)}u_{xx}(x_i, y_j, t_n) - Q_n^{(\beta)}(\delta_x^2 U_{ij})$.

Lemma 9 *Assume that* $u(x, y, t) \in C_{x,y,t}^{4,4,2}\big([0, L_1]\times[0, L_2]\times[0, T]\big)$. *Then, for* $n = 1, \ldots, N$ *and* $(i, j) \in \chi$, *we get*

$$\big|(R_1)_{ij}^n\big| = \big|I^{(\beta)}\Delta u(x_i, y_j, t_n) - Q_n^{(\beta)}(\Delta_h U_{ij})\big| \le C(\tau^2 + h_1^2 + h_2^2). \tag{16}$$

Proof Using the triangle inequality, we have

$$\begin{aligned}&\big|I^{(\beta)}u_{xx}(x_i, y_j, t_n) - Q_n^{(\beta)}(\delta_x^2 U_{ij})\big| \\ &\le \big|I^{(\beta)}u_{xx}(x_i, y_j, t_n) - Q_n^{(\beta)}\big(u_{xx}(x_i, y_j, \cdot)\big)\big| + \big|Q_n^{(\beta)}\big(u_{xx}(x_i, y_j, \cdot)\big) - Q_n^{(\beta)}(\delta_x^2 U_{ij})\big|.\end{aligned}$$

In other hand, for the estimate of $\big|Q_n^{(\beta)}\big(u_{xx}(x_i, y_j, \cdot)\big) - Q_n^{(\beta)}(\delta_x^2 U_{ij})\big|$, we use Lemma 8 and (formula (2.4), Qiao et al. 2022) to get

$$\begin{aligned}&\big|Q_n^{(\beta)}\big(u_{xx}(x_i, y_j, \cdot)\big) - Q_n^{(\beta)}(\delta_x^2 U_{ij})\big| \\ &\le C\Big(\tau^\beta \sum_{p=1}^{n} |\omega_{n-p}^{(\beta)}|\Big)h_1^2 + C\tau^\beta|\omega_n^{(\beta)}|h_1^2 \\ &\le C\Big(\int_0^{t_n} s^{\beta-1}\mathrm{d}s\Big)h_1^2 + C\tau^\beta \frac{\tau t_n^{\beta-1}}{\tau^\beta}h_1^2 \\ &\le Ct_n^\beta h_1^2 + t_n^\beta n^{-1}h_1^2 \\ &\le C(T)h_1^2.\end{aligned}$$

Regarding Lemma 7 and Remark 2, we get

$$\begin{aligned}\big|I^{(\beta)}u_{xx}(x_i, y_j, t_n) - Q_n^{(\beta)}(\delta_x^2 U_{ij})\big| &\le \big|I^{(\beta)}u_{xx}(x_i, y_j, t_n) - Q_n^{(\beta)}\big(u_{xx}(x_i, y_j, \cdot)\big)\big| + Ch_1^2 \\ &\le C\big(\tau^2 + h_1^2\big).\end{aligned}$$

In the same way, we can prove $\big|I^{(\beta)}u_{yy}(x_i, y_j, t_n) - Q_n^{(\beta)}(\delta_y^2 U_{ij})\big| \le C(\tau^2 + h_2^2)$. To sum up, the proof is completed. □

2.2 The derivation of the ADI difference approach in two dimensions

Firstly, we can establish the ADI difference approach for Eqs. (1)–(3). Considering Eq. (1) at the nodal point (x_i, y_j, t_n) for $(i, j) \in \chi$, $n = 1, \ldots, N$, we have

$$\begin{aligned}\mathbb{D}_t^\omega u(x_i, y_j, t_n) - \mu\Delta u(x_i, y_j, t_n) - I^{(\beta)}\Delta u(x_i, y_j, t_n) = f(x_i, y_j, t_n), \\ (x, y) \in \Omega, \qquad n = 1, \ldots, N.\end{aligned} \tag{17}$$

From Lemma 1, we can get

$$\mathbb{D}_t^\omega u_{ij}^n = \int_0^1 \omega(\alpha) D_t^\alpha u_{ij}^n \mathrm{d}\alpha = \Delta\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l) D_t^{\alpha_l} u_{ij}^n + \mathcal{O}(\Delta\alpha^2).$$

Observing the equivalence of the CFD $D_t^\alpha\varphi(t)$ and the RLFD ${}_{-\infty}D_t^\alpha\varphi(t)$ with $\varphi(t) = 0$ at $t \le 0$ and employing Lemma 3 as well as the above formula, we can obtain

$$\mathbb{D}_t^\omega u_{ij}^n = \Delta\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l)\tau^{-\alpha_l} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} u_{ij}^{n-k} + \mathcal{O}(\tau^2 + \Delta\alpha^2). \tag{18}$$

From Lemmas 8 and 9 leads to

$$I^{(\beta)}\Delta u(x_i, y_j, t_n) = Q_n^{(\beta)}(\Delta_h u_{ij}) + (R_1)_{ij}^n, \quad (i, j) \in \chi, \quad n = 1, \ldots, N. \tag{19}$$

Inserting relations (14), (15), (18) and (19) in (17) yields that

$$\begin{aligned}&\triangle\alpha\sum_{l=0}^{2J}c_l\omega(\alpha_l)\tau^{-\alpha_l}\sum_{k=0}^{n}\lambda_k^{(\alpha_l)}u_{ij}^{n-k}-\mu\Delta_h u_{ij}^n-(\tau^\beta\sum_{p=1}^{n}\omega_{n-p}^{(\beta)}\Delta_h u_{ij}^p+\tau^\beta\tilde{\omega}_n^{(\beta)}\Delta_h u_{ij}^0)\\&=f_{ij}^n+(R_1)_{ij}^n,\qquad (i,j)\in\chi,\quad n=1,\ldots,N,\end{aligned}\tag{20}$$

in which

$$\left|(R_1)_{ij}^n\right|\le C\Big(\tau^2+h_1^2+h_2^2+\triangle\alpha^2\Big).\tag{21}$$

Then adding the small term $\tau\mu_1\mu_2^2\delta_x^2\delta_y^2\delta_t u_{ij}^{n-\frac{1}{2}}=(R_2)_{ij}^n$ to both sides of Eq. (20), we can get

$$\begin{aligned}&\triangle\alpha\sum_{l=0}^{2J}c_l\omega(\alpha_l)\tau^{-\alpha_l}\sum_{k=0}^{n}\lambda_k^{(\alpha_p)}u_{ij}^{n-k}-\mu\Delta_h u_{ij}^n-(\tau^\beta\sum_{p=1}^{n}\omega_{n-p}^{(\beta)}\Delta_h u_{ij}^p+\tau^\beta\tilde{\omega}_n^{(\beta)}\Delta_h u_{ij}^0)\\&+\tau\mu_1\mu_2^2\delta_x^2\delta_y^2\delta_t u_{ij}^{n-\frac{1}{2}}=f_{ij}^n+R_{ij}^n,\qquad (i,j)\in\chi,\quad n=1,\ldots,N,\end{aligned}\tag{22}$$

in which

$$\left|R_{ij}^n\right|=\left|(R_1)_{ij}^n+(R_2)_{ij}^n\right|\le C\Big(\tau^2+h_1^2+h_2^2+\triangle\alpha^2\Big),$$

from which, if $u\in C_{x,y,t}^{4,4,2}\big([0,L_1]\times[0,L_2]\times[0,T]\big)$ with $\tau\mu_1\mu_2^2=\mathcal{O}(\tau^2|\ln\tau|)$, then $|(R_2)_{ij}^n|\le C\tau^2$. Noting the IC and BC in (2)–(3), we obtain

$$u_{ij}^0=\kappa(x_i,y_j),\qquad (i,j)\in\chi,\quad u_{ij}^n=0,\quad (i,j)\in\gamma,\quad n=1,\ldots,N.\tag{23}$$

Ignoring the truncation error R_{ij}^n and using the substitution of U_{ij}^n instead of u_{ij}^n in Eqs. (22)–(23), we can provide the ADI difference approach for Eqs. (1)–(3) as

$$\begin{aligned}&\triangle\alpha\sum_{l=0}^{2J}c_l\omega(\alpha_l)\tau^{-\alpha_l}\sum_{k=0}^{n}\lambda_k^{(\alpha_p)}U_{ij}^{n-k}-\mu\Delta_h U_{ij}^n-(\tau^\beta\sum_{p=1}^{n}\omega_{n-p}^{(\beta)}\Delta_h U_{ij}^p+\tau^\beta\tilde{\omega}_n^{(\beta)}\\&\quad\Delta_h U_{ij}^0)+\tau\mu_1\mu_2^2\delta_x^2\delta_y^2\delta_t U_{ij}^{n-\frac{1}{2}}=f_{ij}^n,\qquad (i,j)\in\chi,\quad n=0,\,1,\ldots,N,\end{aligned}\tag{24}$$

$$\begin{aligned}&U_{ij}^0=\kappa(x_i,y_j),\qquad (i,j)\in\chi,\\&U_{ij}^n=0,\qquad (i,j)\in\gamma,\quad n=0,\,1,\ldots,N.\end{aligned}\tag{25}$$

Let us introduce $\mu_2=\mu_1^{-1}(\mu+\tau^\beta\omega_0^{(\beta)})$, where μ_1 is defined in (11). It is not hard to get that $\mu_2=\mathcal{O}(\tau|\ln\tau|)$. At the same time, we notice that (24) can be restated as

$$\mu_1 U_{ij}^n-(\mu+\tau\omega_0^{(\beta)})\Delta_h U_{ij}^n+\tau\mu_1\mu_2^2\delta_x^2\delta_y^2\delta_t U_{ij}^{n-\frac{1}{2}}=\mathcal{F}_{ij}^n,\tag{26}$$

where

$$\mathcal{F}_{ij}^n=-\triangle\alpha\sum_{l=0}^{2J}c_l\omega(\alpha_l)\tau^{-\alpha_l}\sum_{k=1}^{n}\lambda_k^{(\alpha_p)}U_{ij}^{n-k}+\tau^\beta\sum_{p=1}^{n-1}\omega_{n-p}^{(\beta)}\Delta_h U_{ij}^p+\tau^\beta\tilde{\omega}_n^{(\beta)}\Delta_h U_{ij}^0+f_{ij}^n.$$

Denoting the notation $E^n=U^n-U^{n-1}$, after simplification, we can obtain the following ADI scheme

$$(\mathcal{I}-\mu_2\delta_x^2)(\mathcal{I}-\mu_2\delta_y^2)E_{ij}^n=\tilde{\mathcal{F}}_{ij}^n,\tag{27}$$

where $\mathcal{I}$ is an identity operator, $\tilde{\mathcal{F}}_{ij}^n$ is given as follows

$$\tilde{\mathcal{F}}_{ij}^n = -U_{ij}^{n-1} + \mu_2 \Delta_h U_{ij}^{n-1} - \mu_1^{-1} \Delta\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l) \tau^{-\alpha_l} \sum_{k=1}^{n} \lambda_k^{(\alpha_l)} U_{ij}^{n-k}$$
$$+ \mu_1^{-1} \tau^\beta \sum_{p=1}^{n-1} \omega_{n-p}^{(\beta)} \Delta_h U_{ij}^p + \mu_1^{-1} \tau^\beta \tilde{\omega}_n^{(\beta)} \Delta_h U_{ij}^0 + \mu_1^{-1} f_{ij}^n.$$

Solving two sets of independent 1D problem, we can determine U_{ij}^n. Let us define

$$E_{ij}^{n-\frac{1}{2}} = \big(\mathcal{I} - \mu_2 \delta_y^2\big) E_{ij}^n, \qquad (x_i, y_j) \in \Omega_h, \quad n = 1, \ldots, N.$$

Therefore, we give the following computational steps:

Step 1 Firstly, for fixed $j \in \{1, 2, \ldots, M_2 - 1\}$ we solve the following system to calculate $\{E_{ij}^{n-\frac{1}{2}}\}$:

$$\begin{cases} \big(\mathcal{I} - \mu_2 \delta_x^2\big) E_{ij}^{n-\frac{1}{2}} = \tilde{\mathcal{F}}_{ij}^n, \quad 1 \le i \le M_1 - 1, \quad n = 1, \ldots, N, \\ E_{0,j}^{n-\frac{1}{2}} = E_{M_1,j}^{n-\frac{1}{2}} = 0. \end{cases} \tag{28}$$

Step 2 Once $\{E_{ij}^{n-\frac{1}{2}}\}$ is available, fixed $i \in \{1, 2, \ldots, M_1 - 1\}$, we can solve the system as follows:

$$\begin{cases} \big(\mathcal{I} - \mu_2 \delta_y^2\big) E_{ij}^n = E_{ij}^{n-\frac{1}{2}}, \quad 1 \le j \le M_2 - 1, \quad n = 1, \ldots, N, \\ E_{i,0}^n = E_{i,M_2}^n = 0 \end{cases} \tag{29}$$

to compute $\{E_{ij}^n\}$, and we can get the desired solution $\{U_{ij}^n\}$ further.

2.3 Analysis of the ADI difference approach

This subsection only examines the convergence analysis of the proposed algorithm (24)–(25). In what follows, we introduce some useful lemmas.

Lemma 10 (Xu 1997) *For $t \in \{t \in \mathbb{C}, \mathbf{Re}(t) > 0\}$, $\beta(t) \in L^{1,loc}(0, \infty)$ denote a positive value if and only if $\mathbf{Re}\big(\hat{\beta}(t)\big) \ge 0$, in which $\hat{\beta}(\xi) = \int_{-\infty}^{+\infty} e^{i\xi t} \beta(t) \mathrm{d}t$ indicates the Laplace transform of $\beta(t)$ introduced in (6), and $\mathbf{Re}(\cdot)$ represents the real part.*

Lemma 11 (Lopez-Marcos 1990; Xu 1997) *Suppose that a real-valued sequence $\{a_0, a_1, \ldots, a_s, \ldots\}$ satisfies: for any vector $(L^1, L^2, \ldots, L^N) \in \mathbb{R}^N$, positive integer N, and $\hat{a}(z) = \sum_{s=0}^{\infty} a_s z^s$ is analytic in $\mathcal{S} = \{z \in \mathbb{C} : |z| \le 1\}$, for $z \in \mathcal{S}$, it follows that*

$$\sum_{n=1}^{N} \Big(\sum_{p=1}^{n} a_{n-p} L^p \Big) L^n \ge 0,$$

if and only if $\mathbf{Re}\big(\hat{a}(z)\big) \ge 0$.

Lemma 12 (Chen et al. 2016; Lopez-Marcos 1990) *The functions $w, v \in \mathring{\Omega}_h$ have the following properties:*

$$(i) |(\delta_x^2 w, v)| \le \frac{4}{h_1^2} \|w\| \|v\|,$$
$$(ii) (\delta_x^2 w, v) = -h_1 \sum_{j=0}^{M_1 - 1} (\delta_x w_{j+1})(\delta_x v_{j+1}).$$

In the same way, we can denote the notation $(\delta_y^2 w, v)$ *and so on.*

Define

$$e_{ij}^n := u(x_i, y_j, t_n) - U_{ij}^n = u_{ij}^n - U_{ij}^n, \quad (i,j) \in \chi, \quad n = 0, 1, \ldots, N.$$

Subtracting (24)–(25) from (22)–(23), respectively, and denoting the notation $\triangle\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l)\tau^{-\alpha_l} = \Phi_{l,J}$, we can compute the error system of equations as follows

$$\begin{aligned} &\Phi_{l,J} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} e_{ij}^{n-k} - \mu \Delta_h e_{ij}^n - (\tau^\beta \sum_{p=1}^{n} \omega_{n-p}^{(\beta)} \Delta_h e_{ij}^p + \tau^\beta \tilde{\omega}_n^{(\beta)} \Delta_h e_{ij}^0) \\ &+\tau\mu_1\mu_2^2 \delta_x^2 \delta_y^2 \delta_t e_{ij}^{n-\frac{1}{2}} = R_{ij}^n, \qquad (i,j) \in \chi, \quad n = 1, \ldots, N, \end{aligned} \tag{30}$$

$$\begin{aligned} e_{ij}^0 &= 0, \qquad (i,j) \in \chi, \\ e_{ij}^n &= 0, \qquad (i,j) \in \gamma, \quad n = 0, 1, \ldots, N. \end{aligned} \tag{31}$$

Theorem 1 (Convergence). *Suppose that* $\{u^n\}_{n=0}^N$ *and* $\{U^n\}_{n=0}^N$ *are the solutions of* (1)–(3) *and* (24)–(25)*, respectively. Let* $u(x,y,t) \in C_{x,y,t}^{4,4,2}\big([0,L_1] \times [0,L_2] \times [0,T]\big)$*. Then, we obtain*

$$\sqrt{\tau \sum_{n=1}^{N} \|e^n\|^2} \le C(T)\Big(\tau^2 + h^2 + \triangle\alpha^2\Big).$$

Proof We establish the following weak formulation by taking the inner product of (30) by τe^n, summing from $n = 1$ to N, adding small term $\tau\mu_1(e^0, e^0)$ to the both sides of (30) and denoting $\tilde{\varepsilon} = \tau\mu_1$ as

$$\begin{aligned} &\tau\Phi_{l,J} \sum_{n=0}^{N}\sum_{k=0}^{n} \lambda_k^{(\alpha_l)} (e^{n-k}, e^n) - \mu\tau \sum_{n=1}^{N} (\Delta_h e^n, e^n) - \tau^{1+\beta} \sum_{n=1}^{N}\sum_{p=1}^{n} \omega_{n-p}^{(\beta)} (\Delta_h e^p, e^n) \\ &-\tau^{1+\beta} \sum_{n=1}^{N} \tilde{\omega}_n^{(\beta)} (\Delta_h e^0, e^n) + \tau \sum_{n=1}^{N} \tau\mu_1\mu_2^2 (\delta_x^2\delta_y^2\delta_t e^{n-\frac{1}{2}}, e^n) = \tau \sum_{n=1}^{N} (R^n, e^n) + \tilde{\varepsilon}\|e^0\|^2. \end{aligned} \tag{32}$$

Each term in (32) will be analyzed below. Firstly, based on the Lemma 5, we have

$$\tau\Phi_{l,J} \sum_{n=0}^{N}\sum_{k=0}^{n} \lambda_k^{(\alpha_l)} (e^{n-k}, e^n) \ge 0. \tag{33}$$

Secondly, we can get

$$\begin{aligned} -(\Delta_h e^n, e^n) &= -(\delta_x^2 e^n + \delta_y^2 e^n, e^n) \\ &= (\delta_x e^n, \delta_x e^n) + (\delta_y e^n, \delta_y e^n) \\ &= \|\delta_x e^n\|^2 + \|\delta_y e^n\|^2 \\ &:= \|\nabla_h e^n\|^2. \end{aligned} \tag{34}$$

Thirdly, from Lemmas 10–12, we get

$$\begin{aligned} -\sum_{n=1}^{N}\sum_{p=1}^{n} \omega_{n-p}^{(\beta)} (\Delta_h e^p, e^n) &= \sum_{n=1}^{N}\sum_{p=1}^{n} \omega_{n-p}^{(\beta)} \big[(\delta_x e^p, \delta_x e^n) + (\delta_y e^p, \delta_y e^n)\big] \\ &\ge 0. \end{aligned} \tag{35}$$

Fourthly, applying Lemma 12 (i), we arrive at

$$\begin{aligned}\tau^{1+\beta}\sum_{n=1}^{N}\tilde{\omega}_n^{(\beta)}(\Delta_h e^0, e^n) &\le \tau^{1+\beta}\sum_{n=1}^{N}|\tilde{\omega}_n^{(\beta)}|(\Delta_h e^0, e^n)\\ &\le \tau^{1+\beta}\Big[\frac{4\|e^0\|\|e^n\|}{h_1^2}+\frac{4\|e^0\|\|e^n\|}{h_2^2}\Big]\sum_{n=1}^{N}|\tilde{\omega}_n^{(\beta)}| \\ &= 4\tau^{1+\beta}\Big(\frac{1}{h_1^2}+\frac{1}{h_2^2}\Big)\sum_{n=1}^{N}|\tilde{\omega}_n^{(\beta)}|\|e^0\|\|e^n\|.\end{aligned}\tag{36}$$

Moreover, we have

$$\begin{aligned}\tau^2\mu_1\mu_2^2\sum_{n=1}^{N}(\delta_x^2\delta_y^2\delta_t e^{n-\frac{1}{2}}, e^n) &= \tau^2\mu_1\mu_2^2\sum_{n=1}^{N}\Big(\delta_x^2\delta_y^2\frac{e^n-e^{n-1}}{\tau}, e^n\Big)\\ &= \tau^2\mu_1\mu_2^2\sum_{n=1}^{N}\Big(\delta_x\delta_y\frac{e^n-e^{n-1}}{\tau}, \delta_x\delta_y e^n\Big)\\ &= \tau^2\mu_1\mu_2^2\sum_{n=1}^{N}\Big(\delta_x\delta_y\frac{e^n-e^{n-1}}{\tau}, \delta_x\delta_y\frac{e^n-e^{n-1}+e^{n-1}+e^n}{2}\Big)\\ &\ge \tau^2\mu_1\mu_2^2\sum_{n=1}^{N}\Big(\delta_x\delta_y\frac{e^n-e^{n-1}}{\tau}, \delta_x\delta_y\frac{e^n-e^{n-1}+e^{n-1}+e^n}{2}\Big)\\ &\ge \frac{\tau\mu_1\mu_2^2}{2}\big(\|\delta_x\delta_y e^N\|^2-\|\delta_x\delta_y e^0\|^2\big).\end{aligned}\tag{37}$$

Finally, employing the Cauchy–Schwarz inequality arrives at

$$\tau\sum_{n=1}^{N}(R^n, e^n)\le\tau\sum_{n=1}^{N}\|R^n\|\|e^n\|.\tag{38}$$

Inserting (33)–(38) in (32), we have

$$\begin{aligned}\tau\mu\sum_{n=1}^{N}\|\nabla_h e^n\|^2+\frac{\tau\mu_1\mu_2^2}{2}\|\delta_x\delta_y e^N\|^2 &\le \frac{\tau\mu_1\mu_2^2}{2}\|\delta_x\delta_y e^0\|^2+\tilde{\varepsilon}\|e^0\|^2\\ &+4\tau^{1+\beta}\Big(\frac{1}{h_1^2}+\frac{1}{h_2^2}\Big)\sum_{n=1}^{N}|\tilde{\omega}_n^{(\beta)}|\|e^0\|\|e^n\|+\tau\sum_{n=1}^{N}\|R^n\|\|e^n\|.\end{aligned}\tag{39}$$

Next, using the Young inequality $ab\le\varepsilon a^2+\frac{1}{4\varepsilon}b^2 (a, b\in\mathbb{R}, \varepsilon>0)$ and Poincaré inequality $\|e^n\|\le C_0\|\nabla e^n\|$ to get

$$\begin{aligned}\frac{\mu\tau}{C^2}\sum_{n=1}^{N}\|e^n\|^2+\frac{\tau\mu_1\mu_2^2}{2}\|\delta_x\delta_y e^N\|^2 &\le \frac{\tau\mu_1\mu_2^2}{2}\|\delta_x\delta_y e^0\|^2+\tilde{\varepsilon}\|e^0\|^2\\ &+4\tau^{1+\beta}\Big(\frac{1}{h_1^2}+\frac{1}{h_2^2}\Big)\sum_{n=1}^{N}|\tilde{\omega}_n^{(\beta)}|\|e^0\|\|e^n\|+\frac{\tau C^2}{\mu}\sum_{n=1}^{N}\|R^n\|^2+\frac{\mu\tau}{4C^2}\sum_{n=1}^{N}\|e^n\|^2.\end{aligned}\tag{40}$$

Then, because of $e^0 = 0$, we have

$$\frac{3\mu\tau}{4C^2}\sum_{n=1}^{N}\|e^n\|^2 \le \frac{\tau C^2}{\mu}\sum_{n=1}^{N}\|R^n\|^2. \tag{41}$$

Finally, we obtain the convergence results as

$$\begin{aligned}\tau\sum_{n=1}^{N}\|e^n\|^2 &\le C\tau\sum_{n=1}^{N}\|R^n\|^2\\ &\le C(T)\Big(\tau^2+h_1^2+h_2^2+\triangle\alpha^2\Big)^2.\end{aligned} \tag{42}$$

This completes the proof □

3 Numerical method and error analysis for the three-dimensional case

This section presents the numerical scheme and analysis of the three-dimensional problem (1)–(3) with $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$. Except for special definitions, other signs are the same as the two-dimensional case.

3.1 The derivation of the ADI difference scheme in three dimensions

Let $h_1 = \frac{L_1}{M_1}, h_2 = \frac{L_2}{M_2}, h_3 = \frac{L_3}{M_3}, h = \max\{h_1, h_2, h_3\}$, where M_1, M_2, M_3 are the number of divisions in the x, y and z dimensions, respectively. The nodal points $x_i = ih_1, y_j = jh_2, z_m = mh_3, \varrho = \{1 \le i \le M_1 - 1, 1 \le j \le M_2 - 1, 1 \le m \le M_3 - 1\}, \iota = \{(i, j, m)|(x_i, y_j, z_m) \in \partial\Omega\}, \overline{\Omega}_h = \{(x_i, y_j, z_m)|0 \le i \le M_1, 0 \le j \le M_2, 0 \le m \le M_3\}, \mathring{\Omega} = \{w|w \in \Omega_h, w_{ijm} = 0, when(i, j, m) \in \iota\}, \Omega_h = \overline{\Omega}_h \cap \Omega$ and $\partial\Omega_h = \Omega_h \cap \partial\Omega$. Let us introduce the following grid functions

$$u_{ijm}^n := u(x_i, y_j, z_m, t_n), \quad f_{ijm}^n := f(x_i, y_j, z_m, t_n), \quad (x_i, y_j, z_m) \in \Omega_h, \quad n = 0, 1, \ldots, N.$$

For any grid function $w = \{w_{ijm}|0 \le i \le M_1, 0 \le j \le M_2, 0 \le m \le M_3\}$ over $\overline{\Omega}_h$, we define

$$\begin{aligned}\delta_x w_{ijm}^n &:= \frac{1}{h_1}(w_{ijm}^n - w_{i-1,j,m}^n),\\ \delta_x^2 w_{ijm}^n &:= \frac{1}{h_1}(\delta_x w_{ijm}^n - \delta_x w_{i-1,j,m}^n),\\ \Delta_h w_{ijm}^n &:= \delta_x^2 w_{ijm}^n + \delta_y^2 w_{ijm}^n + \delta_z^2 w_{ijm}^n.\end{aligned}$$

In like manner, we define other symbols, e.g., $\delta_y w_{ijm}^n, \delta_z w_{ijm}^n, \delta_y^2 w_{ijm}^n, \delta_z^2 w_{ijm}^n$, etc.

In addition, for grid functions $w = \{w_{ijm}|0 \le i \le M_1, 0 \le j \le M_2, 0 \le m \le M_3\}$ and $v = \{v_{ijm}|0 \le i \le M_1, 0 \le j \le M_2, 0 \le m \le M_3\}$, let us introduce the inner product and norms as

$$\begin{aligned}(w, v) &:= h_1h_2h_3\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}\sum_{m=1}^{M_3-1} w_{ijm}v_{ijm}, \quad \|v\| = \sqrt{(v, v)},\\ \|v\|_\infty &= \max_{\substack{1\le i\le M_1-1\\ 1\le j\le M_2-1\\ 1\le m\le M_3-1}} |v_{ijm}|.\end{aligned}$$

We obtain the following expression by considering (1) at the nodal point (x_i, y_j, z_m, t_n) for $(i, j, m) \in \varrho, n = 1, \ldots, N$, as

$$\begin{aligned}\mathbb{D}_t^{\omega} u(x_i, y_j, z_m, t_n) - \mu \Delta u(x_i, y_j, z_m, t_n) - I^{(\beta)} \Delta u(x_i, y_j, z_m, t_n) = f(x_i, y_j, z_m, t_n),\\ (x, y, z) \in \Omega, \qquad n = 1, \ldots, N.\end{aligned} \tag{43}$$

Similarly to what was considered in Sect. 2, from Lemmas 1 and 3, we can obtain

$$\mathbb{D}_t^{\omega} u_{ijm}^n = \Phi_{l,J} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} u_{ijm}^{n-k} + \mathcal{O}(\tau^2 + \triangle\alpha^2). \tag{44}$$

Meanwhile, in virtue of Lemmas 8 and 9, we have

$$I^{(\beta)} \Delta u(x_i, y_j, z_m, t_n) = Q_n^{(\beta)}(\Delta_h u_{ijm}) + (R_3)_{ijm}^n, \qquad (i, j, m) \in \varrho, \quad n = 1, \ldots, N. \tag{45}$$

Bring (44)–(45) into (43) and according to Lemma 8, we can get

$$\begin{aligned}&\Phi_{l,J} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} u_{ijm}^{n-k} - \mu \Delta_h u_{ijm}^n - (\tau^\beta \sum_{p=1}^{n} \omega_{n-p}^{(\beta)} \Delta_h u_{ijm}^p + \tau^\beta \tilde{\omega}_n^{(\beta)} \Delta_h u_{ijm}^0)\\ &= f_{ijm}^n + (R_3)_{ijm}^n, \qquad (i, j, m) \in \varrho, \quad n = 1, \ldots, N,\end{aligned} \tag{46}$$

where

$$\left|(R_3)_{ijm}^n\right| \le C\Big(\tau^2 + h_1^2 + h_2^2 + h_3^2 + \triangle\alpha^2\Big).$$

Adding the small term $\mathcal{L}U_{ijm}^{n-\frac{1}{2}} = \tau\mu_1\mu_2^2(\delta_x^2\delta_y^2 + \delta_x^2\delta_z^2 + \delta_y^2\delta_z^2)\delta_t U_{ijm}^{n-\frac{1}{2}} - \tau\mu_1\mu_3^3\delta_x^2\delta_y^2\delta_z^2\delta_t U_{ijm}^{n-\frac{1}{2}} = (R_4)_{ijm}^n$ to both sides of (46), we can get

$$\begin{aligned}&\Phi_{l,J} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} u_{ijm}^{n-k} - \mu \Delta_h u_{ijm}^n - \tau^\beta \sum_{p=1}^{n} \omega_{n-p}^{(\beta)} \Delta_h u_{ijm}^p + \mathcal{L}u_{ijm}^{n-\frac{1}{2}}\\ &= \tau^\beta \tilde{\omega}_n^{(\beta)} \Delta_h u_{ijm}^0 + f_{ijm}^n + (\hat{R})_{ijm}^n, \qquad (i, j, m) \in \varrho, \quad n = 1, \ldots, N,\end{aligned} \tag{47}$$

where

$$|(\hat{R})_{ijm}^n| = |(R_3)_{ijm}^n + (R_4)_{ijm}^n| \le C\Big(\tau^2 + h_1^2 + h_2^2 + h_3^2 + \triangle\alpha^2\Big),$$

with the IC and BC as follow

$$\begin{aligned}u_{ijm}^0 &= \kappa(x_i, y_j, z_m), \qquad (i, j, m) \in \varrho,\\ u_{ijm}^n &= 0, \qquad (i, j, m) \in \iota, \quad n = 1, \ldots, N.\end{aligned} \tag{48}$$

Dropping the truncation errors $(\hat{R})_{ijm}^n$, with U_{ijm}^n instead of u_{ijm}^n in (47)–(48), we obtain the difference scheme as follow

$$\begin{aligned}&\Phi_{l,J} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} U_{ijm}^{n-k} - \mu \Delta_h U_{ijm}^n - \tau^\beta \sum_{p=1}^{n} \omega_{n-p}^{(\beta)} \Delta_h U_{ijm}^p + \mathcal{L}U_{ijm}^{n-\frac{1}{2}}\\ &\quad = \tau^\beta \tilde{\omega}_n^{(\beta)} \Delta_h U_{ijm}^0 + f_{ijm}^n, \qquad (i, j, m) \in \varrho, \quad n = 1, \ldots, N.\end{aligned} \tag{49}$$

$$\begin{aligned}U_{ijm}^0 &= \kappa(x_i, y_j, z_m), \qquad (i, j, m) \in \varrho,\\ U_{ijm}^n &= 0, \qquad (i, j, m) \in \iota, \quad n = 1, \ldots, N.\end{aligned} \tag{50}$$

This moment, by observing that (49) can rewritten as

$$\mu_1 U_{ijm}^n - (\mu + \tau\omega_0^{(\beta)})\Delta_h U_{ijm}^n + \mathcal{L}U_{ijm}^{n-\frac{1}{2}} = \bar{\mathcal{F}}_{ijm}^n, \tag{51}$$

in which

$$\bar{\mathcal{F}}^{n}_{ijm} = -\triangle\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l)\tau^{-\alpha_l} \sum_{k=1}^{n} \lambda_k^{(\alpha_p)} U_{ijm}^{n-k} + \tau^{\beta} \sum_{p=1}^{n-1} \omega_{n-p}^{(\beta)} \Delta_h U_{ijm}^{p} + \tau^{\beta} \tilde{\omega}_n^{(\beta)} \Delta_h U_{ijm}^{0} + f_{ijm}^{n}.$$

Let the notation E^n is defined in front. After simplification, we get the following ADI difference scheme

$$(\mathcal{I} - \mu_2 \delta_x^2)(\mathcal{I} - \mu_2 \delta_y^2)(\mathcal{I} - \mu_2 \delta_z^2) E_{ijm}^{n} = \hat{\mathcal{F}}_{ijm}^{n}, \tag{52}$$

where $\mathcal{I}$ is an identity operator, and $\hat{\mathcal{F}}_{ijm}^{n}$ is presented as follows

$$\begin{aligned}\hat{\mathcal{F}}_{ijm}^{n} = &-U_{ijm}^{n-1} + \mu_2 \Delta_h U_{ijm}^{n-1} - \mu_1^{-1} \triangle\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l)\tau^{-\alpha_l} \sum_{k=1}^{n} \lambda_k^{(\alpha_l)} U_{ijm}^{n-k} \\ &+ \mu_1^{-1} \tau^{\beta} \sum_{p=1}^{n-1} \omega_{n-p}^{(\beta)} \Delta_h U_{ijm}^{p} + \mu_1^{-1} \tau^{\beta} \tilde{\omega}_n^{(\beta)} \Delta_h U_{ijm}^{0} + \mu_1^{-1} f_{ijm}^{n}.\end{aligned}$$

Next, we present several intermediate variables to determine at U_{ijm}^{n}:

$$E_{ijm}^{n-\frac{1}{3}} = \big(\mathcal{I} - \mu_2 \delta_z^2\big) E_{ijm}^{n}, \quad 0 \le i \le M_1, \quad 0 \le j \le M_2, \quad 0 \le m \le M_3 - 1, \quad n = 1, \ldots, N$$

and

$$E_{ijm}^{n-\frac{2}{3}} = \big(\mathcal{I} - \mu_2 \delta_y^2\big) E_{ijm}^{n-\frac{1}{3}}, \quad 0 \le i \le M_1, \quad 1 \le j \le M_2 - 1, \quad 0 \le m \le M_3, \quad n = 1, \ldots, N.$$

From the above formulae, we can calculate the E_{ijm}^{n} through the following three steps.

Step 1 Firstly, solve the following system to compute $\{E_{ijm}^{n-\frac{2}{3}}\}$ in the x-dimension by fixing $m \in \{1, 2, \ldots, M_3 - 1\}$ and $j \in \{1, 2, \ldots, M_2 - 1\}$ as

$$\begin{cases} \big(\mathcal{I} - \mu_2 \delta_x^2\big) E_{ijm}^{n-\frac{2}{3}} = \hat{\mathcal{F}}_{ijm}^{n}, \quad 1 \le i \le M_1 - 1, \quad n = 1, \ldots, N, \\ E_{0jm}^{n-\frac{2}{3}} = (\mathcal{I} - \mu_2 \delta_y^2) E_{0jm}^{n-\frac{1}{3}} = (\mathcal{I} - \mu_2 \delta_y^2)(\mathcal{I} - \mu_2 \delta_z^2) E_{0jm}^{n}, \\ E_{M_1 jm}^{n-\frac{2}{3}} = (\mathcal{I} - \mu_2 \delta_y^2) E_{M_1 jm}^{n-\frac{1}{3}} = (\mathcal{I} - \mu_2 \delta_y^2)(\mathcal{I} - \mu_2 \delta_z^2) E_{M_1 jm}^{n}. \end{cases} \tag{53}$$

Step 2 Solve the following system in the y-dimension by fixing $m \in \{1, 2, \ldots, M_3 - 1\}$ and $i \in \{1, 2, \ldots, M_1 - 1\}$, we can

$$\begin{cases} \big(\mathcal{I} - \mu_2 \delta_y^2\big) E_{ijm}^{n-\frac{1}{3}} = E_{ijm}^{n-\frac{2}{3}}, \quad 1 \le j \le M_2 - 1, \quad n = 1, \ldots, N, \\ E_{i0m}^{n-\frac{1}{3}} = (\mathcal{I} - \mu_2 \delta_z^2) E_{i0m}^{n}, \\ E_{iM_2 m}^{n-\frac{1}{3}} = (\mathcal{I} - \mu_2 \delta_z^2) E_{iM_2 m}^{n}. \end{cases} \tag{54}$$

Step 3 Solve the following system in the z-dimension when once $\{E_{ijm}^{n-\frac{2}{3}}\}$ and $\{E_{ijm}^{n-\frac{1}{3}}\}$ are determined by fixing $j \in \{1, 2, \ldots, M_2 - 1\}$ and $i \in \{1, 2, \ldots, M_1 - 1\}$ as

$$\begin{cases} \big(\mathcal{I} - \mu_2 \delta_z^2\big) E_{ijm}^{n} = E_{ijm}^{n-\frac{1}{3}}, \quad 1 \le m \le M_3 - 1, \quad n = 1, \ldots, N, \\ E_{ij0}^{n} = 0, E_{ijM_3}^{n} = 0. \end{cases} \tag{55}$$

3.2 Analysis of the ADI difference approach in three case

Following a similar process used in the 2D case, we only consider the convergence analysis of the proposed scheme (49)–(50). To begin with, we present the following significant lemmas.

Lemma 13 (Sun 2009; Wang and Vong 2014b) *Assume that $w, v \in \mathring{\Omega}$ and w, v are grid functions. Then it holds that*

$$(\delta_x^2\delta_y^2 w, v) = (\delta_x\delta_y w, \delta_x\delta_y v),$$
$$(\delta_x^2\delta_z^2 w, v) = (\delta_x\delta_z w, \delta_x\delta_z v),$$
$$(\delta_y^2\delta_z^2 w, v) = (\delta_y\delta_z w, \delta_y\delta_z v).$$

Lemma 14 (Sun 2009) *Let us define grid functions $w, v \in \mathring{\Omega}$. We obtain*

$$(\delta_x^2\delta_y^2\delta_z^2 w, v) = -(\delta_x\delta_y\delta_z w, \delta_x\delta_y\delta_z v).$$

At first, define

$$e_{ijm}^n := u(x_i, y_j, z_m, t_n) - U_{ijm}^n = u_{ijm}^n - U_{ijm}^n, \quad (i, j, m) \in \varrho, \quad n = 0,\ 1, \ldots, N.$$

Now, we obtain the error system of equations by subtracting (49)–(50) from (47) and (48), respectively, as

$$\begin{aligned} &\Phi_{l,J}\sum_{k=0}^{n}\lambda_k^{(\alpha_l)}e_{ijm}^{n-k} - \mu\Delta_h e_{ijm}^n - \tau^\beta\sum_{p=1}^{n}\omega_{n-p}^{(\beta)}\Delta_h e_{ijm}^p + \mathcal{L}e_{ijm}^{n-\frac{1}{2}} \\ &\quad = \tau^\beta\tilde{\omega}_n^{(\beta)}\Delta_h e_{ijm}^0 + f_{ijm}^n + (\hat{R})_{ijm}^n, \qquad (i, j, m) \in \varrho, \quad n = 1, \ldots, N, \end{aligned} \tag{56}$$

$$\begin{aligned} &e_{ijm}^0 = 0, \qquad (i, j, m) \in \varrho, \\ &e_{ijm}^n = 0, \qquad (i, j, m) \in \iota, \quad n = 0,\ 1, \ldots, N. \end{aligned} \tag{57}$$

Theorem 2 (Convergence). *Assume that $\{u^n\}_{n=0}^N$ and $\{U^n\}_{n=0}^N$ represent the solutions of (1)–(3) and (49)–(50), respectively. If $u(x, y, z, t) \in C_{x,y,z,t}^{4,4,4,2}\big([0, L_1]\times[0, L_2]\times[0, L_3]\times[0, T]\big)$, then we can arrive at*

$$\sqrt{\tau\sum_{n=1}^{N}\|e^n\|^2} \le C(T)\Big(\tau^2 + h^2 + \Delta\alpha^2\Big).$$

Proof We can obtain the following weak formulation employing the inner product of (56) by τe^n and the summing from $n = 1$ to $k = M$ as well as adding small term $\tau\mu_1(e^0, e^0)$ to the both sides of (56) as

$$\begin{aligned} &\tau\Phi_{l,J}\sum_{n=0}^{N}\sum_{k=0}^{n}\lambda_k^{(\alpha_l)}(e^{n-k}, e^n) - \mu\tau\sum_{n=1}^{N}(\Delta_h e^n, e^n) - \tau^{1+\beta}\sum_{n=1}^{N}\sum_{p=1}^{n}\omega_{n-p}^{(\beta)}(\Delta_h e^p, e^n) \\ &-\tau^{1+\beta}\sum_{n=1}^{N}\tilde{\omega}_n^{(\beta)}(\Delta_h e^0, e^n) + \tau(\mathcal{L}e_{ijm}^{n-\frac{1}{2}}, e^n) := \sum_{s=1}^{5}\Xi_s = \tau\sum_{n=1}^{N}(\hat{R}^n, e^n) + \tilde{\varepsilon}\|e^0\|^2. \end{aligned} \tag{58}$$

Below we shall estimate the terms in (58). For the first term Ξ_1, according to Lemma 5, we have

$$\tau\Phi_{l,J}\sum_{n=0}^{N}\sum_{k=0}^{n}\lambda_k^{(\alpha_l)}(e^{n-k}, e^n) \ge 0. \tag{59}$$

Then for Ξ_2, utilizing Poincaré inequality $\tilde{\lambda}\|e^n\| \le \|\nabla e^n\|$, we yield

$$
\begin{aligned}
-(\Delta_h e^n, e^n) &= -\Big((\delta_x^2 + \delta_2^2 + \delta_z^2)e^n, e^n\Big) \\
&= (\delta_x e^n, \delta_x e^n) + (\delta_y e^n, \delta_y e^n) + (\delta_z e^n, \delta_z e^n) \\
&= \|\delta_x e^n\|^2 + \|\delta_y e^n\|^2 + \|\delta_z e^n\|^2 \\
:&= \|\nabla_h e^n\|^2 \ge \tilde{\lambda}^2 \|e^n\|^2.
\end{aligned} \tag{60}
$$

For Ξ_3, from Lemmas 10–12, we have

$$
\begin{aligned}
-\sum_{n=1}^{N}\sum_{p=1}^{n}\omega_{n-p}^{(\beta)}(\Delta_h e^p, e^n) &= \sum_{n=1}^{N}\sum_{p=1}^{n}\omega_{n-p}^{(\beta)}\big[(\delta_x e^p, \delta_x e^n) + (\delta_y e^p, \delta_y e^n) + (\delta_z e^p, \delta_z e^n)\big] \\
&\ge 0.
\end{aligned} \tag{61}
$$

Next for Ξ_4, applying Lemma 12 (i), we arrive at

$$
\begin{aligned}
\tau^{1+\beta}\sum_{n=1}^{N}\tilde{\omega}_n^{(\beta)}(\Delta_h e^0, e^n) &\le \tau^{1+\beta}\sum_{n=1}^{N}|\tilde{\omega}_n^{(\beta)}|(\Delta_h e^0, e^n) \\
&\le \tau^{1+\beta}\Big[\frac{4\|e^0\|\|e^n\|}{h_1^2} + \frac{4\|e^0\|\|e^n\|}{h_2^2} + \frac{4\|e^0\|\|e^n\|}{h_3^2}\Big]\sum_{n=1}^{N}|\tilde{\omega}_n^{(\beta)}| \\
&= 4\tau^{1+\beta}\Big(\frac{1}{h_1^2} + \frac{1}{h_2^2} + \frac{1}{h_3^2}\Big)\sum_{n=1}^{N}|\tilde{\omega}_n^{(\beta)}|\|e^0\|\|e^n\|.
\end{aligned} \tag{62}
$$

Moreover, for Ξ_5, in view of Lemmas 13–14, we have

$$
\begin{aligned}
2\tau(\delta_x^2\delta_y^2\delta_t e^n, e^n) &\ge \|\delta_x\delta_y e^n\|^2 - \|\delta_x\delta_y e^{n-1}\|^2, \\
2\tau(\delta_x^2\delta_z^2\delta_t e^n, e^n) &\ge \|\delta_x\delta_z e^n\|^2 - \|\delta_x\delta_z e^{n-1}\|^2, \\
2\tau(\delta_y^2\delta_z^2\delta_t e^n, e^n) &\ge \|\delta_y\delta_z e^n\|^2 - \|\delta_y\delta_z e^{n-1}\|^2, \\
-2\tau(\delta_x^2\delta_y^2\delta_z^2\delta_t e^n, e^n) &\ge \|\delta_x\delta_y\delta_z e^n\|^2 - \|\delta_x\delta_y\delta_z e^{n-1}\|^2.
\end{aligned} \tag{63}
$$

Let us denote $\tau\mu_1\mu_2^2 := \mu_3 = \mathcal{O}(\tau^2|\ln\tau|)$, $\tau\mu_1\mu_2^3 := \mu_4 = \mathcal{O}(\tau^3|\ln\tau|^2)$, then we obtain

$$
\begin{aligned}
\tau\sum_{n=1}^{N}(\mathcal{L}e_{ijm}^{n-\frac{1}{2}}, e^n) &= \tau\mu_3\sum_{n=1}^{N}\Big((\delta_x^2\delta_y^2 + \delta_x^2\delta_z^2 + \delta_y^2\delta_z^2)\delta_t e^{n-\frac{1}{2}}, e^n\Big) \\
&\quad - \tau\mu_4\sum_{n=1}^{N}(\delta_x^2\delta_y^2\delta_z^2\delta_t e^{n-\frac{1}{2}}, e^n) \\
&= \mu_3\sum_{n=1}^{N}\Big((\delta_x^2\delta_y^2 + \delta_x^2\delta_z^2 + \delta_y^2\delta_z^2)(e^n - e^{n-1}), e^n\Big) \\
&\quad - \mu_4\sum_{n=1}^{N}\Big(\delta_x^2\delta_y^2\delta_z^2(e^n - e^{n-1}), e^n\Big) \\
&= \mu_3\sum_{n=1}^{N}\Big((\delta_x^2\delta_y^2 + \delta_x^2\delta_z^2 + \delta_y^2\delta_z^2)(e^n - e^{n-1}), \frac{e^n - e^{n-1} + e^{n-1} + e^n}{2}\Big)
\end{aligned}
$$

Table 1 Maximum absolute errors $E(\tau, h, \varDelta\alpha)$, associated time convergence orders Order_τ^1 and CPU run times (in s) by c $h = \frac{1}{64}$, $\triangle\alpha = \frac{1}{128}$ and $q = 2$

β	N	$E(\tau, h, \varDelta\alpha)$	Order_τ^1	CPU(s)
	8	2.1994e−2	–	0.37
0.25	16	6.0727e−3	1.86	0.75
	32	1.5688e−3	1.95	1.64
	64	3.7238e−4	2.07	4.27
	8	1.5099e−2	–	0.39
0.50	16	3.9846e−3	1.92	0.76
	32	1.0176e−3	1.97	1.98
	64	2.3685e−4	2.10	4.60
	8	1.3064e−2	–	0.35
0.75	16	3.5761e−3	1.87	0.73
	32	9.5618e−4	1.90	1.64
	64	2.3383e−4	2.03	4.41

$$
\begin{aligned}
&- \mu_4 \sum_{n=1}^{N} \left(\delta_x^2 \delta_y^2 \delta_z^2 (e^n - e^{n-1}), \frac{e^n - e^{n-1} + e^{n-1} + e^n}{2} \right) \\
&\geq \frac{\mu_3}{2} \Big[\|\delta_x \delta_y e^N\|^2 - \|\delta_x \delta_y e^0\|^2 + \|\delta_x \delta_z e^N\|^2 - \|\delta_x \delta_z e^0\|^2 + \|\delta_y \delta_z e^N\|^2 \\
&- \|\delta_y \delta_z e^0\|^2 \Big] + \frac{\mu_4}{2} \Big[\|\delta_x \delta_y \delta_z e^N\|^2 - \|\delta_x \delta_y \delta_z e^0\|^2 \Big]. \qquad (64)
\end{aligned}
$$

Finally, employing Cauchy–Schwarz inequality and Young inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 (a, b \in \mathbb{R}, \varepsilon > 0)$, we arrive at

$$
\tau \sum_{n=1}^{N} (R^n, e^n) \leq \tau \sum_{n=1}^{N} \| R^n \| \|e^n\| \leq \frac{\mu \tilde{\lambda}^2}{3} \sum_{n=1}^{N} \|e^n\|^2 + \frac{3}{4\mu\tilde{\lambda}^2} \sum_{n=1}^{N} \|R^n\|^2. \qquad (65)
$$

Substituting (59)–(65) into (58) and noticing $e^0 = 0$, we can obtain

$$
\tilde{\lambda}^2 \tau \mu \sum_{n=1}^{N} \|e^n\|^2 \leq \frac{\tau \mu \tilde{\lambda}^2}{3} \sum_{n=1}^{N} \|e^n\|^2 + \frac{3\tau}{4\mu\tilde{\lambda}^2} \sum_{n=1}^{N} \|R^n\|^2. \qquad (66)
$$

After simplification, we can get

$$
\frac{2\mu\tilde{\lambda}^2}{3} \tau \sum_{n=1}^{N} \|e^n\|^2 \leq \frac{3}{4\mu\tilde{\lambda}^2} \tau \sum_{n=1}^{N} \|R^n\|^2. \qquad (67)
$$

Thus, we obtain

$$
\begin{aligned}
\tau \sum_{n=1}^{N} \|e^n\|^2 &\leq C\tau \sum_{n=1}^{N} \|R^n\|^2 \\
&\leq C(T) \left(\tau^2 + h_1^2 + h_2^2 + h_3^2 + \triangle\alpha^2 \right)^2, \qquad (68)
\end{aligned}
$$

which finishes the proof. □

Table 2 Maximum absolute errors $E(\tau, h, \Delta\alpha)$, associated time convergence orders Order_τ^1 and CPU run times (in s) by taking $h = \frac{1}{64}$, $\triangle\alpha = \frac{1}{128}$ and $q = 3$

β	N	$E(\tau, h, \Delta\alpha)$	Order_τ^1	CPU(s)
	8	6.6218e−3	–	0.96
0.20	16	1.9702e−3	1.75	1.29
	32	5.3269e−4	1.89	2.80
	64	1.2885e−4	2.05	7.51
	8	4.2271e−3	–	0.36
0.50	16	1.1896e−3	1.83	0.73
	32	3.1289e−4	1.93	1.65
	64	7.2402e−5	2.11	4.13
	8	3.4720e−3	–	0.35
0.80	16	1.0309e−3	1.75	0.71
	32	2.8730e−4	1.84	1.63
	64	7.0872e−5	2.02	4.19

4 Numerical results and discussion

This section presents two test problems to show the accuracy and computational efficiency of the proposed algorithm. Here, the ADI schemes (24)–(25) and (49)–(50) are adopted to approximate the problem (1)–(3). Let $M_1 = M_2 = M_3 = M = \frac{L}{h}$ with $L = 1$, $T = 0.5$ and $\mu = 0.5$. For this aim, we calculate the maximum error and associated convergence orders as

$$E(\tau, h, \Delta\alpha) := \max_{1\le n\le N} \|u^n - U^n\|_\infty, \qquad \text{Order}_\tau^1 := \log_2\left(\frac{E(\tau,h,\Delta\alpha)}{E(\tau/2,h,\Delta\alpha)}\right),$$
$$\text{Order}_h^2 := \log_2\left(\frac{E(\tau,h,\Delta\alpha)}{E(\tau,h/2,\Delta\alpha)}\right), \qquad \text{Order}_{\Delta\alpha}^3 := \log_2\left(\frac{E(\tau,h,\Delta\alpha)}{E(\tau,h,\Delta\alpha/2)}\right).$$

Numerical computations have been done in Matlab environment with a desktop computer with Windows 10 and RAM 16 GB.

Example 1 Let us consider the two-dimensional problem (1)–(3) including an analytic solution

$$u(x, y, t) = t^q \sin(\pi x)\sin(\pi y),$$

such that the weight function and the source term are

$$\omega(\alpha) = \Gamma(1 + q - \alpha),$$
$$f(x, y, t) = t^{q-1}\left(\Gamma(q + 1)(1 - t)(\ln(\tfrac{1}{t}))^{-1} + 2\mu t\pi^2 + \frac{2t^{1+\beta}\Gamma(q+1)\pi^2}{\Gamma(1+\beta+q)}\right)\sin(\pi x)\sin(\pi y),$$

respectively.

We solve this example with various values of parameters at total time T based on the proposed method in the temporal and spatial dimensions. Tables 1 and 2 report the maximum absolute errors, associated time convergence orders and CPU run times (in s) when the space and distribution step sizes are fixed. It is seen that the proposed algorithm (49)–(50) is second-order convergent in the time direction. Tables 3 and 4 list the maximum absolute errors, associated time convergence orders and CPU run times (in s) when the time and distributed-order step sizes are fixed. We observe that the proposed method (49)–(50) is second-order convergent in the spatial direction. Table 5 displays the maximum absolute

Table 3 Maximum absolute errors $E(\tau, h, \Delta\alpha)$, associated space convergence orders Order_h^2 and CPU run times (in s) by choosing $\tau = \frac{1}{256}$, $\Delta\alpha = \frac{1}{256}$ and $q = 2$

β	M	$E(\tau, h, \Delta\alpha)$	Order_h^2	CPU(s)
	2	5.0548e−2	–	0.30
0.25	4	1.1622e−2	2.12	0.61
	8	2.7704e−3	2.07	1.38
	16	6.0912e−4	2.19	3.46
	2	4.9076e−2	–	0.24
0.50	4	1.1354e−2	2.11	0.61
	8	2.7332e−3	2.06	1.38
	16	6.2403e−4	2.13	3.12
	2	4.7521e−2	–	0.24
0.75	4	1.1041e−2	2.11	0.60
	8	2.6564e−3	2.06	1.36
	16	6.0393e−4	2.14	3.10

Table 4 Maximum absolute errors $E(\tau, h, \Delta\alpha)$, associated space convergence orders Order_h^2 and CPU run times (in s) by considering $\tau = \frac{1}{256}$, $\Delta\alpha = \frac{1}{256}$ and $q = 3$

β	M	$E(\tau, h, \Delta\alpha)$	Order_h^2	CPU(s)
	2	1.9100e−2	–	0.27
0.30	4	4.5511e−3	2.07	0.61
	8	1.1031e−3	2.04	1.40
	16	2.5300e−4	2.12	3.17
	2	1.7760e−2	–	0.25
0.50	4	4.2680e−3	2.06	0.62
	8	1.0391e−3	2.04	1.39
	16	2.4122e−4	2.11	3.20
	2	1.6601e−2	–	0.24
0.70	4	4.0151e−3	2.05	0.59
	8	9.7821e−4	2.04	1.41
	16	2.2622e−4	2.11	3.19

errors, distributed orders and CPU run times (s) and reflects the second order in distributed-order. Looking at Tables 1, 2, 3, 4 and 5 as a whole, we see that the proposed method has less time-consuming in the case of the two-dimensional problem. Figure 2 depicts the temporal convergence order when $h = \frac{1}{64}$, $\Delta\alpha = \frac{1}{128}$ and $q = 2$, while Fig. 3 represents the spatial convergence order when fixed $\tau = \frac{1}{256}$ and $\Delta\alpha = \frac{1}{256}$. Finally, Fig. 4 demonstrates the distributed convergence order for fixed $\tau = \frac{1}{380}$, $h = \frac{1}{55}$ and $q = 3$.

To show the efficiency of the ADI algorithm, we show the maximum errors, spatial convergence orders and CPU run times for the ADI FD scheme and the standard finite difference (SFD) scheme in Table 6. From Table 6 we can see that the errors do not have much difference between the two methods, at the same time, our ADI method has a better spatial convergence order and a shorter running time. Then we present Fig. 1, which intuitively illustrate the efficiency of the proposed method. In summary, these demonstrate the competitiveness of the ADI algorithm (Figs. 2, 3, 4).

Table 5 Maximum absolute errors $E(\tau, h, \Delta\alpha)$, distributed orders $\text{Order}^3_{\triangle\alpha}$ and CPU run times (in s) with $\tau = \frac{1}{380}$, $h = \frac{1}{55}$ and $q = 3$

β	$2J$	$E(\tau, h, \Delta\alpha)$	$\text{Order}^3_{\triangle\alpha}$	CPU(s)
	2	3.8236e−4	–	1.96
0.25	4	9.2675e−5	2.04	2.28
	8	2.2105e−5	2.07	2.96
	16	5.6439e−6	1.97	4.19
	2	4.5539e−4	–	2.07
0.50	4	1.0998e−4	2.05	2.55
	8	2.5598e−5	2.10	2.98
	16	5.8311e−6	2.13	4.24
	2	5.3440e−4	–	1.96
0.75	4	1.2979e−4	2.04	2.24
	8	3.0661e−5	2.08	2.89
	16	7.0517e−6	2.12	4.12

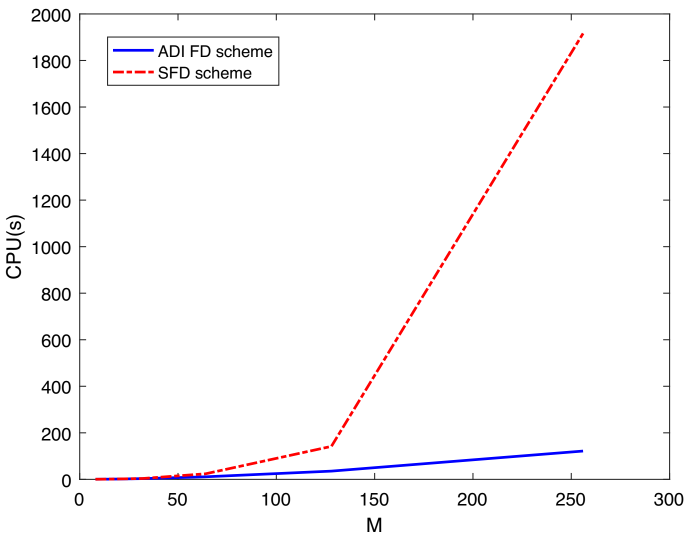

Fig. 1 The CPU run times with $\tau = \frac{1}{400}$, $\triangle\alpha = \frac{1}{32}$, $\mu = 0.1$ and $q = 2$ for the ADI FD scheme and the SFD scheme

Table 6 Maximum absolute errors $E(\tau, h, \Delta\alpha)$, the corresponding spatial orders Order^2_h and CPU run times (in s) with $\tau = \frac{1}{400}$, $\triangle\alpha = \frac{1}{32}$, $q = 2$ and $\mu = 0.1$ for the ADI FD scheme and the SFD scheme

β	M	$E(\tau, h, \Delta\alpha)$	$^{ADIFD}\text{Order}^2_h$	CPU(s)	$E(\tau, h, \Delta\alpha)$	$^{SFD}\text{Order}^2_h$	CPU(s)
	4	1.0415e−2	–	0.24	1.0418e−2	–	0.41
	8	2.5580e−3	2.03	0.45	2.5611e−3	2.02	0.54
0.50	16	6.2889e−4	2.02	1.04	6.3199e−4	2.02	1.18
	32	1.4880e−4	2.08	3.21	1.5190e−4	2.06	3.72
	64	2.9271e−5	2.35	11.04	3.2333e−5	2.23	23.98
	128	5.8827e−6	2.31	35.46	5.0315e−6	2.68	141.76

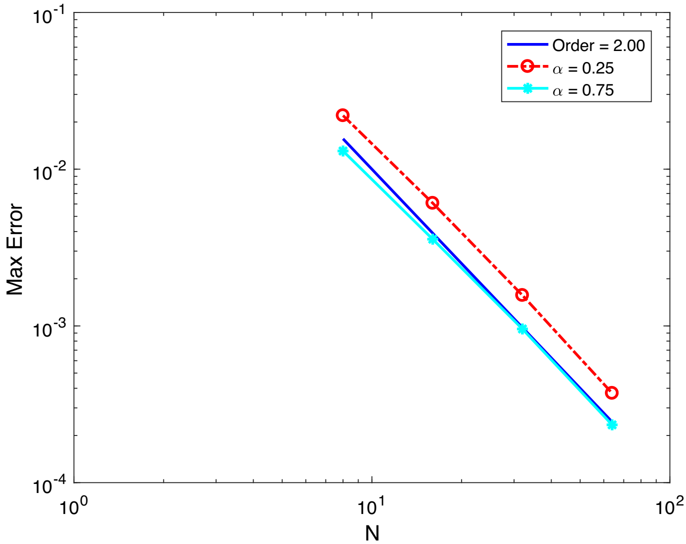

Fig. 2 The time convergence order with $h = \frac{1}{64}$, $\triangle\alpha = \frac{1}{128}$ and $q = 2$

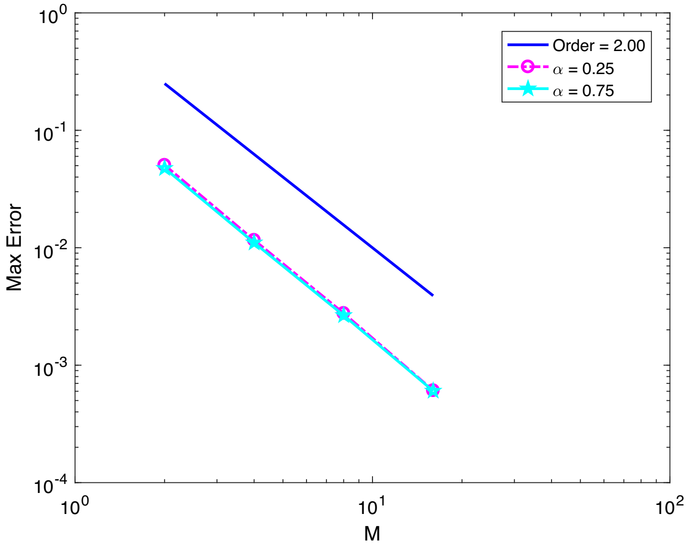

Fig. 3 The space convergence order with $\tau = \frac{1}{256}$, $\triangle\alpha = \frac{1}{256}$ and $q = 2$

Example 2 Consider the three-dimensional problem (1)–(3) including an analytic solution $u(x, y, z, t) = t^q \sin(\pi x) \sin(\pi y) \sin(\pi z)$ such that the weight function and the source term are

$$\omega(\alpha) = \Gamma(1 + q - \alpha)$$

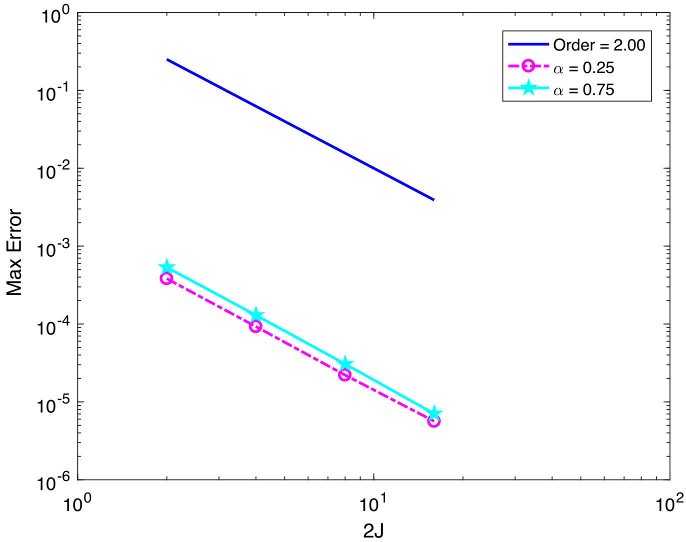

Fig. 4 The distributed convergence order when $\tau = \frac{1}{380}$, $h = \frac{1}{55}$ and $q = 3$

and

$$f(x, y, z, t) = t^{q-1}\Big(\Gamma(q+1)(1-t)(\ln(\tfrac{1}{t}))^{-1} + 3\mu t\pi^2 + \tfrac{3t^{1+\beta}\Gamma(q+1)\pi^2}{\Gamma(1+\beta+q)}\Big)$$
$$\sin(\pi x)\sin(\pi y)\sin(\pi z),$$

respectively.

We simulate this example with different values of parameters at total time T based on the proposed method in the temporal and spatial dimensions. Tables 7 and 8 extract the maximum absolute errors, associated time convergence orders and CPU run times (in s) when the space and distribution step sizes are fixed. It is observed that the proposed method (49)–(50) is second-order convergent in the time direction. Tables 9 and 10 report the maximum absolute errors, associated time convergence orders and CPU run times (in s) when the time and distributed-order step sizes are fixed. It is seen that the proposed method (49)–(50) is second-order convergent in the space direction. Table 11 shows the maximum absolute errors, distributed orders and CPU run times (s) and reflects the second order in distributed-order. Looking at Tables 7, 8, 9, 10 and 11 as a whole, we observe that the proposed method has less time-consuming in the case of the three-dimensional problem.

5 Concluding remarks

This paper analyzed and constructed the ADI difference approaches in two/three dimensions for distributed-order integrodifferential equations. The proposed method computed the unknown solution in two parts. First, the distributed-order time-fractional derivative and the RLFI term were approximated by using the weighted and shifted Grünwald–Letnikov expansion and second-order CQ, respectively. Second, the spatial discretization was obtained by the general centered FD method. The convergence of the ADI difference approaches was

Table 7 Maximum absolute errors $E(\tau, h, \Delta\alpha)$, associated time convergence orders Order_τ^1 and CPU run times (in s) when $h = \frac{1}{50}$, $\triangle\alpha = \frac{1}{64}$ and $q = 2$

β	N	$E(\tau, h, \Delta\alpha)$	Order_τ^1	CPU(s)
	8	5.5624e−2	–	18.84
0.25	16	1.4655e−2	1.92	44.01
	32	3.5595e−3	2.04	92.88
	64	8.2279e−4	2.11	218.58
	8	3.6510e−2	–	21.23
0.50	16	9.0771e−3	2.01	43.84
	32	2.2130e−3	2.04	95.91
	64	5.1091e−4	2.11	223.00
	8	3.0199e−2	–	20.20
0.75	16	7.8655e−3	1.94	43.42
	32	2.0293e−3	1.95	94.35
	64	4.9437e−4	2.04	221.56

Table 8 Maximum absolute errors $E(\tau, h, \Delta\alpha)$, associated time convergence orders Order_τ^1 and CPU run times (in s) by considering $h = \frac{1}{50}$, $\triangle\alpha = \frac{1}{64}$ and $q = 3$

β	N	$E(\tau, h, \Delta\alpha)$	Order_τ^1	CPU(s)
	8	1.4204e−2	–	20.65
0.25	16	4.0466e−3	1.81	43.66
	32	1.0497e−3	1.95	93.92
	64	2.4798e−4	2.08	220.18
	8	9.6892e−3	–	19.39
0.50	16	2.6718e−3	1.86	44.42
	32	6.9237e−4	1.95	94.78
	64	1.6210e−4	2.09	217.15
	8	8.0609e−3	–	19.39
0.75	16	2.3303e−3	1.79	47.54
	32	6.3695e−4	1.87	106.71
	64	1.5756e−4	2.02	235.36

Table 9 Maximum absolute errors $E(\tau, h, \Delta\alpha)$, associated space convergence orders Order_h^2 and CPU run times (in s) by taking $\tau = \frac{1}{420}$, $\triangle\alpha = \frac{1}{64}$ and $q = 2$

β	M	$E(\tau, h, \Delta\alpha)$	Order_h^2	CPU(s)
	4	1.2116e−2	–	2.44
0.25	8	2.9004e−3	2.06	13.17
	12	1.2343e−3	2.11	33.55
	16	6.5411e−4	2.21	68.66
	4	1.1929e−2	–	2.20
0.50	8	2.8740e−3	2.05	12.71
	12	1.2357e−3	2.08	33.12
	16	6.6513e−4	2.15	68.62
	4	1.1704e−2	–	2.41
0.75	8	2.8185e−3	2.05	12.73
	12	1.2096e−3	2.09	37.92
	16	6.4909e−4	2.16	72.50

Table 10 Maximum absolute errors $E(\tau, h, \Delta\alpha)$, associated space convergence orders Order_h^2 and CPU run times (in s) with $\tau = \frac{1}{420}$, $\triangle\alpha = \frac{1}{64}$ and $q = 3$

β	M	$E(\tau, h, \Delta\alpha)$	Order_h^2	CPU(s)
	4	5.1548e−3	–	2.30
0.25	8	1.2485e−3	2.05	12.79
	12	5.3770e−4	2.08	35.23
	16	2.8984e−4	2.15	69.22
	4	4.8648e−3	–	2.24
0.50	8	1.1846e−3	2.04	13.54
	12	5.1347e−4	2.06	34.65
	16	2.7933e−4	2.12	73.87
	4	4.5991e−3	–	2.19
0.75	8	1.1204e−3	2.04	14.52
	12	4.8469e−4	2.07	35.63
	16	2.6283e−4	2.13	76.62

Table 11 Maximum absolute errors $E(\tau, h, \Delta\alpha)$, distributed orders $\text{Order}_{\triangle\alpha}^3$ and CPU run times (in s) by choosing $\tau = \frac{1}{502}$, $h = \frac{1}{52}$ and $q = 3$

β	$2J$	$E(\tau, h, \Delta\alpha)$	$\text{Order}_{\triangle\alpha}^3$	CPU(s)
	2	2.6350e−4	–	478.10
0.25	4	6.3720e−5	2.05	533.91
	8	1.5686e−5	2.02	624.31
	16	4.6653e−6	1.75	830.67
	2	3.1671e−4	–	424.66
0.50	4	7.6038e−5	2.06	469.31
	8	1.7817e−5	2.09	566.37
	16	4.4774e−6	1.99	815.06
	2	3.7715e−4	–	436.54
0.75	4	9.1408e−5	2.04	461.00
	8	2.1820e−5	2.07	567.59
	16	5.5278e−6	1.98	740.13

thoroughly proven and verified numerically. Numerical experiments highlighted the validity of the method and supported the theoretical predictions.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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