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Decision analysis review on the concept of class for bipolar soft set theory

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Abstract

To eliminate uncertainty, all data expressed by decision-makers must be processed correctly. The most useful mathematical models developed for this purpose are hybrid set types. Because they collect all the features of the set types they contain under a single model. In this paper, the bipolar soft rough sets, which are a combination of bipolar soft sets and rough sets, which have been actively preferred in studies aimed at eliminating uncertainty in recent years, have been taken into account. To use the bipolar soft rough sets discussed more actively in the decision-making process, the focus is on handling the data expressed by different decision-makers together. The aim of this paper is to develop the concept of "bipolar soft rough classes" to aim to highlight this contribution to bipolar soft rough set theory. Thus, the concept of bipolar soft rough classes is introduced and a more efficient decision-making algorithm for uncertainty problems is built. Moreover, many novel concepts such as bipolar soft class, bipolar soft partition and bipolar soft cover were proposed and some properties are examined in detail.

Keywords Bipolar soft classes · Bipolar soft rough classes · Algorithm · Decision making

Mathematics Subject Classification 03E72 · 03E75 · 91B06

1 Introduction

Many mathematical approaches have been proposed to express many uncertainty problems encountered in daily life in the most accurate way and thus to manage the decision-making process in an ideal way. The first proposed mathematical approach was the fuzzy set theory introduced into the literature by Zadeh (1965). This theory, which expresses the belonging of an element to a set by the degree of membership, is a very successful approach. A great

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deal of articles has been done on these sets and some of their extensions (Asmus et al. 2017; Rodrigues et al. 2013; Tang et al. 2020; Kamacı et al. 2021; Xian et al. 2020). In the following years, the rough set theory (Pawlak 1982), which was introduced as another important set type, was proposed. In this theory given by Pawlak, equivalence relations are used to overcome uncertainty. Although rough set theory is one of the oldest mathematical approaches to overcome uncertainty, many studies are being conducted on it even today (Sharma et al. 2020; Cekik and Uysal 2020; Zhang et al. 2020a, b; Luo et al. 2020; Hamed et al. 2021; Demirtaş et al. 2020).

Until 1999, fuzzy set and rough set theories, which were the most important theories to overcome uncertainty, were not practical in expressing decision-making approaches. Molodtsov (1999), who thinks that the most important reason for this is due to the lack of a parameterization tool, proposed soft (s-)sets as a new mathematical approach. To express uncertainty problems in a practical way has enabled the development of more successful approaches in decision-making processes and thus the results were obtained in a more ideal way. Hence, s-sets have been successfully applied by many researchers to many areas such as Riemann integration, smoothness of functions, theory of measurement, game theory, and so on. Moreover, to overcome uncertainty, many different types of hybrid sets have been constructed using s-set theory (Dalkılıç and Demirtas 2020a; Dalkılıç 2020; Khalil et al. 2020; Mukherjee and Das 2020; Wang et al. 2020; Riaz et al. 2021b; Kong et al. 2011).

To solve uncertainty problems in the most accurate way, fuzzy set, rough set and s-set theories are the most important mathematical approaches and the relationships between these theories have been discussed by Aktaş and Çağman (2007). Dubois and Prade (1990) extended the notion of rough set to rough fuzzy set and fuzzy rough set. After Herawan and Deris (2009) explained the connection between the soft set and the rough set, Feng and Liu (2009) proposed soft rough sets as a new approach to overcoming uncertainty. Soft rough set theory is a highly adopted mathematical approach in the literature, and studies (Riaz et al. 2019a, b; Feng et al. 2011) can be examined to learn more about this theory. To further develop this successful hybrid set type, Meng et al. (2011) proposed a soft rough fuzzy set. We can say that the construction of more complex hybrid cluster types such as soft multi-rough set (Riaz et al. 2021a), intuitionistic fuzzy soft rough set (Zhang 2012), soft fuzzy rough set (Sun and Ma 2014), interval-valued neutrosophic soft rough set (Broumi and Smarandache 2015), Z-soft fuzzy rough set (Zhan et al. 2017), soft rough q-rung orthopair m-polar fuzzy set (Ping et al. 2021), q-rung orthopair m-polar fuzzy soft rough set (Ping et al. 2021), linear Diophantine fuzzy soft rough set (Riaz et al. 2020) and spherical linear diophantine fuzzy soft rough set (Hashmi et al. 2021) in the following years has been important steps to overcome the uncertainty.

Another successful mathematical model is the bipolar soft (bs-)set theory, which is a generalization of s-sets proposed by Shabir and Naz (Shabir and Naz 2013). This theory is built as a combination of two different s-sets, taking into account the NOT parameter set of the existing parameter set. Moreover, a new definition has been proposed by Karaaslan and Karataş (2015), allowing topological structures to be studied on bs-sets. Especially in recent years, the studies on this set theory have increased with the realization that more successful results are obtained by considering both aspects of the parameters (Kamacı and Petchimuthu 2020; Demirtaş and Dalkılıç 2019; Mukherjee and Das 2020; Dalkılıç and Demirtaş 2020b). Many different types of hybrid sets have been proposed by considering soft sets together with other mathematical approaches mentioned above (Deli and Karaaslan 2020; Ali et al. 2017; Khan et al. 2019; Jana and Pal 2018). Besides these, uncertainty problems can be quite different from each other, as well as the decision-making processes encountered. Karaaslan (2016), who developed a decision-making algorithm for uncertainty problems focused on the selection of decision-makers, proposed the concept of s-rough classes. It is important to be able to process all the data expressed by the decision-makers to remove the uncertainty correctly. Therefore, it is more advantageous to use hybrid set types. In this paper, bs-rough sets, which is a hybrid set type that has been widely used in data processing recently, are examined. Thus, s-rough classes are generalized to bs-rough classes and some of their associated properties are analyzed. The most important advantages of these classes for decision-making problems encountered in uncertain environments can be given as follows:

- It allows data expressed by different decision-makers to be processed together.
- Taking into account the NOT parameters of each parameter, it determines the selection between objects better than s-rough classes.
- It is used as a tool to determine how effectively the current uncertainty can be expressed by decision-makers.

The paper is structured as follows: In Sect. 2, we recall some basic notions in s-set, bs-set and bs-S-rough set. Next, Sect. 3 is built to analyze the bs-classes and some required properties needed to define bs-rough classes. In Sect. 4, bipolar soft rough classes are defined and some of their associated properties are analyzed. Also, basic set operations such as subset, complement, intersection, union are examined. In Sect. 5, we use bs-rough classes to manage the decision-making process for uncertainty problems. For this, we build a decision-making algorithm based on bs-rough classes, then we illustrate how this algorithm can be applied to an uncertainty problem. Finally, we conclude the study in Sect. 6.

2 Preliminaries

In this section, we recall some basic notions in s-set, bs-set and bs-S-rough set.

Throughout this paper, let $U = \{u_1, u_2, ..., u_n\}$ be an initial universe, 2^U denotes the power set of U, $P = \{p_1, p_2, ..., p_m\}$ be the universe of all possible parameters related to the objects in U and K, L, M be non-empty subsets of P. Also, let $D = \{d_1, d_2, ..., d_r\}$ be a set of decision-makers.

Definition 2.1 (Molodtsov 1999) A pair $\widetilde{\Phi_K}$ is called a s-set over U, where Φ_K is a mapping given by $\Phi_K : K \to 2^U$. It can be written as a set of ordered pairs:

$$\widetilde{\Phi_K} = \{(p, \Phi_K(p)) : p \in K\}. \tag{2.1}$$

Definition 2.2 (Maji et al. 2003) The NOT set of P denoted by $\neg P$ is defined by $\neg P = \{\neg p_1, \neg p_2, ..., \neg p_n\}$ where, $\neg p_i = not\ p_i$; $\forall i$. Moreover, $\neg(\neg K) = K$ and $\neg(K \circ L) = \neg K \circ \neg L$ for $\circ \in \{\cap, \cup\}$.

Definition 2.3 (Shabir and Naz 2013) A $\widehat{\Phi_K}$ is called a bs-set over U, where Φ_K and ϕ_K are mappings, given by $\Phi_K : K \to 2^U$ and $\phi_K : \neg K \to 2^U$ such that $\Phi_K(p) \cap \phi_K(\neg p) = \emptyset$; $\forall p \in K$. A bs-set is expressed as a set of ordered triples:

$$\widehat{\Phi_K} = \left\{(p, \Phi_K(p), \phi_K(\neg p)) : p \in K, \neg p \in \neg K; \Phi_K(p), \phi_K(\neg p) \in 2^U\right\}. \tag{2.2}$$

State that the set of all bs-sets over U will be denoted by $\mho(U)$.

Example 2.4 Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the set of hybrid cars available in a gallery and $P = \{p_1 : comfortable, p_2 : cheap, p_3 : economic\}$ be the set of parameters specifying the characteristics of the cars in this gallery. Then, $\neg P = \{\neg p_1 : comfortless, \neg p_2 : expensive, \neg p_3 : not\ economic\}$. Thus, the following bs-set is described how Mr. Q wants to buy a hybrid car:

$$\widehat{\Phi_K} = \{(p_1, \{u_1, u_4, u_5\}, \{u_3\}), (p_3, \{u_2, u_3\}, \{u_1, u_4\})\}.$$

Definition 2.5 (Shabir and Naz 2013) A bs-set over U is said to be

(i) a relative null bs-set, denoted by $\widehat{\Phi_{K\emptyset}}$ if $\Phi_K(p) = \emptyset$; $\forall p \in K$ and $\phi_K(\neg p) = U$; $\forall \neg p \in \neg K$.
(ii) a relative absolute bs-set, denoted by $\widehat{\Phi_{KU}}$ if $\Phi_K(p) = U$; $\forall p \in K$ and $\phi_K(\neg p) = \emptyset$; $\forall \neg p \in \neg K$.

Definition 2.6 (Shabir and Naz 2013) A bs-set $\widehat{\Phi_K}$ is said to be a bs-subset of a bs-set $\widehat{\Psi_L}$, denoted by $\widehat{\Phi_K} \widetilde{\subseteq} \widehat{\Psi_L}$, provided that

(i) $K \subseteq L$ and
(ii) $\Phi_K(p) \subseteq \Psi_L(p)$ and $\phi_K(\neg p) \subseteq \psi_K(\neg p)$; $\forall p \in K, \neg p \in \neg P$.

The bs-sets $\widehat{\Phi_K}$ and $\widehat{\Psi_L}$ are said to be bs-equal if $\widehat{\Phi_K} \widetilde{\subseteq} \widehat{\Psi_L}$ and $\widehat{\Psi_L} \widetilde{\subseteq} \widehat{\Phi_K}$.

Definition 2.7 (Shabir and Naz 2013) The relative complement of a bs-set $\widehat{\Phi_K}$ is a bs-set $\widehat{\Phi_K}^c$, where $\Phi_K^c : K \to 2^U$ and $\phi_K^c : \neg K \to 2^U$ are defined as $\Phi_K^c(p) = \phi_K(\neg p)$ and $\phi_K^c(\neg p) = \Phi_K(p)$; $\forall p \in P$, $\forall \neg p \in \neg P$.

Definition 2.8 (Shabir and Naz 2013) Let $\widehat{\Phi_K}, \widehat{\Psi_L}, \widehat{\Upsilon_M} \in \mho(U)$. Then,

(i) the union of $\widehat{\Phi_K}$ and $\widehat{\Psi_L}$ is $\widehat{\Upsilon_M} = \widehat{\Phi_K} \widetilde{\cup} \widehat{\Psi_L}$ where $M = K \cup L$ and the two mappings $\Upsilon_M : M \to 2^U$ and $\upsilon_M : \neg M \to 2^U$ are given by

$$\Upsilon_M(p) = \begin{cases} \Phi_K(p) : & p \in K - L, \\ \Psi_L(p) : & p \in L - K, \\ \Phi_K(p) \cup \Psi_L(p) : & p \in K \cap L, \end{cases}$$

$$\upsilon_M(\neg p) = \begin{cases} \phi_K(\neg p) : & \neg p \in K - L, \\ \psi_L(\neg p) : & \neg p \in L - K, \\ \phi_K(\neg p) \cap \psi_L(\neg p) : & \neg p \in K \cap L. \end{cases}$$

(ii) the intersection of $\widehat{\Phi_K}$ and $\widehat{\Psi_L}$ is $\widehat{\Upsilon_M} = \widehat{\Phi_K} \widetilde{\cap} \widehat{\Psi_L}$ such that $M = K \cap L \neq \emptyset$ and the two mappings $\Upsilon_M : M \to 2^U$ and $\upsilon_M : \neg M \to 2^U$ are given by $\Upsilon_M(p) = \Phi_K(p) \cap \Psi_L(p)$ and $\upsilon_M(\neg p) = \phi_K(\neg p) \cup \psi_L(\neg p)$.

Proposition 2.9 (Karaaslan and Çağman 2018) *Each bs-set is an information system.*

Definition 2.10 (Karaaslan and Çağman 2018) *Let* $\widehat{\Phi_K} \in \mho(U)$. Then, the pair $S = (U, \widehat{\Phi_K})$ is called bs-approximation (bsa-)space. For $X \subseteq U$;

$$\underline{\Delta}_{S^+}(X) = \{u \in U : \exists p \in K, [u \in \Phi_K(p) \subseteq X]\}, \quad (2.3)$$
$$\underline{\Delta}_{S^-}(X) = \{u \in U : \exists \neg p \in \neg K, [u \in \phi_K(\neg p), \phi_K(\neg p) \cap (U - X) \neq \emptyset]\}, \quad (2.4)$$
$$\overline{\Delta}_{S^+}(X) = \{u \in U : \exists p \in K, [u \in \Phi_K(p), \Phi_K(p) \cap X \neq \emptyset]\}, \quad (2.5)$$
$$\overline{\Delta}_{S^-}(X) = \{u \in U : \exists \neg p \in \neg K, [u \in \phi_K(\neg p) \subseteq U - X]\} \quad (2.6)$$

are called S-lower positive bsa, S-lower negative bsa, S-upper positive bsa and S-upper negative bsa of X, respectively. In addition, $\underline{\Theta}_S(X) = \left(\underline{\Delta}_{S^+}(X), \underline{\Delta}_{S^-}(X)\right)$ and $\overline{\Theta}_S(X) =$

Table 1 The tabular representation of the bs-set $\widehat{\Phi_K}$

$\widehat{\Phi_K}$	$(p_2, \neg p_2)$	$(p_3, \neg p_3)$	$(p_5, \neg p_5)$	$(p_7, \neg p_7)$
u_1	(1, 0)	(1, 0)	(0, 1)	(1, 0)
u_2	(0, 1)	(1, 0)	(0, 0)	(1, 0)
u_3	(0, 0)	(0, 0)	(0, 0)	(0, 0)
u_4	(1, 0)	(0, 1)	(1, 0)	(1, 0)
u_5	(0, 1)	(0, 1)	(0, 1)	(0, 1)

$\left(\overline{\Delta}_{S^+}(X), \overline{\Delta}_{S^-}(X)\right)$ are called bs-rough approximations of X. Moreover,

$$BP_S(X) = \left(\underline{\Delta}_{S^+}(X), \overline{\Delta}_{S^-}(X)\right), \tag{2.7}$$

$$BN_S(X) = \left(U - \overline{\Delta}_{S^+}(X), U - \underline{\Delta}_{S^-}(X)\right), \tag{2.8}$$

$$BB_S(X) = \left(\overline{\Delta}_{S^+}(X) - \underline{\Delta}_{S^+}(X), \underline{\Delta}_{S^-}(X) - \overline{\Delta}_{S^-}(X)\right) \tag{2.9}$$

are called bs-S-positive region, bs-S-negative region and bs-S-boundary region of X, respectively. If $\underline{\Theta}_S(X) = \overline{\Theta}_S(X)$, X is said to be bs-S-definable set; otherwise X is called a bs-S-rough set.

Example 2.11 Let $U = \{u_1, u_2, u_3, u_4, u_5\}$, $P = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ and $K = \{p_2, p_3, p_5, p_7\} \subseteq P$. Also, let $\widehat{\Phi_K}$ be a bs-set over U given by Table 1 and $S = (U, \widehat{\Phi_K})$ be the bsa-space.

For $X = \{u_1, u_3, u_4\} \subseteq U$, we have S-lower positive bsa $\underline{\Delta}_{S^+} = \{u_1, u_4\}$ and S-lower negative bsa $\underline{\Delta}_{S^-} = \{u_1, u_2, u_4, u_5\}$, i.e., $\underline{\Theta}_S(X) = (\{u_1, u_4\}, \{u_1, u_2, u_4, u_5\})$. Moreover we have S-upper positive bsa $\overline{\Delta}_{S^+} = \{u_1, u_2, u_4\}$ and S-upper negative bsa $\overline{\Delta}_{S^-} = \{u_2, u_5\}$, i.e., $\overline{\Theta}_S(X) = (\{u_1, u_2, u_4\}, \{u_2, u_5\})$. Since $\underline{\Theta}_S(X) \neq \overline{\Theta}_S(X)$, then X is a bs-S-rough set. Thus, it is easy to see that $BP_S(X) = (\{u_1, u_4\}, \{u_2, u_5\})$, $BN_S(X) = (\{u_3, u_5\}, \{u_3\})$, $BB_S(X) = (\{u_2\}, \{u_1, u_4\})$. If $Y = \{u_3\} \subseteq U$, then

$$\underline{\Theta}_S(Y) = (\emptyset, \{u_1, u_2, u_4, u_5\}) = \overline{\Theta}_S(Y),$$

i.e., Y is a bs-S-definable set.

3 Preparation for bipolar soft rough classes

This section was built to analyze the bs-classes and some required properties needed to define bs-rough classes.

Definition 3.1 Indexed class of bs-sets

$$\left\{\widehat{\Phi_{K\,d_i}} : \Phi_{K\,d_i} : K \to 2^U, \phi_{K\,d_i} : \neg K \to 2^U; 1 \leq i \leq r\right\} \tag{3.1}$$

is called a bs-class and is denoted by $\widehat{\Phi_{K\,D}}$. Here, the bs-set $\widehat{\Phi_{K\,d_i}}$ does not appear in bs-class $\widehat{\Phi_{K\,D}}$ for any $d_i \in D$, $\widehat{\Phi_{K\,d_i}} = \widehat{\Phi_{K\,\emptyset}}$.

State that the all bs-classes over U, P, D will be denoted by $\mathbb{BSC}_D^P(U)$.

Table 2 The tabular representation of $\widehat{\Phi_{KD}}$

$\widehat{\Phi_{KD}}$	$(p_1, \neg p_1)$	$(p_3, \neg p_3)$	$(p_4, \neg p_4)$
$\widehat{\Phi_{Kd_1}}$	$(\{u_2, u_4, u_6\}, \{u_1, u_5\})$	$(\{u_1, u_3, u_5\}, \{u_2\})$	$(\{u_1, u_4, u_5, u_6\}, \{u_3\})$
$\widehat{\Phi_{Kd_2}}$	$(\{u_1, u_5\}, \{u_2, u_4, u_6\})$	$(\{u_3, u_4\}, \{u_1, u_5\})$	$(\{u_2, u_6\}, \{u_1, u_3\})$
$\widehat{\Phi_{Kd_3}}$	$(U, \{\})$	$(\{u_2, u_3, u_5, u_6\}, \{u_1, u_4\})$	$(\{u_1, u_3, u_5\}, \{u_4, u_6\})$
$\widehat{\Phi_{Kd_4}}$	$(\{\}, \{u_2, u_4, u_5\})$	$(\{u_3, u_5\}, \{u_1, u_6\})$	$(\{u_3, u_6\}, \{u_2, u_4\})$

Example 3.2 Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $P = \{p_1, p_2, p_3, p_4\}$, $D = \{d_1, d_2, d_3, d_4\}$. For $K = \{p_1, p_3, p_4\} \subseteq P$, if we consider bs-sets $\widehat{\Phi_{Kd_1}}, \widehat{\Phi_{Kd_2}}, \widehat{\Phi_{Kd_3}}, \widehat{\Phi_{Kd_4}}$ given as

$$\widehat{\Phi_{Kd_1}} = \{(p_1, \{u_2, u_4, u_6\}, \{u_1, u_5\}), (p_3, \{u_1, u_3, u_5\}, \{u_2\}), (p_4, \{u_1, u_4, u_5, u_6\}, \{u_3\})\},$$
$$\widehat{\Phi_{Kd_2}} = \{(p_1, \{u_1, u_5\}, \{u_2, u_4, u_6\}), (p_3, \{u_3, u_4\}, \{u_1, u_5\}), (p_4, \{u_2, u_6\}, \{u_1, u_3\})\},$$
$$\widehat{\Phi_{Kd_3}} = \{(p_1, U, \{\}), (p_3, \{u_2, u_3, u_5, u_6\}, \{u_1, u_4\}), (p_4, \{u_1, u_3, u_5\}, \{u_4, u_6\})\},$$
$$\widehat{\Phi_{Kd_4}} = \{(p_1, \{\}, \{u_2, u_4, u_5\}), (p_3, \{u_3, u_5\}, \{u_1, u_6\}), (p_4, \{u_3, u_6\}, \{u_2, u_4\})\},$$

then $\widehat{\Phi_{KD}} = \left\{\widehat{\Phi_{Kd_1}}, \widehat{\Phi_{Kd_2}}, \widehat{\Phi_{Kd_3}}, \widehat{\Phi_{Kd_4}}\right\}$ is a bs-class. Moreover, we can represent a bs-class in tabular form as shown in Table 2.

Definition 3.3 Let $\widehat{\Phi_{KD}} \in \mathbb{BSC}_D^P(U)$. Then,

(i) if $\widehat{\Phi_{Kd_i}} = \widehat{\Phi_{K\emptyset}}$; $\forall d_i \in D$, then $\widehat{\Phi_{KD}}$ is called an empty bs-class and is denoted by $\widehat{\emptyset}_{BS}$.

(ii) if $\widehat{\Phi_{Kd_i}} = \widehat{\Phi_{KU}}$; $\forall d_i \in D$, then $\widehat{\Phi_{KD}}$ is called an universal bs-class and is denoted by $\widehat{U}_{BS}$.

Definition 3.4 Let $\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}} \in \mathbb{BSC}_D^P(U)$. Then, $\widehat{\Phi_{KD}}$ is a bs-subclass of $\widehat{\Psi_{LD}}$, denoted by $\widehat{\Phi_{KD}} \widehat{\subseteq} \widehat{\Psi_{LD}}$, if $\widehat{\Phi_{Kd_i}} \widetilde{\subseteq} \widehat{\Psi_{Ld_i}}$; $\forall d_i \in D$.

The bs-classes $\widehat{\Phi_{KD}}$ and $\widehat{\Psi_{LD}}$ are said to be equal bs-classes if and only if $\widehat{\Phi_{KD}} \widehat{\subseteq} \widehat{\Psi_{LD}}$ and $\widehat{\Psi_{LD}} \widehat{\subseteq} \widehat{\Phi_{KD}}$. This relation is denoted by $\widehat{\Phi_{KD}} = \widehat{\Psi_{LD}}$.

Proposition 3.5 *Let* $\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}}, \widehat{\Upsilon_{MD}} \in \mathbb{BSC}_D^P(U)$. *Then;*

(i) $\widehat{\emptyset}_{BS} \widehat{\subseteq} \widehat{\Phi_{KD}} \widehat{\subseteq} \widehat{U}_{BS}$,

(ii) $\widehat{\Phi_{KD}} \widehat{\subseteq} \widehat{\Phi_{KD}}$,

(iii) $\widehat{\Phi_{KD}} \widehat{\subseteq} \widehat{\Psi_{LD}}$ *and* $\widehat{\Psi_{LD}} \widehat{\subseteq} \widehat{\Upsilon_{MD}} \Rightarrow \widehat{\Phi_{KD}} \widehat{\subseteq} \widehat{\Upsilon_{MD}}$.

Proof For all $d_i \in D$;

(i) $\widehat{\Phi_{K\emptyset}} \widetilde{\subseteq} \widehat{\Phi_{Kd_i}} \Rightarrow \widehat{\emptyset}_{BS} \widehat{\subseteq} \widehat{\Phi_{KD}}$ and $\widehat{\Phi_{Kd_i}} \widetilde{\subseteq} \widehat{\Phi_{KU}} \Rightarrow \widehat{\Phi_{KD}} \widehat{\subseteq} \widehat{U}_{BS}$,

(ii) $\widehat{\Phi_{Kd_i}} \widetilde{\subseteq} \widehat{\Phi_{Kd_i}} \Rightarrow \widehat{\Phi_{KD}} \widehat{\subseteq} \widehat{\Phi_{KD}}$,

(iii) if $\widehat{\Phi_{Kd_i}} \widetilde{\subseteq} \widehat{\Psi_{Ld_i}}$ and $\widehat{\Psi_{Ld_i}} \widetilde{\subseteq} \widehat{\Upsilon_{Md_i}} \Rightarrow \widehat{\Phi_{Kd_i}} \widetilde{\subseteq} \widehat{\Upsilon_{Md_i}}$, then $\widehat{\Phi_{KD}} \widehat{\subseteq} \widehat{\Psi_{LD}}$ and $\widehat{\Psi_{LD}} \widehat{\subseteq} \widehat{\Upsilon_{MD}} \Rightarrow \widehat{\Phi_{KD}} \widehat{\subseteq} \widehat{\Upsilon_{MD}}$.

□

Definition 3.6 Let $\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}} \in \mathbb{BSC}_D^P(U)$, $\widehat{\Phi_{KD}} \neq \emptyset$ and $D^1, D^2 \subseteq D$ such that $D^1 \cup D^2 = D$ and $D^1 \cap D^2 = \emptyset$. If $\widehat{\Phi_{Kd_i}} \widetilde{\subseteq} \widehat{\Psi_{Ld_i}}$, $\forall d_i \in D^1$ and $\widehat{\Phi_{Kd_i}} \widetilde{\nsubseteq} \widehat{\Psi_{Ld_i}}$, $\forall d_i \in D^2$; then $\widehat{\Phi_{KD}}$ is called almost-subclass of $\widehat{\Psi_{LD}}$ and is denoted by $\widehat{\Phi_{KD}} \widehat{\subseteq}_A \widehat{\Psi_{LD}}$.

Definition 3.7 Let $\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}} \in \mathbb{BSC}_D^P(U)$ and $\widehat{\Phi_{KD}} \widehat{\subseteq}_A \widehat{\Psi_{LD}}$. Then, according to $\widehat{\Psi_{LD}}$, subclasshood degree of $\widehat{\Phi_{KD}}$, denoted by $\Omega(\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}})$, is defined as follows:

$$\Omega\left(\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}}\right) = \frac{\left|D^1\right|}{|D||K \cup L|} \sum_{d_i \in D_1} \sum_{p \in K \cup L} \chi_{\left[\widehat{\Phi_{Kd_i}}, \widehat{\Psi_{Ld_i}}\right]}(p) \tag{3.2}$$

such that

$$\chi_{\left[\widehat{\Phi_{Kd_i}}, \widehat{\Psi_{Ld_i}}\right]}(p) = \begin{cases} \left|\Phi_{Kd_i}(p)\right| - \left|\phi_{Kd_i}(\neg p)\right| : & p \in K - L, \\ 1/\left[\left|\Psi_{Ld_i}(p)\right| - \left|\psi_{Ld_i}(\neg p)\right|\right] : & p \in L - K, \\ \left[\left|\Phi_{Kd_i}(p)\right| - \left|\phi_{Kd_i}(\neg p)\right|\right] / \left[\left|\Psi_{Ld_i}(p)\right| - \left|\psi_{Ld_i}(\neg p)\right|\right] : & p \in K \cap L, \end{cases} \tag{3.3}$$

Here, $\widehat{\Phi_{Kd_i}} \widetilde{\subseteq} \widehat{\Psi_{Ld_i}}$ and $|\Psi_{Ld_i}(p)| \neq |\psi_{Ld_i}(\neg p)|, \forall d_i \in D^1$.

Example 3.8 Consider Example 3.2 and bs-class $\widehat{\Psi_{LD}}$ given as follows:

$$\widehat{\Psi_{Ld_1}} = \left\{ \begin{array}{c} (p_1, \{u_2, u_3, u_4, u_6\}, \{u_1, u_5\}), (p_2, \{u_1, u_3, u_5\}, \{u_6\}), \\ (p_3, \{u_1, u_3, u_4, u_5\}, \{u_2, u_6\}), (p_4, \{u_1, u_4, u_5, u_6\}, \{u_2, u_3\}) \end{array} \right\},$$

$$\widehat{\Psi_{Ld_2}} = \left\{ \begin{array}{c} (p_1, \{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}), (p_2, \{u_1, u_4, u_6\}, \{u_2, u_3, u_5\}), \\ (p_3, \{u_3, u_5\}, \{u_1, u_2, u_6\}), (p_4, \{u_1, u_5, u_6\}, \{u_2, u_4\}) \end{array} \right\},$$

$$\widehat{\Psi_{Ld_3}} = \left\{ \begin{array}{c} (p_1, U, \{\}), (p_2, \{u_1, u_3, u_4, u_6\}, \{u_2, u_5\}), \\ (p_3, \{u_2, u_3, u_5, u_6\}, \{u_1, u_4\}), (p_4, \{u_1, u_2, u_3, u_5\}, \{u_4, u_6\}) \end{array} \right\},$$

$$\widehat{\Psi_{Ld_4}} = \left\{ \begin{array}{c} (p_1, \{u_1\}, \{u_3, u_4, u_5\}), (p_2, \{u_1, u_4\}, \{u_2, u_3, u_6\}), \\ (p_3, \{u_2, u_4, u_5\}, \{u_3, u_6\}), (p_4, \{u_1, u_5\}, \{u_3, u_4, u_6\}) \end{array} \right\},$$

Since $\widehat{\Phi_{Kd_1}} \widetilde{\subseteq} \widehat{\Psi_{Ld_1}}$, $\widehat{\Phi_{Kd_2}} \widetilde{\nsubseteq} \widehat{\Psi_{Ld_1}}$, $\widehat{\Phi_{Kd_3}} \widetilde{\subseteq} \widehat{\Psi_{Ld_3}}$, $\widehat{\Phi_{Kd_4}} \widetilde{\nsubseteq} \widehat{\Psi_{Ld_4}}$; then $|D^1| = |\{d_1, d_3\}| = 2$ and thus,

$$\begin{aligned} \Omega\left(\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}}\right) &= \frac{2}{4 \cdot 4} \sum_{d_i \in D_1} \sum_{p \in P} \chi_{\left[\widehat{\Phi_{Kd_i}}, \widehat{\Psi_{Ld_i}}\right]}(p) \\ &= \frac{2}{4 \cdot 4}\left[\left(\frac{1}{2} + \frac{1}{2} + \frac{2}{2} + \frac{3}{2}\right) + \left(\frac{6}{6} + \frac{1}{2} + \frac{2}{2} + \frac{1}{2}\right)\right] \\ &= 0.8125 \end{aligned}$$

and $\widehat{\Phi_{KD}} \widehat{\subseteq}_A \widehat{\Psi_{LD}}$.

Remark 3.9 Let $\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}} \in \mathbb{BSC}_D^P(U)$.If $\widehat{\Phi_{Kd_i}} = \widehat{\Psi_{Ld_i}}, \forall d_i \in D$; then $\Omega(\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}}) = 1$. Moreover, if $\widehat{\Phi_{KD}} \widehat{\subseteq} \widehat{\Psi_{LD}}$, then $\widehat{\Phi_{KD}}$ may be almost-subclass of $\widehat{\Psi_{LD}}$ and if $\widehat{\Phi_{KD}} \widehat{\subseteq}_A \widehat{\Psi_{LD}}$, then $\widehat{\Phi_{KD}}$ may not be a subclass of $\widehat{\Psi_{LD}}$.

Definition 3.10 Let $\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}} \in \mathbb{BSC}_D^P(U)$. Then,

(i) the union of $\widehat{\Phi_{KD}}$ and $\widehat{\Psi_{LD}}$, denoted by $\widehat{\Phi_{KD}} \widehat{\cup} \widehat{\Psi_{LD}}$, is defined as

$$\widehat{\Phi_{KD}} \widehat{\cup} \widehat{\Psi_{LD}} = \left\{\widehat{\Phi_{Kd_i}} \widetilde{\cup} \widehat{\Psi_{Ld_i}} : d_i \in D\right\} \tag{3.4}$$

(ii) the intersection of $\widehat{\Phi_{KD}}$ and $\widehat{\Psi_{LD}}$, denoted by $\widehat{\Phi_{KD}} \widehat{\cap} \widehat{\Psi_{LD}}$, is defined as

$$\widehat{\Phi_{KD}} \widehat{\cap} \widehat{\Psi_{LD}} = \left\{\widehat{\Phi_{Kd_i}} \widetilde{\cap} \widehat{\Psi_{Ld_i}} : d_i \in D\right\} \tag{3.5}$$

Table 3 The tabular representation of $\widehat{\Phi_{K}}_{D}$

$\widehat{\Phi_{K}}_{D}$	$(p_1, \neg p_1)$	$(p_3, \neg p_3)$
$\widehat{\Phi_{K}}_{d_1}$	$(\{u_1, u_3, u_5\}, \{u_2, u_7\})$	$(\{u_4, u_6\}, \{u_2, u_7\})$
$\widehat{\Phi_{K}}_{d_2}$	$(\{u_2, u_4\}, \{u_1, u_3, u_7\})$	$(\{u_4, u_6\}, \{u_1, u_7\})$
$\widehat{\Phi_{K}}_{d_3}$	$(\{u_1\}, \{u_4, u_7\})$	$(\{u_3, u_5, u_6\}, \{u_2, u_4\})$
$\widehat{\Phi_{K}}_{d_4}$	$(\{u_7\}, \{u_4, u_6\})$	$(\{u_2, u_6\}, \{u_3, u_7\})$

Table 4 The tabular representation of $\widehat{\Psi_{L}}_{D}$

$\widehat{\Psi_{L}}_{D}$	$(p_2, \neg p_2)$	$(p_3, \neg p_3)$
$\widehat{\Psi_{L}}_{d_1}$	$(\{u_2, u_4, u_5\}, \{u_1, u_6\})$	$(\{u_2, u_5, u_6\}, \{u_1, u_6, u_7\})$
$\widehat{\Psi_{L}}_{d_2}$	$(\{u_1, u_4\}, \{u_3, u_6\})$	$(\{u_2, u_6, u_7\}, \{u_5\})$
$\widehat{\Psi_{L}}_{d_3}$	$(\{u_1, u_4, u_7\}, \{u_5\})$	$(\{u_5, u_6\}, \{u_3, u_4\})$
$\widehat{\Psi_{L}}_{d_4}$	$(\{u_4, u_7\}, \{u_1, u_6\})$	$(\{u_6\}, \{u_3, u_5\})$

Table 5 The tabular representation of $\widehat{\Phi_{K}}_{D} \widehat{\cap} \widehat{\Psi_{L}}_{D}$

$\widehat{\Phi_{K}}_{D} \widehat{\cap} \widehat{\Psi_{L}}_{D}$	$(p_3, \neg p_3)$
$\widehat{\Phi_{K}}_{d_1} \widetilde{\cap} \widehat{\Psi_{L}}_{d_1}$	$(\{u_6\}, \{u_1, u_2, u_6, u_7\})$
$\widehat{\Phi_{K}}_{d_2} \widetilde{\cap} \widehat{\Psi_{L}}_{d_2}$	$(\{u_6\}, \{u_1, u_{5,7}\})$
$\widehat{\Phi_{K}}_{d_3} \widetilde{\cap} \widehat{\Psi_{L}}_{d_3}$	$(\{u_5, u_6\}, \{u_2, u_3, u_4\})$
$\widehat{\Phi_{K}}_{d_4} \widetilde{\cap} \widehat{\Psi_{L}}_{d_4}$	$(\{u_6\}, \{u_3, u_{5,7}\})$

Table 6 The tabular representation of $\widehat{\Phi_{K}}_{D} \widehat{\cup} \widehat{\Psi_{L}}_{D}$

$\widehat{\Phi_{K}}_{D} \widehat{\cup} \widehat{\Psi_{L}}_{D}$	$(p_1, \neg p_1)$	$(p_2, \neg p_2)$	$(p_3, \neg p_3)$
$\widehat{\Phi_{K}}_{d_1} \widetilde{\cup} \widehat{\Psi_{L}}_{d_1}$	$(\{u_1, u_3, u_4\}, \{u_2, u_7\})$	$(\{u_2, u_4, u_5\}, \{u_1, u_6\})$	$(\{u_2, u_4, u_5, u_6\}, \{u_7\})$
$\widehat{\Phi_{K}}_{d_2} \widetilde{\cup} \widehat{\Psi_{L}}_{d_2}$	$(\{u_2, u_4\}, \{u_1, u_{3,7}\})$	$(\{u_1, u_4\}, \{u_3, u_6\})$	$(\{u_2, u_4, u_6, u_7\}, \{\})$
$\widehat{\Phi_{K}}_{d_3} \widetilde{\cup} \widehat{\Psi_{L}}_{d_3}$	$(\{u_1\}, \{u_4, u_7\})$	$(\{u_1, u_4, u_7\}, \{u_5\})$	$(\{u_3, u_5, u_6\}, \{u_4\})$
$\widehat{\Phi_{K}}_{d_4} \widetilde{\cup} \widehat{\Psi_{L}}_{d_4}$	$(\{u_7\}, \{u_4, u_6\})$	$(\{u_4, u_7\}, \{u_1, u_6\})$	$(\{u_2, u_6\}, \{u_3\})$

Example 3.11 Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$, $P = \{p_1, p_2, p_3, p_4\}$, $D = \{d_1, d_2, d_3, d_4\}$. For $K = \{p_1, p_3\} \subseteq P$ and $L = \{p_2, p_3\} \subseteq P$, if

and

then $\widehat{\Phi_{K}}_{D}$ and $\widehat{\Psi_{L}}_{D}$ are bs-classes. Thus,

and

Definition 3.12 Let $\widehat{\Phi_{K}}_{D} \in \mathbb{BSC}_{D}^{P}(U)$. Then, the bs-complement of $\widehat{\Phi_{K}}_{D}$, denoted by $\widehat{\Phi_{K}}_{D}^{c}$, is defined as

$$\widehat{\Phi_{K}}_{D}^{c} = \left\{ \widehat{\Phi_{K}}_{d_i}^{c} : d_i \in D \right\}. \tag{3.6}$$

Obviously, $\left(\widehat{\Phi_{K}}_{D}^{c}\right)^{c} = \widehat{\Phi_{K}}_{D}$ and $\widehat{\emptyset}_{BS}^{c} = \widehat{U}_{BS}$.

Proposition 3.13 *Let* $\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}}, \widehat{\Upsilon_{MD}} \in \mathbb{BSC}^P_D(U)$*. Then, for* $\star \in \{\widehat{\cap}, \widehat{\cup}\}$,

(i) $\widehat{\Phi_{KD}} \star \widehat{\Phi_{KD}} = \widehat{\Phi_{KD}}$.
(ii) $\widehat{\Phi_{KD}} \widehat{\cup} \widehat{\emptyset}_{BS} = \widehat{\Phi_{KD}}$ *and* $\widehat{\Phi_{KD}} \widehat{\cap} \widehat{\emptyset}_{BS} = \widehat{\emptyset}_{BS}$.
(iii) $\widehat{\Phi_{KD}} \widehat{\cup} \widehat{U}_{BS} = \widehat{U}_{BS}$ *and* $\widehat{\Phi_{KD}} \widehat{\cap} \widehat{U}_{BS} = \widehat{\Phi_{KD}}$.
(iv) $\widehat{\Phi_{KD}} \widehat{\cup} \widehat{\Phi_{KD}}^c = \widehat{U}_{BS}$ *and* $\widehat{\Phi_{KD}} \widehat{\cap} \widehat{\Phi_{KD}}^c = \widehat{\emptyset}_{BS}$.
(v) $\widehat{\Phi_{KD}} \star \widehat{\Psi_{LD}} = \widehat{\Psi_{LD}} \star \widehat{\Phi_{KD}}$.
(vi) $\left(\widehat{\Phi_{KD}} \star \widehat{\Psi_{LD}}\right) \star \widehat{\Upsilon_{MD}} = \widehat{\Phi_{KD}} \star \left(\widehat{\Psi_{LD}} \star \widehat{\Upsilon_{MD}}\right)$.

Proof For all $d_i \in D$,

(i) Since $\widehat{\Phi_{Kd_i}} \widetilde{\cup} \widehat{\Phi_{Kd_i}} = \widehat{\Phi_{Kd_i}}$ and $\widehat{\Phi_{Kd_i}} \widetilde{\cap} \widehat{\Phi_{Kd_i}} = \widehat{\Phi_{Kd_i}}$, then $\widehat{\Phi_{KD}} \widehat{\cup} \widehat{\Phi_{KD}} = \widehat{\Phi_{KD}}$ and $\widehat{\Phi_{KD}} \widehat{\cap} \widehat{\Phi_{KD}} = \widehat{\Phi_{KD}}$, respectively.
(ii) Since $\widehat{\Phi_{Kd_i}} \widetilde{\cup} \widehat{\Phi_{K\emptyset}} = \widehat{\Phi_{Kd_i}}$ and $\widehat{\Phi_{Kd_i}} \widetilde{\cap} \widehat{\Phi_{K\emptyset}} = \widehat{\Phi_{K\emptyset}}$, then $\widehat{\Phi_{KD}} \widehat{\cup} \widehat{\emptyset}_{BS} = \widehat{\Phi_{KD}}$ and $\widehat{\Phi_{KD}} \widehat{\cap} \widehat{\emptyset}_{BS} = \widehat{\emptyset}_{BS}$, respectively.
(iii) Since $\widehat{\Phi_{Kd_i}} \widetilde{\cup} \widehat{\Phi_{KU}} = \widehat{\Phi_{KU}}$ and $\widehat{\Phi_{Kd_i}} \widetilde{\cap} \widehat{\Phi_{KU}} = \widehat{\Phi_{Kd_i}}$, then $\widehat{\Phi_{KD}} \widehat{\cup} \widehat{U}_{BS} = \widehat{U}_{BS}$ and $\widehat{\Phi_{KD}} \widehat{\cap} \widehat{U}_{BS} = \widehat{\Phi_{KD}}$, respectively.
(iv) Since $\widehat{\Phi_{Kd_i}} \widetilde{\cup} \widehat{\Phi_{Kd_i}}^c = \widehat{\Phi_{KU}}$ and $\widehat{\Phi_{Kd_i}} \widetilde{\cap} \widehat{\Phi_{Kd_i}}^c = \widehat{\Phi_{K\emptyset}}$, then $\widehat{\Phi_{KD}} \widehat{\cup} \widehat{\Phi_{KD}}^c = \widehat{U}_{BS}$ and $\widehat{\Phi_{KD}} \widehat{\cap} \widehat{\Phi_{KD}}^c = \widehat{\emptyset}_{BS}$, respectively.
(v) Since $\widehat{\Phi_{Kd_i}} \widetilde{\cup} \widehat{\Psi_{Ld_i}} = \widehat{\Psi_{Ld_i}} \widetilde{\cup} \widehat{\Phi_{Kd_i}}$ and $\widehat{\Phi_{Kd_i}} \widetilde{\cap} \widehat{\Psi_{Ld_i}} = \widehat{\Psi_{Ld_i}} \widetilde{\cap} \widehat{\Phi_{Kd_i}}$, then $\widehat{\Phi_{KD}} \widehat{\cup} \widehat{\Psi_{LD}} = \widehat{\Psi_{LD}} \widehat{\cup} \widehat{\Phi_{KD}}$ and $\widehat{\Phi_{KD}} \widehat{\cap} \widehat{\Psi_{LD}} = \widehat{\Psi_{LD}} \widehat{\cap} \widehat{\Phi_{KD}}$, respectively.
(vi) Since

$$\left(\widehat{\Phi_{Kd_i}} \widetilde{\cup} \widehat{\Psi_{Ld_i}}\right) \widetilde{\cup} \widehat{\Upsilon_{Md_i}} = \widehat{\Phi_{Kd_i}} \widetilde{\cup} \left(\widehat{\Psi_{Ld_i}} \widetilde{\cup} \widehat{\Upsilon_{Md_i}}\right),$$
$$\left(\widehat{\Phi_{Kd_i}} \widetilde{\cap} \widehat{\Psi_{Ld_i}}\right) \widetilde{\cap} \widehat{\Upsilon_{Md_i}} = \widehat{\Phi_{Kd_i}} \widetilde{\cap} \left(\widehat{\Psi_{Ld_i}} \widetilde{\cap} \widehat{\Upsilon_{Md_i}}\right)$$

then

$$\left(\widehat{\Phi_{KD}} \widehat{\cup} \widehat{\Psi_{LD}}\right) \widehat{\cup} \widehat{\Upsilon_{MD}} = \widehat{\Phi_{KD}} \widehat{\cup} \left(\widehat{\Psi_{LD}} \widehat{\cup} \widehat{\Upsilon_{MD}}\right),$$
$$\left(\widehat{\Phi_{KD}} \widehat{\cap} \widehat{\Psi_{LD}}\right) \widehat{\cap} \widehat{\Upsilon_{MD}} = \widehat{\Phi_{KD}} \widehat{\cap} \left(\widehat{\Psi_{LD}} \widehat{\cap} \widehat{\Upsilon_{MD}}\right)$$

respectively.

□

Proposition 3.14 *Let* $\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}}, \widehat{\Upsilon_{MD}} \in \mathbb{BSC}^P_D(U)$*. Then, for* $\star, * \in \{\widehat{\cap}, \widehat{\cup}\}$ *and* $\star \neq *$,

(i) $\widehat{\Phi_{KD}} \star \left(\widehat{\Psi_{LD}} * \widehat{\Upsilon_{KD}}\right) = \left(\widehat{\Phi_{KD}} \star \widehat{\Psi_{LD}}\right) * \left(\widehat{\Phi_{KD}} \star \widehat{\Upsilon_{MD}}\right)$.
(ii) $\left(\widehat{\Phi_{KD}} \star \widehat{\Psi_{LD}}\right)^c = \widehat{\Phi_{KD}}^c * \widehat{\Psi_{LD}}^c$.

Proof Straightforward. □

Proposition 3.15 *Let* $\left[\widehat{\Phi^j_{K_j}}\right]_D \in \mathbb{BSC}^P_D(U)$ *for* $j = 1, 2, ...m$*. Then, for* $\star, * \in \{\widetilde{\cap}, \widetilde{\cup}\}$, $\circ, \bullet \in \{\widehat{\cap}, \widehat{\cup}\}$ *and* $\star \neq *$, $\circ \neq \bullet$

(i) $\left(\star_{j=1}^m \left[\widehat{\Phi^j_{K_j}}\right]_{d_i}\right)^c = *_{j=1}^m \left(\left[\widehat{\Phi^j_{K_j}}\right]_{d_i}\right)^c; \forall d_i \in D.$
(ii) $\left(\circ_{j=1}^m \left[\widehat{\Phi^j_{K_j}}\right]_D\right)^c = \bullet_{j=1}^m \left(\left[\widehat{\Phi^j_{K_j}}\right]_D\right)^c.$

Proof The proof is obvious. □

Definition 3.16 Let $\widehat{\Phi_{KD}} \in \mathbb{BSC}_D^P(U)$, $\widehat{\Psi_L} \in \mho(U)$. Then, $\widehat{\Phi_{KD}}$ is called bs-partition of $\widehat{\Psi_{LD}}$ if and only if all of the following conditions hold:

(i) $\widehat{\Phi_{K\emptyset}} \notin \widehat{\Phi_{KD}}$.
(ii) $\bigcup_{d_i \in D} \Phi_{Kd_i}(p) = \Psi_L(p); \forall p \in P$ and $\bigcup_{d_i \in D} \phi_{Kd_i}(\neg p) = \psi_L(\neg p); \forall \neg p \in \neg P$.
(iii) If $\widehat{\Phi_{Kd_i}}, \widehat{\Phi_{Kd_j}} \in \widehat{\Phi_{KD}}$ for $i \neq j$; then $\widehat{\Phi_{Kd_i}} \widehat{\cap} \widehat{\Phi_{Kd_j}} = \widehat{\Phi_{K\emptyset}}$.

Moreover, if $\Psi_L(p) \subseteq \bigcup_{d_i \in D} \Phi_{Kd_i}(p), \forall p \in P$ and $\psi_L(\neg p) \subseteq \bigcup_{d_i \in D} \phi_{Kd_i}(p), \forall \neg p \in \neg P$; then $\widehat{\Phi_{KD}}$ is called bs-cover of $\widehat{\Psi_L}$. Then, if $\bigcup_{d_i \in D} \left(\Phi_{Kd_i}(p), \phi_{Kd_i}(\neg p)\right) = \widehat{\Phi_{KU}}$, $\forall p \in P, \neg p \in \neg P, d_i \in D$; then $\widehat{\Phi_{KD}}$ is called full bs-class and is denoted by $\widehat{\Phi_{KD}}'$. Therefore, $\widehat{\Phi_{KD}}'$ is a bs-cover of $\widehat{\Psi_L}$; $\forall \widehat{\Psi_L} \in \mho(U)$.

Example 3.17 Consider Example 3.8. Then, $\widehat{\Psi_{LD}}$ is bs-cover of $\widehat{\Upsilon_M}$ given as follows:

$$\widehat{\Upsilon_M} = \left\{ \begin{array}{l} (p_1, \{u_1, u_3, u_6\}, \{u_2, u_4\}), (p_2, \{u_1, u_4, u_5\}, \{u_3, u_6\}), \\ (p_3, \{u_2, u_3, u_5\}, \{u_1, u_6\}), (p_4, \{u_4, u_6\}, \{u_2, u_3, u_4\}) \end{array} \right\}.$$

Proposition 3.18 *Let $\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}} \in \mathbb{BSC}_D^P(U)$ be two bs-covers of $\widehat{\Upsilon_M} \in \mho(U)$. Then, $\widehat{\Phi_{KD}} \star \widehat{\Psi_{LD}}$ is a bs-cover of $\widehat{\Upsilon_M}$ for $\star \in \{\widehat{\cap}, \widehat{\cup}\}$.*

Proof Since $\widehat{\Phi_{KD}}$, $\widehat{\Psi_{LD}}$ be two bs-covers of $\widehat{\Upsilon_M}$; then $\Upsilon_M(p) \subseteq \bigcup_{d_i \in D} \Phi_{Kd_i}(p)$, $\upsilon_M(\neg p) \subseteq \bigcup_{d_i \in D} \phi_{Kd_i}(\neg p)$, $\Upsilon_M(p) \subseteq \bigcup_{d_i \in D} \Psi_{Ld_i}(p)$, $\upsilon_M(\neg p) \subseteq \bigcup_{d_i \in D} \psi_{Ld_i}(\neg p)$ $\forall p \in P, \neg p \in \neg P$. Thus,

$$\Upsilon_M(p) \subseteq \left(\bigcup_{d_i \in D} \Phi_{Kd_i}(p) \right) * \left(\bigcup_{d_i \in D} \Psi_{Ld_i}(p) \right) = \bigcup_{d_i \in D} \left(\Phi_{Kd_i}(p) * \Psi_{Ld_i}(p) \right),$$

$$\upsilon_M(\neg p) \subseteq \left(\bigcup_{d_i \in D} \phi_{Kd_i}(\neg p) \right) * \left(\bigcup_{d_i \in D} \psi_{Ld_i}(\neg p) \right) = \bigcup_{d_i \in D} \left(\phi_{Kd_i}(\neg p) * \psi_{Ld_i}(\neg p) \right),$$

$\forall p \in P, \neg p \in \neg P$ and $* \in \{\cap, \cup\}$. Hence, $\widehat{\Phi_{KD}} \star \widehat{\Psi_{LD}}$ is a bs-cover of $\widehat{\Upsilon_M}$ for $\star \in \{\widehat{\cap}, \widehat{\cup}\}$. □

4 Bipolar soft rough classes

In this section, bipolar soft rough classes are defined and some of their associated properties are analyzed.

Definition 4.1 Let $\widehat{\Phi_{KD}} \in \mathbb{BSC}_D^P(U)$. Then, the parametrized classes of $\widehat{\Phi_{KD}}$ for $p \in P$ and $\neg p \in \neg P$, denoted by $\mathfrak{E}_{\widehat{\Phi_{KD}}}(p)$ and $\mathfrak{E}_{\widehat{\Phi_{KD}}}(\neg p)$, are defined as $\mathfrak{E}_{\widehat{\Phi_{KD}}}(p) = \left\{\Phi_{Kd_i}(p) : d_i \in D\right\}$ and $\mathfrak{E}_{\widehat{\Phi_{KD}}}(\neg p) = \left\{\phi_{Kd_i}(\neg p) : d_i \in D\right\}$, respectively.

Now, for $\widehat{\Psi_L} \in \mho(U)$ and $p \in P, \neg p \in \neg P$,

$$\underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p) = \left\{u \in U : \exists \Phi_{Kd_i}(p) \in \mathfrak{E}_{\widehat{\Phi_{KD}}}(p), \left[u \in \Phi_{Kd_i}(p) \subseteq \Psi_L(p)\right]\right\}, \quad (4.1)$$

$$\underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p) = \left\{u \in U : \begin{array}{c} \exists \phi_{Kd_i}(\neg p) \in \mathfrak{E}_{\widehat{\Phi_{KD}}}(\neg p), \\ \left[u \in \phi_{Kd_i}(\neg p), \phi_{Kd_i}(\neg p) \cap \psi_L(\neg p) \neq \emptyset\right] \end{array}\right\}, \quad (4.2)$$

$$\overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p) = \left\{u \in U : \begin{array}{c} \exists \Phi_{Kd_i}(p) \in \mathfrak{E}_{\widehat{\Phi_{KD}}}(p), \\ \left[u \in \Phi_{Kd_i}(p), \Phi_{Kd_i}(p) \cap \Psi_L(p) \neq \emptyset\right] \end{array}\right\}, \quad (4.3)$$

$$\overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p) = \left\{u \in U : \exists \phi_{Kd_i}(\neg p) \in \mathfrak{E}_{\widehat{\Phi_{KD}}}(\neg p), \left[u \in \phi_{Kd_i}(\neg p) \subseteq \psi_L(p)\right]\right\}. \quad (4.4)$$

are called p-lower bsa, p-lower NOT bsa, p-upper bsa and p-upper NOT bsa of $\widehat{\Psi_L}$, respectively. Then,

$$BP_{\widehat{\Phi_{KD}}}\widehat{\Psi_L}(p) = \left(\underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p), \overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p)\right), \quad (4.5)$$

$$BN_{\widehat{\Phi_{KD}}}\widehat{\Psi_L}(p) = \left(U - \overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p), U - \underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p)\right), \quad (4.6)$$

$$BB_{\widehat{\Phi_{KD}}}\widehat{\Psi_L}(p) = \left(\overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p) - \underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p), \underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p) - \overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p)\right) \quad (4.7)$$

are called bs-p-positive region, bs-p-negative region and bs-p-boundary region of $\widehat{\Psi_L}$, respectively. If

$$\overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p) - \underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p) = \underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p) - \overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p),$$

$\widehat{\Psi_L}$ is said to be bs-p-definable set; otherwise $\widehat{\Psi_L}$ is called a bs-p-rough set.

Proposition 4.2 *Let $\widehat{\Phi_{KD}} \in \mathbb{BSC}_D^P(U)$, $\widehat{\Psi_L} \in \mho(U)$. Then,*

$$\underline{\widehat{\Psi_L}}_{\left[\widehat{\Phi_{KD}}\right]}(p) = \bigcup_{d_i \in D} \left\{\Phi_{Kd_i}(p) : \Phi_{Kd_i}(p) \subseteq \Psi_L(p)\right\}, \quad (4.8)$$

$$\underline{\widehat{\Psi_L}}_{\left[\widehat{\Phi_{KD}}\right]}(\neg p) = \bigcup_{d_i \in D} \left\{\phi_{Kd_i}(\neg p) : \phi_{Kd_i}(\neg p) \cap \psi_L(\neg p) \neq \emptyset\right\}, \quad (4.9)$$

$$\overline{\widehat{\Psi_L}}_{\left[\widehat{\Phi_{KD}}\right]}(p) = \bigcup_{d_i \in D} \left\{\Phi_{Kd_i}(p) : \Phi_{Kd_i}(p) \cap \Psi_L(p) \neq \emptyset\right\}, \quad (4.10)$$

$$\overline{\widehat{\Psi_L}}_{\left[\widehat{\Phi_{KD}}\right]}(\neg p) = \bigcup_{d_i \in D} \left\{\phi_{Kd_i}(\neg p) : \phi_{Kd_i}(\neg p) \subseteq \psi_L(\neg p)\right\} \quad (4.11)$$

Definition 4.3 Let $\widehat{\Phi_{KD}} \in \mathbb{BSC}_D^P(U)$, $\widehat{\Psi_L} \in \mho(U)$. Then,

$$\underline{\Xi}_{[\widehat{\Phi_{KD}}]}\widehat{\Psi_L} = \left\{\left(p, \underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p), \overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p)\right) : p \in P, \neg p \in \neg P\right\} \quad (4.12)$$

and

$$\overline{\Xi}_{[\widehat{\Phi_{KD}}]}\widehat{\Psi_L} = \left\{\left(p, \overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p), \underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p)\right) : p \in P, \neg p \in \neg P\right\} \quad (4.13)$$

are called bs-$\widehat{\Phi_{KD}}$-lower approximation and bs-$\widehat{\Phi_{KD}}$-upper approximation of $\widehat{\Psi_L}$, respectively. Moreover,

$$
\begin{aligned}
BP_{\widehat{\Phi_{KD}}}\widehat{\Psi_L} &= \underline{\Xi}_{[\widehat{\Phi_{KD}}]}\widehat{\Psi_L} \\
&= \left\{\left(p, \underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p), \overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p)\right) : \begin{array}{c} p \in P, \\ \neg p \in \neg P \end{array}\right\}, \\
BN_{\widehat{\Phi_{KD}}}\widehat{\Psi_L} &= U - \overline{\Xi}_{[\widehat{\Phi_{KD}}]}\widehat{\Psi_L} \\
&= \left\{\left(p, U - \overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p), U - \underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p)\right) : \begin{array}{c} p \in P, \\ \neg p \in \neg P \end{array}\right\}, \\
BB_{\widehat{\Phi_{KD}}}\widehat{\Psi_L} &= \overline{\Xi}_{[\widehat{\Phi_{KD}}]}\widehat{\Psi_L} - \underline{\Xi}_{[\widehat{\Phi_{KD}}]}\widehat{\Psi_L} \\
&= \left\{\left(p, \begin{array}{c} \overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p) - \underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(p), \\ \underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p) - \overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}(\neg p) \end{array}\right) : \begin{array}{c} p \in P, \\ \neg p \in \neg P \end{array}\right\}.
\end{aligned}
$$

are called bs-$\widehat{\Phi_{KD}}$-positive region, bs-$\widehat{\Phi_{KD}}$-negative region and bs-S-boundary region of $\widehat{\Psi_L}$, respectively. If $\overline{\Xi}_{[\widehat{\Phi_{KD}}]}\widehat{\Psi_L} = \underline{\Xi}_{[\widehat{\Phi_{KD}}]}\widehat{\Psi_L}$, $\widehat{\Psi_L}$ is said to be bs-$\widehat{\Phi_{KD}}$-definable set; otherwise $\widehat{\Psi_L}$ is called a bs-$\widehat{\Phi_{KD}}$-rough set.

Example 4.4 Reconsider Example 3.2. Then, all of parametrized classes of $\widehat{\Phi_{KD}}$ for $p \in P$ and $\neg p \in \neg P$ are as follows:

$$
\begin{aligned}
\mathfrak{C}_{\widehat{\Phi_{KD}}}(p_1) &= \{\{u_2, u_4, u_6\}, \{u_1, u_5\}, U, \{\}\}, \\
\mathfrak{C}_{\widehat{\Phi_{KD}}}(\neg p_1) &= \{\{u_1, u_5\}, \{u_2, u_4, u_6\}, \{\}, \{u_2, u_4, u_6\}\}, \\
\mathfrak{C}_{\widehat{\Phi_{KD}}}(p_3) &= \{\{u_1, u_3, u_5\}, \{u_3, u_4\}, \{u_2, u_3, u_5, u_6\}, \{u_3, u_5\}\}, \\
\mathfrak{C}_{\widehat{\Phi_{KD}}}(\neg p_3) &= \{\{u_2\}, \{u_1, u_5\}, \{u_1, u_4\}, \{u_1, u_6\}\}, \\
\mathfrak{C}_{\widehat{\Phi_{KD}}}(p_4) &= \{\{u_1, u_4, u_5, u_6\}, \{u_2, u_6\}, \{u_1, u_3, u_5\}, \{u_3, u_6\}\}, \\
\mathfrak{C}_{\widehat{\Phi_{KD}}}(\neg p_4) &= \{\{u_3\}, \{u_1, u_3\}, \{u_4, u_6\}, \{u_2, u_4\}\}.
\end{aligned}
$$

Now, let

$$
\widehat{\Psi_L} = \left\{\begin{array}{c} (p_1, \{u_2, u_4, u_6\}, \{u_5\}), (p_2, \{u_2, u_3, u_4\}, \{u_1, u_5\}), \\ (p_3, \{u_3, u_4, u_5\}, \{u_1, u_2\}), (p_4, \{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}) \end{array}\right\} \in \mho(U).
$$

Then,

$$
\begin{aligned}
\underline{\Xi}_{[\widehat{\Phi_{KD}}]}\widehat{\Psi_L} &= \left\{\begin{array}{c} (p_1, \{u_2, u_4, u_6\}, \{u_1, u_5\}), (p_2, \{\}, \{\}), \\ (p_3, \{u_3, u_4, u_5\}, \{u_1, u_2, u_4, u_5, u_6\}), (p_4, \{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}) \end{array}\right\}, \\
\overline{\Xi}_{[\widehat{\Phi_{KD}}]}\widehat{\Psi_L} &= \left\{\begin{array}{c} (p_1, U, \{\}), (p_2, \{\}, \{\}), \\ (p_3, U, \{u_2\}), (p_4, \{u_1, u_3, u_4, u_5, u_6\}, \{u_2, u_4, u_6\}) \end{array}\right\}, \\
BP_{\widehat{\Phi_{KD}}}\widehat{\Psi_L} &= \left\{\begin{array}{c} (p_1, \{u_2, u_4, u_6\}, \{u_1, u_5\}), (p_2, \{\}, \{\}), \\ (p_3, \{u_3, u_4, u_5\}, \{u_1, u_2, u_4, u_5, u_6\}), (p_4, \{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}) \end{array}\right\}, \\
BN_{\widehat{\Phi_{KD}}}\widehat{\Psi_L} &= \left\{\begin{array}{c} (p_1, \{\}, U), (p_2, U, U), \\ (p_3, \{\}, \{u_1, u_3, u_4, u_5, u_6\}), (p_4, \{u_2\}, \{u_1, u_3, u_5\}) \end{array}\right\}, \\
BB_{\widehat{\Phi_{KD}}}\widehat{\Psi_L} &= \left\{\begin{array}{c} (p_1, \{u_1, u_3, u_5\}, \{\}), (p_2, \{\}, \{\}), \\ (p_3, \{u_1, u_2, u_6\}, \{\}), (p_4, \{u_4, u_6\}, \{\}) \end{array}\right\}
\end{aligned}
$$

Theorem 4.5 *Let* $\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}} \in \mathbb{BSC}_D^P(U)$, $\widehat{\Upsilon_M} \in \mho(U)$. *Then, for all* $p \in P$, $\neg p \in \neg P$ *and* $d_i \in D$,

(i) $\underline{\widehat{\Upsilon_M}}_{[\widehat{U}_{BS}]}(p) = \emptyset$, $\underline{\widehat{\Upsilon_M}}_{[\widehat{U}_{BS}]}(\neg p) = U$ *and* $\overline{\widehat{\Upsilon}}_{M[\widehat{U}_{BS}]}(p) = U$, $\overline{\widehat{\Upsilon}}_{M[\widehat{U}_{BS}]}(\neg p) = \emptyset$ *for* $\widehat{\Upsilon}_{M\emptyset} \neq \widehat{\Upsilon}_M \neq \widehat{\Upsilon}_{MU}$.

(ii) $\underline{\widehat{\Upsilon_M}}_{[\widehat{U}_{BS}]}(p) = \overline{\widehat{\Upsilon}}_{M[\widehat{U}_{BS}]}(p) = U$ *and* $\underline{\widehat{\Upsilon_M}}_{[\widehat{U}_{BS}]}(\neg p) = \overline{\widehat{\Upsilon}}_{M[\widehat{U}_{BS}]}(\neg p) = \emptyset$ *for* $\widehat{\Upsilon}_M \neq \widehat{\Upsilon}_{MU}$.

(iii) $\underline{\widehat{\Upsilon_M}}_{[\widehat{\emptyset}_{BS}]}(p) = \overline{\widehat{\Upsilon}}_{M[\widehat{\emptyset}_{BS}]}(p) = \emptyset$ *and* $\underline{\widehat{\Upsilon_M}}_{[\widehat{\emptyset}_{BS}]}(\neg p) = \overline{\widehat{\Upsilon}}_{M[\widehat{\emptyset}_{BS}]}(\neg p) = U$.

(iv) $\underline{\widehat{\Upsilon_M}}_{\left[\widehat{\Phi}_{KD}\widehat{\cup}\widehat{\Psi}_{LD}\right]}(p) = \underline{\widehat{\Upsilon_M}}_{\left[\widehat{\Phi}_{KD}\right]}(p) \cup \underline{\widehat{\Upsilon_M}}_{[\widehat{\Psi}_{LD}]}(p)$ *and* $\underline{\widehat{\Upsilon_M}}_{\left[\widehat{\Phi}_{KD}\right]}(\neg p) \cup \underline{\widehat{\Upsilon_M}}_{[\widehat{\Psi}_{LD}]}(\neg p) \subseteq \underline{\widehat{\Upsilon_M}}_{\left[\widehat{\Phi}_{KD}\widehat{\cup}\widehat{\Psi}_{LD}\right]}(\neg p)$.

(v) $\overline{\widehat{\Upsilon}}_{M\left[\widehat{\Phi}_{KD}\right]}(p) \cup \overline{\widehat{\Upsilon}}_{M[\widehat{\Psi}_{LD}]}(p) \subseteq \overline{\widehat{\Upsilon}}_{M\left[\widehat{\Phi}_{KD}\widehat{\cup}\widehat{\Psi}_{LD}\right]}(p)$ *and* $\overline{\widehat{\Upsilon}}_{M\left[\widehat{\Phi}_{KD}\widehat{\cup}\widehat{\Psi}_{LD}\right]}(\neg p) = \overline{\widehat{\Upsilon}}_{M\left[\widehat{\Phi}_{KD}\right]}(\neg p) \cup \overline{\widehat{\Upsilon}}_{M[\widehat{\Psi}_{LD}]}(\neg p)$.

(vi) $\underline{\widehat{\Upsilon_M}}_{\left[\widehat{\Phi}_{KD}\right]}(p) \cap \underline{\widehat{\Upsilon_M}}_{[\widehat{\Psi}_{LD}]}(p) \subseteq \underline{\widehat{\Upsilon_M}}_{\left[\widehat{\Phi}_{KD}\widehat{\cap}\widehat{\Psi}_{LD}\right]}(p)$ *and* $\underline{\widehat{\Upsilon_M}}_{\left[\widehat{\Phi}_{KD}\widehat{\cap}\widehat{\Psi}_{LD}\right]}(\neg p) = \underline{\widehat{\Upsilon_M}}_{\left[\widehat{\Phi}_{KD}\right]}(\neg p) \cap \underline{\widehat{\Upsilon_M}}_{[\widehat{\Psi}_{LD}]}(\neg p)$.

(vii) $\overline{\widehat{\Upsilon}}_{M\left[\widehat{\Phi}_{KD}\widehat{\cap}\widehat{\Psi}_{LD}\right]}(p) = \overline{\widehat{\Upsilon}}_{M\left[\widehat{\Phi}_{KD}\right]}(p) \cap \overline{\widehat{\Upsilon}}_{M[\widehat{\Psi}_{LD}]}(p)$ *and* $\overline{\widehat{\Upsilon}}_{M\left[\widehat{\Phi}_{KD}\right]}(\neg p) \cap \overline{\widehat{\Upsilon}}_{M[\widehat{\Psi}_{LD}]}(\neg p) \subseteq \overline{\widehat{\Upsilon}}_{M\left[\widehat{\Phi}_{KD}\widehat{\cap}\widehat{\Psi}_{LD}\right]}(\neg p)$.

Proof The proofs of (i), (ii) and (iii) are clear from Definition 4.1 and 4.3.

(iv) For the first inequality,

$(\Rightarrow)$: if $u \in \underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}\widehat{\cup}\widehat{\Psi}_{LD}]}(p)$, then $u \in \widehat{\Phi}_{Kd_i} \cup \widehat{\Psi}_{Ld_i} \subseteq \widehat{\Upsilon}_M(p)$; $\exists d_i \in D$, i.e., $u \in \widehat{\Phi}_{Kd_i} \subseteq \widehat{\Upsilon}_M(p)$ or $u \in \widehat{\Psi}_{Ld_i} \subseteq \widehat{\Upsilon}_M(p)$. Thus, $u \in \underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}]}(p) \cup \underline{\widehat{\Upsilon_M}}_{[\widehat{\Psi}_{LD}]}(p)$. So, $\underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}\widehat{\cup}\widehat{\Psi}_{LD}]}(p) \subseteq \underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}]}(p) \cup \underline{\widehat{\Upsilon_M}}_{[\widehat{\Psi}_{LD}]}(p)$–(*).

$(\Leftarrow)$: if $u \in \underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}]}(p) \cup \underline{\widehat{\Upsilon_M}}_{[\widehat{\Psi}_{LD}]}(p)$, then $u \in \underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}]}(p)$ or $u \in \underline{\widehat{\Upsilon_M}}_{[\widehat{\Psi}_{LD}]}(p)$. Thus, $u \in \widehat{\Phi}_{Kd_i}(p) \subseteq \widehat{\Upsilon}_M(p)$ or $u \in \widehat{\Psi}_{Ld_i}(p) \subseteq \widehat{\Upsilon}_M(p)$; $\exists d_i \in D$, and $u \in \widehat{\Phi}_{Kd_i}(p) \cup \widehat{\Psi}_{Ld_i}(p) \subseteq \widehat{\Upsilon}_M(p)$, i.e., $u \in \underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}\widehat{\cup}\widehat{\Psi}_{LD}]}(p)$. So, $\underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}]}(p) \cup \underline{\widehat{\Upsilon_M}}_{[\widehat{\Psi}_{LD}]}(p) \subseteq \underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}\widehat{\cup}\widehat{\Psi}_{LD}]}(p)$–(**).

Hence, from (*) and (**), $\underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}\widehat{\cup}\widehat{\Psi}_{LD}]}(p) = \underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}]}(p) \cup \underline{\widehat{\Upsilon_M}}_{[\widehat{\Psi}_{LD}]}(p)$.

For the second inequality, if $u \in \underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}]}(\neg p) \cup \underline{\widehat{\Upsilon_M}}_{[\widehat{\Psi}_{LD}]}(\neg p)$, then $u \in \underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}]}(\neg p)$ or $u \in \underline{\widehat{\Upsilon_M}}_{[\widehat{\Psi}_{LD}]}(\neg p)$. In that case, $u \in \widehat{\Phi}_{Kd_i}(\neg p)$ or $u \in \widehat{\Psi}_{Ld_i}(\neg p)$; $\exists d_i \in D$, i.e., $u \in \widehat{\Phi}_{Kd_i}(\neg p) \cap \widehat{\Upsilon}_M(\neg p) \neq \emptyset$ or $u \in \widehat{\Psi}_{Ld_i}(\neg p) \cap \widehat{\Upsilon}_M(\neg p) \neq \emptyset$. So, $\left(\widehat{\Phi}_{Kd_i}(\neg p) \cup \widehat{\Psi}_{Ld_i}(\neg p)\right) \cap \widehat{\Upsilon}_M(\neg p) \neq \emptyset$. Thus, $u \in \underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}\widehat{\cup}\widehat{\Psi}_{LD}]}(\neg p)$ and hence $\underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}]}(\neg p) \cup \underline{\widehat{\Upsilon_M}}_{[\widehat{\Psi}_{LD}]}(\neg p) \subseteq \underline{\widehat{\Upsilon_M}}_{[\widehat{\Phi}_{KD}\widehat{\cup}\widehat{\Psi}_{LD}]}(\neg p)$.

The proofs of (v), (vi) and (vii) can be proved similarly to (iv). □

Theorem 4.6 *Let* $\widehat{\Phi}_{KD}, \widehat{\Psi}_{LD} \in \mathbb{BSC}^P_D(U)$, $\widehat{\Upsilon}_M \in \mho(U)$. *Then,*

(i) $\underline{\Xi}_{[\widehat{\emptyset}_{BS}]}\widehat{\Upsilon}_M = \widehat{\Upsilon}_{M\emptyset} = \overline{\Xi}_{[\widehat{\emptyset}_{BS}]}\widehat{\Upsilon}_M$.

(ii) $\underline{\Xi}_{[\widehat{U}_{BS}]}\widehat{\Upsilon}_M = \widehat{\Upsilon}_{M\emptyset}$ *and* $\overline{\Xi}_{[\widehat{U}_{BS}]}\widehat{\Upsilon}_M = \widehat{\Upsilon}_{MU}$ *for* $\widehat{\Upsilon}_M \neq \widehat{\Upsilon}_{MU}$*;* $\forall d_i \in D$.

(iii) $\underline{\Xi}_{[\widehat{U}_{BS}]}\widehat{\Upsilon}_M = \widehat{\Upsilon}_{MU}$ *and* $\overline{\Xi}_{[\widehat{U}_{BS}]}\widehat{\Upsilon}_M = \widehat{\Upsilon}_{MU}$ *for* $\widehat{\Upsilon}_M = \widehat{\Upsilon}_{MU}$*;* $\forall d_i \in D$.

(iv) $\left(\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\widehat{\Upsilon_M}\right)\widetilde{\cap}\left(\underline{\Xi}_{[\widehat{\Psi_{LD}}]}\widehat{\Upsilon_M}\right)\subseteq\left(\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\widehat{\cap}\widehat{\Psi_{LD}}\right]}\widehat{\Upsilon_M}\right)$ *and* $\left(\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\widehat{\Upsilon_M}\right)$ $\widetilde{\cap}\left(\overline{\Xi}_{[\widehat{\Psi_{LD}}]}\widehat{\Upsilon_M}\right)\subseteq\left(\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\widehat{\cap}\widehat{\Psi_{LD}}\right]}\widehat{\Upsilon_M}\right)$.

(v) $\left(\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\widehat{\Upsilon_M}\right)\widetilde{\cup}\left(\underline{\Xi}_{[\widehat{\Psi_{LD}}]}\widehat{\Upsilon_M}\right)\subseteq\left(\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\widehat{\cup}\widehat{\Psi_{LD}}\right]}\widehat{\Upsilon_M}\right)$ *and* $\left(\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\widehat{\Upsilon_M}\right)\widetilde{\cup}$ $\left(\overline{\Xi}_{[\widehat{\Psi_{LD}}]}\widehat{\Upsilon_M}\right)\subseteq\left(\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\widehat{\cup}\widehat{\Psi_{LD}}\right]}\widehat{\Upsilon_M}\right)$

Proof From Theorem 4.5, the proofs are clear. □

Theorem 4.7 *Let* $\widehat{\Phi_{KD}}\in\mathbb{BSC}_D^P(U)$, $\widehat{\Psi_L},\widehat{\Upsilon_M}\in\mho(U)$. *Then,*

(i) $\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\widehat{\Phi_{K\emptyset}}=\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\widehat{\Phi_{K\emptyset}}=\widehat{\Phi_{K\emptyset}}$ *and* $\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\widehat{\Phi_{KU}}=\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\widehat{\Phi_{KU}}=\widehat{\Phi_{KU}}$.

(ii) $\widehat{\Psi_L}\widetilde{\subseteq}\widehat{\Upsilon_M}\Rightarrow\left(\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\widehat{\Psi_L}\right)\widetilde{\subseteq}\left(\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\widehat{\Upsilon_M}\right)$ *and* $\left(\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\widehat{\Psi_L}\right)\widetilde{\subseteq}\left(\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\widehat{\Upsilon_M}\right)$.

(iii) $\left(\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\left(\widehat{\Psi_L}\widetilde{\cap}\widehat{\Upsilon_M}\right)\right)\widetilde{\subseteq}\left(\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\left(\widehat{\Psi_L}\right)\right)\widetilde{\cap}\left(\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\left(\widehat{\Upsilon_M}\right)\right)$.

(iv) $\left(\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\left(\widehat{\Psi_L}\widetilde{\cap}\widehat{\Upsilon_M}\right)\right)\widetilde{\subseteq}\left(\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\left(\widehat{\Psi_L}\right)\right)\widetilde{\cap}\left(\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\left(\widehat{\Upsilon_M}\right)\right)$.

(v) $\left(\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\left(\widehat{\Psi_L}\right)\right)\widetilde{\cup}\left(\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\left(\widehat{\Upsilon_M}\right)\right)\widetilde{\subseteq}\left(\underline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\left(\widehat{\Psi_L}\widetilde{\cup}\widehat{\Upsilon_M}\right)\right)$.

(vi) $\left(\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\left(\widehat{\Psi_L}\right)\right)\widetilde{\cup}\left(\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\left(\widehat{\Upsilon_M}\right)\right)\widetilde{\subseteq}\left(\overline{\Xi}_{\left[\widehat{\Phi_{KD}}\right]}\left(\widehat{\Psi_L}\widetilde{\cup}\widehat{\Upsilon_M}\right)\right)$.

Proof The proofs of (i) and (ii) is simple.

(iii) Since $\widehat{\Psi_L}\widetilde{\cap}\widehat{\Upsilon_M}\subseteq\widehat{\Psi_L}$ and $\widehat{\Psi_L}\widetilde{\cap}\widehat{\Upsilon_M}\subseteq\widehat{\Upsilon_M}$; then, from (ii),

$$\left(\underline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Psi_L}\widetilde{\cap}\widehat{\Upsilon_M}\right)\right)\widetilde{\subseteq}\left(\underline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Psi_L}\right)\right)$$

and

$$\left(\underline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Psi_L}\widetilde{\cap}\widehat{\Upsilon_M}\right)\right)\widetilde{\subseteq}\left(\underline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Upsilon_M}\right)\right),$$

respectively. Hence,

$$\left(\underline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Psi_L}\widetilde{\cap}\widehat{\Upsilon_M}\right)\right)\widetilde{\subseteq}\left(\underline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Psi_L}\right)\right)\widetilde{\cap}\left(\underline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Upsilon_M}\right)\right).$$

The proofs of (iv), (v) and (vi) can be proved similarly to (iii). □

Definition 4.8 Let $\widehat{\Phi_{KD}}\in\mathbb{BSC}_D^P(U)$, $\widehat{\Psi_L},\widehat{\Upsilon_M}\in\mho(U)$. Then,

$$\left(\widehat{\Psi_L}\right)\perp_{\widehat{\Phi_{KD}}}\left(\widehat{\Upsilon_M}\right)\Leftrightarrow\left(\underline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Psi_L}\right)\right)=\left(\underline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Upsilon_M}\right)\right) \tag{4.14}$$

$$\left(\widehat{\Psi_L}\right)\top_{\widehat{\Phi_{KD}}}\left(\widehat{\Upsilon_M}\right)\Leftrightarrow\left(\overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Psi_L}\right)\right)=\left(\overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Upsilon_M}\right)\right) \tag{4.15}$$

are called the bs-lower class rough equal relation and bs-upper class rough equal relation, respectively. Moreover,

$$\left(\widehat{\Psi_L}\right)\diamond_{\widehat{\Phi_{KD}}}\left(\widehat{\Upsilon_M}\right)\Leftrightarrow\left(\widehat{\Psi_L}\right)\perp_{\widehat{\Phi_{KD}}}\left(\widehat{\Upsilon_M}\right)=\left(\widehat{\Psi_L}\right)\top_{\widehat{\Phi_{KD}}}\left(\widehat{\Upsilon_M}\right) \tag{4.16}$$

Theorem 4.9 *Let* $\widehat{\Phi_{KD}} \in \mathbb{BSC}_D^P(U)$, $\widehat{\Psi_L}, \widehat{\Upsilon_M}$. *Then,*

(i) $\widehat{\Psi_L}\,\widetilde{\subseteq}\,\widehat{\Upsilon_M},\ \left(\widehat{\Upsilon_M}\right) \top_{\widehat{\Phi_{KD}}} \left(\widehat{\Phi_{K\emptyset}}\right) \Rightarrow \left(\widehat{\Psi_L}\right) \top_{\widehat{\Phi_{KD}}} \left(\widehat{\Phi_{K\emptyset}}\right)$.

(ii) $\widehat{\Psi_L}\,\widetilde{\subseteq}\,\widehat{\Upsilon_M},\ \left(\widehat{\Psi_L}\right) \top_{\widehat{\Phi_{KD}}} \left(\widehat{\Phi_{KU}}\right) \Rightarrow \left(\widehat{\Upsilon_M}\right) \top_{\widehat{\Phi_{KD}}} \left(\widehat{\Phi_{KU}}\right)$.

Proof (i) From Theorem 4.7, we have $\overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Psi_L}\right) \widetilde{\subseteq} \overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Upsilon_M}\right) = \overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Phi_{K\emptyset}}\right) = \widehat{\Phi_{K\emptyset}}$. Therefore, $\overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Psi_L}\right) = \widehat{\Phi_{K\emptyset}} = \overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Upsilon_M}\right)$ and hence $\left(\widehat{\Psi_L}\right) \top_{\widehat{\Phi_{KD}}} \left(\widehat{\Phi_{K\emptyset}}\right)$.

(ii) From Theorem 4.7, we have $\overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Upsilon_M}\right) \widetilde{\supseteq} \overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Psi_L}\right) = \overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Phi_{KU}}\right)$. Moreover, since $\widehat{\Upsilon_M}\,\widetilde{\subseteq}\,\widehat{\Phi_{KU}}$, then $\overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Upsilon_M}\right) \widetilde{\subseteq} \overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Phi_{KU}}\right)$. Thus, $\overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Upsilon_M}\right) = \overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Phi_{KU}}\right)$ and hence $\left(\widehat{\Upsilon_M}\right) \top_{\widehat{\Phi_{KD}}} \left(\widehat{\Phi_{KU}}\right)$. □

Definition 4.10 Let $\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}} \in \mathbb{BSC}_D^P(U)$, $\widehat{\Upsilon_M} \in \mho(U)$. Then,

$$\left(\widehat{\Phi_{KD}}\right) \perp_{\widehat{\Upsilon_M}} \left(\widehat{\Psi_{LD}}\right) \Leftrightarrow \left(\underline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Upsilon_M}\right)\right) = \left(\underline{\Xi}_{[\widehat{\Psi_{LD}}]}\left(\widehat{\Upsilon_M}\right)\right) \tag{4.17}$$

$$\left(\widehat{\Phi_{KD}}\right) \top_{\widehat{\Upsilon_M}} \left(\widehat{\Psi_{LD}}\right) \Leftrightarrow \left(\overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Upsilon_M}\right)\right) = \left(\overline{\Xi}_{[\widehat{\Psi_{LD}}]}\left(\widehat{\Upsilon_M}\right)\right) \tag{4.18}$$

are called the bs-lower class rough $\widehat{\Upsilon_M}$-equal relation and bs-upper class rough $\widehat{\Upsilon_M}$-equal relation, respectively. Moreover,

$$\left(\widehat{\Phi_{KD}}\right) \Diamond_{\widehat{\Upsilon_M}} \left(\widehat{\Psi_{LD}}\right) \Leftrightarrow \left(\widehat{\Phi_{KD}}\right) \perp_{\widehat{\Upsilon_M}} \left(\widehat{\Psi_{LD}}\right) = \left(\widehat{\Phi_{KD}}\right) \top_{\widehat{\Upsilon_M}} \left(\widehat{\Psi_{LD}}\right) \tag{4.19}$$

Theorem 4.11 *Let* $\widehat{\Phi_{KD}}, \widehat{\Psi_{LD}} \in \mathbb{BSC}_D^P(U)$, $\widehat{\Upsilon_M} \in \mho(U)$. *Then,*

(i) $\widehat{\Phi_K}\,\widehat{\subseteq}\,\widehat{\Psi_L},\ \left(\widehat{\Psi_L}\right) \top_{\widehat{\Upsilon_M}} \left(\widehat{\emptyset}_{BS}\right) \Rightarrow \left(\widehat{\Phi_K}\right) \top_{\widehat{\Upsilon_M}} \left(\widehat{\emptyset}_{BS}\right)$.

(ii) $\widehat{\Phi_K}\,\widehat{\subseteq}\,\widehat{\Psi_L},\ \left(\widehat{\Phi_K}\right) \top_{\widehat{\Upsilon_M}} \left(\widehat{U}_{BS}\right) \Rightarrow \left(\widehat{\Psi_L}\right) \top_{\widehat{\Upsilon_M}} \left(\widehat{U}_{BS}\right)$.

Proof (i) Since $\overline{\Xi}_{[\widehat{\Psi_{LD}}]}\left(\widehat{\Upsilon_M}\right) = \overline{\Xi}_{[\widehat{\emptyset}_{BS}]}\left(\widehat{\Upsilon_M}\right)$ and $\widehat{\Phi_K}\,\widehat{\subseteq}\,\widehat{\Psi_L}$, then $\overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Upsilon_M}\right) \widetilde{\subseteq} \overline{\Xi}_{[\widehat{\emptyset}_{BS}]}\left(\widehat{\Upsilon_M}\right) = \widehat{\emptyset}_{BS}$. Thus, $\overline{\Xi}_{[\widehat{\Phi_{KD}}]}\left(\widehat{\Upsilon_M}\right) \widetilde{\subseteq} \overline{\Xi}_{[\widehat{\emptyset}_{BS}]}\left(\widehat{\Upsilon_M}\right)$ and hence $\left(\widehat{\Phi_K}\right) \top_{\widehat{\Upsilon_M}} \left(\widehat{\emptyset}_{BS}\right)$.

The proof (ii) can be proved similarly to (i). □

5 Decision making under uncertainty using bipolar soft rough classes

In this section, we use bs-rough classes to manage the decision-making process for uncertainty problems. First, we build a decision-making algorithm based on bs-rough classes, then we illustrate how this algorithm can be applied to an uncertainty problem.

Definition 5.1 Let $\widehat{\Phi_{KD}} \in \mathbb{BSC}_D^P(U)$, $\widehat{\Psi_L} \in \mho(U)$ and $d_i, d_j \in D$. Then,

$$\Im^{\widehat{\Psi_L}}_{\widehat{\Phi_{KD}}}(d_i) = \frac{1}{|K \cup L|}\left(\sum_{p \in K \cup L}\left[\frac{\left|\underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}^{(p)}\right| - \left|\overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}^{(\neg p)}\right|}{\left|\overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}^{(p)}\right| - \left|\underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}}]}^{(\neg p)}\right|} - \frac{\left|\underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}} - \{\widehat{\Phi_{Kd_i}}\}]}^{(p)}\right| - \left|\overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}} - \{\widehat{\Phi_{Kd_i}}\}]}^{(\neg p)}\right|}{\left|\overline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}} - \{\widehat{\Phi_{Kd_i}}\}]}^{(p)}\right| - \left|\underline{\widehat{\Psi_L}}_{[\widehat{\Phi_{KD}} - \{\widehat{\Phi_{Kd_i}}\}]}^{(\neg p)}\right|}\right]\right) \tag{5.1}$$

is called effectiveness degree of the decision maker d_i compared to other decision makers. Moreover, when we compare the relations between d_i and d_j according to their effectiveness degrees, the following expressions are defined:

(i) $\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_i) >_{\widehat{\Phi_L}} \Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_j) \Rightarrow d_i$ is more effectiveness degree than d_j.
(ii) $\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_i) =_{\widehat{\Phi_L}} \Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_j) \Rightarrow d_i$ is same effectiveness degree than d_j.
(iii) $\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_i) <_{\widehat{\Phi_L}} \Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_j) \Rightarrow d_j$ is more effectiveness degree than d_i.

Now, the decision making algorithm aiming to identify the best decision maker based on bs-rough classes has been created as follows:

Algorithm 1 Algorithm for bs-rough classes

Require: U is the initial universe, P is the universe of all possible parameters related to the objects in U and $D = \{d_1, d_2, ..., d_r\}$ is the set of decision-makers for $1 \leq i \leq r$
Input: Input $\widehat{\Phi_{KD}} \in \mathbb{BSC}_D^P(U)$ and reference $\widehat{\Psi_L} \in \mho(U)$.
Output: Decision Making.
1. Calculate the effectiveness degrees $\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_i)$ of decision makers d_i for $d_i \in D$.
2. The calculated efficiency effectiveness degree of each decision-maker are compared.
3. Find s, for which $\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_s) = max\left\{\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_i) : d_i \in D\right\}$.

Let us now illustrate how the decision-making algorithm we proposed can be applied to an uncertainty problem:

Example 5.2 Suppose a college wanted to predict how successful the current teachers could be in a year. For this, the college administration requires six inspectors to inspect the college. Also, the set of parameters that determine the success criteria of teachers is expressed as $K = \{p_2, p_3, p_4\} \subseteq P = \{p_1 : hardworking, p_2 : disciplined, p_3 : successful, p_4 : honest\}$ by the college administration. In this case, $\neg P = \{\neg p_1 : lazy, \neg p_2 : undisciplined, \neg p_3 : unsuccessful, \neg p_4 : dishonest\}$. Moreover, let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$ be the set of current teachers and $D = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ be the set of inspectors who apply to inspect the college.

The opinions of each inspector about the teachers are expressed with the help of bs-sets $\widehat{\Phi_{Kd_i}}$ for $d_i \in D$ as follows:

$$\widehat{\Phi_{Kd_1}} = \left\{\begin{array}{c}(p_2, \{u_1, u_2, u_5, u_7\}, \{u_3, u_4, u_6, u_8\}), (p_3, \{u_2, u_5, u_7, u_8\}, \{u_1, u_4, u_{10}\}), \\ (p_4, \{u_3\}, \{u_1, u_4, u_9, u_{10}\})\end{array}\right\},$$

$$\widehat{\Phi_{Kd_2}} = \left\{\begin{array}{c}(p_2, \{u_3, u_4, u_6\}, \{u_5, u_7, u_9, u_{10}\}), (p_3, \{u_5, u_7\}, \{u_3, u_6\}), \\ (p_4, \{u_1, u_3, u_9, u_{10}\}, \{u_2, u_6, u_7\})\end{array}\right\},$$

$$\widehat{\Phi_{Kd_3}} = \left\{\begin{array}{c}(p_2, \{u_3, u_5, u_7\}, \{u_2, u_4\}), (p_3, \{u_1, u_4, u_9\}, \{u_6, u_8\}), \\ (p_4, \{u_8, u_9\}, \{u_1, u_5, u_9\})\end{array}\right\},$$

$$\widehat{\Phi_{Kd_4}} = \left\{\begin{array}{c}(p_2, \{u_8, u_{10}\}, \{u_1, u_2\}), (p_3, \{u_2, u_5\}, \{u_8, u_9, u_{10}\}), \\ (p_4, \{u_2, u_4, u_5\}, \{u_1, u_3, u_7\})\end{array}\right\},$$

$$\widehat{\Phi_{Kd_5}} = \left\{\begin{array}{c}(p_2, \{u_1, u_5, u_8\}, \{u_3, u_7, u_9\}), (p_3, \{u_1, u_3, u_6, u_{10}\}, \{u_1, u_5, u_7\}), \\ (p_4, \{u_1, u_2, u_6, u_7, u_9\}, \{u_3, u_8, u_{10}\})\end{array}\right\},$$

$$\widehat{\Phi_{Kd_6}} = \left\{\begin{array}{c}(p_2, \{u_2, u_6, u_7\}, \{u_4, u_9\}), (p_3, \{u_2, u_4, u_6\}, \{u_1, u_3, u_8, u_9\}), \\ (p_4, \{u_4, u_6, u_8\}, \{u_1, u_2, u_9\})\end{array}\right\}.$$

Table 7 Effectiveness degree of each decision maker

Decision maker	Effectiveness degrees
$\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_1)$	0.296
$\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_2)$	0.333
$\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_3)$	0.2644
$\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_4)$	0.2298
$\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_5)$	0.141
$\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_6)$	0.1026

Hence, $\widehat{\Phi_{KD}} = \left\{\widehat{\Phi_{Kd_1}}, \widehat{\Phi_{Kd_2}}, \widehat{\Phi_{Kd_3}}, \widehat{\Phi_{Kd_4}}\right\}$ is a bs-class. Then, taking into account each parameter, i.e., for $L = P$, the achievements of teachers at the end of one year are expressed by the school administration with the help of the bs-set $\widehat{\Psi_L}$ as follows:

$$\widehat{\Psi_L} = \left\{ \begin{array}{c} (p_1, \{u_1, u_2, u_4, u_6, u_9\}, \{u_3, u_5, u_7, u_{10}\}), (p_2, \{u_3, u_4, u_6\}, \{u_1, u_2, u_4, u_9\}), \\ (p_3, \{u_2, u_5, u_7\}, \{u_3, u_6, u_8\}), (p_4, \{u_3, u_8, u_{10}\}, \{u_2, u_6, u_7\}) \end{array} \right\}.$$

According to these data, the following values are easily obtained by making use of (5.1) to determine which inspector makes the most accurate decision:

According to the values obtained, $\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_2) = max\left\{\Im_{\widehat{\Phi_{KD}}}^{\widehat{\Psi_L}}(d_i) : d_i \in D\right\} = 0.333$ is obtained, and therefore we recommend that the d_2 inspector be more preferable than other inspectors to inspect the college for later years.

As seen in Example 5.2, the values expressed by different decision-makers are considered together in the decision-making process. This situation is quite different from the decision-making processes that are customary in the literature. A similar approach can be observed in Karaaslan (2016). In the current uncertainty problem, it is aimed to determine the best decision-maker by trying to determine the difference between the values expressed by the decision-makers and the real situation. Thus, we aimed to find out which decision maker we should use in solving a similar uncertainty problem. The advantages of bipolar soft rough classes used in Algorithm 1 are as follows:

- We cannot express the uncertainty problem in Example 5.2, which also includes NOT parameters of parameters, using soft rough classes. In this respect, bipolar soft rough classes should be preferred in uncertain environments.
- Modeling the values expressed by different decision-makers with a single set type allows the data to be processed more easily.
- It enables us to make a ranking among decision-makers by enabling the construction of a formulation that provides the effectiveness degrees among decision-makers.

6 Conclusion

The aim of this paper is to provide a more effective approach to uncertainty problems that focus on decision-makers. For this, bipolar soft rough classes have been suggested. Thanks to these classes, it is possible to handle the data expressed by different decision-makers together. Moreover, it is used as a tool to determine how effectively the current uncertainty can be expressed by decision-makers. To construct bipolar soft rough classes, bipolar soft classes have been defined and some of their operations subset, complement, intersection, union are examined. Then, for uncertainty problems, a decision-making algorithm is proposed using bipolar soft rough classes. Moreover, how this algorithm can be applied in solving an uncertainty problem is exemplified. We think that the proposed mathematical approach can be very useful in every field where the best decision-maker should be identified. In addition to these, the proposed classes can be generalized to theories such as fuzzy bipolar soft set Naz and Shabir (2014), rough fuzzy bipolar soft set Malik and Shabir (2019), m-polar fuzzy bipolar soft set Akram et al. (2021), modified rough bipolar soft set Shabir and Gul (2020). Moreover, it may be considered to develop better approaches by developing a reduction method for the proposed classes.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical standard This article does not contain any studies with human participants or animals performed by any of the authors.

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