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Viscosity S-iteration method with inertial technique and self-adaptive step size for split variational inclusion, equilibrium and fixed point problems

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Abstract

Several efficient methods have been developed in the literature for approximating solutions of fixed point and optimization problems. However, the S-iteration process has been shown to outperform many of these existing methods. In this paper, we study the problem of finding the common solution of split variational inclusion problem, equilibrium problem and common fixed point of nonexpansive mappings. We introduce an improved S-iteration method, which combines inertial and viscosity techniques with self-adaptive step size for approximating the solution of the problem in the framework of Hilbert spaces. Moreover, under some mild conditions we prove strong convergence theorem for the proposed algorithm without the knowledge of the operator norm and we apply our result to study split minimization problem, split feasibility problem and relaxed split feasibility problem. Finally, we present some numerical experiments with graphical illustrations to demonstrate the implementability and efficiency of our proposed method in comparison with some existing state of the art methods in the literature.

Keywords S-iteration method · Inertial technique · Self-adaptive step size · Split variational inclusion problem · Equilibrium problem · Nonexpansive mappings

Mathematics Subject Classification 65K15 · 47J25 · 65J15 · 90C33

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1 Introduction

Throughout this paper, let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$ denote the set of all positive numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $|| \cdot ||$, and let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a nonlinear mapping. A point $\hat{x} \in C$ is called a fixed point of S if $S\hat{x} = \hat{x}$. We denote by $F(S)$, the set of all fixed points of S, i.e.

$$F(S) = \{\hat{x} \in C : S\hat{x} = \hat{x}\}. \tag{1.1}$$

A mapping $S : C \to C$ is called a nonexpansive mapping if

$$||Sx - Sy|| \leq ||x - y|| \quad \forall\, x, y \in C.$$

The study of fixed point theory for nonexpansive mappings has flourished in recent years due to its vast applications in fields like economics, compressed sensing, and other applied sciences. In particular, certain problems such as variational inequalities problems, convex optimization problems, convex feasibility problems, monotone inclusion problems and image restoration problems can be formulated as finding the fixed points of nonexpansive mapping (see Bauschke and Borwein 1996; Chen et al. 2013). Several researchers have put considerable efforts in the study and formulation of iterative methods to approximate the fixed points of nonexpansive mappings (for example, see Halpern 1967; Moudafi 2000 and the reference therein).

In 2007, Agarwal et al. (2007) introduced the following iterative method known as the *S-iteration*. Let C be a convex subset of a linear space X and S a mapping of C into itself. The sequence $\{x_n\}$ is generated as follows:

Algorithm 1.1

$$\begin{aligned} &x_1 \in C, \\ &y_n = (1 - \beta_n)x_n + \beta_n Sx_n, \\ &x_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Sy_n, \quad n \in \mathbb{N}, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. The authors showed that the S-iteration process has a better rate of convergence than Mann and Ishikawa iteration processes.

Some authors have used the S-iteration process and its modifications to find common fixed points of two mappings; see for example (Bussaban and Kettapun 2018; Pandey et al. 2019) and the references therein.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* (EP) for the bifunction F on C is formulated as finding a point $\hat{x} \in C$ such that

$$F(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The solution set of EP (1.2) is denoted by $EP(F)$. The EP covers a wide range of topics that have emerged from the social sciences, economics, finance, image restoration, ecology, transport, networking, elasticity and optimization problems (see Olona et al. 2021a; Patriksson 2015). The problem is a generalized concept that unifies several mathematical problems as special cases, namely minimization problems, variational inequality problems, mathematical programming problems, complementarity problems, saddle point problems, Nash equilibrium problems in noncooperative games, minimax inequality problems, fixed point

problems, scalar and vector minimization problems, and others; see (Alakoya et al. 2021; Blum 1994) and the references therein. Recently, the EP and its various generalizations have attracted considerable research efforts and various iterative methods have been proposed for approximating their solutions (see Alakoya et al. 2021; Jolaoso et al. 2020; Ogwo et al. 2021; Olona et al. 2021b; Oyewole et al. 2021; Taiwo et al. 2021a, b and the references therein).
In the recent time, the Split Inverse Problem (SIP) has attracted the attention of several authors (see Alakoya et al. 2021; Oyewole et al. 2021; Taiwo et al. 2021 and the references therein) due to its wide areas of applications, for example, in phase retrieval, image recovery, signal processing, data compression, intensity-modulated radiation therapy, among others (see Censor et al. 2006; Censor and Elfving 1994 and the references therein). The SIP model is formulated as follows: Find a point

$$\hat{x} \in H_1 \quad \text{that solves } \mathrm{IP}_1 \tag{1.3}$$

such that

$$\hat{y} := A\hat{x} \in H_2 \quad \text{solves } \mathrm{IP}_2, \tag{1.4}$$

where H_1 and H_2 are real Hilbert spaces, IP_1 denotes an inverse problem formulated in H_1 and IP_2 denotes an inverse problem formulated in H_2, and $A : H_1 \to H_2$ is a bounded linear operator.
Censor and Elfving (1994) introduced the first instance of the SIP called the *split feasibility problem* (SFP) in 1994 for modelling inverse problems which arise from medical image reconstruction. Since then, the SFP has been studied intensively by several authors due to its wide areas of application such as in signal processing, control theory, approximation theory, geophysics, biomedical engineering, communications, etc (Byrne 2002; Censor et al. 2006; Godwin et al. 2020). Let C and Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ be a bounded linear operator. The SFP is formulated as follows:

$$\text{find a point } \hat{x} \in C \text{ such that } \hat{y} = A\hat{x} \in Q. \tag{1.5}$$

Moudafi (2011) introduced another instance of the SIP known as the *split monotone variational inclusion problem* (SMVIP). Let H_1, H_2 be real Hilbert spaces, $f_1 : H_1 \to H_1$, $f_2 : H_2 \to H_2$, are inverse strongly monotone mappings, $A : H_1 \to H_2$ is a bounded linear operator, $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ are multivalued maximal monotone mappings. The SMVIP is formulated as follows:

$$\text{find a point } \hat{x} \in H_1 \text{ such that } 0 \in f_1(\hat{x}) + B_1(\hat{x}) \tag{1.6}$$

and

$$\hat{y} = A\hat{x} \in H_2 \text{ such that } 0 \in f_2(\hat{y}) + B_2(\hat{y}). \tag{1.7}$$

If $f_1 \equiv 0 \equiv f_2$, then the SMVIP (1.6)-(1.7) reduces to the following *split variational inclusion problem* (SVIP):

$$\text{find a point } \hat{x} \in H_1 \text{ such that } 0 \in B_1(\hat{x}) \tag{1.8}$$

and

$$\hat{y} = A\hat{x} \in H_2 \text{ such that } 0 \in B_2(\hat{y}). \tag{1.9}$$

The SVIP (1.8)-(1.9) constitutes a pair of variational inclusion problems which have to be solved so that the image $\hat{y} = A\hat{x}$ under a given bounded linear operator A of the solution of the SVIP (1.8) in H_1 is the solution of the other SVIP (1.9) in another Hilbert space H_2. Moudafi (2011), showed that the SVIP (1.8)-(1.9) includes as a special case the SFP (1.11). The SVIP is at the core of modelling many inverse problems arising from phase retrieval and other real world problems, for instance, in sensor networks in computerized and data compression (Byrne 2002; Combettes 1996). We denote the solution set of SVIP (1.8) by SOLVIP(B_1) while the solution set of SVIP (1.9) is denoted by SOLVIP(B_2). Hence, the solution set of the SVIP (1.8)–(1.9) is denoted by

$$\mathcal{F} = \{x^* \in H_1 : x^* \in \text{SOLVIP}(B_1) \text{ and } Ax^* \in \text{SOLVIP}(B_2)\}. \tag{1.10}$$

To solve the SVIP for two maximal monotone operators B_1 and B_2 in Hilbert spaces, Byrne et al. (2012) proposed the following algorithm:

Algorithm 1.2

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \end{cases} \tag{1.11}$$

for $\lambda > 0$ and A^* is the adjoint operator of the bounded linear operator A, $\gamma \in (0, \frac{2}{L})$, $L = ||A^*A||$, $J_\lambda^{B_1} := (I + \lambda B_1)^{-1}$, $J_\lambda^{B_2} := (I + \lambda B_2)^{-1}$ are the resolvent operators of B_1 and B_2 respectively, and $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. The authors obtained a strong convergence result for the proposed algorithm under some mild conditions.

In the recent time, the problems of finding common solutions of the set of fixed points of nonlinear mappings and the set of solutions of optimization problems have been considered by some authors (for instance, see Alakoya et al. 2021; Cholamjiak and Suantai 2013; Khan et al. 2020 and the references therein). The importance and motivation for studying such a common solution problem lies in its potential application to mathematical models whose constraints can be expressed as fixed point problems and optimization problems. In this article we will be studying the problems of finding common solutions of the set of fixed points of nonlinear mappings and the set of solutions of certain optimization problems.

Recently, Wangkeeree et al. (2018) introduced the following general iterative scheme for approximating a common solution of SVIP and FPP for a nonexpansive mapping in the setting of real Hilbert spaces.

Algorithm 1.3

$$\begin{cases} x_0 \in H_1, \\ u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n \beta f(x_n) + (I - \alpha_n D)Su_n, \end{cases}$$

where $f : H_1 \to H_1$ is a contraction with constant $k \in (0, 1)$, $S : H_1 \to H_1$ is a nonexpansive mapping, $D : H_1 \to H_1$ is a strongly positive bounded linear operator with constant μ and $0 < \beta < \frac{\mu}{k}$, $\lambda > 0$, $\gamma \in (0, \frac{1}{L})$, where L is the spectral radius of the operator A^*A, $\{\alpha_n\} \subset (0, 1)$ and $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ are two multi-valued maximal monotone operators on H_1 and H_2 respectively. Under certain conditions, the sequence generated by the proposed Algorithm 1.3 was proved to converge strongly to a common solution of split variational inclusion problem and fixed point problem for a nonexpansive mapping.

Remark 1.4 Here, we remark that the step size γ of Algorithms 1.2 and 1.3 above plays an essential role in the convergence properties of the algorithms. Many of the existing iterative methods for solving SVIP involve step size that depends on the norm of the bounded linear operator A. Such algorithms are usually not easy to implement because they require computation of the operator norm which oftentimes is difficult to compute. In addition, the step size defined by these methods are often very small and deteriorates the convergence rate of the algorithm. A larger step size can often be used in practice to yield better numerical results.

Very recently, Tang (2020) proposed the following Halpern-type algorithm with self-adaptive step size for solving SVIP in the framework of Hilbert spaces.

Algorithm 1.5

Choose a positive sequence $\{\rho_n\}$ *satisfying* $0 < \rho_n < 4$ *and* $\inf \rho_n(4-\rho_n) > 0$. *Select arbitrary starting point* x_0 *and set* $n = 0$.
Iterative Step: *Given the iterates* $x_n (n \geq 0)$. *Compute*

$$\tau_n = \frac{\rho_n g(x_n)}{||G(x_n)||^2 + ||H(x_n)||^2},$$

and calculate the next iterate as

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n) J_\lambda^{B_1}(I - \tau_n A^*(I - J_\lambda^{B_2})A)x_n.$$

Stop Criterion: *If* $x_{n+1} = x_n$, *then stop. Otherwise, set* $n := n+1$ *and return to Iterative Step,*

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where $g(x) = \frac{1}{2}||(I - J_\lambda^{B_2})Ax||^2$, $G(x) = A^*(I - J_\lambda^{B_2})Ax$, $H(x) = (I - J_\lambda^{B_1})x$, and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Under mild conditions, strong convergence result was obtained for the proposed algorithm.
Based on the heavy ball methods of a two-order time dynamical system, Polyak (1964) first introduced an inertial extrapolation as an acceleration process to solve the smooth convex minimization problem. The inertial algorithm is a two-step iteration where the next iterate is defined by making use of the previous two iterates. In the recent time, there has been an increasing interest in the study of inertial type algorithms and several authors have constructed some fast iterative methods by employing the inertial technique (see, e.g., Alakoya et al. 2021, 2020, 2021; Izuchukwu et al. 2020; Ogwo et al. 2021; Owolabi et al. 2021).
For approximating the zero point of a maximal monotone operator B, Alvarez and Attouch (2001) introduced the following inertial proximal algorithm:

Algorithm 1.6

$$x_{n+1} = J_{\lambda_n}^B(x_n + \alpha_n(x_n - x_{n-1})), \quad n \geq 1.$$

The authors obtained a weak convergence result for the algorithm under the following conditions:

(B1) There exists $\lambda > 0$ such that for all $n \in \mathbb{N}$, $\lambda_n \geq \lambda$.
(B2) There exists $\alpha \in [0, 1)$ such that for all $n \in \mathbb{N}$, $0 \leq \alpha_n \leq \alpha$.
(B3) $\sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\|^2 < \infty$.

Recently, authors have pointed out one of the drawbacks of the summability condition (B3) of the Algorithm 1.6, that is, to satisfy the summability condition, it is necessary to first calculate α_n at each step (see Moudafi and Oliny 2003).
Very recently, Shehu et al. (2021), Iyiola et al. (2018), Shehu et al. (2020), and Shehu and Iyiola (2020) proposed some efficient inertial iterative methods with self-adaptive step size for approximating solutions of certain classes of optimization problems and the authors were able to establish convergence of the proposed methods under some mild conditions imposed on the control parameters.
From the above review, the following natural question arises:
Question: *Can we construct an inertial iterative method with self-adaptive step size for approximating a common solution of split variational inclusion problem, equilibrium problem and fixed point problem in Hilbert spaces such that condition (B3) of Algorithm 1.6 is dispensed with?*
Our interest in this paper is to provide an affirmative answer to the above question.
Inspired by the above results and the ongoing research interest in this direction, in this paper, we introduce a new inertial iterative scheme which employs the viscosity S-iteration technique with self-adaptive step size for approximating a common element of the set of solutions of split variational inclusion problem, equilibrium problem and common fixed point problem for nonexpansive mappings in Hilbert spaces. Our motivation for studying such a common solution problem lies in its potential application to mathematical models whose constraints can be expressed as split variational inclusion problem, equilibrium problem and common fixed point problem. This arises in practical problems such as signal processing, network resource allocation, image recovery. A scenario is in network bandwidth allocation problem for two services in a heterogeneous wireless access networks in which the bandwidth of the services are mathematically related (see, for instance, Iiduka 2012; Luo et al. 2009 and the references therein). Unlike in Algorithms 1.2 and 1.3 and several other algorithms in the literature, our algorithm is designed such that its implementation does not require the knowledge of the norm of the bounded linear operator. Moreover, our work extend the results in Agarwal et al. (2007), Wangkeeree et al. (2018), Tang (2020), Alvarez and Attouch (2001) to the problem of finding a common solution of split variational inclusion problem, equilibrium problem and common fixed point problem for nonexpansive mappings and the inertial technique employed is more efficient than that used in Alvarez and Attouch (2001). Under some mild conditions, we prove strong convergence theorem for the proposed algorithm. Furthermore, we apply our result to study other optimization problems and we provide some numerical experiments with graphical illustrations to demonstrate the efficiency of the proposed algorithm in comparison with some existing state of the art algorithms in the literature.
The outline of the paper is as follows: In Sect. 2, we recall some basic definitions and existing results which are needed for the convergence analysis of the proposed algorithm. In Sect. 3, we present the proposed algorithm and highlight some of its important features while in Sect. 4 we discuss its convergence. In Sect. 5 we apply our results to study split minimization problem and split feasibility problems. In Sect. 6, numerical examples and comparison with some related algorithms are presented to demonstrate the performance of our new algorithm. Finally, we give the concluding remarks in Sect. 7.

2 Preliminaries

In this section, we recall some concepts and results which will be employed in the sequel. Let H be a real Hilbert space, for a nonempty closed and convex subset C of H, the *metric projection* $P_C : H \to C$ is defined, for each $x \in H$, as the unique element $P_C x \in C$ such that

$$||x - P_C x|| = \inf\{||x - z|| : z \in C\}.$$

It is known that P_C is firmly nonexpansive, i.e.,

$$||P_C x - P_C y||^2 \leq \langle P_C x - P_C y, x - y\rangle, \tag{2.1}$$

for all $x, y \in H$. Moreover, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if (see [?])

$$\langle x - z, z - y\rangle \geq 0 \quad \forall\, y \in C. \tag{2.2}$$

In what follows, we denote the weak and strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively and $w_\omega(x_n)$ denotes set of weak limits of $\{x_n\}$, that is,

$$\omega_w(x_n) := \{x \in H : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\}.$$

Definition 2.1 Let H be a real Hilbert space, $C \subset H$ be a subset of H and $h : C \to H$ be an operator from C onto H. The operator h is said to be

(1) *firmly nonexpansive* if

$$\langle h(x) - h(y), x - y\rangle \geq ||h(x) - h(y)||^2, \quad \forall x, y \in C;$$

(2) *L-Lipschitz continuous*, where $L > 0$, if

$$||hx - hy|| \leq L||x - y||, \quad \forall\ x, y \in C;$$

if $L \in [0, 1)$, then T is called a *contraction mapping;*

(3) *nonexpansive* if T is 1$-$Lipschitz continuous;
(4) *hemicontinuous* if it is continuous along each line segment in C.

Lemma 2.2 (Zhao et al. 2018) *(Demiclosedness Principle). Let T be a nonexpansive mapping on a closed convex subset C of a real Hilbert space H. Then $I - T$ is demiclosed at any point $y \in H$, that is, if $x_n \rightharpoonup x$ and $x_n - Tx_n \to y \in H$, then $x - Tx = y$.*

Definition 2.3 Let H be a real Hilbert space. A function $f : H \to \mathbb{R} \cup \{+\infty\}$ is said to be weakly lower semicontinuous (w-lsc) at $x \in H$, if

$$f(x) \leq \liminf_{n\to\infty} f(x_n)$$

holds for an arbitrary sequence $\{x_n\}_{n=0}^{\infty}$ in H satisfying $x_n \rightharpoonup x$.

Definition 2.4 Let H be a real Hilbert space and $\lambda > 0$. The operator $B : H \to 2^H$ is said to be

- *monotone* if

$$\langle u - v, x - y\rangle \geq 0 \text{ for all } u \in B(x), v \in B(y).$$

- *maximal monotone* mapping if the graph $G(B)$ of B,

$$G(B) := \{(x, u) \in H \times H | u \in B(x)\},$$

  is not properly contained in the graph of any other monotone mapping.
- The *resolvent* of B with parameter $\lambda > 0$ denoted by J_λ^B is given by

$$J_\lambda^B := (I + \lambda B)^{-1},$$

  where I is the identity operator.

Remark 2.5 For $\lambda > 0$, the following results hold (Tang 2020):

(1) B is maximal monotone if and only if J_λ^B is single-valued, firmly nonexpansive and $dom(J_\lambda^B) = H$, where $dom(B) := \{x \in H | B(x) \neq \emptyset\}$.
(2) The point $x^* \in B^{-1}(0)$ if and only if $x^* = J_\lambda^B x^*$.
(3) The solution set $\mathcal{F}$ of the SVIP (1.8)-(1.9) is equivalent to the following:

$$\text{Find } x^* \in H_1 \text{ with } x^* = J_\lambda^{B_1} x^* \text{ such that } y^* = Ax^* \in H_2 \text{ and } y^* = J_\lambda^{B_2} y^* \tag{2.3}$$

Assumption 2.6 For solving the EP, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) F is upper hemicontinuous, that is, for all $x, y, z \in C$, $\lim_{t \downarrow 0} F\big(tz + (1-t)x, y\big) \leq F(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.7 Ma et al. (2015) *Let C be a nonempty closed convex subset of a Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.6. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to C$ as follows:*

$$T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall\, y \in C\}. \tag{2.4}$$

Then T_r^F is well defined and the following hold:

(1) *for each $x \in H$, $T_r^F(x) \neq \emptyset$;*
(2) *T_r^F is single-valued;*
(3) *T_r^F is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle;$$

(4) *$F(T_r^F) = EP(F)$;*
(5) *$EP(F)$ is closed and convex.*

Lemma 2.8 Chuang (2013), Ogwo et al. (2021) *Let H be a real Hilbert space. Then the following results hold for all $x, y \in H$ and $\delta \in \mathbb{R}$:*

(i) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle$;
(ii) $||x + y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2$;
(iii) $||x - y||^2 = ||x||^2 - 2\langle x, y\rangle + ||y||^2$.
(iv) $||\delta x + (1-\delta)y||^2 = \delta||x||^2 + (1-\delta)||y||^2 - \delta(1-\delta)||x - y||^2$.

Lemma 2.9 Maingé (2007) *Let* $\{a_n\}, \{c_n\} \subset \mathbb{R}_+$, $\{\sigma_n\} \subset (0, 1)$ *and* $\{b_n\} \subset \mathbb{R}$ *be sequences such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + b_n + c_n \ \textit{for all } n \geq 0.$$

Assume $\sum_{n=0}^{\infty} |c_n| < \infty$. *Then the following results hold:*

(1) *If* $b_n \leq \beta\sigma_n$ *for some* $\beta \geq 0$, *then* $\{a_n\}$ *is a bounded sequence.*
(2) *If we have*

$$\sum_{n=0}^{\infty} \sigma_n = \infty \ \textit{ and } \ \limsup_{n\to\infty} \frac{b_n}{\sigma_n} \leq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10 Saejung and Yotkaew (2012) *Let* $\{a_n\}$ *be a sequence of non-negative real numbers,* $\{\alpha_n\}$ *be a sequence in* $(0, 1)$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\{b_n\}$ *be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, \quad \textit{for all } \ n \geq 1,$$

if $\limsup_{k\to\infty} b_{n_k} \leq 0$ *for every subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying* $\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \geq 0$, *then* $\lim_{n\to\infty} a_n = 0$.

3 Proposed method

In this section, we present the proposed algorithm. First, we define the following functions:

$$g(x) = \frac{1}{2}||(I - J_{\lambda_2}^{B_2})Ax||^2, \qquad h(x) = \frac{1}{2}||(I - J_{\lambda_1}^{B_1})x||^2$$

and

$$G(x) = A^*(I - J_{\lambda_2}^{B_2})Ax, \qquad H(x) = (I - J_{\lambda_1}^{B_1})x.$$

From Aubin (1993), it can easily be verified that g and h are weak lower semi-continuous and convex differentiable. Moreover, G and H are Lipschitz continuous (see Tang 2020). In what follows, we assume that C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are maximal monotone mappings, $F : C \times C \to \mathbb{R}$ is a bifunction satisfying Assumption 2.6, and $A : H_1 \to H_2$ is a bounded linear operator with A^* being its adjoint operator ($A^* = A^T$ in finite dimensional spaces). Let $S, T : H_1 \to H_1$ be nonexpansive mappings and $f : H_1 \to H_1$ be a contraction with coefficient $k \in (0, 1)$. We denote the solution set by $\Omega = F(S)\cap F(T)\cap\mathcal{F}\cap EP(F) \neq \emptyset$. We establish the convergence of the algorithm under the following conditions on the control parameters:

(C1) Let $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C2) Let $\{\beta_n\}, \{\sigma_n\}, \{\delta_n\}, \{\xi_n\} \subset [a, b]$, $a, b \in (0, 1)$ and such that $\alpha_n + \delta_n + \xi_n = 1$;
(C3) Let $\theta > 0$, $\{\mu_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\mu_n}{\alpha_n} = 0$;
(C4) $0 < a \leq \rho_n \leq b < 4$, $\{r_n\} \subset (0, \infty)$ such that $\liminf_{n\to\infty} r_n > 0$, and $\lambda_i > 0$, $i = 1, 2$.

Now, our main algorithm is presented as follows:

Algorithm 3.1

Step 0. *Let $x_0, x_1 \in H$ be two arbitrary initial points and set $n = 1$.*

Step 1. *Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by*

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\mu_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. *Compute*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. *Find $u_n \in C$ such that*

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \geq 0. \tag{3.2}$$

Step 4. *Compute*

$$v_n = \beta_n w_n + (1 - \beta_n)u_n.$$

Step 5. *Compute*

$$t_n = J_{\lambda_1}^{B_1}(I - \gamma_n A^*(I - J_{\lambda_2}^{B_2})A)v_n,$$

where

$$\gamma_n := \begin{cases} \frac{\rho_n g(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2}, & \text{if } ||G(v_n)||^2 + ||H(v_n)||^2 \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 6. *Compute*

$$z_n = (1 - \sigma_n)v_n + \sigma_n St_n.$$

Step 7. *Compute*

$$x_{n+1} = \alpha_n f(x_n) + \delta_n St_n + \xi_n Tz_n.$$

*Set $n := n + 1$ and return to **Step 1**.*

Remark 3.2 By conditions (C1) and (C3), one can easily verify from (3.1) that

$$\lim_{n\to\infty} \theta_n||x_n - x_{n-1}|| = 0 \text{ and } \lim_{n\to\infty} \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| = 0.$$

Remark 3.3 Observe that the step size of our proposed Algorithm (3.1) is constructed such that its implementation does not require knowledge of the operator norm. Moreover, the implementation of the inertial term does not require the very stringent summability condition (B3) of Algorithm 1.6 (Alvarez and Attouch 2001). These important features make our proposed method easily implementable.

Remark 3.4 We remark that since the split variational inclusion problem generalizes the split feasibility problem and variational inequality problem, our proposed method could be viewed as an extension of the methods proposed in Shehu and Ogbuisi (2015), Cai et al. (2018), Gibali and Shehu (2019).

Remark 3.5 Observe that according to Lemma 2.7, the operator defined in (2.4) is single-valued. To compute u_n in (3.2) is equivalent to evaluating $T_{r_n}^F(w_n)$. In doing this, we find an element $u_n \in C$ such that inequality (3.2) holds. By applying the given definition of $F(u_n, y)$ and simplifying the resulting inequality, we will obtain a quadratic function in variable y. Since the operator $T_{r_n}^F$ is single-valued, then the quadratic function will have at most one solution in $\mathbb{R}$. Hence, the value of u_n for which the discriminant of the quadratic function is zero is determined and this gives the value of the operator $T_{r_n}^F$ at w_n.

4 Convergence analysis

Next, we state the strong convergence theorem for the proposed algorithm as follows.

Theorem 4.1 *Let H_1 and H_2 be real Hilbert spaces, and $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. Suppose $S, T : H_1 \to H_1$ are nonexpansive mappings, and $f : H_1 \to H_1$ is a contraction with coefficient $k \in (0, 1)$. Let $\{x_n\}$ be the sequence generated by Algorithm 3.1 such that conditions (A1)-(A4) and (C1)-(C4) are satisfied. Then $\{x_n\}$ converges strongly to a point $\hat{x} \in \Omega$, where $\hat{x} = P_\Omega \circ f(\hat{x})$.*

We divide the proof of the strong convergence theorem into the following lemmas.

Lemma 4.2 *Suppose that $\{x_n\}$ is the sequence generated by Algorithm 3.1. Then $\{x_n\}$ is bounded.*

Proof Observe that by (3.2), $u_n = T_{r_n}^F w_n$. Also, we note that the mapping $P_\Omega \circ f$ is a contraction. Then by the Banach Contraction Principle, there exists $p \in H_1$ such that $p = P_\Omega \circ f(p)$ and in particular $p \in \Omega$. Thus, it follows that $Sp = p = Tp$, $T_{r_n} p = p$, $J_{\lambda_1}^{B_1} p = p$ and $J_{\lambda_2}^{B_2}(Ap) = Ap$. Since $T_{r_n}^F$ is nonexpansive, then we have

$$||u_n - p|| = ||T_{r_n}^F w_n - p|| \leq ||w_n - p||. \tag{4.1}$$

Applying (4.1), we get

$$\begin{aligned} ||v_n - p|| &= ||\beta_n w_n + (1 - \beta_n)u_n - p|| \\ &\leq \beta_n ||w_n - p|| + (1 - \beta_n)||u_n - p|| \\ &\leq \beta_n ||w_n - p|| + (1 - \beta_n)||w_n - p|| \\ &= ||w_n - p||. \end{aligned} \tag{4.2}$$

Next, by the definition of $G(x)$ and the firmly nonexpansivity of $I - J_{\lambda_2}^{B_2}$, we have

$$\begin{aligned} \langle G(v_n), v_n - p\rangle &= \langle A^*(I - J_{\lambda_2}^{B_2})Av_n, v_n - p\rangle \\ &= \langle (I - J_{\lambda_2}^{B_2})Av_n, Av_n - Ap\rangle \\ &= \langle (I - J_{\lambda_2}^{B_2})Av_n - (I - J_{\lambda_2}^{B_2})Ap, Av_n - Ap\rangle \end{aligned}$$

$$\geq ||(I - J_{\lambda_2}^{B_2})Av_n||^2$$
$$= 2g(v_n). \tag{4.3}$$

Then by Lemma 2.8(iii) and applying (4.3) together with the nonexpansivity of $J_{\lambda_1}^{B_1}$, it follows that

$$\begin{aligned}
||t_n - p||^2 &= ||J_{\lambda_1}^{B_1}(I - \gamma_n A^*(I - J_{\lambda_2}^{B_2})A)v_n - p||^2 \\
&\leq ||v_n - \gamma_n A^*(I - J_{\lambda_2}^{B_2})Av_n - p||^2 \\
&= ||v_n - p - \gamma_n G(v_n)||^2 \\
&= ||v_n - p||^2 + \gamma_n^2||G(v_n)||^2 - 2\gamma_n\langle G(v_n), v_n - p\rangle \\
&\leq ||v_n - p||^2 + \gamma_n^2||G(v_n)||^2 - 4\gamma_n g(v_n) \\
&= ||v_n - p||^2 + \frac{\rho_n^2 g^2(v_n)}{(||G(v_n)||^2 + ||H(v_n)||^2)^2}||G(v_n)||^2 - \frac{4\rho_n g^2(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2} \\
&\leq ||v_n - p||^2 - \frac{(4 - \rho_n)\rho_n g^2(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2}.
\end{aligned} \tag{4.4}$$

By the condition on ρ_n, we have that

$$||t_n - p|| \leq ||v_n - p||. \tag{4.5}$$

Applying (4.5), we obtain

$$\begin{aligned}
||z_n - p|| &= ||(1 - \sigma_n)v_n + \sigma_n St_n - p|| \\
&\leq (1 - \sigma_n)||v_n - p|| + \sigma_n||St_n - p|| \\
&\leq (1 - \sigma_n)||v_n - p|| + \sigma_n||t_n - p|| \\
&\leq (1 - \sigma_n)||v_n - p|| + \sigma_n||v_n - p|| \\
&= ||v_n - p||.
\end{aligned} \tag{4.6}$$

Next, applying the triangle inequality, we get

$$\begin{aligned}
||w_n - p|| &= ||x_n + \theta_n(x_n - x_{n-1}) - p|| \\
&\leq ||x_n - p|| + \theta_n||x_n - x_{n-1}|| \\
&= ||x_n - p|| + \alpha_n\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||.
\end{aligned} \tag{4.7}$$

By Remark 3.2, $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| = 0$. Then, it follows that there exists a constant $M_1 > 0$ such that $\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \leq M_1$ for all $n \geq 1$. Hence, from (4.7) we obtain

$$||w_n - p|| \leq ||x_n - p|| + \alpha_n M_1. \tag{4.8}$$

Applying (4.2), (4.5), (4.6) and (4.8), we have

$$\begin{aligned}
||x_{n+1} - p|| &= ||\alpha_n f(x_n) + \delta_n St_n + \xi_n Tz_n - p|| \\
&= ||\alpha_n(f(x_n) - fp) + \alpha_n(f(p) - p) + \delta_n(St_n - p) + \xi_n(Tz_n - p)|| \\
&\leq \alpha_n k||x_n - p|| + \alpha_n||f(p) - p|| + \delta_n||t_n - p|| + \xi_n||z_n - p|| \\
&\leq \alpha_n k||x_n - p|| + \alpha_n||f(p) - p|| + \delta_n(||x_n - p|| + \alpha_n M_1) \\
&\quad + \xi_n(||x_n - p|| + \alpha_n M_1) \\
&= (\alpha_n k + (1 - \alpha_n))||x_n - p|| + \alpha_n||f(p) - p|| + (1 - \alpha_n)\alpha_n M_1
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n(1-k))||x_n - p|| + \alpha_n(1-k)\Big\{\frac{||f(p)-p||}{1-k} + \frac{(1-\alpha_n)M_1}{1-k}\Big\} \\
&\leq (1-\alpha_n(1-k))||x_n - p|| + \alpha_n(1-k)M^*,
\end{aligned}
$$

where $M^* := \sup_{n\in\mathbb{N}}\left\{\frac{||f(p)-p||}{1-k} + \frac{(1-\alpha_n)M_1}{1-k}\right\}$. Setting $a_n := ||x_n - p||$; $b_n := \alpha_n(1-k)M^*$; $c_n := 0$, and $\sigma_n := \alpha_n(1-k)$. By invoking Lemma 2.9 together with the assumptions on the control parameters, we have that $\{||x_n - p||\}$ is bounded and this implies that $\{x_n\}$ is bounded. Consequently, $\{w_n\}$, $\{u_n\}$, $\{v_n\}$, $\{t_n\}$ and $\{z_n\}$ are all bounded. □

Lemma 4.3 *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1 and $p \in \Omega$. Then, under conditions* (C1)-(C4) *and for all $n \in \mathbb{N}$, we have*

$$
\begin{aligned}
||x_{n+1} - p||^2 \leq & \Big(1 - \frac{2\alpha_n(1-k)}{(1-\alpha_n k)}\Big)||x_n - p||^2 + \frac{2\alpha_n(1-k)}{(1-\alpha_n k)}\Big\{\frac{\alpha_n M_3}{2(1-k)} \\
& + \frac{3M_2(1-\alpha_n)^2}{2(1-k)}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \\
& + \frac{1}{(1-k)}\langle f(p) - p, x_{n+1} - p\rangle\Big\} - \frac{\xi_n(1-\alpha_n)}{(1-\alpha_n k)}\Big\{\beta_n(1-\beta_n)||w_n - u_n||^2 \\
& + \frac{(4-\rho_n)\sigma_n\rho_n g^2(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2} \\
& + \sigma_n(1-\sigma_n)||v_n - St_n||^2\Big\}.
\end{aligned}
$$

Proof Let $p \in \Omega$. Then, applying Lemma 2.8(iv) and (4.1), we get

$$
\begin{aligned}
||v_n - p||^2 &= ||\beta_n w_n + (1-\beta_n)u_n - p||^2 \\
&= \beta_n||w_n - p||^2 + (1-\beta_n)||u_n - p||^2 - \beta_n(1-\beta_n)||w_n - u_n||^2 \\
&\leq \beta_n||w_n - p||^2 + (1-\beta_n)||w_n - p||^2 - \beta_n(1-\beta_n)||w_n - u_n||^2 \\
&\leq ||w_n - p||^2 - \beta_n(1-\beta_n)||w_n - u_n||^2. \qquad (4.9)
\end{aligned}
$$

Again, by invoking Lemma 2.8(ii) and applying Cauchy-Schwartz inequality we have

$$
\begin{aligned}
||w_n - p||^2 &= ||x_n + \theta_n(x_n - x_{n-1}) - p||^2 \\
&= ||x_n - p||^2 + \theta_n^2||x_n - x_{n-1}||^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle \\
&\leq ||x_n - p||^2 + \theta_n^2||x_n - x_{n-1}||^2 + 2\theta_n||x_n - x_{n-1}||||x_n - p|| \\
&= ||x_n - p||^2 + \theta_n||x_n - x_{n-1}||(\theta_n||x_n - x_{n-1}|| + 2||x_n - p||) \\
&\leq ||x_n - p||^2 + 3M_2\theta_n||x_n - x_{n-1}|| \\
&= ||x_n - p||^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||, \qquad (4.10)
\end{aligned}
$$

where $M_2 := \sup_{n\in\mathbb{N}}\{||x_n - p||, \theta_n||x_n - x_{n-1}||\} > 0$.
Next, applying Lemma 2.8(iv), (4.4), (4.9) and (4.10), we obtain

$$
\begin{aligned}
||z_n - p||^2 &= ||(1-\sigma_n)v_n + \sigma_n St_n - p||^2 \\
&= (1-\sigma_n)||v_n - p||^2 + \sigma_n||St_n - p||^2 - \sigma_n(1-\sigma_n)||v_n - St_n||^2 \\
&\leq (1-\sigma_n)||v_n - p||^2 + \sigma_n||t_n - p||^2 - \sigma_n(1-\sigma_n)||v_n - St_n||^2 \\
&\leq (1-\sigma_n)||v_n - p||^2 + \sigma_n\Big\{||v_n - p||^2 - \frac{(4-\rho_n)\rho_n g^2(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&- \sigma_n(1-\sigma_n)||v_n - St_n||^2 \\
&= ||v_n - p||^2 - \frac{(4-\rho_n)\sigma_n\rho_n g^2(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2} \\
&\quad - \sigma_n(1-\sigma_n)||v_n - St_n||^2 \\
&\le ||w_n - p||^2 - \beta_n(1-\beta_n)||w_n - u_n||^2 - \frac{(4-\rho_n)\sigma_n\rho_n g^2(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2} \\
&\quad - \sigma_n(1-\sigma_n)||v_n - St_n||^2. \qquad (4.11)
\end{aligned}
$$

Now, invoking Lemma 2.8 and applying (4.2), (4.5) and (4.11) we have

$$
\begin{aligned}
||x_{n+1} - p||^2 &= ||\alpha_n f(x_n) + \delta_n St_n + \xi_n Tz_n - p||^2 \\
&\le ||\delta_n(St_n - p) + \xi_n(Tz_n - p)||^2 + 2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle \\
&\le \delta_n^2||St_n - p||^2 + \xi_n^2||Tz_n - p||^2 + 2\delta_n\xi_n||St_n - p||||Tz_n - p|| \\
&\quad + 2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle \\
&\le \delta_n^2||St_n - p||^2 + \xi_n^2||Tz_n - p||^2 + \delta_n\xi_n\big(||St_n - p||^2 + ||Tz_n - p||^2\big) \\
&\quad + 2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle \\
&= \delta_n(\delta_n + \xi_n)||St_n - p||^2 + \xi_n(\xi_n + \delta_n)||Tz_n - p||^2 \\
&\quad + 2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle \\
&\le \delta_n(1-\alpha_n)||t_n - p||^2 + \xi_n(1-\alpha_n)||z_n - p||^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le \delta_n(1-\alpha_n)||w_n - p||^2 + \xi_n(1-\alpha_n)\Big\{||w_n - p||^2 \\
&\quad - \beta_n(1-\beta_n)||w_n - u_n||^2 - \frac{(4-\rho_n)\sigma_n\rho_n g^2(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2} \\
&\quad - \sigma_n(1-\sigma_n)||v_n - St_n||^2\Big\} + 2\alpha_n k||x_n - p||||x_{n+1} - p|| \\
&\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le (1-\alpha_n)^2||w_n - p||^2 - \xi_n(1-\alpha_n)\Big\{\beta_n(1-\beta_n)||w_n - u_n||^2 \\
&\quad + \frac{(4-\rho_n)\sigma_n\rho_n g^2(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2} \\
&\quad + \sigma_n(1-\sigma_n)||v_n - St_n||^2\Big\} + \alpha_n k\big(||x_n - p||^2 + ||x_{n+1} - p||^2\big) \\
&\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le (1-\alpha_n)^2\Big(||x_n - p||^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||\Big) \\
&\quad - \xi_n(1-\alpha_n)\Big\{\beta_n(1-\beta_n)||w_n - u_n||^2 \\
&\quad + \frac{(4-\rho_n)\sigma_n\rho_n g^2(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2} + \sigma_n(1-\sigma_n)||v_n - St_n||^2\Big\} \\
&\quad + \alpha_n k\big(||x_n - p||^2 + ||x_{n+1} - p||^2\big) \\
&\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&= \big((1-\alpha_n)^2 + \alpha_n k\big)||x_n - p||^2 + \alpha_n k||x_{n+1} - p||^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 3M_2(1-\alpha_n)^2\alpha_n\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \\
&- \xi_n(1-\alpha_n)\Big\{\beta_n(1-\beta_n)||w_n - u_n||^2 + \frac{(4-\rho_n)\sigma_n\rho_n g^2(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2} \\
&+ \sigma_n(1-\sigma_n)||v_n - St_n||^2\Big\} \\
&+ 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle
\end{aligned}
$$

From this, we obtain

$$
\begin{aligned}
||x_{n+1} - p||^2 \leq{}& \frac{(1-2\alpha_n + \alpha_n^2 + \alpha_n k)}{(1-\alpha_n k)}||x_n - p||^2 + \frac{\alpha_n}{(1-\alpha_n k)}\Big\{3M_2(1-\alpha_n)^2\frac{\theta_n}{\alpha_n}||x_n \\
&- x_{n-1}|| + 2\langle f(p) - p, x_{n+1} - p\rangle\Big\} \\
&- \frac{\xi_n(1-\alpha_n)}{(1-\alpha_n k)}\Big\{\beta_n(1-\beta_n)||w_n - u_n||^2 + \frac{(4-\rho_n)\sigma_n\rho_n g^2(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2} \\
&+ \sigma_n(1-\sigma_n)||v_n - St_n||^2\Big\} \\
={}& \frac{(1-2\alpha_n + \alpha_n k)}{(1-\alpha_n k)}||x_n - p||^2 + \frac{\alpha_n^2}{(1-\alpha_n k)}||x_n - p||^2 \\
&+ \frac{\alpha_n}{(1-\alpha_n k)}\Big\{3M_2(1-\alpha_n)^2\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \\
&+ 2\langle f(p) - p, x_{n+1} - p\rangle\Big\} - \frac{\xi_n(1-\alpha_n)}{(1-\alpha_n k)}\Big\{\beta_n(1-\beta_n)||w_n - u_n||^2 \\
&+ \frac{(4-\rho_n)\sigma_n\rho_n g^2(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2} \\
&+ \sigma_n(1-\sigma_n)||v_n - St_n||^2\Big\} \\
\leq{}& \Big(1 - \frac{2\alpha_n(1-k)}{(1-\alpha_n k)}\Big)||x_n - p||^2 + \frac{2\alpha_n(1-k)}{(1-\alpha_n k)}\Big\{\frac{\alpha_n M_3}{2(1-k)} \\
&+ \frac{3M_2(1-\alpha_n)^2}{2(1-k)}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \\
&+ \frac{1}{(1-k)}\langle f(p) - p, x_{n+1} - p\rangle\Big\} - \frac{\xi_n(1-\alpha_n)}{(1-\alpha_n k)}\Big\{\beta_n(1-\beta_n)||w_n - u_n||^2 \\
&+ \frac{(4-\rho_n)\sigma_n\rho_n g^2(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2} \\
&+ \sigma_n(1-\sigma_n)||v_n - St_n||^2\Big\},
\end{aligned}
$$

where $M_3 := \sup\{||||x_n - p||^2 : n \in \mathbb{N}\}$. This completes the proof. □

Lemma 4.4 *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such that conditions (C1)-(C4) hold. Then, the following inequality holds for all $p \in \Omega$ and $n \in \mathbb{N}$:*

$$\begin{aligned}||x_{n+1}-p||^2 \leq &(1-\alpha_n)||x_n-p||^2+\alpha_n||f(x_n)-p||^2\\ &+3M_2(1-\alpha_n)\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||-\delta_n||t_n-v_n||^2\\ &+2M_4\delta_n||A^*(I-J_{\lambda_2}^{B_2})Av_n||-\delta_n\xi_n||St_n-Tz_n||^2.\end{aligned}$$

Proof Let $p\in\Omega$. From (4.4), observe that

$$||v_n-\gamma_nA^*(I-J_{\lambda_2}^{B_2})Av_n-p||^2\leq||v_n-p||^2.$$

Applying Lemma 2.8 and the firmly nonexpansivity of $J_{\lambda_1}^{B_1}$, we have

$$\begin{aligned}||t_n-p||^2 &= ||J_{\lambda_1}^{B_1}(I-\gamma_nA^*(I-J_{\lambda_2}^{B_2})A)v_n-p||^2\\ &\leq\langle t_n-p, v_n-\gamma_nA^*(I-J_{\lambda_2}^{B_2})Av_n-p\rangle\\ &=\frac{1}{2}\big(||t_n-p||^2+||v_n-\gamma_nA^*(I-J_{\lambda_2}^{B_2})Av_n-p||^2-||t_n-v_n\\ &\quad+\gamma_nA^*(I-J_{\lambda_2}^{B_2})Av_n||^2\big)\\ &\leq\frac{1}{2}\big(||t_n-p||^2+||v_n-p||^2-||t_n-v_n+\gamma_nA^*(I-J_{\lambda_2}^{B_2})Av_n||^2\big)\\ &=\frac{1}{2}\big(||t_n-p||^2+||v_n-p||^2-\big(||t_n-v_n||^2+\gamma_n^2||A^*(I-J_{\lambda_2}^{B_2})Av_n||^2\\ &\quad-2\gamma_n\langle v_n-t_n, A^*(I-J_{\lambda_2}^{B_2})Av_n\rangle\big)\big)\\ &\leq\frac{1}{2}\big(||t_n-p||^2+||v_n-p||^2-||t_n-v_n||^2-\gamma_n^2||A^*(I-J_{\lambda_2}^{B_2})Av_n||^2\\ &\quad+2\gamma_n||v_n-t_n||||A^*(I-J_{\lambda_2}^{B_2})Av_n||\big)\\ &\leq\frac{1}{2}\big(||t_n-p||^2+||v_n-p||^2-||t_n-v_n||^2+2\gamma_n||v_n-t_n||||A^*\\ &\quad\times(I-J_{\lambda_2}^{B_2})Av_n||\big).\end{aligned}$$

Consequently, we have that

$$\begin{aligned}||t_n-p||^2 &\leq||v_n-p||^2-||t_n-v_n||^2+2\gamma_n||v_n-t_n||||A^*(I-J_{\lambda_2}^{B_2})Av_n||\\ &\leq||v_n-p||^2-||t_n-v_n||^2+2M_4||A^*(I-J_{\lambda_2}^{B_2})Av_n||\\ &\leq||w_n-p||^2-||t_n-v_n||^2+2M_4||A^*(I-J_{\lambda_2}^{B_2})Av_n||,\end{aligned}\tag{4.12}$$

where $M_4:=\sup\{\gamma_n||v_n-t_n||:n\in\mathbb{N}\}$.
Next, by invoking Lemma 2.8(iv) and applying (4.2), (4.6), (4.10) and (4.12) we obtain

$$\begin{aligned}||x_{n+1}-p||^2 &= ||\alpha_nf(x_n)+\delta_nSt_n+\xi_nTz_n-p||^2\\ &=\alpha_n||f(x_n)-p||^2+\delta_n||St_n-p||^2+\xi_n||Tz_n-p||^2-\delta_n\xi_n||St_n-Tz_n||^2\\ &\leq\alpha_n||f(x_n)-p||^2+\delta_n||t_n-p||^2+\xi_n||z_n-p||^2-\delta_n\xi_n||St_n-Tz_n||^2\\ &\leq\alpha_n||f(x_n)-p||^2+\delta_n\big(||w_n-p||^2-||t_n-v_n||^2\\ &\quad+2M_4||A^*(I-J_{\lambda_2}^{B_2})Av_n||\big)+\xi_n||w_n-p||^2\\ &\quad-\delta_n\xi_n||St_n-Tz_n||^2\\ &=\alpha_n||f(x_n)-p||^2+(1-\alpha_n)||w_n-p||^2-\delta_n||t_n-v_n||^2\end{aligned}$$

$$
\begin{aligned}
&+ 2M_4\delta_n||A^*(I - J_{\lambda_2}^{B_2})Av_n|| \\
&- \delta_n\xi_n||St_n - Tz_n||^2 \\
\leq\ & \alpha_n||f(x_n) - p||^2 + (1-\alpha_n)\Big(||x_n - p||^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||\Big) \\
&- \delta_n||t_n - v_n||^2 \\
&+ 2M_4\delta_n||A^*(I - J_{\lambda_2}^{B_2})Av_n|| - \delta_n\xi_n||St_n - Tz_n||^2 \\
=\ & (1-\alpha_n)||x_n - p||^2 + \alpha_n||f(x_n) - p||^2 \\
&+ 3M_2(1-\alpha_n)\alpha_n\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| - \delta_n||t_n - v_n||^2 \\
&+ 2M_4\delta_n||A^*(I - J_{\lambda_2}^{B_2})Av_n|| - \delta_n\xi_n||St_n - Tz_n||^2,
\end{aligned}
$$

which is the required inequality. □

Lemma 4.5 *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1 such that conditions (A1)-(A4) and (C1)-(C4) are satisfied. Then $\{x_n\}$ converges strongly to a point $\hat{x} \in \Omega$, where $\hat{x} = P_\Omega \circ f(\hat{x})$.*

Proof Let $\hat{x} = P_\Omega \circ f(\hat{x})$. It then follows from Lemma 4.3 that

$$
\begin{aligned}
||x_{n+1} - \hat{x}||^2 \leq \Big(1 - \frac{2\alpha_n(1-k)}{(1-\alpha_n k)}\Big)||x_n - \hat{x}||^2 + \frac{2\alpha_n(1-k)}{(1-\alpha_n k)}\Big\{\frac{\alpha_n M_3}{2(1-k)} \\
+ \frac{3M_2(1-\alpha_n)^2}{2(1-k)}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \\
+ \frac{1}{(1-k)}\langle f(\hat{x}) - \hat{x}, x_{n+1} - \hat{x}\rangle\Big\}. \quad (4.13)
\end{aligned}
$$

Now, we claim that the sequence $\{||x_n - \hat{x}||\}$ converges to zero. To do this, by Lemma 2.10 it suffices to show that $\limsup_{k\to\infty}\langle f(\hat{x}) - \hat{x}, x_{n_k+1} - \hat{x}\rangle \leq 0$ for every subsequence $\{||x_{n_k} - \hat{x}||\}$ of $\{||x_n - \hat{x}||\}$ satisfying

$$\liminf_{k\to\infty}(||x_{n_k+1} - \hat{x}|| - ||x_{n_k} - \hat{x}||) \geq 0.$$

Now, suppose that $\{||x_{n_k} - \hat{x}||\}$ is a subsequence of $\{||x_n - \hat{x}||\}$ such that

$$\liminf_{k\to\infty}(||x_{n_k+1} - \hat{x}|| - ||x_{n_k} - \hat{x}||) \geq 0. \quad (4.14)$$

From Lemma 4.3 we have

$$
\begin{aligned}
&\frac{\xi_{n_k}(1-\alpha_{n_k})}{(1-\alpha_{n_k}k)}\beta_{n_k}(1-\beta_{n_k})||w_{n_k} - u_{n_k}||^2 \\
&\leq \Big(1 - \frac{2\alpha_{n_k}(1-k)}{(1-\alpha_{n_k}k)}\Big)||x_{n_k} - p||^2 - ||x_{n_k+1} - p||^2 + \frac{2\alpha_{n_k}(1-k)}{(1-\alpha_{n_k}k)}\Big\{\frac{\alpha_{n_k}M_3}{2(1-k)} \\
&+ \frac{3M_2(1-\alpha_{n_k})^2}{2(1-k)}\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}|| + \frac{1}{(1-k)}\langle f(p) - p, x_{n_k+1} - p\rangle\Big\}.
\end{aligned}
$$

By (4.14) together with the fact that $\lim_{k\to\infty}\alpha_{n_k} = 0$, we obtain

$$\frac{\xi_{n_k}(1-\alpha_{n_k})}{(1-\alpha_{n_k}k)}\beta_{n_k}(1-\beta_{n_k})||w_{n_k} - u_{n_k}||^2 \to 0, \quad k \to \infty.$$

Consequently, we have

$$||w_{n_k} - u_{n_k}|| \to 0, \quad k \to \infty. \tag{4.15}$$

Following similar argument, from Lemma 4.3 we obtain

$$||v_{n_k} - St_{n_k}| \to 0, \quad k \to \infty, \tag{4.16}$$

and

$$\frac{(4-\rho_{n_k})\sigma_{n_k}\rho_{n_k}g^2(v_{n_k})}{||G(v_{n_k})||^2 + ||H(v_{n_k})||^2} \to 0, \quad k \to \infty.$$

Since G and H are Lipschitz continuous, then by the condition on ρ_n it follows that

$$g^2(v_{n_k}) \to 0, \quad k \to \infty.$$

From this, we obtain

$$\lim_{k\to\infty} g(v_{n_k}) = \lim_{k\to\infty} \frac{1}{2}||(I - J_{\lambda_2}^{B_2})Av_{n_k}||^2 = 0. \tag{4.17}$$

Consequently, we have

$$||(I - J_{\lambda_2}^{B_2})Av_{n_k}|| \to 0, \quad k \to \infty. \tag{4.18}$$

From this, we get

$$||A^*(I - J_{\lambda_2}^{B_2})Av_{n_k}|| \leq ||A^*||||(I - J_{\lambda_2}^{B_2})Av_{n_k}|| = ||A||||(I - J_{\lambda_2}^{B_2})Av_{n_k}|| \to 0, \quad k \to \infty. \tag{4.19}$$

Also, from Lemma 4.4 we have

$$\begin{aligned}\delta_{n_k}||t_{n_k} - v_{n_k}||^2 \leq &(1-\alpha_{n_k})||x_{n_k} - p||^2 - ||x_{n_k+1} - p||^2 + \alpha_{n_k}||f(x_{n_k}) - p||^2 \\ &+ 3M_2(1-\alpha_{n_k})\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}|| + 2M_4\delta_{n_k}||A^*(I - J_{\lambda_2}^{B_2})Av_{n_k}||.\end{aligned}$$

By (4.14), and by applying (4.19) together with Remark 3.2 and the fact that $\lim_{k\to\infty}\alpha_{n_k} = 0$, we obtain

$$||t_{n_k} - v_{n_k}|| \to 0, \quad k \to \infty. \tag{4.20}$$

Similarly, from Lemma 4.4 we get

$$||St_{n_k} - Tz_{n_k}|| \to 0, \quad k \to \infty. \tag{4.21}$$

By Remark 3.2, we obtain

$$||w_{n_k} - x_{n_k}|| = \theta_{n_k}||x_{n_k} - x_{n_k-1}|| \to 0, \quad k \to \infty. \tag{4.22}$$

Applying (4.15) and (4.22), we get

$$||x_{n_k} - u_{n_k}|| \to 0, \quad k \to \infty; \qquad ||v_{n_k} - x_{n_k}|| \to 0, \quad k \to \infty. \tag{4.23}$$

On the other hand, by applying (4.16), (4.20), (4.21) and (4.23) we obtain

$$||x_{n_k} - t_{n_k}|| \to 0, \quad k \to \infty; \quad ||x_{n_k} - St_{n_k}|| \to 0, \quad k \to \infty; \quad ||x_{n_k} - Tz_{n_k}|| \to 0, \quad k \to \infty. \tag{4.24}$$

Also, by applying (4.23)–(4.25) we get

$$||z_{n_k} - x_{n_k}|| \to 0, \quad k \to \infty; \quad ||t_{n_k} - St_{n_k}|| \to 0, \quad k \to \infty; \quad ||z_{n_k} - Tz_{n_k}|| \to 0, \\ k \to \infty. \tag{4.25}$$

Now, by using (4.24) together with the fact that $\lim_{k\to\infty} \alpha_{n_k} = 0$, we have

$$||x_{n_k+1} - x_{n_k}|| \le \alpha_{n_k}||f(x_{n_k}) - x_{n_k}|| + \delta_{n_k}||St_{n_k} - x_{n_k}|| + \xi_{n_k}||Tz_{n_k} - x_{n_k}|| \to 0, \\ k \to \infty. \tag{4.26}$$

To complete the proof, we need to show that $w_\omega(x_n) \subset \Omega$. First, we claim that $w_\omega(x_n) \subset EP(F)$. Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $x^* \in w_\omega(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$. By (4.23), it follows that $u_{n_k} \rightharpoonup x^*$ as $k \to \infty$. By the definition of $T^F_{r_{n_k}} w_{n_k}$, we have that

$$F(u_{n_k}, y) + \frac{1}{r_{n_k}}\langle y - u_{n_k}, u_{n_k} - w_{n_k}\rangle \ge 0, \quad \forall y \in C.$$

It follows from the monotonicity of F that

$$\frac{1}{r_{n_k}}\langle y - u_{n_k}, u_{n_k} - w_{n_k}\rangle \ge F(y, u_{n_k}), \quad \forall\, y \in C.$$

By (4.15), $\liminf_{k\to\infty} r_{n_k} > 0$, and condition (A4), we have that

$$F(y, x^*) \le 0, \quad \forall\, y \in C. \tag{4.27}$$

Let $y_t = ty + (1-t)x^*, \quad \forall\, t \in (0, 1]$ and $y \in C$. This implies that $y_t \in C$, and it follows from (4.27) that $F(y_t, x^*) \le 0$. So, by applying conditions (A1)-(A4), we have

$$\begin{aligned} 0 &= F(y_t, y_t) \\ &\le tF(y_t, y) + (1-t)F(y_t, x^*) \\ &\le tF(y_t, y). \end{aligned}$$

Hence, we have

$$F(y_t, y) \ge 0, \quad \forall\, y \in C.$$

Letting $t \to 0$, by condition (A3), we get

$$F(x^*, y) \ge 0, \quad \forall\, y \in C.$$

This implies that $x^* \in EP(F)$.
Next, we show that $x^* \in \mathcal{F}$. By the lower semicontinuity of g, it follows from (4.17) that

$$0 \le g(x^*) \le \lim_{k\to\infty} g(v_{n_k}) = \lim_{n\to\infty} g(v_n) = 0,$$

which implies that

$$g(x^*) = \frac{1}{2}||(I - J^{B_2}_{\lambda_2})Ax^*||^2 = 0.$$

Thus, by Remark 2.5 we have that

$$Ax^* \in B_2^{-1}(0) \quad \text{or} \quad 0 \in B_2(Ax^*). \tag{4.28}$$

Since $t_{n_k} = J_{\lambda_1}^{B_1}(v_{n_k} - \gamma_{n_k} A^*(I - J_{\lambda_2}^{B_2})Av_{n_k})$ can be rewritten as

$$v_{n_k} - \gamma_{n_k} A^*(I - J_{\lambda_2}^{B_2})Av_{n_k} \in t_{n_k} + \lambda_1 B_1(t_{n_k})$$

or, equivalently

$$\frac{(v_{n_k} - t_{n_k}) - \gamma_{n_k} A^*(I - J_{\lambda}^{B_2})Av_{n_k}}{\lambda_1} \in B_1(t_{n_k}). \tag{4.29}$$

By passing to limit as $k \to \infty$ in (4.29), applying (4.19), (4.20) and (4.24), and taking into consideration the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(x^*)$. This together with (4.28) implies that $x^* \in \Gamma$.
Next, we show that $x^* \in F(S) \cap F(T)$. By (4.24) and (4.25), we have $t_{n_k} \rightharpoonup x^*$ and $z_{n_k} \rightharpoonup x^*$ as $k \to \infty$. Since S and T are nonexpansive, by the demiclosedness principle, it follows from (4.25) that $x^* \in F(S) \cap F(T)$. Consequently, we have that $w_\omega(x_n) \subset \Omega$.
Moreover, from (4.24) and (4.25) it follows that $w_\omega\{t_n\} = w_\omega\{x_n\} = w_\omega\{z_n\}$. By the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup x^\dagger$ and

$$\lim_{j\to\infty} \langle f(\hat{x}) - \hat{x}, x_{n_{k_j}} - \hat{x}\rangle = \limsup_{k\to\infty} \langle f(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\rangle = \limsup_{k\to\infty} \langle f(\hat{x}) - \hat{x}, t_{n_k} - \hat{x}\rangle.$$

Since $\hat{x} = P_\Omega \circ f(\hat{x})$, then it follows that

$$\limsup_{k\to\infty} \langle f(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\rangle = \lim_{j\to\infty} \langle f(\hat{x}) - \hat{x}, x_{n_{k_j}} - \hat{x}\rangle = \langle f(\hat{x}) - \hat{x}, x^\dagger - \hat{x}\rangle \le 0. \tag{4.30}$$

Now, from (4.26) and (4.30), we obtain

$$\limsup_{k\to\infty} \langle f(\hat{x}) - \hat{x}, x_{n_k+1} - \hat{x}\rangle = \limsup_{k\to\infty} \langle f(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\rangle = \langle f(\hat{x}) - \hat{x}, x^\dagger - \hat{x}\rangle \le 0. \tag{4.31}$$

Applying Lemma 2.10 to (4.13), and using (4.31) together with the fact that $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| = 0$ and $\lim_{n\to\infty} \alpha_n = 0$, we deduce that $\lim_{n\to\infty} ||x_n - \hat{x}|| = 0$ as desired. □

5 Applications

In this section we apply our results to study some related optimization problems.

5.1 Split minimization problem

Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Given some proper, lower semicontinuous and convex functions $f_1 : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $f_2 : H_2 \to \mathbb{R} \cup \{+\infty\}$, the *Split Minimization Problem* (SMP) is defined as

$$\text{Find} \quad \bar{x} \in H_1 \quad \text{such that} \quad \bar{x} \in \arg\min_{x\in H_1} f_1(x) \quad \text{and} \quad A\bar{x} \in \arg\min_{y\in H_2} f_2(y). \tag{5.1}$$

We denote the set of solution of the SMP (5.1) by Γ_{SMP}. The SMP was first introduced by Moudafi and Thakur (2014). It has attracted lots of attention in recent years and has been

applied in the study of many applied science problems such as multi-resolution sparse regularization, Fourier regularization, hard-constrained inconsistent feasibility and alternating projection signal synthesis problems (see (Abbas et al. 2018) and the references therein).
In a real Hilbert space H, the proximal operator of f is defined by

$$prox_{\lambda,f}(x) := \arg\min_{z\in H}\left\{f(z)+\frac{1}{2\lambda}\|x-z\|^2\right\} \quad \forall\, x\in H,\ \ \lambda>0.$$

It is well known that

$$prox_{\lambda,f}(x) = (I+\lambda\partial f)^{-1}(x) = J_{\lambda}^{\partial f}(x), \tag{5.2}$$

where ∂f is the subdifferential of f defined by

$$\partial f(x) = \{z\in H : f(x)-f(y)\le \langle z, x-y\rangle, \forall y\in H\},$$

for each $x\in H$. From [?], ∂f is a maximal monotone operator and $prox_{\lambda,f}$ is firmly nonexpansive.
By setting $B_1=\partial f_1$ and $B_2=\partial f_2$ in Theorem 4.1, we obtain the following result for approximating a common solution of split minimization problem, equilibrium problem and and common fixed point of nonexpansive mappings in Hilbert spaces.

Theorem 5.1 *Let H_1 and H_2 be real Hilbert spaces, and $A: H_1\to H_2$ be a bounded linear operator with adjoint A^*. Let $f_1: H_1\to\mathbb{R}\cup\{+\infty\}$ and $f_2: H_2\to\mathbb{R}\cup\{+\infty\}$ be proper, lower semicontinuous and convex functions, $S, T: H_1\to H_1$ be nonexpansive mappings, and $f: H_1\to H_1$ be a contraction with coefficient $k\in(0,1)$. Suppose that $\Omega = F(S)\cap F(T)\cap\Gamma_{SMP}\cap EP(F)\ne\emptyset$, and conditions (A1)-(A4) and (C1)-(C4) are satisfied. Then the sequence $\{x_n\}$ generated by the following algorithm converges strongly to a point $\hat{x}\in\Omega$, where $\hat{x}=P_\Omega\circ f(\hat{x})$.*

Algorithm 5.2

Step 0. *Let $x_0, x_1\in H$ be two arbitrary initial points and set $n=1$.*
Step 1. *Given the $(n-1)th$ and nth iterates, choose θ_n such that $0\le\theta_n\le\hat{\theta}_n$ with $\hat{\theta}_n$ defined by*

$$\hat{\theta}_n = \begin{cases}\min\left\{\theta,\ \frac{\mu_n}{||x_n-x_{n-1}||}\right\}, & \text{if } x_n\ne x_{n-1},\\ \theta, & \text{otherwise.}\end{cases} \tag{5.3}$$

Step 2. *Compute*

$$w_n = x_n+\theta_n(x_n-x_{n-1}).$$

Step 3. *Compute*

$$F(u_n, y)+\frac{1}{r_n}\langle y-u_n, u_n-w_n\rangle\ge 0. \tag{5.4}$$

Step 4. *Compute*

$$v_n = \beta_n w_n+(1-\beta_n)u_n.$$

Step 5. *Compute*

$$t_n = prox_{\lambda_1, f_1}(I - \gamma_n A^*(I - prox_{\lambda_2, f_2})A)v_n,$$

where

$$\gamma_n := \begin{cases} \frac{\rho_n g(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2}, & \text{if } ||G(v_n)||^2 + ||H(v_n)||^2 \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 6. *Compute*

$$z_n = (1 - \sigma_n)v_n + \sigma_n S t_n.$$

Step 7. *Compute*

$$x_{n+1} = \alpha_n f(x_n) + \delta_n S t_n + \xi_n T z_n.$$

Set $n := n + 1$ *and return to* ***Step 1,***

where

$$g(x) = \frac{1}{2}\|(I - prox_{\lambda_2, f_2})Ax\|^2, \qquad h(x) = \frac{1}{2}\|(I - prox_{\lambda_1, f_1})x\|^2$$

and

$$G(x) = A^*(I - prox_{\lambda_2, f_2})Ax, \qquad H(x) = (I - prox_{\lambda_1, f_1})x.$$

5.2 Split feasibility problem

Let H_1 and H_2 be two real Hilbert spaces and let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The *Split Feasibility Problem* (SFP) is defined as follows:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q, \tag{5.5}$$

where $A : H_1 \to H_2$ is a bounded linear operator. Let the solution set of SFP (5.5) be denoted by Γ_{SFP}. In 1994, the SFP was introduced by Censor and Elfving (1994) in finite dimensional Hilbert spaces for modelling inverse problems which arise from phase retrievals and in medical image reconstruction (Byrne 2004). Furthermore, the problem (5.5) is also useful in various disciplines such as computer tomography, image restoration, and radiation therapy treatment planning (Censor et al. 2006, 2005). The problem has been studied by numerous researchers, (see Byrne 2004; Censor et al. 2006, 2005 and the references therein). Let C be a nonempty closed convex subset of a real Hilbert space H and i_c be the indicator function on C, that is

$$i_c(x) = \begin{cases} 0 & \text{if } x \in C; \\ \infty & \text{if } x \notin C. \end{cases}$$

Moreover, we define the *normal cone* $N_C u$ of C at $u \in C$ as follows:

$$N_C u = \{z \in H : \langle z, v - u \rangle \leq 0, \forall v \in C\}.$$

It is known that i_C is a proper, lower semicontinuous and convex function on H. Hence, the subdifferential ∂i_C of i_C is a maximal monotone operator. Therefore, we define the resolvent $J_r^{\partial i_C}$ of ∂i_C, $\forall r > 0$ as follows:

$$J_r^{\partial i_C} x = (I + r\partial i_C)^{-1} x, \forall x \in H.$$

Moreover, for each $x \in C$, we have

$$\begin{aligned}\partial i_C x &= \{z \in H : i_C x + \langle z, u - x\rangle \le i_C u, \forall u \in H\}\\ &= \{z \in H : \langle z, u - x\rangle \le 0, \forall u \in C\}\\ &= N_C x.\end{aligned}$$

Hence, for all $\alpha > 0$, we derive

$$\begin{aligned}u = J_r^{\partial i_C} x &\Leftrightarrow x \in u + r\partial i_C u\\ &\Leftrightarrow x - u \in r\partial i_C u\\ &\Leftrightarrow \langle x - u, z - u\rangle \le 0 \;\; \forall z \in C\\ &\Leftrightarrow u = P_C x.\end{aligned}$$

Now, by applying Theorem 4.1 we obtain the following result for approximating a common solution of split feasibility problem, equilibrium problem and common fixed point of nonexpansive mappings in Hilbert spaces.

Theorem 5.3 *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. Let $S, T : H_1 \to H_1$ be nonexpansive mappings, and $f : H_1 \to H_1$ be a contraction with coefficient $k \in (0, 1)$. Suppose that $\Omega = F(S) \cap F(T) \cap \Gamma_{SFP} \cap EP(F) \neq \emptyset$, and conditions (A1)-(A4) and (C1)-(C4) are satisfied. Then the sequence $\{x_n\}$ generated by the following algorithm converges strongly to a point $\hat{x} \in \Omega$, where $\hat{x} = P_\Omega \circ f(\hat{x})$.*

Algorithm 5.4

Step 0. *Let $x_0, x_1 \in H$ be two arbitrary initial points and set $n = 1$.*

Step 1. *Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \le \theta_n \le \hat{\theta}_n$ with $\hat{\theta}_n$ defined by*

$$\hat{\theta}_n = \begin{cases}\min\left\{\theta, \frac{\mu_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1},\\ \theta, & \text{otherwise.}\end{cases} \tag{5.6}$$

Step 2. *Compute*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. *Compute*

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \ge 0. \tag{5.7}$$

Step 4. *Compute*

$$v_n = \beta_n w_n + (1 - \beta_n)u_n.$$

Step 5. *Compute*

$$t_n = P_C(I - \gamma_n A^*(I - P_Q)A)v_n,$$

where

$$\gamma_n := \begin{cases}\frac{\rho_n g(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2}, & \text{if } ||G(v_n)||^2 + ||H(v_n)||^2 \neq 0,\\ 0, & \text{otherwise.}\end{cases}$$

Step 6. *Compute*

$$z_n = (1-\sigma_n)v_n + \sigma_n St_n.$$

Step 7. *Compute*

$$x_{n+1} = \alpha_n f(x_n) + \delta_n St_n + \xi_n T z_n.$$

Set $n := n+1$ *and return to* ***Step 1,***

---

where

$$g(x) = \frac{1}{2}\|(I-P_Q)Ax\|^2, \qquad h(x) = \frac{1}{2}\|(I-P_C)x\|^2$$

and

$$G(x) = A^*(I-P_Q)Ax, \qquad H(x) = (I-P_C)x.$$

5.3 Relaxed split feasibility problem

Next, we study the *Relaxed Split Feasibility Problem* (RSFP) which is a special case of the split feasibility problem when the sets C and Q are defined as follows:

$$C := \{u \in H_1 : c(u) \leq 0\} \quad \text{and} \quad Q := \{v \in H_2 : q(v) \leq 0\}, \tag{5.8}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex and lower semicontinuous functions such that ∂c and ∂q are bounded on bounded sets. We denote the solution set of the RSFP by Γ_{RSFP}. Now, by applying Theorem 4.1 we obtain the following result for approximating a common solution of relaxed split feasibility problem, equilibrium problem and common fixed point of nonexpansive mappings in Hilbert spaces.

Theorem 5.5 *Let H_1 and H_2 be real Hilbert spaces H_1 and H_2, and let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. Let $S, T : H_1 \to H_1$ be nonexpansive mappings, and $f : H_1 \to H_1$ be a contraction with coefficient $k \in (0,1)$. Suppose that $\Omega = F(S) \cap F(T) \cap \Gamma_{RSFP} \cap EP(F) \neq \emptyset$, and conditions (A1)-(A4) and (C1)-(C4) are satisfied. Then the sequence $\{x_n\}$ generated by the following algorithm converges strongly to a point $\hat{x} \in \Omega$, where $\hat{x} = P_\Omega \circ f(\hat{x})$.*

Algorithm 5.6

Step 0. *Let $x_0, x_1 \in H$ be two arbitrary initial points and set $n = 1$.*
Step 1. *Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by*

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \ \frac{\mu_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{5.9}$$

Step 2. *Compute*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. *Compute*

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \geq 0. \tag{5.10}$$

Step 4. *Compute*

$$v_n = \beta_n w_n + (1-\beta_n)u_n.$$

Step 5. *Compute*

$$t_n = P_{C_n}(I - \gamma_n A^*(I - P_{Q_n})A)v_n,$$

where

$$\gamma_n := \begin{cases} \frac{\rho_n g(v_n)}{||G(v_n)||^2 + ||H(v_n)||^2}, & \text{if } ||G(v_n)||^2 + ||H(v_n)||^2 \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$
$$C_n = \{v \in H_1 : c(v_n) + \langle c_n, v - v_n\rangle \leq 0\}, \quad c_n \in \partial c(v_n),$$
$$Q_n = \{w \in H_2 : q(Av_n) + \langle q_n, w - Av_n\rangle \leq 0\}, \quad q_n \in \partial q(Av_n).$$

Step 6. *Compute*

$$z_n = (1-\sigma_n)v_n + \sigma_n St_n.$$

Step 7. *Compute*

$$x_{n+1} = \alpha_n f(x_n) + \delta_n St_n + \xi_n T z_n.$$

Set $n := n+1$ and return to ***Step 1,***

where

$$g(x) = \frac{1}{2}\|(I - P_{Q_n})Ax\|^2, \qquad h(x) = \frac{1}{2}\|(I - P_{C_n})x\|^2$$

and

$$G(x) = A^*(I - P_{Q_n})Ax, \qquad H(x) = (I - P_{C_n})x.$$

6 Numerical examples

In this section, we present some numerical experiments to illustrate the performance of our method, Algorithm 3.1 in comparison with Algorithms 1.2, Algorithm 1.3, Algorithm 1.5 and Algorithm 7.2 in the literature. All numerical computations were carried out using Matlab version R2019(b).
In our computations, we choose $\rho_n = 3 - \frac{1}{2n+1}$, $\beta_n = \frac{n}{n+3}$, $\sigma_n = \frac{n}{2n+1}$, $\alpha_n = \frac{1}{2n+3}$, $\delta_n = \xi_n = \frac{n+1}{2n+3}$, $\mu_n = \frac{1}{(2n+3)^3}$, $r_n = \frac{n}{n+3}$, $\lambda = \lambda_1 = \lambda_2 = 0.5$, $\theta = 0.8$ for each $n \in \mathbb{N}$. It can easily be checked that all the conditions of Theorem 4.1 are satisfied. We take $\gamma = 0.0001$ in Algorithms 1.2, 1.3 and 7.2, and $Dx = \frac{1}{3}x$, $\beta = 0.5$ in Algorithm 1.3.

Example 6.1 Let $H_1 = \mathbb{R}^3 = H_2$ and $C = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 | \langle a, x\rangle \geq b\}$. For $r > 0$, $T_r^F x = P_C x = \frac{b - \langle a, x\rangle}{\|a\|_2^2} a + x$. Here we take $a = (8, -3, 1)$, $b = -1$ and define

Table 1 Numerical results for Example 6.1

	Alg. 1.2	Alg. 1.3	Alg. 1.5	Alg. 7.2	Alg. 3.1
No. of Iter.	12	7	30	7	8
No. of Iter.	11	6	30	6	7
No. of Iter.	11	7	30	7	7
No. of Iter.	12	7	30	7	7

$S, T : H_1 \to H_1$ by $Sx = \frac{1}{2}x, Tx = \frac{1}{3}x$ for all $x \in H_1$. Define the operators A, B_1 and B_2 as follows:

$$Ax = \begin{pmatrix} 6 & 3 & 1 \\ 8 & 7 & 5 \\ 3 & 6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B_1x = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B_2x = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and set $f(x) = \frac{1}{5}x$. We choose different initial values as follows:
Case I: $x_0 = (-3, 2, 5)^T, x_1 = (2, 1, -1)^T$;
Case II: $x_0 = (7, 2.1, 3.5)^T, x_1 = (5, 1, 2)^T$;
Case III: $x_0 = (2.3, 4.7, -3.5)^T, x_1 = (3, 1, 0)^T$;
Case IV: $x_0 = (8, 2, 5)^T, x_1 = (-5, 1, -1)^T$.
Using MATLAB 2019(b), we compare the performance of Algorithm 3.1 with Algorithm 1.2, Algorithm 1.3, Algorithm 1.5 and Algorithm 7.2. The stopping criterion used for our computation is $||x_{n+1} - x_n|| < 10^{-3}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Fig. 1 and Table 1.

Example 6.2 Let $H_1 = (l_2(\mathbb{R}), \|\cdot\|_2) = H_2$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_n, \ldots), x_j \in \mathbb{R} : \sum_{j=1}^{\infty} |x_j|^2 < \infty\}$, $||x||_2 = (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}}$ for all $x \in l_2(\mathbb{R})$. We set $f(x) = \frac{1}{2}x$ and define $S, T : H_1 \to H_1$ by $Sx = \frac{1}{3}x, Tx = \frac{1}{5}x$, $A : H_1 \to H_2$ be defined by $Ax = \frac{x}{3}$ for all $x \in H_1$, then $A^*y = \frac{y}{3}$ for all $y \in H_2$. Define $B_1 : H_1 \to H_1$ by $B_1x = \frac{5}{2}x$, and $B_2 : H_2 \to H_2$ by $B_2x = \frac{3}{2}x$. Then B_1 and B_2 are maximal monotone operators. Define the bifunction F by $F(x, y) = x(y - x)$. It can be verified that

$$T_r^F x = \frac{x}{1 + r} \quad \text{for all } x \in H_1.$$

We choose different initial values as follows:
Case I: $x_0 = (0, -3, 7, \ldots), x_1 = (-1, 2, 3, \ldots)$,
Case II: $x_0 = (5, -1, \frac{1}{5}, \ldots), x_1 = (3, 0.3, 0.03, \ldots)$,
Case III: $x_0 = (0, 3, -7, \ldots), x_1 = (1, -2, 3, \ldots)$,
Case IV: $x_0 = (7, -3, -\frac{1}{7}, \ldots), x_1 = (4, 0.4, 0.04, \ldots)$.
Using MATLAB 2019(b), we compare the performance of Algorithm 3.1 with Algorithm 1.2, Algorithm 1.3, Algorithm 1.5 and Algorithm 7.2. The stopping criterion used for our computation is $||x_{n+1} - x_n|| < 10^{-3}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Fig. 2 and Table 2.

Remark 6.3 By using different starting points and plotting the graphs of errors against the number of iterations in each example (Examples 6.1–6.2), we obtain the numerical results displayed in Tables 1 and 2 and Figs. 1 and 2. We compared our proposed Algorithm 3.1

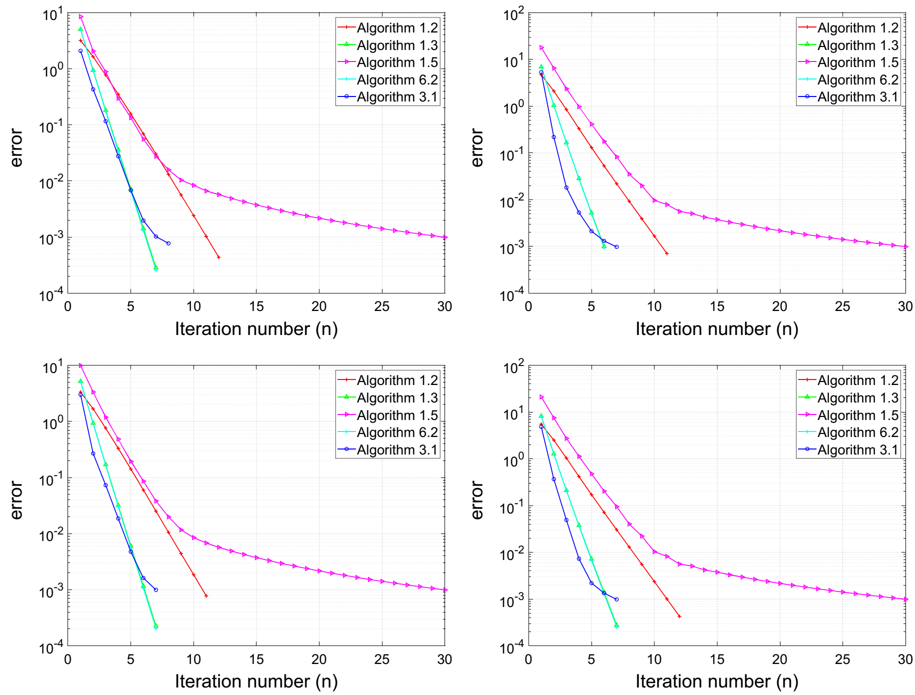

Fig. 1 Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV

Table 2 Numerical results for Example 6.2

	Alg. 1.2	Alg. 1.3	Alg. 1.5	Alg. 7.2	Alg. 3.1
No. of Iter.	13	7	37	7	6
No. of Iter.	13	7	37	7	6
No. of Iter.	13	7	37	7	6
No. of Iter.	13	7	37	7	6

with Algorithm 1.2, Algorithm 1.3, Algorithm 1.5 and Algorithm 7.2. Furthermore, we note the following from our numerical experiments:

- We observe that the different choices of the starting point and key parameters does not have a significant effects on the output of our method with respect to the performance of the proposed algorithm.
- In all the examples, we can see from the tables and figures that the number of iterations for our proposed method remain consistent (well-behaved).
- From the Table 2 and Fig. 2, we can see clearly that in terms of number of iterations, our proposed Algorithm 3.1 outperforms the other four existing methods. Table 1 and Fig. 1 also show that our method performs favourable well compared with the four existing methods.

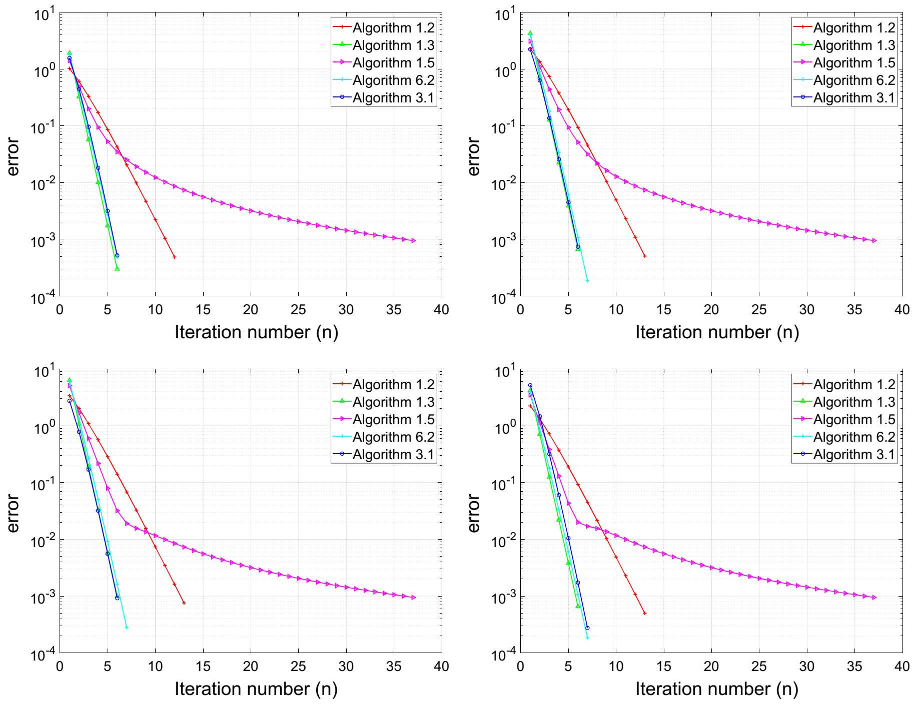

Fig. 2 Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV

7 Conclusion

In this paper, we studied the problem of finding the common solution of split variational inclusion problem, equilibrium problem and common fixed point of nonexpansive mappings. We introduced a new inertial viscosity S-iteration method for approximating the solution of the problem and we proved strong convergence theorem for the proposed algorithm without the knowledge of the operator norm. Finally, we applied our results to study other optimization problems and provided some numerical experiments with graphical illustrations to demonstrate the efficiency of our method in comparison with some existing methods in the current literature.

Appendix 7.1 *(Algorithm 3.1 in Kazmi and Rizvi 2014)*

Algorithm 7.2

$$\begin{cases} x_0 \in H_1, \\ u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, \end{cases}$$

where $f : H_1 \to H_1$ is a contraction with constant $k \in (0, 1)$, $S : H_1 \to H_1$ is a nonexpansive mapping, $\gamma \in (0, \frac{1}{L})$, where L is the spectral radius of the operator A^*A, and A^* is the adjoint of A, $\{\alpha_n\} \subset (0, 1)$ and $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ are two multi-valued maximal monotone operators on H_1 and H_2, respectively.

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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