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Milne type inequality and interval orders

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Abstract
In this paper, we prove some Milne type inequalities for interval-valued functions and, along with it, we explore some connections with other inequalities. More precisely, using the Aumann integral and the Kulisch–Miranker order and including-order on the space of real and compact intervals, we establish some Milne type inequalities for interval-valued functions. Also, using different orders, we obtain some connections with Chebyshev, Cauchy–Schwarz, and Hölder inequality. Finally, some new ideas and results based on submodular measures are explored as well as some examples and applications are presented for illustrating our results.

Keywords Integral inequalities · Interval-valued functions · Milne's inequality · Interval orders

Mathematics Subject Classification 65G40 · 26D15

1 Introduction

The importance of the study of set-valued analysis from a theoretical point of view as well as from their application is well known (Aubin and Cellina 1984; Aubin and Franskowska 2000). Also, many advances in set-valued analysis have been motivated by control theory and dynamical games and, in addition, optimal control theory and mathematical programming were a motivating force behind set-valued analysis since the 60s (Aubin and Franskowska

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1990). Interval analysis is a particular case and it was introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. The first monograph dealing with interval analysis was given by Moore (1966). Moore is recognized as the first to use intervals in computational mathematics, now called numerical analysis. He also extended and implemented the arithmetic of intervals to computers. One of his major achievements was to show that Taylor series methods for solving differential equations not only are more tractable, but also more accurate (see Moore 1979, 1985; Moore et al. 2009) .

Several generalizations of classical integral inequalities were obtained in the recent years by the authors (Agahi et al. 2011; Agahi 2020; Flores-Franulič and Román-Flores 2007; Flores-Franulič et al. 2008, 2009; Román-Flores and Chalco-Cano 2006; Román-Flores et al. 2007a, b, 2008a, b, 2013, 2018, 2020), in the context of non-additive measures and Sugeno's integral. Additionally, also see the following related references: (Pap 1995) and (Wang and Klir 2009) which also contain some aplications to non-deterministic problems.

On the other hand, several integral inequalities involving functions and their integrals and derivatives have been extended by the authors to the interval and/or fuzzy-interval context, including Minkowski, Radon and Beckenbach inequalities (Román-Flores et al. 2018; Costa and Román-Flores 2017), Ostrowski's inequality (Chalco-Cano et al. 2012), Gauss. Opial and Wirtinger-type inequalities (Costa and Román-Flores 2019a; Costa et al. 2019b, 2020) respectively, among others.

In general, any integral inequality can be a very powerful tool for applications and, in particular, when we think an integral operators as a predictive tool, then an integral inequality can be very important in measuring, computing errors, and delineating such processes. In most cases, when we want to model a real problems, it is necessary to know the dynamics given by a certain real function f, and the problem is that this function f is difficult to know explicitly due to phenomena of uncertainty. However, if we knows its ranges of minimum and maximum variation, then we can approximate it by an interval function. In that way, interval-valued functions (or fuzzy-interval-valued functions) may provide an alternative choice in the modeling of real problems with uncertainty, along with the Aumann integral for interval-valued functions (the natural associated expectation) and the knowledge and management of Aumann integral inequalities, could be a power tool for measuring and quantify the uncertainty involved in the modeling processes.

Finally, one of the most important things we want to highlight is that, in the interval context, the Milne integral inequality is related to other relevant inequalities through different partial orders on the class of compact-convex intervals.

This work generalizes Milne's inequality for interval-valued functions and, also, some connection with other classical inequalities and interval orders are explored.

2 Preliminaries and basic results

2.1 Interval operations

Let $\mathbb{R}$ be the one-dimensional Euclidean space, and consider $\mathcal{K}_C$ the family of all non-empty compact convex subsets of $\mathbb{R}$, that is

$$\mathcal{K}_C = \{[a, b] \mid a, b \in \mathbb{R} \text{ and } a \leq b\}. \tag{1}$$

The Hausdorff metric on $\mathcal{K}_C$ is defined by

$$H(A, B) = \max \{d(A, B), d(B, A)\}, \tag{2}$$

where $d(A, B) = \max\limits_{a \in A} d(a, B)$ and $d(a, B) = \min\limits_{b \in B} d(a, b) = \min\limits_{b \in B} |a - b|$.

Remark 1 An equivalent form for the Hausdorff metric defined in (2) is:

$$M\left(\left[\underline{a}, \overline{a}\right], \left[\underline{b}, \overline{b}\right]\right) = max \left\{\left|\underline{a} - \underline{b}\right|, \left|\overline{a} - \overline{b}\right|\right\},$$

which is also known as the Moore metric on the space of intervals (Moore et al. 2009, eq. (6.3), pp. 52).

It is well known that $(\mathcal{K}_C, H)$ is a complete metric space (see Aubin and Cellina 1984).

If $A \in \mathcal{K}_C$, then we define the norm of A as $\|A\| = H(A, \mathbf{0})$.

The Minkowski sum and scalar multiplication are defined on $\mathcal{K}_C$ by means

$$A + B = \{a + b \mid a \in A,\ b \in B\} \text{ and } \lambda A = \{\lambda a \mid a \in A\}. \tag{3}$$

Also, if $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ are two compact intervals, then we define the difference

$$A - B = \left[\underline{a} - \overline{b}, \overline{a} - \underline{b}\right], \tag{4}$$

the product

$$A \cdot B = \left[min\left\{\underline{ab}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{ab}\right\}, max\left\{\underline{ab}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{ab}\right\}\right], \tag{5}$$

and the division

$$\frac{A}{B} = \left[min\left\{\frac{\underline{a}}{\underline{b}}, \frac{\underline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\overline{a}}{\overline{b}}\right\}, max\left\{\frac{\underline{a}}{\underline{b}}, \frac{\underline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\overline{a}}{\overline{b}}\right\}\right], \tag{6}$$

whenever $0 \notin B$.

An order relation "$\leq$" is defined on $\mathcal{K}_C$ as follows (Kulisch and Miranker 1981):

$$[\underline{a}, \overline{a}] \leq [\underline{b}, \overline{b}] \Leftrightarrow \underline{a} \leq \underline{b} \text{ and } \overline{a} \leq \overline{b}. \tag{7}$$

Remark 2 We note that if $[a, b]$, $[c, d]$ and $[x, y]$ are intervals with positive endpoints, then

$$[a, b] \geq [x, y] \Leftrightarrow \frac{[a, b]}{[c, d]} \geq \frac{[x, y]}{[c, d]} \tag{8}$$

$$[c, d] \leq [x, y] \Leftrightarrow \frac{[a, b]}{[c, d]} \geq \frac{[a, b]}{[x, y]}. \tag{9}$$

If $f(x)$ is a monotone and continuous function over an interval $X = [a, b]$, we can define

$$f(X) = f([a, b]) = [min\{f(a), f(b)\}, max\{f(a), f(b)\}]. \tag{10}$$

Example 1 a) If $f(x) = x^r$, $r > 0$, and $0 \leq a \leq b$, then

$$f([a, b]) = [a, b]^r = \left[a^r, b^r\right]. \tag{11}$$

b) If $g(x) = e^x$, then the "exponential" of an interval $[a, b]$ is defined as

$$g([a, b]) = e^{[a,b]} = \left[e^a, e^b\right]. \tag{12}$$

For more details on interval operations and interval analysis, see (Markov 1979; Moore 1966; Rokne 2001).

2.2 Integral of interval-valued functions

If $T = [a, b]$ is a closed interval and $F : T \to \mathcal{K}_C$ is an interval-valued function, then we will denote

$$F(t) = [\underline{f}(t), \overline{f}(t)],$$

where $\underline{f}(t) \leq \overline{f}(t)$, $\forall t \in T$. The functions $\underline{f}$ and $\overline{f}$ are called the lower and the upper (endpoint) functions of F, respectively. For interval-valued functions, it is clear that $F : T \to \mathcal{K}_C$ is continuous at $t_0 \in T$ if

$$lim_{t \to t_0} F(t) = F(t_0), \tag{13}$$

where the limit is taken in the metric space $(\mathcal{K}_C, H)$. Consequently, F is continuous at $t_0 \in T$ if and only if its endpoint functions $\underline{f}$ and $\overline{f}$ are continuous functions at $t_0 \in T$. We denote by $C([a, b], \mathcal{K}_C)$ the family of all continuous interval-valued functions.

Definition 1 Let $\mathcal{M}$ be the class of all Lebesgue measurable sets of T, and then

(a) the function $f : T \to \mathbb{R}$ is measurable if and only if

$$f^{-1}(C) \in \mathcal{M}$$

for all closed subset C of $\mathbb{R}$;

(b) the interval-valued function $F : T \to \mathcal{K}_C$ is measurable if and only if

$$F^{\omega}(C) = \{t \in T \mid F(t) \cap C \neq \emptyset\} \in \mathcal{M}, \ \forall\, C \subseteq \mathbb{R}, \ C \ closed;$$

(c) also, if $F : T \to \mathcal{K}_C$ is an interval-valued function and $f : T \to \mathbb{R}$, then we say that f is a selector (or selection) of F if and only if $f(t) \in F(t)$ for all $t \in T$. In this case if, additionally f is a measurable function, then we say that f is a measurable selector of F. Finally, an integrable selector of F is a measurable selector of F for which there is $\int_T f(t)$.

Definition 2 (Aubin and Cellina 1984) Let $F : T \to \mathcal{K}_C$ be an interval-valued function. The integral (Aumann integral) of F over $T = [a, b]$ is defined as

$$\int_a^b F(t)\mathrm{d}t = \left\{ \int_a^b f(t)\mathrm{d}t \mid f \in S(F) \right\}, \tag{14}$$

where $S(F)$ is the set of all integrable selectors of F, that is

$$S(F) = \{f : T \to \mathbb{R} \mid f \ integrable \ and \ f(t) \in F(t) \ for \ all \ t \in T\} \ .$$

If $S(F) \neq \emptyset$, then the integral exists and F is said to be integrable (Aumann integrable).

Note that if F is integrable, then it has a measurable selector which is integrable and, consequently, $S(F) \neq \emptyset$.

Also, in the above definition, the integral symbols $\int_a^b F(t)\mathrm{d}t$ and/or $\int_a^b f(t)\mathrm{d}t$ denote the integral with respect to the Lebesgue measure.

Definition 3 We say that a mapping $F : T \to \mathcal{K}_C$ is integrally bounded if there exists a positive integrable function $g : T \to \mathbb{R}$, such that $\|F(t)\| \leq g(t)$, for all $t \in T$.

Theorem 1 *(Aubin and Cellina 1984) Let $F : T \to \mathcal{K}_C$ be a measurable and integrally bounded interval-valued function. Then, it is integrable and $\int_a^b F(t)\mathrm{d}t \in \mathcal{K}_C$.*

Corollary 1 *(Aubin and Cellina 1984) A continuous interval-valued function $F : T \to \mathcal{K}_C$ is integrable.*

The Aumann integral satisfies the following properties.

Proposition 1 *(Aubin and Cellina 1984) Let $F, G : T \to \mathcal{K}_C$ be two measurable and integrally bounded interval-valued functions. Then,*
(i) $\int_{t_1}^{t_2} (F(t) + G(t))\,\mathrm{d}t = \int_{t_1}^{t_2} F(t)\mathrm{d}t + \int_{t_1}^{t_2} G(t)\mathrm{d}t$, $a \le t_1 \le t_2 \le b$
(ii) $\int_{t_1}^{t_2} F(t)\mathrm{d}t = \int_{t_1}^{\tau} F(t)\mathrm{d}t + \int_{\tau}^{t_2} F(t)\mathrm{d}t$, $a \le t_1 \le \tau \le t_2 \le b$.

Theorem 2 *(Aubin and Franskowska 1990) Let $F : T \to \mathcal{K}_C$ be a measurable and integrally bounded interval-valued functionm such that $F(t) = [\underline{f}(t), \overline{f}(t)]$. Then, $\underline{f}$ and $\overline{f}$ are integrable functions and*

$$\int_{t_1}^{t_2} F(t)\mathrm{d}t = \left[\int_{t_1}^{t_2} \underline{f}(t)\mathrm{d}t\ ,\ \int_{t_1}^{t_2} \overline{f}(t)\mathrm{d}t \right]. \tag{15}$$

Remark 3 Above Theorem 2 is a direct consecuence of two relevant results:

a) (Aumann 1965, Theorem 1, pp. 2) $\int_T F(t)\mathrm{d}t$ is convex.
b) (Aumann 1965, Theorem 4, pp. 2) If F is closed valued, then $\int_T F(t)\mathrm{d}t$ is compact. In fact, because $\underline{f}, \overline{f} \in S(F)$ then, by convexity of $\int_T F(t)\mathrm{d}t$, we obtain $[\int_T \underline{f}(t)\mathrm{d}t, \int_T \overline{f}(t)\mathrm{d}t] \subseteq \int_T F(t)\mathrm{d}t$.
   On the other hand, if $f \in S(F)$, then $\underline{f}(t) \le f(t) \le \overline{f}(t)$, for all $t \in T$, which implies that

$$\int_T f(t)\mathrm{d}t \in \left[\int_T \underline{f}(t)\mathrm{d}t, \int_T \overline{f}(t)\mathrm{d}t \right],$$

   and, consequently, $\int_T F(t)\mathrm{d}t \subseteq [\int_T \underline{f}(t)\mathrm{d}t, \int_T \overline{f}(t)\mathrm{d}t]$. Therefore, equality (15) in Theorem 2 holds.

In the sequel, we will use the notation $\int_E F(x)\mathrm{d}x$ or $\int_E F\mathrm{d}\mu$ if necessary.

3 Interval Milne's inequality

A problem in astrophysics, specifically the stellar absorption, and through a paper by Rosseland (norwegian astrophycisist) on this subject written in 1924, Edward Arthur Milne established the following interesting integral inequality in 1925:

Theorem 3 (Milne 1925)

$$\int_a^b \frac{f(x)g(x)}{f(x)+g(x)}\mathrm{d}x \int_a^b (f(x)+g(x))\,\mathrm{d}x \le \int_a^b f(x)\mathrm{d}x \int_a^b g(x)\mathrm{d}x, \tag{16}$$

and this inequality holds for all positive and integrable functions f, g on $[a, b]$.

Now, using above theorem and properties of interval integration, we can prove the following interval version of Milne's inequality:

Theorem 4 (Interval Milne's inequality) *If* $F, G : [a, b] \to \mathcal{K}_C$ *are two integrable interval-valued functions, with* $F = [\underline{f}, \overline{f}]$, $G = [\underline{g}, \overline{g}]$, $\underline{f}(x), \underline{g}(x) > 0$, *then*

$$\int_a^b \frac{FG}{F+G}\mathrm{d}\mu \int_a^b (F+G)\,\mathrm{d}\mu \leq \left[\int_a^b \underline{f}\mathrm{d}\mu \int_a^b \underline{g}\mathrm{d}\mu, \int_a^b \frac{\overline{f}\overline{g}}{\underline{f}+\underline{g}}\mathrm{d}\mu \int_a^b (\overline{f}+\overline{g})\mathrm{d}\mu\right]. \tag{17}$$

Proof Using basic properties of sum, product, and division of interval operations, and applying Theorem 3, we have

$$\begin{aligned}
&\int_a^b \frac{FG}{F+G}\mathrm{d}\mu \int_a^b (F+G)\,\mathrm{d}\mu \\
&= \int_a^b \frac{[(\underline{f}\underline{g})(x), (\overline{f}\overline{g})(x)]}{[(\underline{f}+\underline{g})(x), (\overline{f}+\overline{g})(x)]}\mathrm{d}x \times \int_a^b [(\underline{f}+\underline{g})(x), (\overline{f}+\overline{g})(x)]\mathrm{d}x \\
&= \left[\int_a^b \frac{\underline{f}(x)\underline{g}(x)}{\overline{f}(x)+\overline{g}(x)}\mathrm{d}x, \int_a^b \frac{\overline{f}(x)\overline{g}(x)}{\underline{f}(x)+\underline{g}(x)}\mathrm{d}x\right] \times \left[\int_a^b \underline{f}(x)+\underline{g}(x)\mathrm{d}x, \int_a^b \overline{f}(x)+\overline{g}(x)\mathrm{d}x\right] \\
&= \left[\int_a^b \frac{\underline{f}(x)\underline{g}(x)}{\overline{f}(x)+\overline{g}(x)}\mathrm{d}x \int_a^b \underline{f}(x)+\underline{g}(x)\mathrm{d}x, \int_a^b \frac{\overline{f}(x)\overline{g}(x)}{\underline{f}(x)+\underline{g}(x)}\mathrm{d}x \int_a^b \overline{f}(x)+\overline{g}(x)\mathrm{d}x\right] \\
&\leq \left[\int_a^b \frac{\underline{f}(x)\underline{g}(x)}{\underline{f}(x)+\underline{g}(x)}\mathrm{d}x \int_a^b \underline{f}(x)+\underline{g}(x)\mathrm{d}x, \int_a^b \frac{\overline{f}(x)\overline{g}(x)}{\underline{f}(x)+\underline{g}(x)}\mathrm{d}x \int_a^b \overline{f}(x)+\overline{g}(x)\mathrm{d}x\right] \\
&\leq \left[\int_a^b \underline{f}(x)\mathrm{d}x \int_a^b \underline{g}(x)\mathrm{d}x, \int_a^b \frac{\overline{f}(x)\overline{g}(x)}{\underline{f}(x)+\underline{g}(x)}\mathrm{d}x \int_a^b \overline{f}(x)+\overline{g}(x)\mathrm{d}x\right],
\end{aligned} \tag{18}$$

and the proof is completed. □

The next result says us that interval integral inequality in Theorem 4 implies the classical Milne's inequality (16).

Corollary 2 *Let* $f, g : [a, b] \to \mathbb{R}$ *be two integrable real positive functions and consider* $F, G : [a, b] \to \mathcal{K}_C$ *two integrable interval-valued functions, such that* $F = [f, f]$ *and* $G = [g, g]$. *Then*

$$\int_a^b \frac{FG}{F+G}\mathrm{d}\mu \int_a^b (F+G)\,\mathrm{d}\mu \leq \left[\int_a^b f\mathrm{d}\mu \int_a^b g\mathrm{d}\mu, \int_a^b f\mathrm{d}\mu \int_a^b g\mathrm{d}\mu\right]. \tag{19}$$

Proof By hypothesis, we have $\underline{f} = \overline{f} = f$ and $\underline{g} = \overline{g} = g$; therefore, replacing in (18) and using Theorem 16, we have

$$\begin{aligned}
&\int_a^b \frac{FG}{F+G}\mathrm{d}\mu \int_a^b (F+G)\,\mathrm{d}\mu \\
&\leq \left[\int_a^b \underline{f}(x)\mathrm{d}x \int_a^b \underline{g}(x)\mathrm{d}x, \int_a^b \frac{\overline{f}(x)\overline{g}(x)}{\underline{f}(x)+\underline{g}(x)}\mathrm{d}x \int_a^b \overline{f}(x)+\overline{g}(x)\mathrm{d}x\right] \\
&= \left[\int_a^b f\mathrm{d}\mu \int_a^b g\mathrm{d}\mu, \int_a^b \frac{fg}{f+g}\mathrm{d}\mu \int_a^b (f+g)\mathrm{d}\mu\right] \\
&\leq \left[\int_a^b f\mathrm{d}\mu \int_a^b g\mathrm{d}\mu, \int_a^b f\mathrm{d}\mu \int_a^b g\mathrm{d}\mu\right]
\end{aligned} \tag{20}$$

recapturing Milne's inequality in both components, and completing the proof. □

Remark 4 Moore (1979) explore some order relations on intervals and, in particular, mention the order defined by inclusion "$\trianglelefteq$" as a partial order on the class of compact intervals. More precisely, if $[a_l, a_r]$, $[b_l, b_r]$ are two closed intervals, then

$$[a_l, a_r] \trianglelefteq [b_l, b_r] \;\; iff \;\; b_l \leq a_l \;\; and \;\; a_r \leq b_r$$

or, equivalently

$$[a_l, a_r] \trianglelefteq [b_l, b_r] \;\; iff \;\; [a_l, a_r] \subseteq [b_l, b_r]\,.$$

Now, we know that, in general, in the same conditions of Theorem 4, the inequality

$$\int_a^b \frac{FG}{F+G} \mathrm{d}\mu \int_a^b (F+G)\,\mathrm{d}\mu \leq \left[\int_a^b \underline{f}\mathrm{d}\mu \int_a^b \underline{g}\mathrm{d}\mu, \int_a^b \overline{f}\mathrm{d}\mu \int_a^b \overline{g}\mathrm{d}\mu\right] \tag{21}$$

is not verified; however, in this case, the following inequality:

$$\int_a^b \frac{FG}{F+G} \mathrm{d}\mu \int_a^b (F+G)\,\mathrm{d}\mu \trianglerighteq \left[\int_a^b \underline{f}\mathrm{d}\mu \int_a^b \underline{g}\mathrm{d}\mu, \int_a^b \overline{f}\mathrm{d}\mu \int_a^b \overline{g}\mathrm{d}\mu\right] \tag{22}$$

holds.

4 Exploring other connections

In this section, we will explore some connections between Milne's inequality and other classical inequalities such as Chebyshev's and Cauchy–Schwarz inequality in the intervalar setting.

First, we recall that in the classical context, one of the more general versions for Chebyshev's inequality is the following (where, for simplicity, this result is presented on the interval [0,1]):

Theorem 5 (Girotto and Holder 2011) *If $f, g : [0, 1] \to \mathbb{R}$ be two Lebesgue integrable real comonotone functions, then*

$$\int_0^1 fg\mathrm{d}\mu \geq \int_0^1 f\mathrm{d}\mu \int_0^1 g\mathrm{d}\mu\,. \tag{23}$$

We recall that functions $f, g : X \to \mathbb{R}$ are said to be comonotone if for all $x, y \in X$

$$(f(x) - f(y))\,(g(x) - g(y)) \geq 0.$$

With respect to comonotone functions, an useful result is the following (Chateauneuf et al. 1997):

Proposition 2 *If f and g are comonotone on $[a, b]$, then the family*

$$\{f, g, f+g, \phi(f, g)\}\;,$$

for any nondecreasing (in both variables) numerical function ϕ, is also a family of comonotone functions on $[a, b]$.

Example 2 In particular, in above proposition, we can take $\phi(x, y) = \lambda x + \eta y$ with $\lambda, \eta \geq 0$, or $\phi(x, y) = \frac{xy}{x+y}$.

Theorem 6 (Interval Chebyshev's inequality) *Consider* $F, G : [0,1] \to \mathcal{K}_C$ *two integrable interval-valued functions, with* $F = [\underline{f}, \overline{f}]$, $G = [\underline{g}, \overline{g}]$ *and* $\underline{f}, \underline{g} \geq 0$. *If* $\underline{f}, \underline{g}$ *comonotonic and* $\overline{f}$ *comonotonic with* $\overline{g}$, *then*

$$\int_0^1 FG\mathrm{d}\mu \geq \int_0^1 F\mathrm{d}\mu \int_0^1 G\mathrm{d}\mu. \tag{24}$$

Proof Using basic properties of interval operations, Theorem 2 and Theorem 5, and hypothesis, we have

$$\begin{aligned}
\int_0^1 FG\mathrm{d}\mu &= \int_0^1 [\underline{f}, \overline{f}][\underline{g}, \overline{g}]\mathrm{d}\mu \\
&= \int_0^1 [\underline{fg}, \overline{fg}]\mathrm{d}\mu \\
&= [\int_0^1 \underline{fg}\mathrm{d}\mu, \int_0^1 \overline{fg}\mathrm{d}\mu] \\
&\geq [\int_0^1 \underline{f}\mathrm{d}\mu \int_0^1 \underline{g}\mathrm{d}\mu, \int_0^1 \overline{f}\mathrm{d}\mu \int_0^1 \overline{g}\mathrm{d}\mu] \\
&= [\int_0^1 \underline{f}\mathrm{d}\mu, \int_0^1 \overline{f}\mathrm{d}\mu][\int_0^1 \underline{g}\mathrm{d}\mu, \int_0^1 \overline{g}\mathrm{d}\mu] \\
&= \int_0^1 F\mathrm{d}\mu \int_0^1 G\mathrm{d}\mu
\end{aligned}$$

completing the proof. □

The following is a version of the Chebyshev's inequality which is more stronger than (23):

Theorem 7 (Mitrinović et al. 1993) *Let* $f, g, p : [0,1] \to \mathbb{R}$ *be three Lebesgue integrable real functions. If* f *is comonotone with* g *and* $p \geq 0$, *then*

$$\int_0^1 fgp\mathrm{d}\mu \int_0^1 p\mathrm{d}\mu \geq \int_0^1 fp\mathrm{d}\mu \int_0^1 gp\mathrm{d}\mu\ . \tag{25}$$

Now, we can extend this last inequality to the interval context in the following form:

Theorem 8 (Interval Chebyshev's inequality) *Consider* $F, G.P : [0,1] \to \mathcal{K}_C$ *three integrable interval-valued functions, with* $F = [\underline{f}, \overline{f}]$, $G = [\underline{g}, \overline{g}]$, $P = [\underline{p}, \overline{p}]$, *and* $\underline{f}(x), \underline{g}, \underline{p} \geq 0$. *If* $\underline{f}, \underline{g}$ *is comonotonic and* $\overline{f}, \overline{g}$ *are comonotone functions, then*

$$\int_0^1 FGP\mathrm{d}\mu \int_0^1 P\mathrm{d}\mu \geq \int_0^1 FP\mathrm{d}\mu \int_0^1 GP\mathrm{d}\mu. \tag{26}$$

Proof Using basic properties of interval operations, Aumann integral, and Theorem 7, we have

$$\begin{aligned}\int_0^1 FGP\mathrm{d}\mu \int_0^1 P\mathrm{d}\mu &= \int_0^1 [\underline{fgp}, \overline{fgp}]\mathrm{d}\mu \int_0^1 [\underline{p}, \overline{p}]\mathrm{d}\mu \\ &= \left[\int_0^1 \underline{fgp}\mathrm{d}\mu, \int_0^1 \overline{fgp}\mathrm{d}\mu\right]\left[\int_0^1 \underline{p}\mathrm{d}\mu, \int_0^1 \overline{p}\mathrm{d}\mu\right] \\ &= \left[\int_0^1 \underline{fgp}\mathrm{d}\mu \int_0^1 \underline{p}\mathrm{d}\mu, \int_0^1 \overline{fgp}\mathrm{d}\mu \int_0^1 \overline{p}\mathrm{d}\mu\right] \\ &\geq \left[\int_0^1 \underline{fp}\mathrm{d}\mu \int_0^1 \underline{gp}\mathrm{d}\mu, \int_0^1 \overline{fp}\mathrm{d}\mu \int_0^1 \overline{gp}\mathrm{d}\mu\right], \\ &= \int_0^1 FP\mathrm{d}\mu \int_0^1 GP\mathrm{d}\mu\ ,\end{aligned}$$

completing the proof. □

On the other hand, the classical Cauchy–Schwarz inequality (Agahi 2020) establishes that:

Theorem 9 (C-S inequality) *Let $f, g : [0, 1] \to \mathbb{R}$ two Lebesgue integrable and positive functions, and then*

$$\int_0^1 fg\mathrm{d}\mu \leq \left(\int_0^1 f^2\mathrm{d}\mu\right)^{\frac{1}{2}} \left(\int_0^1 g^2\mathrm{d}\mu\right)^{\frac{1}{2}}\ . \tag{27}$$

Also, the extension of this inequality is the following.

Theorem 10 (Hölder's inequality) *Let $f, g : [0, 1] \to \mathbb{R}$ be two Lebesgue integrable and positive functions. If p, q are positive real numbers, such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_0^1 fg\mathrm{d}\mu \leq \left(\int_0^1 f^p\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_0^1 g^q\mathrm{d}\mu\right)^{\frac{1}{q}}\ . \tag{28}$$

It is well known that Milne's inequality has connection with Cauchy–Schwarz and Chebyshev inequality. In fact, the following result is an interesting combination of these inequalities.

Theorem 11 *Let $f, g : [0, 1] \to \mathbb{R}$ be two Lebesgue integrable and positive function. Furthermore, if f, g are comonotone functions, then the followng inequality:*

$$\left(\int_0^1 \sqrt{fg}\mathrm{d}\mu\right)^2 \leq \int_0^1 \frac{fg}{f+g}\mathrm{d}\mu \int_0^1 (f+g)\,\mathrm{d}\mu \leq \int_0^1 f\mathrm{d}\mu \int_0^1 g\mathrm{d}\mu \leq \int_0^1 fg\mathrm{d}\mu \tag{29}$$

holds.

Proof First inequality is the Cauchy–Schwarz's inequality, central inequality corresponds to Milne's inequality and the last one is the Chebyshev's inequality. □

Now, we would like to extend Theorem 11 to the intervalar context; however, the following example shows that central inequality is not true in the setting of interval-valued functions.

Example 3 Consider the interval functions $F, G : [0,1] \to \mathcal{K}_C$, with $F(x) = [x^2, x]$ and $G(x) = [1, x+1]$. Then

$$
\begin{aligned}
a) & \int_0^1 \frac{FG}{F+G} \mathrm{d}\mu \int_0^1 (F+G)\,\mathrm{d}\mu \\
&= \int_0^1 \left[\frac{x^2}{2x+1}, \frac{x^2+x}{x^2+1}\right] \mathrm{d}x \int_0^1 \left[x^2+1, 2x+1\right] \mathrm{d}x \\
&= \left[\frac{x^2}{4} - \frac{x}{4} + \frac{1}{8} ln(2x+1)\Big|_0^1, \frac{ln(x^2+1)}{2} - arctan(x) + x\Big|_0^1\right] \left[\frac{4}{3}, 2\right] \\
&= \left[\frac{1}{8} ln3, \frac{Ln2}{2} - \frac{\pi}{4} + 1\right] \left[\frac{4}{3}, 2\right] \\
&= \left[\frac{1}{6} ln3, ln2 - \frac{\pi}{2} + 2\right] \\
&\approx [0.18310, 1.12235]\ .
\end{aligned}
$$

On the other hand

$$
b) \int_0^1 F\mathrm{d}\mu \int_0^1 G\mathrm{d}\mu = \left[\frac{1}{3}, \frac{1}{2}\right] \left[1, \frac{3}{2}\right] = \left[\frac{1}{3}, \frac{3}{4}\right].
$$

Thus, from a) and b), we can conclude that

$$
\int_0^1 \frac{FG}{F+G} \mathrm{d}\mu \int_0^1 (F+G)\,\mathrm{d}\mu \not\leq \int_0^1 F\mathrm{d}\mu \int_0^1 G\mathrm{d}\mu,
$$

and, consequently, central inequality in Theorem 29 is not necessarily true for interval-valued functions.

Moreover, because

$$
\int_0^1 FG\mathrm{d}\mu = \int_0^1 \left[x^2, x^2+x\right] \mathrm{d}x = \left[\frac{1}{3}, \frac{5}{6}\right],
$$

then by a), we have

$$
\int_0^1 \frac{FG}{F+G} \mathrm{d}\mu \int_0^1 (F+G)\,\mathrm{d}\mu \not\leq \int_0^1 FG\mathrm{d}\mu.
$$

However, in this case, following Remark 4, we obtain

$$
\int_0^1 \frac{FG}{F+G} \mathrm{d}\mu \int_0^1 (F+G)\,\mathrm{d}\mu \trianglerighteq \int_0^1 FG\mathrm{d}\mu.
$$

Thus, from Remark 4 and Example 3, we can see that integral inequalities are very sensitive to the order we are considering.

To finalize this section, we will show Hölder's inequality for interval-valued functions, and then, as a corollary, Cauchy–Schwarz's inequality will be obtained.

Theorem 12 (Interval Hölder's inequality) *If $F, G : [a,b] \to \mathcal{K}_C$ are two integrable interval-valued functions, with $F = [\underline{f}, \overline{f}]$, $G = [\underline{g}, \overline{g}]$, $\underline{f}(x), \underline{g}(x) > 0$. If p, q are positive* *real numbers, such that* $\frac{1}{p}+\frac{1}{q}=1$*, then*

$$\left(\int_0^1 F^p \mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_0^1 G^q \mathrm{d}\mu\right)^{\frac{1}{q}} \geq \int_0^1 FG\mathrm{d}\mu. \tag{30}$$

Proof Because "$()^p$" and "$()^{\frac{1}{p}}$" are increasing functions (see Example 1) and using Theorem 10, then we have

$$\begin{aligned}
\left(\int_0^1 F^p \mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_0^1 G^q \mathrm{d}\mu\right)^{\frac{1}{q}} &= \left(\int_0^1 [\underline{f},\overline{f}]^p \mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_0^1 [\underline{g},\overline{g}]^q \mathrm{d}\mu\right)^{\frac{1}{q}}\\
&= \left(\int_0^1 [\underline{f}^p,\overline{f}^p] \mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_0^1 [\underline{g}^q,\overline{g}^q] \mathrm{d}\mu\right)^{\frac{1}{q}}\\
&= \left(\left[\int_0^1 \underline{f}^p \mathrm{d}\mu,\int_0^1 \overline{f}^p \mathrm{d}\mu\right]\right)^{\frac{1}{p}}\left(\left[\int_0^1 \underline{g}^q \mathrm{d}\mu,\int_0^1 \overline{g}^q \mathrm{d}\mu\right]\right)^{\frac{1}{q}}\\
&= \left[\left(\int_0^1 \underline{f}^p \mathrm{d}\mu\right)^{\frac{1}{p}},\left(\int_0^1 \overline{f}^p \mathrm{d}\mu\right)^{\frac{1}{p}}\right]\left[\left(\int_0^1 \underline{g}^q \mathrm{d}\mu\right)^{\frac{1}{q}},\left(\int_0^1 \overline{g}^q \mathrm{d}\mu\right)^{\frac{1}{q}}\right]\\
&= \left[\left(\int_0^1 \underline{f}^p \mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_0^1 \underline{g}^q \mathrm{d}\mu\right)^{\frac{1}{q}},\left(\int_0^1 \overline{f}^p \mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_0^1 \overline{g}^q \mathrm{d}\mu\right)^{\frac{1}{q}}\right]\\
&\geq \left[(\int_0^1 \underline{f}\underline{g} \mathrm{d}\mu,\int_0^1 \overline{f}\overline{g} \mathrm{d}\mu\right]\\
&= \int_0^1 FG\mathrm{d}\mu\ .
\end{aligned}$$

□

Now, as a corollary of the above Theorem 12, we have the following.

Corollary 3 (Interval Cauchy–Schwarz's inequality) *If* $F, G : [a, b] \to \mathcal{K}_C$ *are two integrable interval-valued functions, with* $F = [\underline{f}, \overline{f}]$*,* $G = [\underline{g}, \overline{g}]$*,* $\underline{f}(x), \underline{g}(x) > 0$*. Then*

$$\left(\int_0^1 F^2 \mathrm{d}\mu\right)^{\frac{1}{2}}\left(\int_0^1 G^2 \mathrm{d}\mu\right)^{\frac{1}{2}} \geq \int_0^1 FG\mathrm{d}\mu. \tag{31}$$

Remark 5 Very recently, Agahi (2020) has shown a nice result for Choquet integral, where it refines the Hölder inequality as follows.

Theorem 13 *(Agahi 2020) Let p and q be positive real numbers. such that* $\frac{1}{p}+\frac{1}{q}=1$*. Let* (Ω, Σ, ν) *be a monotone measure space and* Z, W *be two non-negative functions. such that* Z^p *and* W^q *are Choquet integrable. Assume one of the following conditions being valid:*

(1) ν *is submodular and continuous from below;*
(2) Z, W *are comonotonic and* ν *is continuous from below. Then, the following Hölder inequality:*

$$\left((C)\int Z^p \mathrm{d}\nu\right)^{\frac{1}{p}}\left((C)\int W^q \mathrm{d}\nu\right)^{\frac{1}{q}} \geq (C)\int_Z W\mathrm{d}\nu \tag{32}$$

holds.

We recall that a set-function ν is submodular (or concave) if

$$\nu(A \cup B) + \nu(A \cap B) \leq \nu(A) + \nu(B).$$

Now, analogously to the above theorem, we can extend this result to the interval context as follows.

Theorem 14 (Submodular-Comonotonic-Interval Hölder inequality) *Let p and q be positive real numbers, such that $\frac{1}{p} + \frac{1}{q} = 1$. Let (Ω, Σ, ν) be a monotone measure space and $F = [\underline{f}, \overline{f}], G = [\underline{g}, \overline{g}] : \Omega \to \mathcal{K}_C$ be two non-negative functions, such that F^p and G^q are Choquet integrable. Assume one of the following conditions being valid:*

(1) ν is submodular and continuous from below;
(2) $\underline{f}$ comonotonic with $\underline{g}$, and $\overline{f}$ comonotonic with $\overline{g}$, and ν is continuous from below.
Then, the following Hölder inequality

$$\left((C)\int F^p \mathrm{d}\nu\right)^{\frac{1}{p}} \left((C)\int G^q \mathrm{d}\nu\right)^{\frac{1}{q}} \geq (C)\int FG\mathrm{d}\nu \tag{33}$$

holds.

As a corollary, taking $p = q = 2$, in the same above Theorem 13, we obtain the interval Cauchy–Schwarz inequality for submodular measures.

Corollary 4 (Submodular-Comonotonic-Interval C–S inequality) *Let (Ω, Σ, ν) be a monotone measure space and $F = [\underline{f}, \overline{f}], G = [\underline{g}, \overline{g}] : \Omega \to \mathcal{K}_C$ be two non-negative functions, such that F^2 and G^2 are Choquet integrable. Assume one of the following conditions being valid:*

(1) ν is submodular and continuous from below;
(2) $\underline{f}$ comonotonic with $\underline{g}$, and $\overline{f}$ comonotonic with $\overline{g}$ (i.e., comonotonic in first and second components), and ν is continuous from below.
Then, the following C–S inequality:

$$\left((C)\int F^2 \mathrm{d}\nu\right)^{\frac{1}{2}} \left((C)\int G^2 \mathrm{d}\nu\right)^{\frac{1}{2}} \geq (C)\int FG\mathrm{d}\nu \tag{34}$$

holds.

The following example shows that C–S inequality is verified by comonotonic property of functions (although without submodularity of measure).

Example 4 Let ν be the distorted Lebesgue measure with $\nu = \mu^2$ on $\Omega = [0, 1]$, and consider $F = \left[x^2, x\right]$ and $\left[x^4, x^3\right]$. We will check that C–S inequality is verified, although we know that ν is a monotone and lower continuous measure, but it is not a submodular measure (Agahi 2020).

On one hand, a straightforward calculus shows that

$$(C)\int_0^1 (\underline{f})^2 \mathrm{d}\nu = (C)\int_0^1 (1 - \sqrt{\alpha})^2 \mathrm{d}\alpha = \tfrac{1}{6}\ ;$$
$$(C)\int_0^1 (\overline{f})^2 \mathrm{d}\nu = (C)\int_0^1 (1 - \alpha)^2 \mathrm{d}\alpha = \tfrac{1}{3}\ .$$

Thus

$$\left((C)\int_0^1 F^2 \mathrm{d}\nu\right)^{\frac{1}{2}} = \left(\left[\frac{1}{6}, \frac{1}{3}\right]\right)^{\frac{1}{2}} = \left[\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right]. \tag{35}$$

Analogously

$$(C)\int_0^1 (\underline{g})^2 \mathrm{d}\nu = (C)\int_0^1 (1-\sqrt[4]{\alpha})^2 \mathrm{d}\alpha = \tfrac{1}{15}\ ;$$
$$(C)\int_0^1 (\overline{g})^2 \mathrm{d}\nu = (C)\int_0^1 (1-\sqrt[3]{\alpha}])^2 \mathrm{d}\alpha = \tfrac{1}{10}\ .$$

Thus

$$\left((C)\int_0^1 G^2 \mathrm{d}\nu\right)^{\frac{1}{2}} = \left(\left[\frac{1}{15}, \frac{1}{10}\right]\right)^{\frac{1}{2}} = \left[\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{10}}\right]. \tag{36}$$

Finally

$$(C)\int_0^1 FG\mathrm{d}\nu = \left[(C)\int_0^1 \left(1-\sqrt[6]{\alpha}\right)^2 \mathrm{d}\alpha, (C)\int_0^1 \left(1-\sqrt[4]{\alpha}\right)^2 \mathrm{d}\alpha\right] = \left[\frac{1}{28}, \frac{1}{15}\right], \tag{37}$$

and, consequently, from (35), (36), and (37), we obtain

$$\begin{aligned}\left((C)\int_0^1 F^2 \mathrm{d}\nu\right)^{\frac{1}{2}} \left((C)\int_0^1 G^2 \mathrm{d}\nu\right)^{\frac{1}{2}} &= \left[\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right]\left[\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{10}}\right] \\ &= \left[\frac{1}{\sqrt{90}}, \frac{1}{\sqrt{30}}\right] \\ &\geq (C)\int_0^1 FG\mathrm{d}\nu\ ,\end{aligned}$$

and interval C–S inequality (34) is verified.

To finalize, in the following example, we want to show that for Cauchy–Schwarz inequality to be valid, it is sufficient to have the submodularity property of the measure (and without necessarily having the comonotonic condition).

Example 5 Let μ be the Lebesgue measure on [0, 1] and consider $F, G : [0, 1] \to \mathcal{K}_C$ defined by $F = \left[x, \sqrt{x}\right]$ and $G = \left[\frac{x}{2}, 1 - \frac{x}{2}\right]$. We note that μ is a monotone, continuous from below and submodular measure, whereas F and G are not comonotonic in the second component. However, we will see that C–S is also verified.

In fact, a simple calculus shows that

$$\left((C)\int_0^1 F^2 \mathrm{d}\mu\right)^{\frac{1}{2}} = \left[\int_0^1 \left(1-\sqrt{\alpha}\right)\mathrm{d}\alpha, \int_0^1 (1-\alpha)\,\mathrm{d}\alpha\right]^{\frac{1}{2}} = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right], \tag{38}$$

and

$$\left((C)\int_0^1 G^2 \mathrm{d}\mu\right)^{\frac{1}{2}} = \left[\int_0^1 \left(1-\sqrt{\alpha}\right)\mathrm{d}\alpha, \int_0^1 \left(1-\sqrt{\alpha}\right)\mathrm{d}\alpha\right]^{\frac{1}{2}} = \left[\frac{1}{\sqrt{12}}, \frac{\sqrt{7}}{\sqrt{12}}\right]. \tag{39}$$

On the other hand, for calculating the integral of FG, due to continuity of functions and addivity of μ, then Choquet integral coincides whit Lebesgue integral (Denneberg 1994). Thus, after a simple calculus, we obtain

$$(C)\int_0^1 FG\mathrm{d}\mu = \left[\int_0^1 \frac{x^2}{2}\mathrm{d}x, \int_0^1 \sqrt{x}(1-\frac{x}{2})\mathrm{d}x\right] = \left[\frac{1}{6}.\frac{7}{15}\right], \tag{40}$$

and, consequently, from (38), (39), and (40), we obtain

$$
\begin{aligned}
&\left((C)\int_0^1 F^2 \mathrm{d}\mu\right)^{\frac{1}{2}}\left((C)\int_0^1 G^2 \mathrm{d}\mu\right)^{\frac{1}{2}} \\
&= \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right]\left[\frac{1}{\sqrt{12}}, \frac{\sqrt{7}}{\sqrt{12}}\right] \\
&\geq \left[\frac{1}{6}, \frac{7}{15}\right] = (C)\int_0^1 FG\mathrm{d}\mu,
\end{aligned}
$$

and interval C–S inequality (34) is verified.

5 Conclusion

In general, any integral inequality can be a very powerful tool for applications and, in particular, considering an integral operator as a predictive tool, then an integral inequality can be very important in measuring, computing errors, and delineating such processes. Interval-valued functions (or fuzzy-interval-valued functions) may provide a good alternative for including the uncertainty into the prediction processes. If, in addition, we consider the Aumann integral for interval-valued function as the natural associated expectation, we have a strong model for handle and quantify the uncertainty. Using this arguments as a basis, we use the Aumann integral and the Kulisch–Miranker order on the space of the real and compact intervals, for demonstrate the Milne's integral inequality for interval-valued functions. Furthermore, we explore other partial-order relationships in the class of compact interval to establish, with the order defined by inclusion, the corresponding version of Milne's inequality. On the other hand, we established the connections between Milne, Chebyshev, Hölder, and Cauchy–Schwarz inequalities in the interval scope. With this, we are contributing to the generalization of the various classical integral inequalities made in the last time, specifically to the set-valued context. To finalyze this work, we show a new version of Hölder and C–S inequality based on submodular measures and comonotone functions, topics that we want to deep in our next articles in preparation. In addition, for clarity of the results achieved, examples and applications are shown.

References

Agahi H, Román-Flores H, Flores-Franulič A (2011) General Barnes-Godunova-Levin type inequalities for Sugeno integral. Inf Sci 181:1072–1079

Agahi H (2020) A refined Hölder's inequality for Choquet expectation by Cauchy-Schwarz's inequality. Inf Sci 512:929–934

Aubin JP, Cellina A (1984) Differential inclusions. Springer, New York

Aubin JP, Franskowska H (1990) Set-valued analysis. Birkhäuser, Boston

Aubin JP, Franskowska H (2000) Introduction: set-valued analysis in control theory. Set Value Anal 8:1–9

Aumann RJ (1965) Integrals of set-valued functions. J Math Anal Appl 12:1–12

Chalco-Cano Y, Flores-Franulič A, Román-Flores H (2012) Ostrowski type inequalities for inteval-valued functions using generalized Hukuhara derivative. Comput Appl Math 31:457–472

Chateauneuf A, Cohen M, Kast R (1997) Comonotone random variables in economics. A review of some results, GREQAM, Working Paper 97A07

Costa TM, Román-Flores H (2017) Some integral inequalities for fuzzy-interval-valued functions. Inf Sci 420:110–125

Costa TM, Román-Flores H (2019a) Gauss-type integral inequalities for interval and fuzzy-interval-valued functions. Comput Appl Math 38:Article No 58
Costa TM, Román-Flores H, Chalco-Cano Y (2019b) Opial-type inequalities for interval-valued functions. Fuzzy Sets Syst 358:48–63
Costa TM, Chalco-Cano Y, Román-Flores H (2020) Wirtinger-type integral inequalities for interval-valued functions. Fuzzy Sets Syst 396:102–114
Denneberg D (1994) Non-additive measure and integral, 1st edn. Springer, Amsterdam, p 1994
Flores-Franulič A, Román-Flores H (2007) A Chebyshev type inequality for fuzzy integrals. Appl Math Comput 190:1178–1184
Flores-Franulič A, Román-Flores H, Chalco-Cano Y (2008) A note on fuzzy integral inequality of Stolarsky type. Appl Math Comput 196:55–59
Flores-Franulič A, Román-Flores H, Chalco-Cano Y (2009) Markov type inequalities for fuzzy integrals. Appl Math Comput 207:242–247
Girotto B, Holder S (2011) Chebyshev type inequality for Choquet integral and comonotonicity. Int J Approx Reason 52:1118–1123
Kulisch U, Miranker W (1981) Computer arithmetic in theory and practice. Academic Press, New York
Markov S (1979) Calculus for interval functions of a real variable. Computing 22:325–337
Milne EA (1925) Note on Rosseland's integral for the stellar absorption coefficient. Monthly Notices R Astron Soc 85:979–984
Mitrinović DS, Pečarić JE, Fink AM (1993) Classical and new inequalities in analysis. Mathematics and its applications. Kluwer Academic Publishers Group, Dordrecht
Moore RE (1966) Interval analysis. Prince-Hall, Englewood Cliffs (NJ)
Moore RE (1979) Method and application of interval analysis. SIAM, New York
Moore RE (1985) Computational functional analysis. Ellis Horwood Limited, England
Moore RE, Kearfott RB, Cloud MJ (2009) Introduction to interval analysis. SIAM, USA
Pap E (1995) Null-additive set functions. Kluwer, Dordrecht
Rokne JG (2001) Interval arithmetic and interval analysis: an introduction. In Granular Computing: An Emerging Paradigm, W. Pedrycz Ed.- Heldelberg; New York; Physica-Verlag, pp 1–22
Román-Flores H, Chalco-Cano Y (2006) H-continuity of fuzzy measures and set defuzzification. Fuzzy Sets Syst 157:230–242
Román-Flores H, Flores-Franulič A, Chalco-Cano Y (2007a) The fuzzy integral for monotone functions. Appl Math Comput 185:492–498
Román-Flores H, Flores-Franulič A, Chalco-Cano Y (2007b) A Jensen type inequality for fuzzy integrals. Inf Sci 177:3192–3201
Román-Flores H, Flores-Franulič A, Chalco-Cano Y (2008a) A convolution type inequality for fuzzy integrals. Appl Math Comput 19:94–99
Román-Flores H, Flores-Franulič A, Chalco-Cano Y (2008b) A Hardy-type inequality for fuzzy integrals. Appl Math Comput 204:178–183
Román-Flores H, Flores-Franulič A, Chalco-Cano Y, Dan R (2013) A two-dimensional Hardy type inequality for fuzzy integrals. Int J Uncertain Fuzzy Knowl Based Syst 21:165–173
Román-Flores H, Chalco-Cano Y, Lodwick W (2018) Some integral inequalities for interval-valued functions. Comput Appl Math 37:1306–1318
Román-Flores H, Flores-Franulič A, Aguirre-Cipe I, Martínez-Romero M (2020) A Sugeno integral inequality of Carleman–Knopp type and some refinements. Fuzzy Sets Syst 396:72–81
Wang Z, Klir G (2009) Generalized measure theory. Springer, New York