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A novel operational matrix for the numerical solution of nonlinear Lane–Emden system of fractional order

A. M. Nagy[1,2] · A. A. El-Sayed[3,4]

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Abstract
In this work, we introduce a numerical method for solving nonlinear fractional system of Lane–Emden type equations. The proposed technique is based on Dickson operational matrix of a fractional derivative. First, we deduce the Dickson operational matrix of the fractional derivative using Dickson polynomial, and then, the obtained matrix is unitized to convert the fractional Lane–Emden system with its initial conditions into a system of nonlinear algebraic equations. This system of algebraic equations can be solved numerically via Newton's iteration method. An error estimate of the proposed method is derived. Numerical examples are provided to demonstrate the validity, applicability, and accuracy of the new technique.

Keywords Dickson polynomials · Caputo differential operator · Spectral collocation method · Nonlinear system of Lane-Emden type in the fractional-order · Operational matrix

Mathematics Subject Classification 41A30 · 65L05 · 65L70

1 Introduction

The branch of fractional-order calculus has achieved noteworthy notoriety and attention during the past 4 decades or so, due chiefly to its expressed applications in numerous apparently various and widespread fields such as physics, mechanics, medicine, chemistry, and engineering (Gürbüz and Sezer 2017; Kilbas et al. 2006; Parand and Pirkhedri 2010; Qureshi and Yusuf

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✉ A. M. Nagy
abdelhameed_nagy@yahoo.com

A. A. El-Sayed
aaa18@fayoum.edu.eg

1 Department of Mathematics, Faculty of Science, Kuwait University, 13060 Safat, Kuwait

2 Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Egypt

3 Department of Mathematics, University of Technology and Applied Sciences-AlRustaq, 329 Rustaq, Sultanate of Oman

4 Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt

2019; Sun et al. 2018). It does really contribute several possibly useful tools for modeling many natural phenomena such as the differential, integral equations, and integro-differential equations (Nagy and El-Sayed 2019; Odibat and Momani 2009; Pinto and Carvalho 2015; Sweilam et al. 2016). Also, for description of some phenomena accurately, we need for a system of linear/nonlinear fractional-order differential equations (Naik et al. 2020; Parand et al. 2010). The Lane–Emden systems of differential equations appear in the modeling of several problems in physical and chemical, such as pattern creation, population growth, chemical reactions, and so on (Flockerzi and Sundmacher 2011; Hao et al. 2018; Muatjetjeja and Khalique 2010). For solving the modeled problems analytically, especially the systems, there is many complexity; therefore, the numerical methods are appropriate in these cases. One of these methods is the spectral method which have many importance and popularity for solving many problems (Abd-Elhameed et al. 2016; Babolian et al. 2015). The operational matrix is one of these methods and it is used also for many applications (Ameen et al. 2021; Bhrawy et al. 2015; Irfan et al. 2014; Nagy et al. 2018; Öztürk and Gülsu 2017; Zaky et al. 2017; Zaky 2019). The main goal of this work is propose the numerical solution of the following fractional-order nonlinear system of the Lane–Emden type:

$$\begin{aligned} \mathfrak{D}^{\alpha} x(t) + \frac{k_1}{t}\mathfrak{D}^{\alpha-1} x(t) + f_1\left(x(t), y(t)\right) = 0, \\ \mathfrak{D}^{\alpha} y(t) + \frac{k_2}{t}\mathfrak{D}^{\alpha-1} y(t) + f_2\left(x(t), y(t)\right) = 0, \end{aligned} \tag{1}$$

with the initial conditions:

$$x(0) = x_0,\ y(0) = y_0,\ x'(0) = x_1,\ y'(0) = y_1, \tag{2}$$

where $t > 0$, $k_1,\ k_2$ are given constants, the fractional parameter α is the fractional-order derivative defined in the Captuo sense, $1 < \alpha \leq 2$, $f_1\left(x(t), y(t)\right)$, $f_2\left(x(t), y(t)\right)$ are given nonlinear functions, and $x_0,\ y_0,\ x_1,\ y_1$ are known initial conditions of the system.

To the best of our knowledge, the operational matrix of fractional derivatives based on Dickon's polynomials has not previously been implemented in the literature. Moreover, the desired system has not been studied either. In case $\alpha = 2$, system (1) becomes the classical Lane–Emden system that has been studied in some articles [see (Rach et al. 2014; Wazwaz et al. 2013)].

Outline of the article: In Sect. 2, some necessary mathematical relations and definitions of the fractional calculus will be presented. In Sect. 3, Dickson polynomial operational matrix (DPOM) will be investigated in addition to converting the nonlinear system of Lane–Emden type of the fractional order into a system of algebraic equations via the DPOM. In Sect. 4, we discuss the error estimate of the proposed technique. In Sect. 5, some numerical examples will be provided. In Sect. 6, some concluding remarks are listed.

2 Preliminaries

In this section, we introduce some mathematical tools which are essential for subsequential our work. These benefits instrument in brief from fractional calculus and Dickson polynomials.

2.1 Notations from fractional calculus

Definition 2.1 Nagy and El-Sayed (2019) The fractional derivative of order α in Caputo sense, $\mathfrak{D}^{\alpha}$, is defined by:

$$\mathfrak{D}^{\alpha} f(t) = \frac{1}{\Gamma(k-\alpha)} \int_0^t f^{(k)}(x)(t-x)^{k-(\alpha+1)} \mathrm{d}x, \quad \alpha > 0, \quad t > 0, \tag{3}$$

where $k-1 < \alpha \leq k,\ k \in \mathbb{N}$.

The linear property of the Caputo operator holds as follows:

$$\mathfrak{D}^{\alpha}(\lambda_1\, h(t) + \lambda_2\, g(t)) = \lambda_1\, \mathfrak{D}^{\alpha} h(t) + \lambda_2\, \mathfrak{D}^{\alpha} g(t), \quad \forall\, \lambda_1, \lambda_2 \in \mathbb{R}. \tag{4}$$

Using the definition 2.1 to claim the following explicitly fractionl derivatives:

$$\mathfrak{D}^{\alpha} K = 0, \quad K \text{ is a constant.} \tag{5}$$

$$\mathfrak{D}^{\alpha} t^m = \begin{cases} 0, & m \in \{0, 1, 2, \ldots, \lceil \alpha \rceil - 1\}, \\ \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} t^{m-\alpha}, & m \in \mathbb{N} \wedge m \geq \lceil \alpha \rceil, \end{cases} \tag{6}$$

where the value of the function $\lceil \alpha \rceil$ is the smallest integer $\geq \alpha$.

2.2 Dickson polynomials

The Dickson polynomials are considered as important tool for obtaining the approximate solutions for differential equations of the integro-differential equations. In this work, we will use it for solving a system of fractional-order Lane–Emden type equations. Hence, some definitions and properties are given as follows:

The Dickson polynomial of the first kind, $D_n(t, a)$, can be generated using the following recurrence relation:

$$D_n(t, a) = t\, D_{n-1}(t, a) - a\, D_{n-2}(t, a), n \geq 2,\ a \in (0, \infty),\ -\infty < t < \infty, \tag{7}$$

with the starting functions $D_0(t, a) = 2,\ D_1(t, a) = t$. Using Eq. 7, we can obtain all $n \geq 2$ polynomials. Moreover, we can obtain its analytical expansion as the following:

Definition 2.2 $D_n(t, a)$ of degree $n \geq 1$ in the indeterminate t with the parameter $a \in (0, \infty)$ is defined as:

$$D_n(t, a) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-a)^i \frac{n\, \Gamma(n-i)}{\Gamma(i+1)\, \Gamma(n-2i+1)}\, t^{n-2i}. \tag{8}$$

Here, the value of $\lfloor n/2 \rfloor$ is the largest integer $\leq n/2$.

Also, $D_n(t, a)$ satisfy the orthogonality relation: (Dominici 2017)

$$\langle\, D_n(t, a),\ D_m(t, a)\,\rangle = \int_{-2\sqrt{a}}^{2\sqrt{a}} \frac{D_n(t, a) D_m(t, a)}{\sqrt{4a - t^2}}\, \mathrm{d}t = \begin{cases} 0, & n \neq m, \\ 4\pi, & n = m = 0, \\ 2\pi a^n, & n = m \neq 0. \end{cases} \tag{9}$$

Remark 2.1 In general, the first kind Dickson polynomials are considered as a generalization of some defined polynomials such as the first kind Chebyshev polynomials $2T_n(\frac{t}{2})$ which obtained at $a = 1$. Also, if $a = -1$, then we obtain the Lucas (w)-polynomials $L_n(t)$ and the Pell–Lucas polynomials $Q_n(\frac{t}{2})$. Moreover, if $a = 2$, the Fermat–Lucas polynomials $FL_n(\frac{t}{3})$ are obtained. Furthermore, these polynomials can be used for solving a large class of the fractional-order differential equations.

For more details of the first kind Dickson polynomials, see Kürkçü et al. (2016) and Wang and Yucas (2012).

3 Function approximation and operational matrix of Dickson polynomials

This section is divided into three sections: the first section provides the desired solution as a power expansion of the Dickson polynomials. The second is related to the operational matrix instructions of $D_n(t, a)$ in Caputo's fractional derivative sense, while the third investigates the application of the operational matrix on the system given in Eq. (1) and its initial conditions given in Eq. (2).

3.1 Function approximation

Consider the solution for the system given in terms of Dickson polynomials as follows:

$$\begin{aligned} x(t) &= \sum_{i=0}^{\infty} c_i \ D_i(t, a), \\ y(t) &= \sum_{i=0}^{\infty} h_i \ D_i(t, a), \end{aligned} \tag{10}$$

where c_i and h_i are the unknown coefficients of the power series expansion. Taking the first $(n + 1)$ terms of Eq. (10):

$$\begin{aligned} x_n(t) &= \sum_{i=0}^{n} c_i \ D_i(t, a) = C^T \phi(t, a), \\ y_n(t) &= \sum_{i=0}^{n} h_i \ D_i(t, a) = H^T \phi(t, a), \end{aligned} \tag{11}$$

where:

$$\phi(t, a) = [D_0(t, a), D_1(t, a), \ldots, D_n(t, a)]^T, \tag{12}$$

and the coefficient vectors C and H for the approximate solutions $x_n(t)$ and $y_n(t)$, are given, respectively, as follows:

$$\begin{aligned} C &= [c_0, c_1, \ldots, c_n]^T, \\ H &= [h_0, h_1, \ldots, h_n]^T. \end{aligned}$$

Now, if we assume that:

$$R(t) = [2, t, t^2, \ldots, t^n]^T, \tag{13}$$

then, $\phi(t,a)$ can be expressed as:

$$\phi(t,a) = B\ R(t), \tag{14}$$

where B is the square matrix of order $n+1$ obtained as:

$$B = \begin{pmatrix} b_{0,0} & 0 & 0 & 0 & \dots & 0 \\ b_{1,0} & b_{1,1} & 0 & 0 & \dots & 0 \\ b_{2,0} & b_{2,1} & b_{2,2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n,0} & b_{n,1} & b_{n,2} & \dots & b_{n-1,n} & b_{n,n} \end{pmatrix},$$

and whose elements are given by:

$$(b_{k,j})_{0\le k,j\le n} = \begin{cases} 1, & k=j, \\ (-a)^{\frac{k}{2}}, & j=0,\ k>j,\ k \text{ even}, \\ (-a)^{\frac{k-j}{2}} \dfrac{k\,\Gamma\left(\frac{k+j}{2}\right)}{\Gamma\left(\frac{k-j}{2}+1\right)\Gamma(j+1)}, & j\neq 0,\ k>j,\ k+j \text{ even}, \\ 0, & \text{otherwise}. \end{cases} \tag{15}$$

If $n=5$, then B is given by:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -a & 0 & 1 & 0 & 0 & 0 \\ 0 & -3a & 0 & 1 & 0 & 0 \\ a^2 & 0 & -4a & 0 & 1 & 0 \\ 0 & 5a^2 & 0 & -5a & 0 & 1 \end{pmatrix}.$$

Therefore, using Eq. (14), we claim:

$$R(t) = B^{-1}\ \phi(t,a). \tag{16}$$

3.2 Operational matrices based on Dikcson polynomials

In this part, our target is derive the operational matrix of $\mathfrak{D}^{\alpha}\phi(t,a)$.

To do that, since $\phi(t,a) = B\ R(t)$, then we obtain:

$$\mathfrak{D}^{\alpha}\phi(t,a) = \mathfrak{D}^{\alpha}(B\ R(t)) = B\ \mathfrak{D}^{\alpha}[2,t,t^2,\dots,t^n]^T. \tag{17}$$

Using the Caputo relation given in Eq. (6), one can find:

$$\begin{aligned} \mathfrak{D}^{\alpha}\phi(t,a) &= B\left[0, \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}, \frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha}, \dots, \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{n-\alpha}\right]^T \\ &= B\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{-\alpha} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{-\alpha} \end{bmatrix}\begin{bmatrix} 2 \\ t \\ t^2 \\ \vdots \\ t^n \end{bmatrix} \\ &= B\ G^{\alpha}(t)\ R(t), \end{aligned} \tag{18}$$

where:

$$G^{\alpha}(t) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)} t^{-\alpha} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} t^{-\alpha} \end{bmatrix}. \tag{19}$$

Using Eq. (16), we have:

$$\mathfrak{D}^{\alpha}\phi(t,a) = B\ G^{\alpha}(t)\ B^{-1}\ \phi(t,a), \tag{20}$$

where $B\ G^{\alpha}(t)\ B^{-1}$ is the fractional-order operational matrix, $B\ G^{\alpha-1}(t)\ B^{-1}$, of $\mathfrak{D}^{\alpha}\phi(t,a)$ in terms of Dickson polynomials.

By the same way, we can obtain the operational matrix, $B\ G^{\alpha-1}(t)\ B^{-1}$, of $\mathfrak{D}^{\alpha-1}\phi(t,a)$, where $G^{\alpha-1}(t)$ is given by:

$$G^{\alpha-1}(t) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-(\alpha-1))} t^{-(\alpha-1)} & 0 & \dots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-(\alpha-1))} t^{-(\alpha-1)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\Gamma(n+1)}{\Gamma(n+1-(\alpha-1))} t^{-(\alpha-1)} \end{bmatrix}. \tag{21}$$

3.3 Applied the operational matrix on the Lane–Emden system

In this subsection, we show how to apply the operational matrix of the fractional-order that is investigated in Sect. 3.2 and the approximate solution given in (11) to solve the given fractional Lane–Emden system. If we use Eqs. (11) and (20), then we can rewrite Eqs. (1) and (2) as follows:

$$\begin{aligned} & C^T\ B\ G^{\alpha}(t)\ B^{-1}\phi(t,a) + \frac{k_1}{t}\ C^T\ B\ G^{\alpha-1}(t)\ B^{-1}\phi(t,a) \\ & \quad + f_1\Big(C^T\ B\ G^{\alpha}(t)\ B^{-1}\phi(t,a),\ H^T\ B\ G^{\alpha}(t)\ B^{-1}\phi(t,a)\Big) = 0, \\ & H^T\ B\ G^{\alpha}(t)\ B^{-1}\phi(t,a) + \frac{k_2}{t}\ H^T\ B\ G^{\alpha-1}(t)\ B^{-1}\phi(t,a) \\ & \quad + f_2\Big(C^T\ B\ G^{\alpha}(t)\ B^{-1}\phi(t,a),\ H^T\ B\ G^{\alpha}(t)\ B^{-1}\phi(t,a)\Big) = 0. \end{aligned} \tag{22}$$

$$\begin{aligned} & C^T\ \phi(0,a) = x_0, \\ & H^T\ \phi(0,a) = y_0, \\ & C^T\ E\ B^{-1}\phi(0,a) = x_1, \\ & H^T\ E\ B^{-1}\phi(0,a) = y_1, \end{aligned} \tag{23}$$

where E is $(n+1)\times(n+1)$ square matrix obtained from the first derivative of the Dickson polynomials $D\phi(t,a) = E\ R(t)$.

Applying this technique leads to a system of nonlinear algebraic equations in $2n+2$ unknown coefficients. Now, let us define the collocation points $t_s = L + \left(\frac{M-L}{n}\right) s,\ t \in [L, M],\ s = 1, 2, \dots, n-1$. To solve this system, we first collocate Eq. (22) at the points t_s

and then use Eq. (23). This establishes a system of $2n+2$ equations which can be solved to obtain the coefficient vectors C and H. Hence, we can use any nonlinear technique such as Newton's iteration to obtain the approximate solutions $x(t)$ and $y(t)$.

4 Error estimate

Theorem 4.1 *Let $u \in C^{\infty}[-1,1]$ and $u_n(t)$ be the best square approximation of $u(t)$ defined by $u_n(t)=\sum_{i=0}^{n} c_i D_i(t,1/4)$, and then, we have:*

$$\|u-u_n\|_w \leq \frac{M_n}{(n+1)!}\sqrt{\pi},$$

where:

$$M_n = \max_{t\in[-1,1]} \left|u^{(n+1)}(t)\right| .$$

Proof By expanding the function $u(t)$ using Taylor expansion, we obtain:

$$u(t)=u(0)+tu'(0)+\ldots+\frac{t^n}{n!}u^{(n)}(0)+\frac{t^{n+1}}{(n+1)!}u^{(n+1)}(\xi), \tag{24}$$

where $\xi \in]-1,1[$. Assume:

$$\tilde{u}_n(t)=u(0)+tu'(0)+\ldots+\frac{t^n}{n!}u^{(n)}(0), \tag{25}$$

then:

$$|u(t)-\tilde{u}_n(t)|=\left|\frac{t^{n+1}}{(n+1)!}u^{(n+1)}(\xi)\right| \leq \frac{M_n}{(n+1)!}. \tag{26}$$

Since $u_n(t)$ is the best square approximation of $u(t)$ and according to Eq. 9, we can claim:

$$\begin{aligned}\|u-u_n\|_w^2 \leq \|u-\tilde{u}_n\|_w^2 &= \int_{-1}^{1} \omega(t)[u(t)-\tilde{u}_n(t)]^2 dt \\ &= \int_{-1}^{1} \omega(t)\left[\frac{M_n}{(n+1)!}\right]^2 dt.\end{aligned} \tag{27}$$

Since, $\omega(t)=\frac{1}{\sqrt{1-t^2}}$, then:

$$\begin{aligned}\|u-u_n\|_w^2 &\leq \left[\frac{M_n}{(n+1)!}\right]^2 \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}dt \\ &= \left[\frac{M_n}{(n+1)!}\right]^2 \cdot \pi.\end{aligned} \tag{28}$$

Hence, by taking the square roots of both sides, the proof is complete. □

Now, if we consider the solution of the system (1) is $(x(t), y(t)), \forall t \in [\epsilon, 1]$, $0<\epsilon<1$, then we have $(x(t), y(t)) \in C^{\infty}[\epsilon, 1]$. Using Borel's theorem in Narasimhan (1985), there exists $(\hat{x}(t), \hat{y}(t)) \in C^{\infty}[-1,1]$ an extension of $(x(t), y(t))$. By applying **Theorem** 4.1, we have:

$$\|\hat{x} - x_n\|_w \leq \frac{U_n}{(n+1)!} \cdot \sqrt{\pi},$$
$$\|\hat{y} - y_n\|_w \leq \frac{V_n}{(n+1)!} \cdot \sqrt{\pi},$$

where $U_n = \max_{t\in[-1,1]} \left|\hat{x}^{(n+1)}(t)\right|$ and $V_n = \max_{t\in[-1,1]} \left|\hat{y}^{(n+1)}(t)\right|$.

5 Numerical examples

In what follows, we present two numerical examples to show the applicability and accuracy of the proposed method.

Example 5.1 Consider the following nonlinear fractional Lane–Emden systems of the form:

$$\begin{aligned} \mathfrak{D}^{\alpha} x(t) + \frac{1}{t}\mathfrak{D}^{\alpha-1} x(t) - y^3(t)(x^2(t)+1) = 0 \\ \mathfrak{D}^{\alpha} y(t) + \frac{3}{t}\mathfrak{D}^{\alpha-1} y(t) + y^5(t)(x^2(t)+3) = 0, \end{aligned} \tag{29}$$

with the initial conditions:

$$x(0) = 1,\ y(0) = 1,\ x'(0) = 0,\ y'(0) = 0. \tag{30}$$

In case $\alpha = 2$, the exact solution of Example 5.1 is given by $x(t) = \sqrt{1+t^2}$ and $y(t) = \frac{1}{\sqrt{1+t^2}}$ (see Wazwaz et al. (2013)).

Using the presented method in this paper with $a = 1/4$, and $n = 6$, we obtain approximate solutions at different values of α. Figure 1 illustrates how α affects on the behavior of the solutions. From the curves obtained in Fig. 1, we observe that the numerical solutions for various values of $\alpha = 1.9,\ 1.7, 1.5$ converge to the exact solution for the classical case, i.e.,

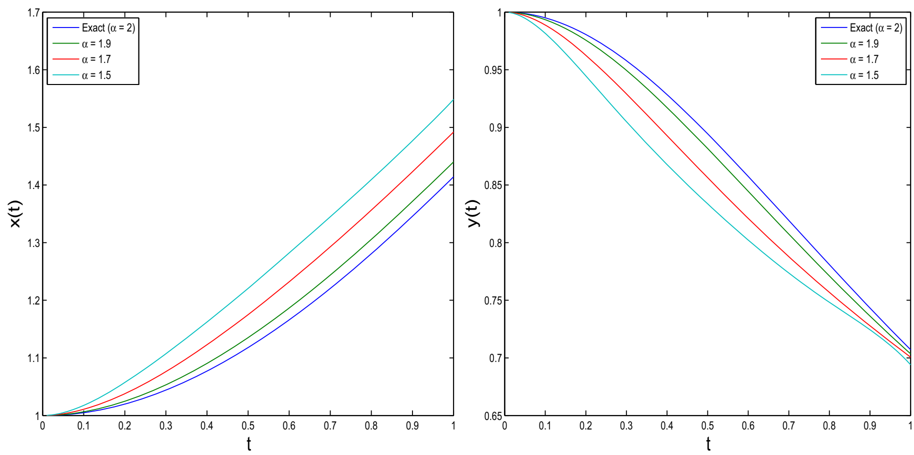

Fig. 1 The behavior of the approximate solutions using the proposed method for various values of α and the exact solution for the classical type for Example 5.1

Table 1 The absolute errors E_x and E_y for Example 5.1 in the classical case

t	E_x				E_y			
	$n=4$	$n=6$	$n=8$	$n=10$	$n=4$	$n=6$	$n=8$	$n=10$
0.1	1.1172e−4	3.3843e−6	6.4447e−9	2.9504e−9	2.6502e−4	5.1417e−6	1.1714e−7	3.7437e−8
0.2	2.3151e−4	4.5873e−7	1.1935e−8	3.7629e−10	4.1510e−4	7.6212e−6	5.0132e−7	2.8077e−8
0.3	2.0055e−4	4.6093e−6	5.5838e−8	2.7877e−9	6.3536e−5	1.7840e−5	6.0377e−7	2.5500e−8
0.4	3.0356e−5	4.7531e−6	8.0093e−8	7.7183e−9	6.2581e−4	1.3324e−5	9.2394e−7	7.3826e−8
0.5	1.7168e−4	7.8962e−7	5.0594e−8	9.4530e−11	1.2528e−3	4.0715e−6	7.1538e−7	1.6123e−8
0.6	2.7780e−4	2.7230e−6	1.1620e−7	5.4360e−9	1.4535e−3	2.1064e−6	1.3553e−6	5.1366e−8
0.7	2.0315e−4	3.3342e−6	4.5422e−8	4.8662e−9	1.0778e−3	7.8524e−6	4.1090e−7	4.8612e−8
0.8	4.9084e−5	2.2625e−6	1.2559e−7	4.8131e−10	2.6478e−4	1.3584e−5	9.6843e−7	6.3366e−9
0.9	3.6074e−4	1.7423e−6	2.1643e−8	1.3643e−9	5.5576e−4	8.9550e−6	6.7861e−7	6.8480e−9
1	4.8636e−4	1.6054e−6	6.3703e−8	1.1576e−9	7.0619e−4	9.7795e−7	1.8360e−7	4.4782e−9

Table 2 Comparison between the absolute error for our results and the results obtained in Öztürk (2019) for Example 5.1 in the classical case

t	E_x in Öztürk (2019)		E_y in Öztürk (2019)		E_x^{ours}		E_y^{ours}	
	$n=4$	$n=6$	$n=4$	$n=6$	$n=4$	$n=6$	$n=4$	$n=6$
0.2	5.09e−4	6.42e−6	8.87e−4	4.67e−6	2.32e−4	4.59e−7	4.15e−4	7.62e−6
0.4	6.28e−4	2.20e−6	2.27e−4	4.05e−5	3.04e−5	4.75e−6	6.26e−4	1.33e−5
0.6	2.77e−4	6.11e−6	1.00e−4	1.85e−6	2.78e−4	2.72e−6	1.45e−3	2.11e−6
0.8	2.72e−4	4.71e−6	6.92e−4	3.27e−5	4.91e−5	2.26e−6	2.65e−4	1.36e−5
1	6.44e−4	5.56e−6	2.62e−4	1.34e−7	4.86e−4	1.61e−6	7.06e−4	9.78e−7

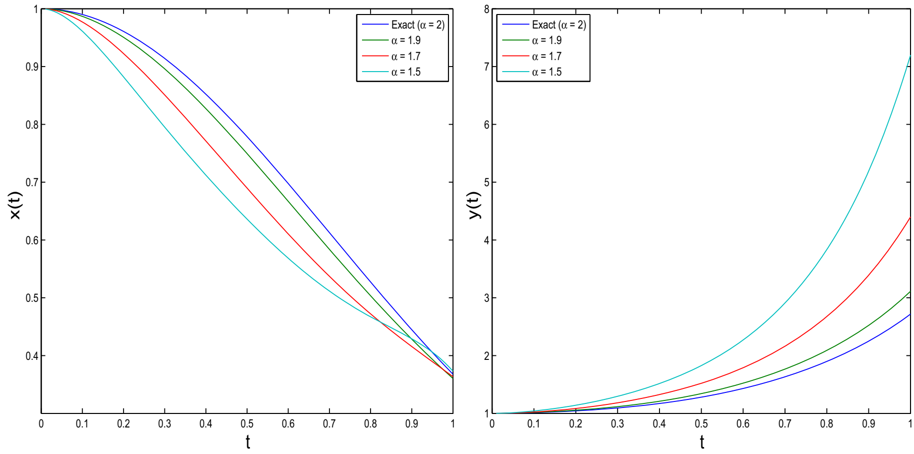

Fig. 2 The behavior of the approximate solutions using the proposed method for various values of α and the exact solution for the classical type for Example 5.2

$\alpha = 2$. In Table 1, to show the accuracy of the proposed method, we have computed the absolute error between the exact and approximate solution, E_x and E_y, for the classical case with different values n. Moreover, in Table 2, we have compared the results obtained by our technique with those obtained in Öztürk (2019).

Example 5.2 Consider the following nonlinear fractional Lane–Emden systems of the form:

$$\begin{aligned} &\mathfrak{D}^{\alpha}x(t)+\frac{8}{t}\mathfrak{D}^{\alpha-1}x(t)+18x(t)-4x(t)\ln(y(t))=0\\ &\mathfrak{D}^{\alpha}y(t)+\frac{4}{t}\mathfrak{D}^{\alpha-1}y(t)+4y(t)\ln(x(t))-10y(t)=0, \end{aligned} \tag{31}$$

with the initial conditions:

$$x(0)=1,\ y(0)=1,\ x'(0)=0,\ y'(0)=0. \tag{32}$$

In case $\alpha = 2$, the exact solution of Example 5.2 is given by $x(t) = e^{-t^2}$ and $y(t) = e^{t^2}$ [see (Wazwaz et al. 2013)].

Numerical solutions of the proposed method at $n = 6$ and different values of α together with the exact solution of the classical case are displayed in Fig. 2. It is obvious that the

Table 3 The absolute errors E_x and E_y for Example 5.2 in the classical case

t	E_x				E_y			
	$n = 4$	$n = 6$	$n = 8$	$n = 10$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
0.1	2.0755e−4	2.3343e−6	5.8434e−8	1.1707e−9	7.9473e−4	1.5508e−5	3.3031e−7	1.2703e−8
0.2	2.4542e−4	1.1639e−5	1.0386e−7	1.6729e−9	1.2487e−3	4.9793e−5	1.1785e−6	1.2819e−8
0.3	2.3119e−4	1.8233e−5	2.2106e−7	2.1711e−10	1.7768e−4	1.0596e−4	1.1418e−6	6.2367e−9
0.4	1.0040e−3	2.0449e−7	1.3674e−7	2.9064e−9	3.3915e−3	4.1900e−5	1.6882e−6	3.1504e−8
0.5	1.5429e−3	2.4929e−5	3.1833e−7	2.1727e−9	7.0238e−3	9.6539e−5	1.4423e−6	1.3458e−8
0.6	1.3106e−3	2.4092e−5	2.5903e−7	1.5925e−9	9.0019e−3	1.4974e−4	2.4583e−6	2.3055e−8
0.7	8.1085e−5	8.3552e−6	3.0032e−7	3.3325e−9	7.4245e−3	2.0470e−5	1.1737e−6	3.5061e−8
0.8	1.7640e−3	3.5224e−5	2.4131e−7	1.1550e−9	1.8814e−3	1.5833e−4	1.8727e−6	9.1426e−9
0.9	2.9920e−3	1.0758e−5	2.7741e−7	3.2129e−11	4.5583e−3	8.7513e−5	2.2015e−6	7.3309e−10
1	1.3494e−3	1.2179e−5	5.9498e−8	2.1942e−10	2.3365e−3	4.8318e−5	3.3571e−7	2.5744e−9

approximate solution approaches the exact solution of the classical case as α approaches 2. Tables 2 and 3 show the absolute error for the solutions $x(t)$ and $y(t)$ with different values of n for the classical case. From Examples 5.1 and 5.2, we can conclude that the introduced method can successfully solve the suggested problems and is easy to implement.

6 Conclusions

Throughout this article, we solved the nonlinear system of Lane–Eden type of fractional order. The proposed method is based on Dickson polynomials. These polynomials are used for constructing the operational matrix of the fractional derivative in Caputo sense. The investigated matrix is used to convert the studied system into a system of algebraic equations. The error estimate of the suggested method is given. Some numerical examples are given to clarify the validity and accuracy of the proposed method for both fractional and classical cases. All results are computed via the MATLAB software.

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