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A new iterative method for solving pseudomonotone variational inequalities with non-Lipschitz operators

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Abstract
The purpose of this paper is to study and analyze a new projection-type algorithm for solving pseudomonotone variational inequality problems in real Hilbert spaces. The advantage of the proposed algorithm is the strong convergence proved without assuming Lipschitz continuity of the associated mapping. In addition, the proposed algorithm uses only two projections onto the feasible set in each iteration. The numerical behaviors of the proposed algorithm on a test problem are illustrated and compared with several previously known algorithms.

Keywords Projection-type method · Viscosity method · Variational inequality · Pseudomonotone mapping

Mathematics Subject Classification 47H09 · 47J20 · 47J05 · 47J25

1 Introduction

We consider the following variational inequality problem (VI) of finding a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \ \ \forall x \in C, \tag{1}$$

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where C is a nonempty closed convex subset in a real Hilbert space H, $A : H \rightarrow H$ is a single-valued mapping, and $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the inner product and the norm in H, respectively.

Let us denote the solution set of VI (1) by $VI(C, A)$. Variational inequality problems are fundamental in a broad range of mathematical and applied sciences; the theoretical and algorithmic foundations as well as the applications of variational inequality problems have been extensively studied in the literature and continue to attract intensive research. For the current state of the art in the finite dimensional setting, see for instance (Facchinei and Pang 2003; Konnov 2001) and the extensive list of references therein.

Many authors have proposed and analyzed several iterative methods for solving the variational inequality (1). The simplest one is the following projection method, which can be seen as an extension of the projected gradient method for optimization problems:

$$x_{n+1} = P_C(x_n - \tau Ax_n), \tag{2}$$

for each $n \geq 1$, where P_C denotes the metric projection from H onto C. Convergence results for this method require some monotonicity properties of A. This method converges under quite strong hypotheses. If A is Lipschitz continuous with Lipschitz constant L and α-strongly monotone, then the sequence generated by (2) converges to an element of $VI(C, A)$, if $\tau \in \left(0, \frac{2\alpha}{L^2}\right)$.

To deal with the weakness of the method defined by (2). Korpelevich 1976 (also independently by Antipin (1976)) proposed the extragradient method in the finite dimensional Euclidean space $\mathbb{R}^m$ for a monotone and L-Lipschitz continuous operator $A : \mathbb{R}^m \rightarrow \mathbb{R}^m$. The algorithm is of the following form:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \tau_n Ax_n), \\ x_{n+1} = P_C(x_n - \tau_n Ay_n), \end{cases} \tag{3}$$

where $\tau_n \in \left(0, \frac{1}{L}\right)$. The sequence $\{x_n\}$ generated by (3) converges to an element of $VI(C, A)$ provided that $VI(C, A)$ is nonempty.

In recent years, the extragradient method was further extended to infinite-dimensional spaces in various ways, see, e.g. (Censor et al. 2011a, b, c; Bello Cruz and Iusem 2009, 2010, 2012, 2015; Bello Cruz et al. 2019; Gibali et al. 2019; Kanzow and Shehu 2018; Konnov 1997, 1998; Maingé and Gobinddass 2016; Malitsky 2015; Thong and Hieu 2018, 2019; Thong et al. 2019c; Thong and Vuong 2019; Thong and Gibali 2019a, b; Thong et al. 2019a, b; Vuong 2018) and the references therein.

Note that, when A is not Lipschitz continuous or the constant L is very difficult to compute, the method of Korpelevich is not applicable (or possible to use) because we cannot determine the step-size τ_n. To overcome this obstacle, Iusem (1994) proposed an iterative algorithm in the finite dimensional Euclidean space $\mathbb{R}^m$ for $VI(C, A)$ as follows:

Algorithm 1.1

Initialization: *Given* $l \in (0, 1)$, $\mu \in (0, 1)$, $\gamma > 0$. *Let* $x_1 \in C$ *be arbitrary*

Iterative Steps: *Given the current iterate* x_n, *calculate* x_{n+1} *as follows:*

Step 1. *Compute*

$$y_n = P_C(x_n - \gamma_n A x_n)$$

where $\gamma_n := \gamma l^{j_n}$ *with* j_n *is the smallest non-negative integer* j *satisfying*

$$\gamma l^j \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|. \tag{4}$$

Step 2. *Compute*

$$x_{n+1} = P_C(x_n - \tau_n A y_n),$$

where $\tau_n = \dfrac{\langle Ay_n, x_n - y_n\rangle}{\|Ay_n\|^2}$.

It is worth noting that this modification allows the author to prove convergence without Lipschitz continuity of the operator A.

Algorithm 1.1 may require many iterations in the iterative procedure used to select the stepsize γ_n and each iterations uses a new projections. This may lead to a large computational effort that should be avoided.

Motivated by this idea, Iusem and Svaiter (1997) proposed a modified extragradient method for solving monotone variational inequalities which requires only two projections onto C at each iteration. Few years later, this method was improved by Solodov and Svaiter (1999). They introduced an algorithm for solving (1) in finite-dimensional spaces. In fact, the method in Solodov and Svaiter (1999) can solve a more general case where A is only continuous and satisfies the following condition:

$$\langle Ax, x - x^*\rangle \geq 0 \;\; \forall x \in C \text{ and } x^* \in VI(C, A). \tag{5}$$

The property (5) holds if A is monotone or more generally pseudomonotone on C in the sense of Karamardian (1976). Very recently, Vuong and Shehu (2019) modified result of Solodov and Svaiter to obtain strong convergence in infinite dimensional real Hilbert spaces. They constructed the algorithm based on Halpern method (Halpern 1967) and method of Solodov and Svaiter. The algorithm is of the following form:

Algorithm 1.2

Initialization: *Given* $\{\alpha_n\} \subset (0, 1)$, $l \in (0, 1)$, $\mu \in (0, 1)$. *Let* $x_1 \in C$ *be arbitrary*

Iterative Steps: *Given the current iterate* x_n, *calculate* x_{n+1} *as follows:*

Step 1. *Compute*

$$z_n = P_C(x_n - Ax_n)$$

and $r(x_n) := x_n - z_n$. *If* $r(x_n) = 0$ *then stop and* x_n *is a solution of* $VI(C, A)$. *Otherwise*
Step 2. *Compute*

$$y_n = x_n - \tau_n r(x_n),$$

where $\tau_n := l^{j_n}$ *and* j_n *is the smallest non-negative integer* j *satisfying*

$$\langle A(x_n - l^j r(x_n)), r(x_n)\rangle \geq \frac{\mu}{2}\|r(x_n)\|^2. \tag{6}$$

Step 3. *Compute*

$$x_{n+1} = \alpha_n x_1 + (1 - \alpha_n) P_{C_n}(x_n),$$

where

$$C_n := \{x \in C : h_n(x) \leq 0\}$$

and

$$h_n(x) = \langle Ay_n, x - y_n \rangle.$$

Set $n := n + 1$ *and go to* ***Step 1***.

They proved that if $A : H \to H$ is a pseudomonotone, uniformly continuous and sequentially weakly continuous on bounded subsets of C and the sequence $\{\alpha_n\}$ satisfies the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ generated by Algorithm 1.2 converges strongly to $p \in VI(C, A)$, where $p = P_C x_1$. Note that when $\alpha_n = 0\ \forall n$ then Algorithm 1.2 reduces to the algorithm of Solodov and Svaiter.

Motivated and inspired by the works in Moudafi (2000), Solodov and Svaiter (1999) and Vuong and Shehu (2019), and by the ongoing research in these directions, in this paper we introduce a modification of the algorithm proposed by Solodov and Svaiter for solving variational inequalities with uniformly continuous pseudomonotone operator. The main modification is to use a different Armijo-type line search to obtain a hyperplane strictly separating current iterate from the solutions of the variational inequalities.

The paper is organized as follows: We first recall some basic definitions and results in Sect. 2. Our algorithm is presented and analyzed in Sect. 3. In Sect. 4 we present some numerical experiments which demonstrate the proposed algorithm performances as well as provide a preliminary computational overview by comparing it with some related algorithms.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \to x$ as $n \to \infty$. For each $x, y \in H$, we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.$$

Definition 2.1 Let $T : H \to H$ be an operator. Then

1. The operator T is called L-`Lipschitz continuous` with $L > 0$ if

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in H.$$

   if $L = 1$ then the operator T is called `nonexpansive` and if $L \in (0, 1)$, T is called `contraction`.
2. T is called `monotone` if

$$\langle Tx - Ty, x - y \rangle \geq 0 \quad \forall x, y \in H.$$

3. T is called `pseudomonotone` if

$$\langle Tx, y - x \rangle \geq 0 \Longrightarrow \langle Ty, y - x \rangle \geq 0 \quad \forall x, y \in H.$$

4. T is called α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha\|x - y\|^2 \ \forall x, y \in H.$$

5. The operator T is called `sequentially weakly continuous` if for each sequence $\{x_n\}$ we have: x_n converges weakly to x implies Tx_n converges weakly to Tx.

It is easy to see that every monotone operator is pseudomonotone but the converse is not true. We next present an academic example of a variational inequality problem (VI) in infinite dimensional Hilbert spaces where the cost function A is pseudomonotone, uniformly continuous and sequentially weakly continuous, but A fails to be Lipschitz continuous on H.

Example 1 Consider the Hilbert space

$$H = l_2 := \left\{ u = (u_1, u_2, \ldots, u_i, \ldots) \mid \sum_{i=1}^{\infty} |u_i|^2 < +\infty \right\}$$

equipped with the inner product and induced norm on H:

$$\langle u, v \rangle = \sum_{i=1}^{\infty} u_i v_i \text{ and } \|u\| = \sqrt{\langle u, u \rangle}$$

for any $u = (u_1, u_2, \ldots, u_i, \ldots), v = (v_1, v_2, \ldots, v_i, \ldots) \in H$.

Consider the set and the mapping:

$$C = \{u = (u_1, u_2, \ldots, u_i, \ldots) \in H : |u_i| \leq \frac{1}{i} \quad \forall i = 1, 2, \ldots\}, \quad Au = \left(\|u\| + \frac{1}{\|u\| + 1} \right) u.$$

With this C and A, it is easy to see that $VI(C, A) \neq \emptyset$ since $0 \in VI(C, A)$ and moreover, A is pseudomonotone, uniformly continuous and sequentially weakly continuous on C, but A fails to be Lipschitz continuous on H.

Now let $u, v \in C$ be such that $\langle Au, v-u \rangle \geq 0$. This implies that $\langle u, v-u \rangle \geq 0$. Consequently,

$$\begin{aligned} \langle Av, v-u \rangle &= \left(\|u\| + \frac{1}{\|u\| + 1} \right) \langle v, v-u \rangle \\ &\geq \left(\|u\| + \frac{1}{\|u\| + 1} \right) (\langle v, v-u \rangle - \langle u, v-u \rangle) \\ &= \left(\|u\| + \frac{1}{\|u\| + 1} \right) \|v-u\|^2 \geq 0 \end{aligned}$$

meaning that A is pseudomonotone. Now, since C is compact, the mapping A is uniformly continuous on C and A is sequentially weakly continuous on C.

Finally, we show that A is not Lipschitz continuous on H. Assume to the contrary that A is Lipschitz continuous on H, i.e., there exists $L > 0$ such that

$$\|Au - Av\| \leq L\|u - v\| \;\; \forall u, v \in H.$$

Let $u = (L, 0, \ldots, 0, \ldots)$ and $v = (0, 0, \ldots, 0, \ldots)$, then

$$\|Au - Av\| = \|Au\| = \left(\|u\| + \frac{1}{\|u\| + 1} \right) \|u\| = \left(L + \frac{1}{L+1} \right) L.$$

Thus, $\|Au - Av\| \leq L\|u - v\|$ is equivalent to

$$\left(L + \frac{1}{L+1} \right) L \leq L^2,$$

equivalently

$$\frac{1}{L+1} \leq 0,$$

this leads to a contradiction and thus A is not Lipschitz continuous on H.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\| \ \forall y \in C$. P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive.

Lemma 2.1 (Goebel and Reich 1984) *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x \iff \langle x - z, z - y \rangle \geq 0 \ \forall y \in C$.*

Lemma 2.2 (Goebel and Reich 1984) *Let C be a closed and convex subset in a real Hilbert space H, $x \in H$. Then*

(i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle \ \forall y \in H$;
(ii) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2 \ \forall y \in C$;
(iii) $\langle (I - P_C)x - (I - P_C)y, x - y \rangle \geq \|(I - P_C)x - (I - P_C)y\|^2 \ \forall y \in H$.

For properties of the metric projection, the interested reader could be referred to Sect. 3 in Goebel and Reich (1984) and Chapter 4 in Cegielski (2012).

The following Lemmas are useful for the convergence of our proposed method.

Lemma 2.3 (Iusem and Garciga 2001) *Let H_1 and H_2 be two real Hilbert spaces. Suppose $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then $A(M)$ is bounded.*

Lemma 2.4 (Cottle and Yao 1992, Lemma 2.1) *Consider the problem $VI(C, A)$ with C being a nonempty, closed, convex subset of a real Hilbert space H and $A : C \to H$ being pseudomonotone and continuous. Then, x^* is a solution of $VI(C, A)$ if and only if*

$$\langle Ax, x - x^* \rangle \geq 0 \ \ \forall x \in C.$$

Lemma 2.5 (He 2006) *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let h be a real-valued function on H and define $K := \{x \in C : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on C with modulus $\theta > 0$, then*

$$\mathrm{dist}(x, K) \geq \theta^{-1} \max\{h(x), 0\} \ \ \forall x \in C,$$

where $\mathrm{dist}(x, K)$ denotes the distance function from x to K.

Lemma 2.6 (Maingé 2008) *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, \ldots, k\}$ such that $a_n < a_{n+1}$.

Lemma 2.7 (Xu 2002) *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{b_n\}$ is a sequence such that

a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
b) $\limsup_{n\to\infty} b_n \leq 0$.
Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section we introduce a new modified method of Solodov and Svaiter for solving (1), and the following conditions are assumed for the convergence of the proposed method:

Condition 1 *The feasible set C is a nonempty, closed, and convex subset of the real Hilbert space H.*

Condition 2 *The VI (1) associated operator $A : H \to H$ is a pseudomonotone, uniformly continuous and sequentially weakly continuous on bounded subsets of C.*

Condition 3 *The solution set of the VI (1) is nonempty, that is $VI(C, A) \neq \emptyset$.*

Condition 4 *We assume that $f : C \to C$ is a contractive mapping with a coefficient $\rho \in [0, 1)$, and we add the following condition*

Condition 5 *Let $\{\alpha_n\}$ be a real sequences in $(0, 1)$ such that*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Algorithm 3.3

Initialization: *Given $l \in (0, 1)$, $\mu > 0$, $\lambda \in (0, \frac{1}{\mu})$. Let $x_1 \in C$ be arbitrary*

Iterative Steps: *Given the current iterate x_n, calculate x_{n+1} as follows:*

Step 1. *Compute*

$$z_n = P_C(x_n - \lambda A x_n)$$

and $r_\lambda(x_n) := x_n - z_n$. If $r_\lambda(x_n) = 0$ then stop and x_n is a solution of $VI(C, A)$. Otherwise

Step 2. *Compute*

$$y_n = x_n - \tau_n r_\lambda(x_n),$$

where $\tau_n := l^{j_n}$ and j_n is the smallest non-negative integer j satisfying

$$\langle Ax_n - A(x_n - l^j r_\lambda(x_n)), r_\lambda(x_n)\rangle \leq \mu \|r_\lambda(x_n)\|^2. \tag{7}$$

Step 3. *Compute*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_{C_n}(x_n),$$

where

$$C_n := \{x \in C : h_n(x) \leq 0\}$$

and

$$h_n(x) = \langle Ay_n, x - y_n\rangle. \tag{8}$$

*Set $n := n + 1$ and go to **Step 1**.*

Now let us compare the above algorithm with Algorithm 1.2. In the step of the Armijo-type linesearch, the above algorithm uses a different procedure which replaces the one described in (6) as follows:

$$\langle Ax_n - A(x_n - l^j r_\lambda(x_n)), r_\lambda(x_n)\rangle \leq \mu \|r_\lambda(x_n)\|^2,$$

where $\mu > 0$ and $\lambda \in \left(0, \frac{1}{\mu}\right)$. Unlike (6), which requires $\mu \in (0, 1)$ and $\lambda = 1$. Such choices are crucial for the convergence analysis in Solodov and Svaiter (1999) and Vuong and Shehu (2019), while the parameter μ in our algorithm can take any positive scalar. In addition, we use viscosity techniques to solve this problem, which increases the speed of the proposed algorithm (see our numerical experiments). So our algorithm can be applied more conveniently in practice.

Remark 3.1 It is easy to see that x_n, y_n, z_n in Algorithm 3.3 belong to C.

We start the analysis of the algorithm's convergence by proving some Lemmas.

Lemma 3.8 *Assume that Conditions 1–2 hold. The Armijo-line search rule (7) is well defined.*

Proof Since $l \in (0, 1)$ and A is continuous on C hence $\langle Ax_n - A(x_n - l^j r_\lambda(x_n)), r_\lambda(x_n)\rangle$ converges to zero as j tends to infinite. On the other hand, as a consequence of Step 1, $\|r_\lambda(x_n)\| > 0$ (otherwise, the procedure stops). Therefore, there exists a non-negative integer j_n satisfying (7). □

Lemma 3.9 *Assume that $\{x_n\}$ is generated by Algorithm 3.3; then we have*

$$\langle Ax_n, r_\lambda(x_n)\rangle \geq \lambda^{-1}\|r_\lambda(x_n)\|^2.$$

Proof By the projection property we have $\|x - P_C y\|^2 \leq \langle x - y, x - P_C y\rangle$ for all $x \in C$ and $y \in H$. Let $y = x_n - \lambda Ax_n$, $x = x_n$; then

$$\|x_n - P_C(x_n - \lambda Ax_n)\|^2 \leq \lambda\langle Ax_n, x_n - P_C(x_n - \lambda Ax_n)\rangle;$$

thus

$$\langle Ax_n, r_\lambda(x_n)\rangle \geq \lambda^{-1}\|r_\lambda(x_n)\|^2.$$

□

Lemma 3.10 *Assume that Conditions 1–3 hold. Let x^* be a solution of problem (1) and the function h_n be defined by (8). Then $h_n(x^*) \leq 0$ and $h_n(x_n) \geq \tau_n \left(\frac{1}{\lambda} - \mu\right)\|r_\lambda(x_n)\|^2$. In particular, if $r_\lambda(x_n) \neq 0$ then $h_n(x_n) > 0$.*

Proof Since x^* is a solution of problem (1), by Lemma 2.4 we have $h_n(x^*) = \langle Ay_n, x^* - y_n\rangle \leq 0$. The first claim of Lemma 3.10 is proved. Now, we prove the second claim. We have

$$h_n(x_n) = \langle Ay_n, x_n - y_n\rangle = \langle Ay_n, \tau_n r_\lambda(x_n)\rangle = \tau_n\langle Ay_n, r_\lambda(x_n)\rangle. \tag{9}$$

On the other hand, from (7) we have

$$\langle Ax_n - Ay_n, r_\lambda(x_n)\rangle \leq \mu\|r_\lambda(x_n)\|^2,$$

thus

$$\langle Ay_n, r_\lambda(x_n)\rangle \geq \langle Ax_n, r_\lambda(x_n)\rangle - \mu\|r_\lambda(x_n)\|^2.$$

Using Lemma 3.9 we get

$$\langle Ay_n, r_\lambda(x_n)\rangle \geq \left(\frac{1}{\lambda} - \mu\right) \|r_\lambda(x_n)\|^2. \tag{10}$$

Combining (9) and (10) we get

$$h_n(x_n) \geq \tau_n \left(\frac{1}{\lambda} - \mu\right) \|r_\lambda(x_n)\|^2.$$

□

Remark 3.2 From Lemma 3.10 we have $C_n \neq \emptyset$.

Lemma 3.11 *Assume that Conditions 1–3 hold. Let $\{x_n\}$ be a sequence generated by Algorithm 3.3. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $z \in C$ and $\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\| = 0$ then $z \in VI(C, A)$.*

Proof We have $z_{n_k} = P_C(x_{n_k} - Ax_{n_k})$ thus,

$$\langle x_{n_k} - Ax_{n_k} - z_{n_k}, x - z_{n_k}\rangle \leq 0 \ \ \forall x \in C.$$

or equivalently

$$\langle x_{n_k} - z_{n_k}, x - z_{n_k}\rangle \leq \langle Ax_{n_k}, x - z_{n_k}\rangle \ \ \forall x \in C.$$

This implies that

$$\langle x_{n_k} - z_{n_k}, x - z_{n_k}\rangle + \langle Ax_{n_k}, z_{n_k} - x_{n_k}\rangle \leq \langle Ax_{n_k}, x - x_{n_k}\rangle \ \ \forall x \in C. \tag{11}$$

Taking $k \to \infty$ in (11) since $\|x_{n_k} - z_{n_k}\| \to 0$ and $\{Ax_{n_k}\}$ is bounded, we get

$$\liminf_{k\to\infty} \langle Ax_{n_k}, x - x_{n_k}\rangle \geq 0. \tag{12}$$

Let us choose a sequence $\{\epsilon_k\}_k$ of positive numbers decreasing and tending to 0. For each ϵ_k, we denote by N_k the smallest positive integer such that

$$\left\langle Ax_{n_j}, x - x_{n_j}\right\rangle + \epsilon_k \geq 0 \quad \forall j \geq N_k, \tag{13}$$

where the existence of N_k follows from (12). Since $\{\epsilon_k\}$ is decreasing, it is clear that the sequence $\{N_k\}$ is increasing. Furthermore, for each k, $Ax_{N_k} \neq 0$ and, setting

$$v_{N_k} = \frac{Ax_{N_k}}{\|Ax_{N_k}\|^2},$$

we have $\left\langle Ax_{N_k}, v_{N_k}\right\rangle = 1$ for each k. Now we can deduce from (13) that for each k

$$\left\langle Ax_{N_k}, x + \epsilon_k v_{N_k} - x_{N_k}\right\rangle \geq 0,$$

and, since A is pseudomonotone on H, that

$$\left\langle A(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - x_{N_k}\right\rangle \geq 0. \tag{14}$$

On the other hand, we have that $\left\{x_{n_k}\right\}$ converges weakly to z when $k \to \infty$. By the fact that A is sequentially weakly continuous on C, we have that $\left\{Ax_{n_k}\right\}$ converges weakly to Az. We assume that $Az \neq 0$ (otherwise, z is a solution). By the sequentially weakly lower semicontinuity of norm, we get

$$0 < \|Az\| \leq \liminf_{k\to\infty} \|Ax_{n_k}\|.$$

Since $\{x_{N_k}\} \subset \{x_{n_k}\}$ and $\epsilon_k \to 0$ as $k \to \infty$, we obtain

$$\begin{aligned} 0 \leq \limsup_{k\to\infty} \|\epsilon_k v_{N_k}\| &= \limsup_{k\to\infty} \left(\frac{\epsilon_k}{\|Ax_{n_k}\|}\right) \\ &\leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|Ax_{n_k}\|} \leq \frac{0}{\|Az\|} = 0, \end{aligned}$$

which implies that $\lim_{k\to\infty} \|\epsilon_k v_{N_k}\| = 0$. Hence, taking the limit as $k \to \infty$ in (14), we obtain

$$\langle Ax, x - z\rangle \geq 0.$$

By Lemma 2.4 we obtain $z \in VI(C, A)$. □

Theorem 3.1 *Assume that Conditions 1–5 hold. Then any sequence $\{x_n\}$ generated by Algorithm 3.3 converges strongly to $p \in VI(C, A)$, where $p = P_{VI(C,A)} \circ f(p)$.*

Proof **Claim 1.** The sequence $\{x_n\}$ is bounded. Indeed, let $w_n = P_{C_n} x_n$. Since $p \in C_n$ we have

$$\begin{aligned} \|w_n - p\|^2 = \|P_{C_n} x_n - p\|^2 &\leq \|x_n - p\|^2 - \|P_{C_n} x_n - x_n\|^2 \\ &= \|x_n - p\|^2 - \text{dist}^2(x_n, C_n). \end{aligned} \tag{15}$$

This implies that

$$\|w_n - p\| \leq \|x_n - p\|. \tag{16}$$

Using (16) we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1 - \alpha_n) w_n - p\| \\ &= \|\alpha_n (f(x_n) - p) + (1 - \alpha_n)(w_n - p)\| \\ &\leq \alpha_n \|f(x_n) - p\| + (1 - \alpha_n)\|w_n - p\| \\ &\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n)\|w_n - p\| \\ &\leq \alpha_n \rho \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\ &\leq [1 - \alpha_n(1 - \rho)]\|x_n - p\| + \alpha_n(1 - \rho)\frac{\|f(p) - p\|}{1 - \rho} \\ &\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1 - \rho}\} \\ &\leq \cdots \leq \max\left\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1 - \rho}\right\}. \end{aligned}$$

Thus, the sequence $\{x_n\}$ is bounded. Consequently, the sequences $\{y_n\}$, $\{f(x_n)\}$, $\{Ay_n\}$ are bounded.

Claim 2.

$$\|w_n - x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n \langle f(x_n) - p, x_{n+1} - p\rangle.$$

We have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n (f(x_n) - p) + (1 - \alpha_n)(w_n - p)\|^2 \\ &\leq (1 - \alpha_n)\|w_n - p\|^2 + 2\alpha_n \langle f(x_n) - p, x_{n+1} - p\rangle \\ &\leq \|w_n - p\|^2 + 2\alpha_n \langle f(x_n) - p, x_{n+1} - p\rangle. \end{aligned} \tag{17}$$

On the other hand, we have

$$\|w_n - p\|^2 = \|P_{C_n} x_n - p\|^2 \leq \|x_n - p\|^2 - \|w_n - x_n\|^2. \tag{18}$$

Substituting (18) into (17) we get

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \|w_n - x_n\|^2 + 2\alpha_n \langle f(x_n) - p, x_{n+1} - p\rangle.$$

This implies that

$$\|w_n - x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n \langle f(x_n) - p, x_{n+1} - p\rangle.$$

Claim 3.

$$(1-\alpha_n)\left[\frac{1}{L}\tau_n\left(\frac{1}{\lambda} - \mu\right)\|r_\lambda(x_n)\|^2\right]^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n \|f(x_n) - p\|^2.$$

We first prove that

$$\|w_n - p\|^2 \leq \|x_n - p\|^2 - \left[\frac{1}{L}\tau_n\left(\frac{1}{\lambda} - \mu\right)\|r_\lambda(x_n)\|^2\right]^2. \tag{19}$$

Since $\{Ay_n\}$ is bounded, there exists $L > 0$ such that $\|Ay_n\| \leq L$ for all n. Using this fact, we have for all $u, v \in C_n$ that

$$\|h_n(u) - h_n(v)\| = \|\langle Ay_n, u - v\rangle\| \leq \|Ay_n\|\|u - v\| \leq L\|u - v\|.$$

This implies that $h_n(\cdot)$ is L-Lipschitz continuous on C_n. By Lemma 2.5 we obtain

$$\text{dist}(x_n, C_n) \geq \frac{1}{L} h_n(x_n),$$

which, together with Lemma 3.10 we get

$$\text{dist}(x_n, C_n) \geq \frac{1}{L}\tau_n\left(\frac{1}{\lambda} - \mu\right)\|r_\lambda(x_n)\|^2. \tag{20}$$

Combining (15) and (20) we obtain

$$\|w_n - p\|^2 \leq \|x_n - p\|^2 - \left[\frac{1}{L}\tau_n\left(\frac{1}{\lambda} - \mu\right)\|r_\lambda(x_n)\|^2\right]^2.$$

Now, we prove Claim 3. From the definition of the sequence $\{x_n\}$ and (19) we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(f(x_n) - p) + (1-\alpha_n)(w_n - p)\|^2 \\
&= \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 - \alpha_n(1-\alpha_n)\|f(x_n) - w_n\|^2 \\
&\leq \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 \\
&\leq \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad - (1-\alpha_n)\left[\frac{1}{L}\tau_n\left(\frac{1}{\lambda} - \mu\right)\|r_\lambda(x_n)\|^2\right]^2 \\
&\leq \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - (1-\alpha_n)\left[\frac{1}{L}\tau_n\left(\frac{1}{\lambda} - \mu\right)\|r_\lambda(x_n)\|^2\right]^2.
\end{aligned}$$

This implies that

$$(1-\alpha_n)\left[\frac{1}{L}\tau_n\left(\frac{1}{\lambda} - \mu\right)\|r_\lambda(x_n)\|^2\right]^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n \|f(x_n) - p\|^2.$$

Claim 4.

$$\|x_{n+1}-p\|^2 \le (1-(1-\rho)\alpha_n)\|x_n-p\|^2 + (1-\rho)\alpha_n \frac{2}{1-\rho}\langle f(p)-p, x_{n+1}-p\rangle.$$

We have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)z_n - p\|^2 \\
&= \|\alpha_n(f(x_n)-f(p)) + (1-\alpha_n)(z_n-p) + \alpha_n(f(p)-p)\|^2 \\
&\le \|\alpha_n(f(x_n)-f(p)) + (1-\alpha_n)(z_n-p)\|^2 + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\le \alpha_n\|f(x_n)-f(p)\|^2 + (1-\alpha_n)\|z_n-p\|^2 + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\le \alpha_n\rho\|x_n-p\|^2 + (1-\alpha_n)\|x_n-p\|^2 + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&= (1-(1-\rho)\alpha_n)\|x_n-p\|^2 + (1-\rho)\alpha_n\frac{2}{1-\rho}\langle f(p)-p, x_{n+1}-p\rangle.
\end{aligned} \tag{21}$$

Claim 5. The sequence $\{\|x_n-p\|^2\}$ converges to zero by considering two possible cases on the sequence $\{\|x_n-p\|^2\}$.

Case 1: There exists an $N \in \mathbb{N}$ such that $\|x_{n+1}-p\|^2 \le \|x_n-p\|^2$ for all $n \ge N$. This implies that $\lim_{n\to\infty}\|x_n-p\|^2$ exists. It implies from **Claim 2** that

$$\lim_{n\to\infty}\|x_n-w_n\| = 0.$$

Since the sequence $\{x_n\}$ is bounded, it implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some $z \in C$ such that

$$\limsup_{n\to\infty}\langle f(p)-p, x_n-p\rangle = \lim_{k\to\infty}\langle f(p)-p, x_{n_k}-p\rangle = \langle f(p)-p, z-p\rangle.$$

Now, according to Claim 3

$$\lim_{k\to\infty}(1-\alpha_{n_k})\left[\frac{1}{L}\tau_{n_k}\left(\frac{1}{\lambda}-\mu\right)\|r_\lambda(x_{n_k})\|^2\right]^2 = 0.$$

This follows that

$$\lim_{k\to\infty}\tau_{n_k}\|r_\lambda(x_{n_k})\|^2 = \lim_{k\to\infty}\tau_{n_k}\|x_{n_k}-z_{n_k}\|^2 = 0. \tag{22}$$

Now, we prove that

$$\lim_{k\to\infty}\|x_{n_k}-z_{n_k}\| = 0. \tag{23}$$

We first consider the case $\liminf_{k\to\infty}\tau_{n_k} > 0$. In this case, there is a constant $\tau > 0$ such that $\tau_{n_k} \ge \tau > 0$ for all $k \in \mathbb{N}$. We have

$$\|x_{n_k}-z_{n_k}\|^2 = \frac{1}{\tau_{n_k}}\tau_{n_k}\|x_{n_k}-z_{n_k}\|^2 \le \frac{1}{\tau}.\tau_{n_k}\|x_{n_k}-z_{n_k}\|^2. \tag{24}$$

Combining (22) and (24) we obtain

$$\lim_{k\to\infty}\|x_{n_k}-z_{n_k}\| = 0.$$

Second, we consider the case $\liminf_{k\to} \tau_{n_k} = 0$. In this case, we take a subsequence $\{n_{k_j}\}$ of $\{n_k\}$ if necessary, we assume without loss of generality that

$$\lim_{k\to\infty} \tau_{n_k} = 0, \tag{25}$$

and

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\| = a > 0. \tag{26}$$

Let $y_k = \frac{1}{l}\tau_{n_k} z_{n_k} + \left(1 - \frac{1}{l}\tau_{n_k}\right) x_{n_k}$. Using (25), we have

$$\lim_{k\to\infty} \|y_k - x_{n_k}\| = \lim_{k\to\infty} \frac{1}{l}\tau_{n_k} \|x_{n_k} - z_{n_k}\| = 0. \tag{27}$$

From the step size rule (6) and the definition of y_k we have

$$\langle Ax_{n_k} - Ay_k, x_{n_k} - z_{n_k}\rangle > \mu\|x_{n_k} - z_{n_k}\|^2. \tag{28}$$

Since A is uniformly continuous on bounded subsets of C and using (27) it implies that

$$\lim_{k\to\infty} \|Ax_{n_k} - Ay_k\| = 0. \tag{29}$$

Combining (28) and (29) we obtain

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\| = 0.$$

This is a contraction to (26). Therefore, the limit (23) is proved.
Since $x_{n_k} \rightharpoonup z$ and (23), Lemma 3.11 shows that $z \in VI(C, A)$.
On the other hand,

$$\|x_{n+1} - w_n\| = \alpha_n \|f(x_n) - w_n\| \to 0 \text{ as } n \to \infty.$$

Thus

$$\|x_{n+1} - x_n\| = \|x_{n+1} - w_n\| + \|x_n - w_n\| \to 0 \text{ as } n \to \infty.$$

Since $p = P_{VI(C,A)} f(p)$ and $x_{n_k} \rightharpoonup z \in VI(C, A)$ we get

$$\limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle = \langle f(p) - p, z - p\rangle \leq 0.$$

This implies that

$$\begin{aligned}\limsup_{n\to\infty} \langle f(p) - p, x_{n+1} - p\rangle &\leq \limsup_{n\to\infty} \langle f(p) - p, x_{n+1} - x_n\rangle + \limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle \\ &= \langle f(p) - p, z - p\rangle \leq 0,\end{aligned}$$

which, together with Claim 4, implies from Lemma 2.7 that

$$x_n \to p \text{ as } n \to \infty.$$

Case 2: There exists a subsequence $\{\|x_{n_j} - p\|^2\}$ of $\{\|x_n - p\|^2\}$ such that $\|x_{n_j} - p\|^2 < \|x_{n_j+1} - p\|^2$ for all $j \in \mathbb{N}$. In this case, it follows from Lemma 2.6 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{m_k} - p\|^2 \leq \|x_{m_k+1} - p\|^2 \text{ and } \|x_k - p\|^2 \leq \|x_{m_k+1} - p\|^2. \tag{30}$$

According to Claim 2 we have

$$\begin{aligned}\|w_{m_k} - x_{m_k}\|^2 &\le \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + 2\alpha_{m_k}\langle f(x_{m_k}) - p, x_{m_k+1} - p\rangle.\\ &\le \alpha_{m_k}\langle f(x_{m_k}) - p, x_{m_k+1} - p\rangle\\ &\le \alpha_{m_k}\|f(x_{m_k}) - p\|\|x_{m_k+1} - p\| \to 0 \text{ as } k \to \infty.\end{aligned}$$

According to Claim 3 we have

$$\begin{aligned}&(1 - \alpha_{m_k})\left[\frac{1}{L}\tau_{m_k}\left(\frac{1}{\lambda} - \mu\right)\|r_\lambda(x_{m_k})\|^2\right]^2\\ &\le \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + \alpha_{m_k}\|f(x_{m_k}) - p\|^2\\ &\le \alpha_{m_k}\|f(x_{m_k}) - p\|^2 \to 0 \text{ as } k \to \infty.\end{aligned}$$

Using the same arguments as in the proof of Case 1, we obtain

$$\lim_{k\to\infty}\|x_{n_k} - z_{n_k}\| = 0, \lim_{k\to\infty}\|x_{m_k+1} - x_{m_k}\| \to 0$$

and

$$\limsup_{k\to\infty}\langle f(p) - p, x_{m_k+1} - p\rangle \le 0.$$

Since (21) we get

$$\begin{aligned}\|x_{m_k+1} - p\|^2 &\le (1 - \alpha_{m_k}(1 - \rho))\|x_{m_k} - p\|^2 + 2\alpha_{m_k}\langle f(p) - p, x_{m_k+1} - p\rangle\\ &\le (1 - \alpha_{m_k}(1 - \rho))\|x_{m_k+1} - p\|^2 + 2\alpha_{m_k}\langle f(p) - p, x_{m_k+1} - p\rangle,\end{aligned}$$

which, together with (30), implies that

$$\|x_k - p\|^2 \le \|x_{m_k+1} - p\|^2 \le 2\langle f(p) - p, x_{m_k+1} - p\rangle.$$

Therefore, $\limsup_{k\to\infty}\|x_k - p\| \le 0$, that is $x_k \to p$. The proof is completed. □

4 Numerical illustrations

In this section, we discuss the numerical behavior of our proposed Algorithm 3.3 using different test examples and compare our method with method (1.2). In all the examples, we take $\alpha_n = 1/(n+1)$ and some choices of μ, λ and l.

Example 2 We take a classical example (see, e.g., Maingé and Gobinddass 2016) for which the usual gradient method does not converge to a solution of the variational inequality. Here, the feasible set is $C := \mathbb{R}^m$ (for some positive even integer m) and $A := (a_{ij})_{1\le i,j\le m}$ is the square matrix $m \times m$ whose terms are given by

$$a_{ij} = \begin{cases} -1, & \text{if } \ j = m + 1 - i \ \text{ and } \ j > i\\ 1, & \text{if } \ j = m + 1 - i \ \text{ and } \ j < i\\ 0 & \text{otherwise}\end{cases}$$

It is clear that the zero vector $z = (0, \ldots, 0)$ is the solution of this test example. Let x_1 be the initial point whose element is randomly chosen in the closed interval $[-1, 1]$. We terminate the iterations if $\|x_n - z_n\|_2 \le \varepsilon$ with $\varepsilon = 10^{-4}$. The results are listed in Table 1 below. We consider different values of m.

We next consider some model in infinite dimensional Hilbert spaces in the following example:

Table 1 Proposed Alg. 3.3 vs Vuong and Shehu Alg. 1.2 with $(\mu, \lambda, l) = (0.1, 9, 0.5)$

m	Proposed Alg. 3.3		Vuong and Shehu Alg. 1.2	
	No. of iter.	CPU (time)	No. of iter.	CPU (time)
50	11	1.4461×10^{-2}	367,520	1730.9830
100	11	6.8007×10^{-2}	530,075	3933.2485
150	11	1.4527×10^{-1}	565,727	7068.8152
200	11	2.8712×10^{-1}	738,056	30484.1555
500	12	1.4298	–	–
1000	12	5.9000	–	–
2000	12	23.3217	–	–
3000	12	53.9247	–	–
5000	12	165.1993	–	–
10000	13	719.1500	–	–

Example 3 Consider $C := \{x \in H : \|x\| \leq 2\}$. Let $g : C \to \mathbb{R}$ be defined by $g(u) := \frac{1}{1+\|u\|^2}$. Observe that g is L_g-Lipschitz continuous with $L_g = \frac{16}{25}$ and $\frac{1}{5} \leq g(u) \leq 1, \ \forall u \in C$. Define the Volterra integral operator $F : L^2([0,1]) \to L^2([0,1])$ by $F(u)(t) := \int_0^t u(s)ds, \ \forall u \in L^2([0,1]), t \in [0,1]$. Then F is bounded linear monotone (see Exercise 20.12 of Bauschke and Combettes 2011) and $\|F\| = \frac{2}{\pi}$. Now, define $A : C \to L^2([0,1])$ by $A(u)(t) := g(u)F(u)(t), \ \forall u \in C, t \in [0,1]$. Suppose $\langle Au, v-u\rangle \geq 0, \ \forall u, v \in C$; then we have that $\langle Fu, v-u\rangle \geq 0$ (noting that $g(u) \geq \frac{1}{5} > 0$). Hence,

$$\begin{aligned}\langle Av, v-u\rangle &= g(v)\langle F(v), v-u\rangle \\ &\geq g(v)\Big[\langle F(v), v-u\rangle - \langle F(u), v-u\rangle\Big] \\ &= g(v)\langle F(v) - F(u), v-u\rangle \geq 0.\end{aligned}$$

Therefore, A is pseudomonotone. Observe that A is not monotone since $\langle Av - Au, v-u\rangle = -\frac{1}{20} < 0$ with $v = 1$ and $u = 2$. Furthermore, for all $u, v \in C$, we have

$$\begin{aligned}\|Au - Av\| &= \|g(u)F(u) - g(v)F(v)\| \\ &= \|g(u)F(u) - g(u)F(v) + g(u)F(v) - g(v)F(v)\| \\ &\leq \|g(u)F(u) - g(u)F(v)\| + \|g(u)F(v) - g(v)F(v)\| \\ &\leq |g(u)|\|F(u) - F(v)\| + \|F(v)\||g(u) - g(v)| \\ &\leq |g(u)|\|F\|\|u-v\| + \|F\|\|v\|L_g\|u-v\| \\ &= (|g(u)|\|F\| + \|F\|\|v\|L_g)\|u-v\| \\ &= \frac{82}{\pi}\|u-v\|.\end{aligned}$$

Hence, A is L_A-Lipschitz-continuous with $L_A = \frac{82}{\pi}$. Therefore, A is uniformly continuous. Observe that $VI(C, A) \neq \emptyset$ since $0 \in VI(C, A)$. We take $\frac{\|x_{n+1}-x_n\|_2}{\max\{1, \|x_n\|_2\}} \leq \epsilon$ as stopping criterion, with $\epsilon = 10^{-2}$. We consider different initial points $x_1 = 1$, $x_1 = 1 + t^2$, $x_1 = e^t$ and $x_1 = sin(t)$ in the following cases:

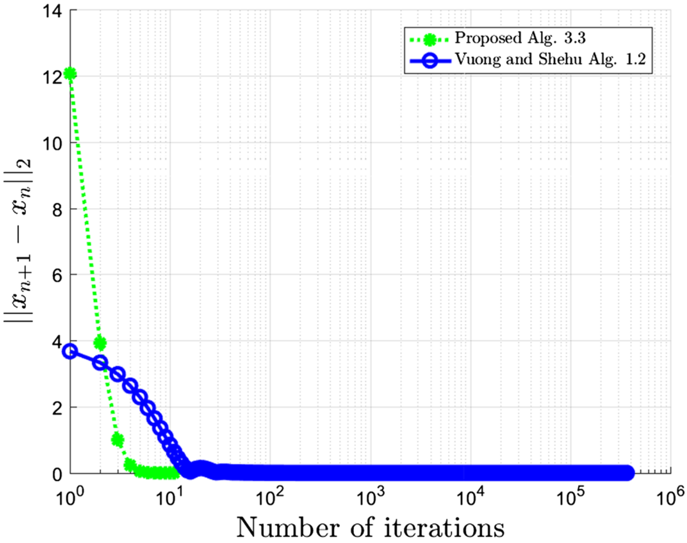

Fig. 1 Comparison with $m = 50$

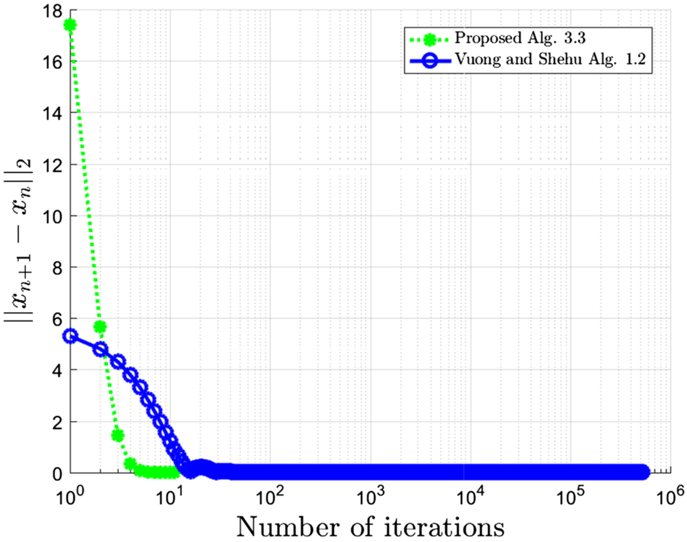

Fig. 2 Comparison with $m = 100$

Case I: $(\mu, \lambda) = (0.3, 3.2)$

Case II: $(\mu, \lambda) = (3.0, 0.3)$
We report the numerical behaviour of this example in Figs. 11, 12, 13 and 14 and Table 2.

Remark 4.3 1. From the numerical results of Examples 2–3, we observe that our proposed Algorithm 3.3 is efficient and easy to implement. See Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14 and Tables 1 and 2.

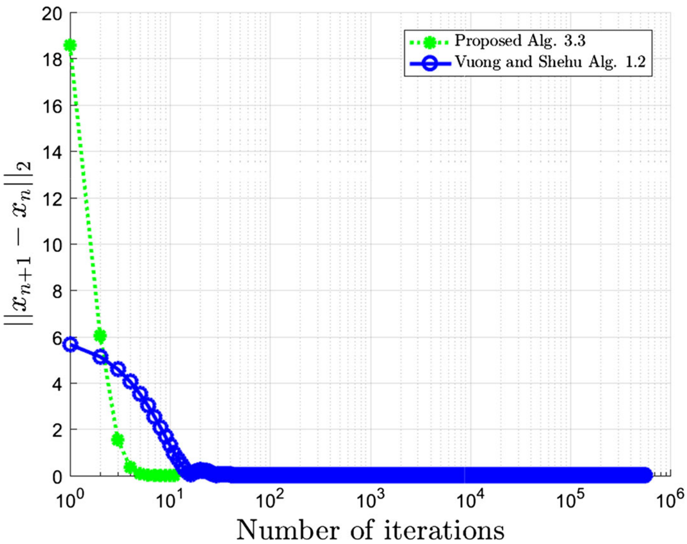

Fig. 3 Comparison with $m = 150$

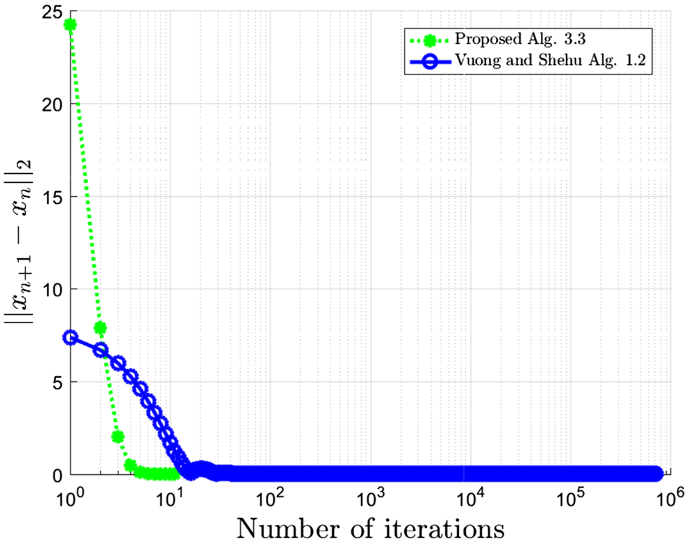

Fig. 4 Comparison with $m = 200$

2. In Example 2, there is no significant change in the number of iterations required as we increase the dimension m. This is a significant improvement since the iteration number required to terminate the proposed Algorithm 3.3 does not depend on the size of the problem. See Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 and Table 1.
3. In Example 3, we observe also that both the choice of initial points and the parameters μ and λ do not have significant effect on the number of iterations and the CPU time. See Figs. 11, 12, 13 and 14 and Table 2.

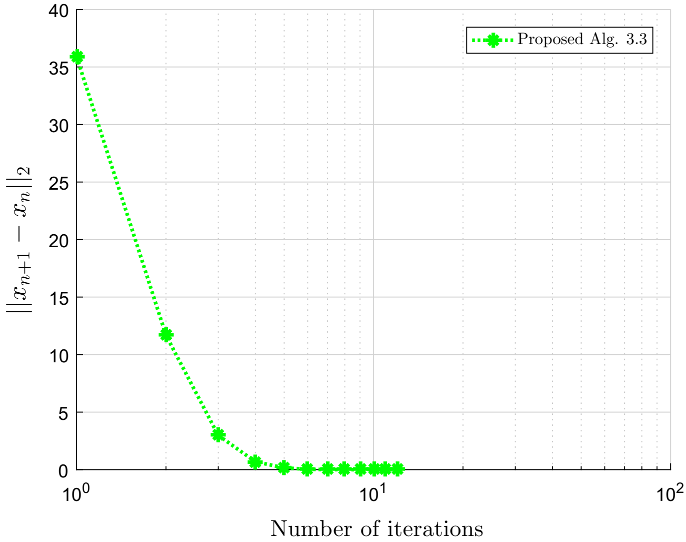

Fig. 5 Proposed Alg. 3.3 with $m = 500$

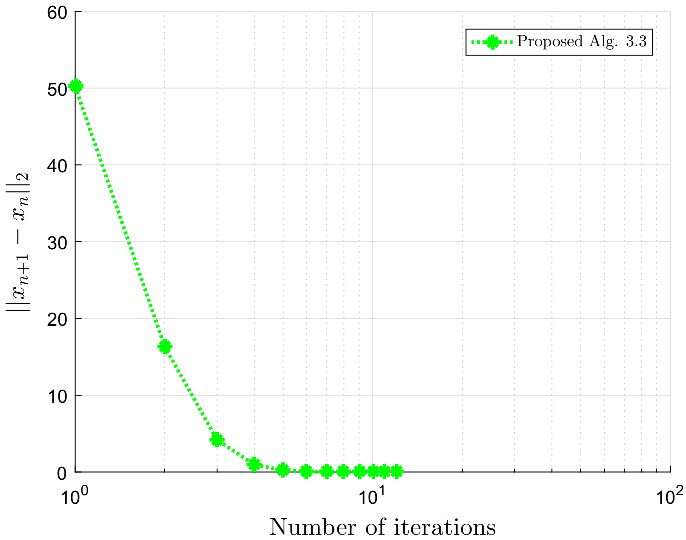

Fig. 6 Proposed Alg. 3.3 with $m = 1000$

4. Clearly from the numerical Examples presented above, our proposed Algorithm 3.3 outperformed Algorithm 1.2 proposed by Vuong & Shehu. See Table 1 and Figs. 1 2, 3 and 4. We have omitted some of the comparison results (part in Example 2 and all in Example 3) due to large amount of time required by Vuong and Shehu Algorithm 1.2 to terminate.

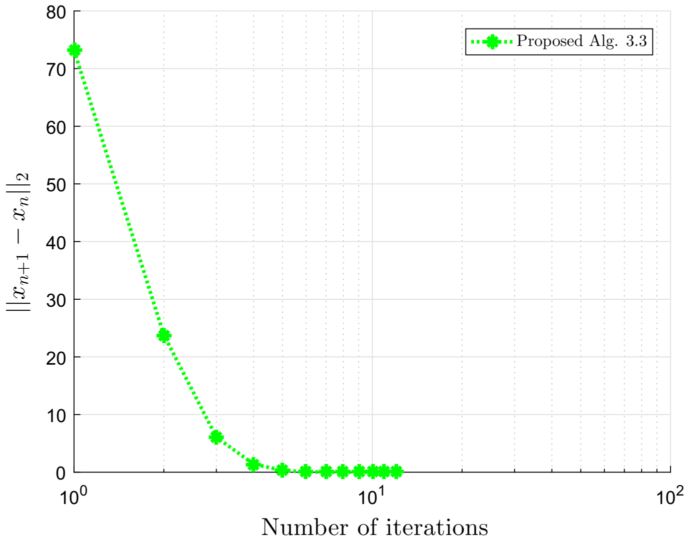

Fig. 7 Proposed Alg. 3.3 with $m = 2000$

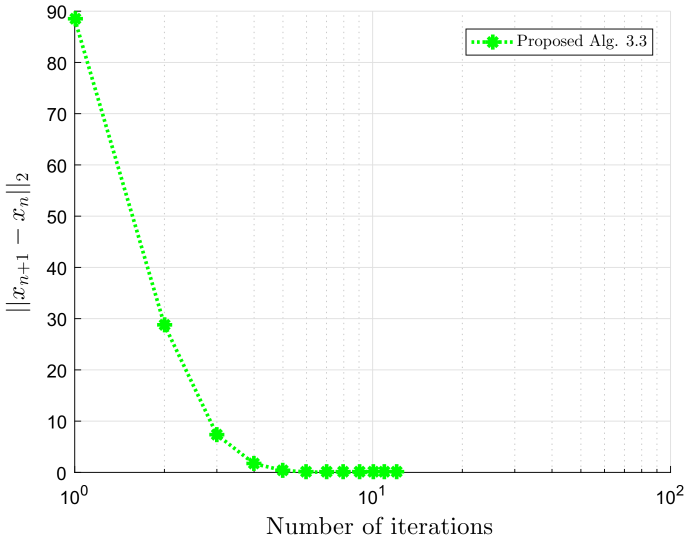

Fig. 8 Proposed Alg. 3.3 with $m = 3000$

Our preliminary examples in this section play a role in illustrating the rationality in studying pseudomonotone variational inequality. In future works, we shall give more interesting examples from applications.

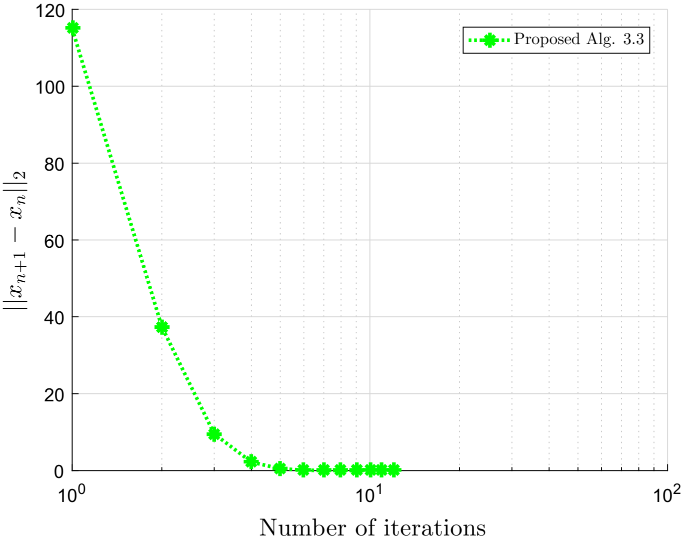

Fig. 9 Proposed Alg. 3.3 with $m = 5000$

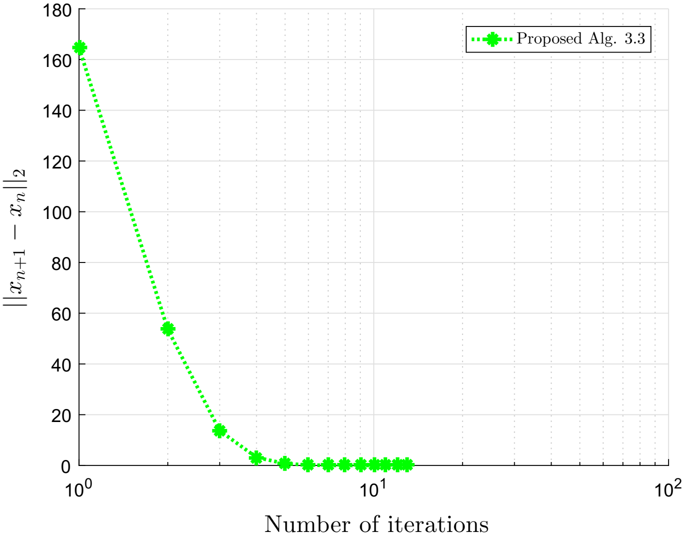

Fig. 10 Proposed Alg. 3.3 with $m = 10000$

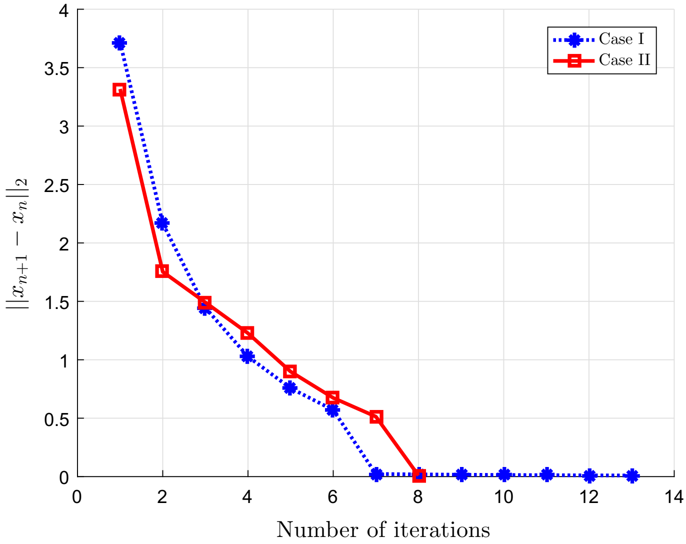

Fig. 11 Proposed Alg. 3.3 with $x_1 = 1$

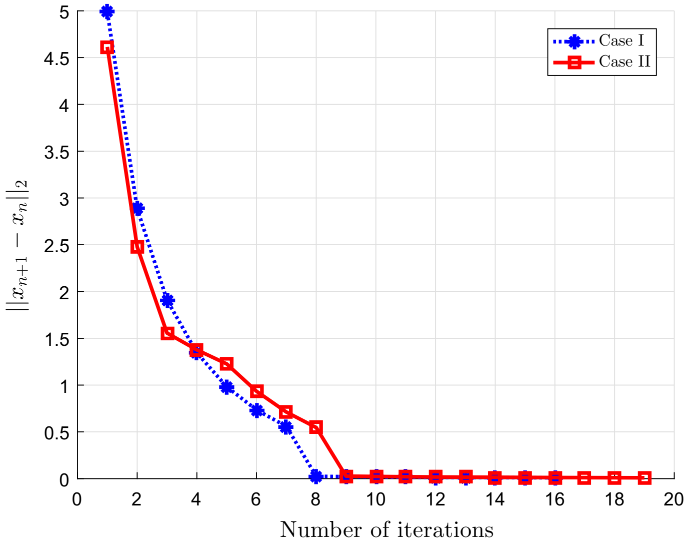

Fig. 12 Proposed Alg. 3.3 with $x_1 = 1 + t^2$

5 Conclusions

In this paper we proposed a new algorithm for solving variational inequalities in real Hilbert spaces. The proposed algorithm shows the strongly convergence property under pseudomonotonicity and non-Lipschitz continuity of the mapping A. The algorithm requires the calculation of only two projections onto the feasible set C per iteration. These two properties pseudomonotonicity and non-Lipschitz continuity of the mapping A emphasize the applica-

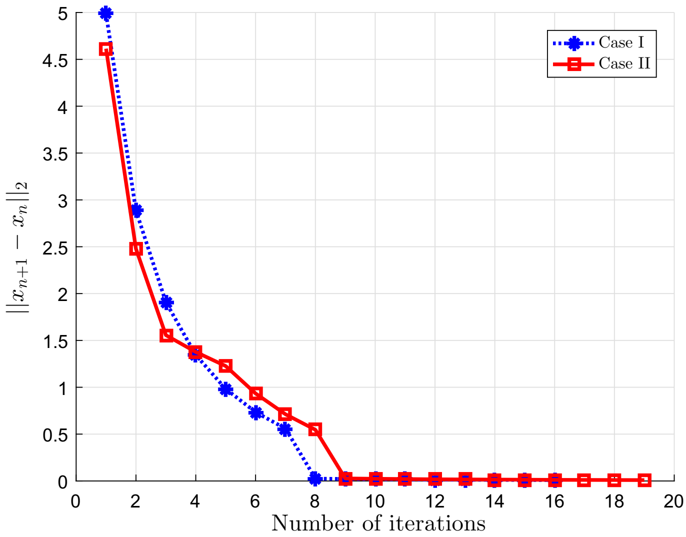

Fig. 13 Proposed Alg. 3.3 with $x_1 = e^t$

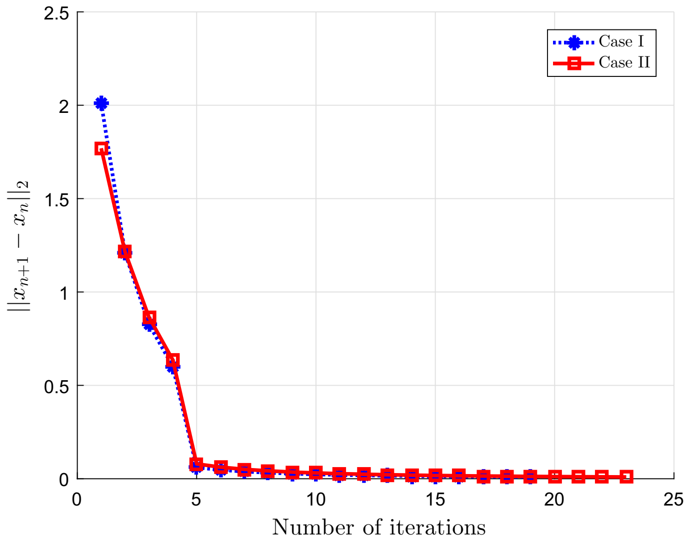

Fig. 14 Proposed Alg. 3.3 with $x_1 = \sin(t)$

Table 2 Proposed algorithm 3.3 with $l = 0.2$

x_1	Case I		Case II	
	No. of iter.	CPU (time)	No. of iter.	CPU (time)
1	13	1.5213×10^{-3}	8	7.2533×10^{-4}
$1 + t^2$	16	2.7737×10^{-3}	19	2.5132×10^{-3}
e^t	18	2.5230×10^{-3}	23	2.2292×10^{-3}
$\sin(t)$	19	2.5944×10^{-3}	23	1.9536×10^{-3}

bility and advantages over several existing results in the literature. Numerical experiments in finite and infinite dimensional spaces illustrate the performance of the new scheme.

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