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On the numerical treatment and analysis of two-dimensional Fredholm integral equations using quasi-interpolant

M. Derakhshan[1] · M. Zarebnia[1]

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Abstract

In this paper, we study the quadratic rule for the numerical solution of linear and nonlinear two-dimensional Fredholm integral equations based on spline quasi-interpolant. Also the convergence analysis of the method is given. We show that the order of the method is $O(h_x^{m+1}) + O(h_y^{m'+1})$. The theoretical behavior is tested on examples and it is shown that the numerical results confirm theoretical part.

Keywords Spline · Quasi-interpolant · Quadrature · Two-dimensional Fredholm · Convergence

Mathematical Subject Classification 65R20 · 41A15 · 41A25

1 Introduction

Integral and integro-differential equations are mathematical tools in many branches of science and engineering. Among these equations, Fredholm integral equations arise from multiple applications, for example biology, medicine, economics, potential theory and many others. Many problems in engineering and mechanics can be transformed into two-dimensional Fredholm integral equations (Kress 1989). However, the analysis of computational methods for two-dimensional Fredholm integral equations has started more recently. During these years, several numerical methods have been developed to approximate the solution of two-dimensional Fredholm integral equations such as the papers (Alipanah and Esmaeili 2011; Avazzadeh and Heydari 2012; Babolian et al. 2011) that are concerned, respectively, with Gaussian functions, Chebyshev collocation and rationalized Haar functions. Han and Wang (2002) approximated the two-dimensional Fredholm integral equations by the Galerkin

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✉ M. Zarebnia
zarebnia@uma.ac.ir

M. Derakhshan
m.derakhshan@uma.ac.ir

1 Department of Mathematics and Applications, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran

method. Han and Wang (2001) have considered this problem by the Nyström's method. Operational method (Rahimi et al. 2010), Gauss quadrature rules (Babolian and Bazm 2012) and Legendre series (Chokri 2013; Tari and Shahmorad 2008) are other works on developing and analyzing numerical methods for solving Fredholm integral equations. Moreover, one can refer to other methods such as Hanson and Phillips (1978); Liang and Lin (2010); Ma et al. (2018, 2017, 2016) and Xie and Lin (2009). The two-dimensional Fredholm integral equation is considered as

$$u - K(Fu) = g, \tag{1}$$

where integral operator K is defined as

$$K(Fu)(x, y) = \int_c^d \int_a^b k(x, y, s, t)F(s, t, u(s, t))\mathrm{d}s\mathrm{d}t, \tag{2}$$

$g(x, y)$ and $k(x, y, s, t)$ are known continuous functions, $F(s, t, u(s, t))$ nonlinear in u and $u(., .)$ is the unknown function to be determined. Integration of a function on bounded interval or on a certain district is an important operation for many physical problems. On the other hand, quasi-interpolation based on a B-spline is a general approach for efficiently constructing approximations. The various methods in the literature for producing quasi-interpolants are excellently documented (Sablonniere 2007, 2005).

Consider the following quadratic spline quasi-interpolant defined in Sablonniere (2007) and Sablonniere (2005) by

$$Q_2 f = \sum_{k \in J} \zeta_k(f) B_k, \tag{3}$$

where

$$\zeta_1(f) = f_1, \quad \zeta_{n+2}(f) = f_{n+2},$$

$$\zeta_2(f) = -1/3 f_1 + 3/2 f_2 - 1/6 f_3 = \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3, \tag{4}$$

$$\zeta_{n+1}(f) = -1/6 f_n + 3/2 f_{n+1} - 1/3 f_{n+2} = \beta_3 f_n + \beta_2 f_{n+1} + \beta_1 f_{n+2}, \tag{5}$$

and for $3 \leq j \leq n$,

$$\zeta_j(f) = -1/8 f_{j-1} + 5/4 f_j - 1/8 f_{j+1} = \gamma_1 f_{j-1} + \gamma_2 f_j + \gamma_3 f_{j+1}, \tag{6}$$

with $f_i = f(t_i)$, $t_1 = a$, $t_{n+2} = b$, $t_i = a + (i - 3/2)h,\ 2 \leq i \leq n + 1$. By integrating this spline quasi-interpolant, the author in Sablonniere (2005) presented a new quadrature rule of convergence order $O(h^4)$. But in this paper, by considering Sablonniere et al. (2012), we study a new class of endpoint corrected rule based on integrating spline quasi-interpolant and using only function values inside the interval of integration. Also, we consider quadrature rule based on this type of spline quasi-interpolant of covergence order $O(h^r), r > 4$ for solution of two-dimensional Fredholm integral equations. For $j = 2, \ldots, n+1$, we appoint the same values of $\mu_j(f)$, given by (4), (5), (6) and for $j = 1$ and $j = n + 2$, we have

$$\mu_1(f) = \sum_{i=1}^{m} \alpha_i f_i, \tag{7}$$

$$\mu_{n+2}(f) = \sum_{i=1}^{m} \alpha_i f_{n+3-i}, \tag{8}$$

where m is an odd integer $3 \leq m \leq n+2$, and $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ are real parameters to be determined later. Frequently, we consider the quadrature rule as

$$\mathcal{I}_{Q_2}^{m}(f) := \int_I Q_2 f(x)\mathrm{d}x. \tag{9}$$

We define error of proposed quadrature rule as follows:

$$E_m(f) = \mathcal{I}_{Q_2}^{m}(f) - \int_a^b f(x)\mathrm{d}x,$$

where

$$E_m(f) = \mathcal{I}_{Q_2}^{m}(f) - \int_a^b f(x)\mathrm{d}x = O(h^{m+1}).$$

But the error bound for the presented quadrature rule in Sablonniere (2007) and Sablonniere (2005) is

$$E(f) = \mathcal{I}_{Q_2}(f) - \int_a^b f(x)\mathrm{d}x = O(h^4).$$

The organization of the paper is as follows. In Sect. 2, we describe quadrature rule based on spline quasi-interpolant. In Sect. 3, we give an application of the quadrature rule of Sect. 2 to the numerical solution of two-dimensional Fredholm integral equations. In Sect. 4, the convergence and error analysis of the numerical solution are provided. At the end we give some numerical examples which confirm our theoretical results.

2 Quadrature rule based on a quadratic spline quasi-interpolant

Let $X_n := \{x_k,\ 0 \leq k \leq n\}$ be the uniform partition of the interval $I = [a, b]$ into n equal subintervals, i.e. $x_k := a + kh$, with $h = \frac{b-a}{n}$. We consider the space $S_2 = S_2(I, X_n)$ of quadratic splines of class C^1 on this partition. Canonical basis is formed by the $n+2$ normalized B-splines, $\{B_k,\ k \in J\}$, $J := \{1, 2, \ldots, n+2\}$. Consider the quadratic spline quasi-interpolant (dQI) of a function f defined on I and given in Sablonniere et al. (2012), that is

$$Q_2 f = \sum_{k \in J} \zeta_k(f) B_k, \tag{10}$$

where

$$\zeta_1(f) = \sum_{i=1}^{m} \alpha_i f_i, \tag{11}$$

$$\zeta_{n+2}(f) = \sum_{i=1}^{m} \alpha_i f_{n+3-i}, \tag{12}$$

$$\zeta_2(f) = -\frac{1}{3} f_1 + \frac{3}{2} f_2 - \frac{1}{6} f_3 = \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3, \tag{13}$$

$$\zeta_{n+1}(f) = -\frac{1}{6} f_n + \frac{3}{2} f_{n+1} - \frac{1}{3} f_{n+2} = \beta_3 f_n + \beta_2 f_{n+1} + \beta_1 f_{n+2}, \tag{14}$$

$$\zeta_j(f) = -\frac{1}{8} f_{j-1} + \frac{5}{4} f_j - \frac{1}{8} f_{j+1} = \gamma_1 f_{j-1} + \gamma_2 f_j + \gamma_3 f_{j+1},\ 3 \leq j \leq n, \tag{15}$$

with $f_i = f(t_i)$, $t_1 = a$, $t_{n+2} = b$, $t_i = a+(i-\frac{3}{2})h,\ 2 \le i \le n+1$. Parameter m is an odd integer $3 \le m \le n+2$, and $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ are determined later. The quadratic B-spline functions at knots are defined as

$$B_i(x) = \begin{cases} \dfrac{(x-x_{i-3})^2}{(x_{i-1}-x_{i-3})(x_{i-2}-x_{i-3})}, & x_{i-3} \le x < x_{i-2}, \\[2ex] \dfrac{(x_i-x)(x-x_{i-2})}{(x_i-x_{i-2})(x_{i-1}-x_{i-2})} + \dfrac{(x-x_{i-3})(x_{i-1}-x)}{(x_{i-1}-x_{i-3})(x_{i-1}-x_{i-2})}, & x_{i-2} \le x < x_{i-1}, \\[2ex] \dfrac{(x_i-x)^2}{(x_i-x_{i-2})(x_i-x_{i-1})}, & x_{i-1} \le x < x_i. \end{cases}$$

We consider the quadrature rule as

$$\mathcal{I}^m_{Q_2}(f) := \int_I Q_2 f(x)\mathrm{d}x. \tag{16}$$

Using $\int_a^b B_j \mathrm{d}x = \frac{1}{3}(x_j - x_{j-3})$, we can get

$$\int_I B_1(x)\mathrm{d}x = \int_I B_{n+2}(x)\mathrm{d}x = h/3,\ \int_I B_2(x)\mathrm{d}x = \int_I B_{n+1}(x)\mathrm{d}x = 2h/3,$$
$$\int_I B_k(x)\mathrm{d}x = h, 3 \le k \le n.$$

We consider error of the quadrature rule (16) on function x^{m-1}, $I = [0, n]$ and $h = 1$ as

$$\begin{aligned} E_m(f) &= \mathcal{I}^m_{Q_2}(f) - \int_a^b f(x)\mathrm{d}x \\ &= \frac{1}{3}\left(\sum_{i=1}^{m} \alpha_i (t_i^{m-1} + t_{n+3-i}^{m-1})\right) + \frac{2}{3}\left(\sum_{i=1}^{3} \beta_i (t_i^{m-1} + t_{n+3-i}^{m-1})\right) \\ &\quad + \sum_{j=3}^{n} (\gamma_1 t_{j-1}^{m-1} + \gamma_2 t_j^{m-1} + \gamma_3 t_{j+1}^{m-1}) - \frac{1}{m} n^m. \end{aligned}$$

This quadrature formula with m correction points and based on integrating quasi-interpolant Q_2 can be given as

$$\mathcal{I}^m_{Q_2}(f) = h \sum_{i=1}^{m} v_i^{(m,2)} (f_i + f_{n+3-i}) + h \sum_{i=m+1}^{n+2-m} f_i. \tag{17}$$

In Table 16, we give correction weights $\{v_i^{(m,2)}, i = 1, \ldots, m\}$. Now, we can obtain the double integration formula concerned with Eq. (17), as follows:

$$x_1 = a,\ x_{n+2} = b,\ x_i = a + (i - \frac{3}{2})h_x,\ h_x = \frac{b-a}{n},$$
$$y_1 = c,\ y_{n'+2} = d,\ y_i = c + (i - \frac{3}{2})h_y,\ h_y = \frac{d-c}{n'},$$

where x_i and y_i are grid points. The double integration formula concerned with Eq. (17) obtained as follows:

$$\int_a^b \int_c^d \varphi(x, y)\mathrm{d}y\mathrm{d}x = \int_a^b g(x)\mathrm{d}x,$$

such that

$$g(x)=\int_c^d \varphi(x,y)\mathrm{d}y=h_y\sum_{j=1}^{m'} v_j^{(m',2)}(\varphi(x,y_j)+\varphi(x,y_{n'+3-j}))+h_y\sum_{j=m'+1}^{n'+2-m'}\varphi(x,y_j).$$

Then by integrating we have

$$\begin{aligned}\int_a^b g(x)\mathrm{d}x &= h_y\sum_{j=1}^{m'} v_j^{(m',2)}\int_a^b \varphi(x,y_j)\mathrm{d}x+h_y\sum_{j=1}^{m'} v_j^{(m',2)}\int_a^b \varphi(x,y_{n'+j-3})\mathrm{d}x\\
&\quad+h_y\sum_{j=m'}^{n'+2-m'}\int_a^b \varphi(x,y_j)\mathrm{d}x\\
&= h_y\sum_{j=1}^{m'} v_j^{(m',2)}\Bigg[h_x\sum_{i=1}^{m} v_i^{(m,2)}(\varphi(x_i,y_j)+\varphi(x_{n+3-i},y_j))\\
&\quad+h_x\sum_{i=m+1}^{n+2-m}\varphi(x_i,y_j)\Bigg]\\
&\quad+h_y\sum_{j=1}^{m'} v_j^{(m',2)}\Bigg[h_x\sum_{i=1}^{m} v_i^{(m,2)}(\varphi(x_i,y_{n'+3-j})\\
&\quad+\varphi(x_{n+3-i},y_{n'+3-j}))+h_x\sum_{i=m+1}^{n+2-m}\varphi(x_i,y_{n'+3-j})\Bigg]\\
&\quad+h_y\sum_{i=m'+1}^{n'+2-m'}\Bigg[h_x\sum_{i=1}^{m} v_i^{(m,2)}(\varphi(x_i,y_j)\\
&\quad+\varphi(x_{n+3-i},y_j))+h_x\sum_{i=m+1}^{n+2-m}\varphi(x_i,y_j)\Bigg].\end{aligned}$$

Finally, we deduce the double integrating formula, as follows:

$$\begin{aligned}\mathcal{I}_{Q_2}^{m,m'}(\varphi(x,y)) &= h_xh_y\sum_{j=1}^{m'}\sum_{i=1}^{m} v_j^{(m',2)}v_i^{(m,2)}\varphi(x_{n+3-i},y_{n'+3-j})\\
&\quad+h_xh_y\sum_{j=1}^{m'}\sum_{i=1}^{m} v_j^{(m',2)}v_i^{(m,2)}\varphi(x_{n+3-i},y_j)\\
&\quad+h_xh_y\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m} v_j^{(m',2)}\varphi(x_i,y_j)\\
&\quad+h_xh_y\sum_{j=1}^{m'}\sum_{i=1}^{m} v_j^{(m',2)}v_i^{(m,2)}\varphi(x_i,y_{n'+3-j})\\
&\quad+h_xh_y\sum_{j=1}^{m'}\sum_{i=1}^{m} v_j^{(m',2)}v_i^{(m,2)}\varphi(x_i,y_j)\end{aligned}$$

$$+ h_x h_y \sum_{j=1}^{m'} \sum_{i=m+1}^{n+2-m} v_j^{(m',2)} \varphi(x_i, y_{n'+3-j})$$
$$+ h_x h_y \sum_{j=m'+1}^{n'+2-m'} \sum_{i=1}^{m} v_i^{(m,2)} \varphi(x_i, y_j)$$
$$+ h_x h_y \sum_{j=m'+1}^{n'+2-m'} \sum_{i=1}^{m} v_i^{(m,2)} \varphi(x_{n+3-i}, y_j)$$
$$+ h_x h_y \sum_{j=m'+1}^{n'+2-m'} \sum_{i=m+1}^{n+2-m} \varphi(x_{n+3-i}, y_j). \qquad (18)$$

Now we obtain the values of $\{\alpha_i\}_{i=1}^m$. For this purpose, we need the following Lemmas.

Lemma 1 *Let m be an odd integer with $3 \leq m \leq n+2$, and let*

$$S = \sum_{j=3}^{n} (\gamma_1 t_{j-1}^{m-1} + \gamma_2 t_j^{m-1} + \gamma_3 t_{j+1}^{m-1}).$$

Then S is a polynomial function of degree m in the variable n. More precisely,

$$S = \sum_{j=0}^{m} \theta_j^{(m)} n^j, \qquad (19)$$

where

$$\theta_0^{(m)} = \sum_{l=1}^{3} \gamma_l (\frac{1}{m}((l-7/2)^m + (1/2-l)^m) + 1/2((l-7/2)^{m-1} + (1/2-l)^{m-1})$$
$$+ \sum_{i=1}^{\frac{m-1}{2}} \frac{\tilde{B}_{2i}}{2i!} (m-1)\dots(m-2i+1)((l-7/2)^{m-2i} + (1/2-l)^{m-2i})),$$
$$\theta_j^{(m)} = \sum_{l=1}^{3} \gamma_l (\frac{1}{m} C_m^j (l-7/2)^{m-j} + 1/2 C_{m-1}^j (l-7/2)^{m-1-j}$$
$$+ \sum_{i=1}^{[\frac{m-j-1}{2}]} \frac{\tilde{B}_{2i}}{2i!} (m-1)\dots(m-2i+1) C_{m-2i}^j (l-7/2)^{m-2i-j}), \quad j = 1, \dots, m,$$

where C_m^j are the binomial coefficients, $\tilde{B}_{2i}$ are the Bernoulli numbers and $[x]$ denotes the integer part of x.

Proof For proof, refer to Sablonniere et al. (2012).

□

Table 1 The values of parameters α_i

α_i	$m=5$	$m=7$	$m=9$	$m=13$
α_1	1.05397	1.08045	1.09145	1.09745
α_2	− 0.108854	− 0.177438	− 0.210955	−0.232091
α_3	0.0904514	0.193017	0.264956	0.327069
α_4	− 0.0432292	− 0.150553	− 0.267354	−0.41376
α_5	0.00766369	0.0723562	0.19614	0.4412
α_6	–	− 0.0204115	− 0.105158	−0.402043
α_7	–	0.00257848	0.0386972	0.302815
α_8	–	–	− 0.00866666	−0.182146
α_9	–	–	0.000886433	0.0846578
α_{10}	–	–	–	−0.0291921
α_{11}	–	–	–	0.00701754
α_{12}	–	–	–	−0.00104865
α_{13}	–	–	–	0.0000733031

Lemma 2 *For m odd and $3 \leq m \leq n+2$, we have $E_m = \sum_{j=0}^{m-1} \lambda_j^{(m)} n^j$, where*

$$\lambda_0^{(m)} = 2\left(\frac{1}{3}\sum_{i=1}^{m} \alpha_i t_i^{m-1} + \frac{2}{3}\sum_{i=1}^{3} \beta_i t_i^{m-1}\right) + \theta_0^m,$$

$$\lambda_j^{(m)} = \frac{1}{3}\sum_{i=1}^{m} \alpha_i C_{m-1}^j (-t_i)^{m-1-j} + \frac{2}{3}\sum_{i=1}^{3} \beta_i C_{m-1}^j (-t_i)^{m-1-j} + \theta_j^{(m)}.$$

According to the Lemma 1, we deduce that imposing $E_m = 0$ for all n is equivalent to solving the following linear system on $(\alpha_1, \alpha_2, \ldots, \alpha_m)$:

$$\lambda_j^{(m)} = 0, \qquad j = 0, \ldots, m-1. \tag{20}$$

Proof For proof, refer to (Sablonniere et al. 2012).

□

Having used the Lemmas 1 and 2, we can get the parameters $\alpha_1, \ldots, \alpha_m$ as Table 1.

3 Application to two-dimensional Fredholm integral equations

In this section, we illustrate an application of the quadrature rule to numerical solution of two-dimensional Fredholm integral equation. We consider two-dimensional Fredholm integral equation as

$$u(x,y) - \int_c^d \int_a^b k(x,y,s,t) F(s,t,u(s,t)) \mathrm{d}s\mathrm{d}t = g(x,y), \tag{21}$$

where $k(.,.,.,.) \in C([a,b]\times[c,d]\times[a,b]\times[c,d])$ and $g(x,y) \in C([a,b]\times[c,d])$ are known functions and $u(x,y)$ is the unknown function to be determined. Using Eq. (18), we get

$$
\begin{aligned}
&u(x,y)-h_xh_y(\sum_{j=1}^{m'}\sum_{i=1}^{m}v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_i,t_j)F(s_i,t_j,u(s_i,t_j))\\
&\quad+\sum_{j=1}^{m'}\sum_{i=1}^{m}v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_{n+3-i},t_j)F(s_{n+3-i},t_j,u(s_{n+3-i},t_j))\\
&\quad+\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m}v_j^{(m',2)}k(x,y,s_i,t_j)F(s_i,t_j,u(s_i,t_j))\\
&\quad+\sum_{j=1}^{m'}\sum_{i=1}^{m}v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_i,t_{n'+3-j})F(s_i,t_{n'+3-j},u(s_i,t_{n'+3-j}))\\
&\quad+\sum_{j=1}^{m'}\sum_{i=1}^{m}v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_{n+3-i},t_{n'+3-j})F(s_{n+3-i},t_{n'+3-j},u(s_{n+3-i},t_{n'+3-j}))\\
&\quad+\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m}v_j^{(m',2)}k(x,y,s_i,t_{n'+3-j})F(s_i,t_{n'+3-j},u(s_i,t_{n'+3-j}))\\
&\quad+\sum_{j=m'+1}^{n'+2-m'}\sum_{i=1}^{m}v_i^{(m,2)}k(x,y,s_i,t_j)F(s_i,t_j,u(s_i,t_j))\\
&\quad+\sum_{j=m'+1}^{n'+2-m'}\sum_{i=1}^{m}v_i^{(m,2)}k(x,y,s_{n+3-i},t_j)F(s_{n+3-i},t_j,u(s_{n+3-i},t_j))\\
&\quad+\sum_{j=m'+1}^{n'+2-m'}\sum_{i=m+1}^{n+2-m}k(x,y,s_{n+3-i},t_j)F(s_{n+3-i},t_j,u(s_{n+3-i},t_j)))=g(x,y).
\end{aligned}
\tag{22}
$$

Equation (22) is rewritten as

$$
u^{m,m'}(x,y)-(K_{mn}(Fu^{m,m'}))(x,y)=g(x,y),
\tag{23}
$$

where

$$
\begin{aligned}
&(K_{mn}(Fu^{m,m'}))(x,y)=h_xh_y\left(\sum_{j=1}^{m'}\sum_{i=1}^{m}v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_i,t_j)F(s_i,t_j,u^{m,m'}(s_i,t_j)\right)\\
&\quad+\sum_{j=1}^{m'}\sum_{i=1}^{m}v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_{n+3-i},t_j)F(s_{n+3-i},t_j,u^{m,m'}(s_{n+3-i},t_j))\\
&\quad+\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m}v_j^{(m',2)}k(x,y,s_i,t_j)F(s_i,t_j,u^{m,m'}(s_i,t_j))\\
&\quad+\sum_{j=1}^{m'}\sum_{i=1}^{m}v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_i,t_{n'+3-j})F(s_i,t_{n'+3-j},u^{m,m'}(s_i,t_{n'+3-j}))\\
&\quad+\sum_{j=1}^{m'}\sum_{i=1}^{m}v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_{n+3-i},t_{n'+3-j})F(s_{n+3-i},t_{n'+3-j},u^{m,m'}(s_{n+3-i},t_{n'+3-j}))
\end{aligned}
$$

$$+\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m} v_j^{(m',2)} k(x,y,s_i,t_{n'+3-j}) F(s_i,t_{n'+3-j},u^{m,m'}(s_i,t_{n'+3-j}))$$
$$+\sum_{j=m'+1}^{n'+2-m'}\sum_{i=1}^{m} v_i^{(m,2)} k(x,y,s_i,t_j) F(s_i,t_j,u^{m,m'}(s_i,t_j))$$
$$+\sum_{j=m'+1}^{n'+2-m'}\sum_{i=1}^{m} v_i^{(m,2)} k(x,y,s_{n+3-i},t_j) F(s_{n+3-i},t_j,u^{m,m'}(s_{n+3-i},t_j))$$
$$+\sum_{j=m'+1}^{n'+2-m'}\sum_{i=m+1}^{n+2-m} k(x,y,s_{n+3-i},t_j) F(s_{n+3-i},t_j,u^{m,m'}(s_{n+3-i},t_j))). \tag{24}$$

By replacing $x = x_i,\ y = y_j,\ i,\ j = 1, 2, \ldots, n+2$, we have

$$u^{m,m'}(x_i, y_j) - (K_{mn}(Fu^{m,m'}))(x_i, y_j) = g(x_i, y_j). \tag{25}$$

The above nonlinear system consists $(n+2)\times(n+2)$ equations with $(n+2)\times(n+2)$ unknown coefficients $\{u_{i,j}^{m,m'}\}_{i,j=1}^{n+2}$. Solving this nonlinear system by Newton's method, we can obtain the values of $\{u_{i,j}^{m,m'}\}_{i,j=1}^{n+2}$. Having used the solution $\{u_{i,j}^{m,m'}\}_{i,j=1}^{n+2}$, we employ a method similar to the Nyström's idea for the two-dimension Fredholm integral equation, i.e. we use

$$u^{m,m'}(x, y) = (K_{mn}(Fu^{m,m'}))(x, y) + g(x, y). \tag{26}$$

4 Convergence analysis

In this section, we shall provide the convergence analysis of the proposed method. For this purpose, we consider the following theorems and lemma.

Theorem 1 *Suppose that m is an odd integer* $3 \le m \le n+2$, *and* (a, b) *is a pair of real numbers such that* $a < b$. *Let* $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ *be a solution of the linear system* (13). *Then the quadrature rule* $\mathcal{I}_{Q_2}^m$ *given by* (17), *is of order* $m+1$, *i.e. for any* $f \in C^{m+1}[a, b]$, *there exists a real number* $c > 0$ *independent of n, such that*

$$\left|\mathcal{I}_{Q_2}^m(f) - \int_a^b f(x)\mathrm{d}x\right| < \frac{c}{n^{m+1}}. \tag{27}$$

Furthermore, the dQI Q_2 *given by* (10) *with modified functionals* (11) *and* (12) *is exact on* $\mathbb{P}_2$.

Proof For proof, refer to Sablonniere et al. (2012). □

Lemma 3 *If* $k(x, y, s, t)$ *be continuous function then the norm of* K_{mn} *is*

$$\|K_{mn}\|_\infty = h_x h_y\Bigg(\max\sum_{j=1}^{m'}\sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)}|k(x,y,s_i,t_j)|$$
$$+\max\sum_{j=1}^{m'}\sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)}|k(x,y,s_{n+3-i},t_j)|$$

$$
\begin{aligned}
&+\max \sum_{j=1}^{m'} \sum_{i=m+1}^{n+2-m} v_j^{(m',2)} |k(x, y, s_i, t_j)| \\
&+\max \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} |k(x, y, s_i, t_{n'+3-j})| \\
&+\max \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} |k(x, y, s_{n+3-i}, t_{n'+3-j})| \\
&+\max \sum_{j=1}^{m'} \sum_{i=m+1}^{n+2-m} v_j^{(m',2)} |k(x, y, s_i, t_{n'+3-j})| \\
&+\max \sum_{j=m'+1}^{n'+2-m'} \sum_{i=1}^{m} v_i^{(m,2)} |k(x, y, s_i, t_j)| \\
&+\max \sum_{j=m'+1}^{n'+2-m'} \sum_{i=1}^{m} v_i^{(m,2)} |k(x, y, s_{n+3-i}, t_j)| \\
&+\max \sum_{j=m'+1}^{n'+2-m'} \sum_{i=m+1}^{n+2-m} |k(x, y, s_{n+3-i}, t_j)| \Bigg). \qquad (28)
\end{aligned}
$$

Proof For $\forall u(x, y) \in C([a, b] \times [c, d])$ and $\|u\|_\infty \le 1$, one has

$$
\begin{aligned}
\|K_{mn}u\|_\infty =& h_x h_y \Bigg(\max \Big| \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} k(x, y, s_i, t_j) u(s_i, t_j) \Big| \\
&+\max \Big| \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} k(x, y, s_{n+3-i}, t_j) u(s_{n+3-i}, t_j) \Big| \\
&+\max \Big| \sum_{j=1}^{m'} \sum_{i=m+1}^{n+2-m} v_j^{(m',2)} k(x, y, s_i, t_j) u(s_i, t_j) \Big| \\
&+\max \Big| \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} k(x, y, s_i, t_{n'+3-j}) u(s_i, t_{n'+3-j}) \Big| \\
&+\max \Big| \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} k(x, y, s_{n+3-i}, t_{n'+3-j}) u(s_{n+3-i}, t_{n'+3-j}) \Big| \\
&+\max \Big| \sum_{j=1}^{m'} \sum_{i=m+1}^{n+2-m} v_j^{(m',2)} k(x, y, s_i, t_{n'+3-j}) u(s_i, t_{n'+3-j}) \Big| \\
&+\max \Big| \sum_{j=m'+1}^{n'+2-m'} \sum_{i=1}^{m} v_i^{(m,2)} k(x, y, s_i, t_j) u(s_i, t_j) \Big|
\end{aligned}
$$

$$
\begin{aligned}
&+ \max \Big| \sum_{j=m'+1}^{n'+2-m'} \sum_{i=1}^{m} v_i^{(m,2)} k(x, y, s_{n+3-i}, t_j) u(s_{n+3-i}, t_j) \Big| \\
&+ \max \Big| \sum_{j=m'+1}^{n'+2-m'} \sum_{i=m+1}^{n+2-m} k(x, y, s_{n+3-i}, t_j) u(s_{n+3-i}, t_j) \Big| \Bigg) \\
&\leq h_x h_y \Bigg(\max \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} |k(x, y, s_i, t_j)| \\
&+ \max \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} |k(x, y, s_{n+3-i}, t_j)| \\
&+ \max \sum_{j=1}^{m'} \sum_{i=m+1}^{n+2-m} v_j^{(m',2)} |k(x, y, s_i, t_j)| \\
&+ \max \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} |k(x, y, s_i, t_{n'+3-j})| \\
&+ \max \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} |k(x, y, s_{n+3-i}, t_{n'+3-j})| \\
&+ \max \sum_{j=1}^{m'} \sum_{i=m+1}^{n+2-m} v_j^{(m',2)} |k(x, y, s_i, t_{n'+3-j})| \\
&+ \max \sum_{j=m'+1}^{n'+2-m'} \sum_{i=1}^{m} v_i^{(m,2)} |k(x, y, s_i, t_j)| \\
&+ \max \sum_{j=m'+1}^{n'+2-m'} \sum_{i=1}^{m} v_i^{(m,2)} |k(x, y, s_{n+3-i}, t_j)| \\
&+ \max \sum_{j=m'+1}^{n'+2-m'} \sum_{i=m+1}^{n+2-m} |k(x, y, s_{n+3-i}, t_j)| \Bigg).
\end{aligned}
$$

Since kernel function $k(x, y, s, t)$ is continuous, there exists $(x_p, y_p) \in ([a, b] \times [c, d])$ such that

$$max|k(x, y, s_i, t_j)| = |k(x_p, y_p, s_i, t_j)|, \quad i, \; j = 1, \ldots, n+2.$$

We choose $u_p \in C([a, b] \times [c, d])$ with $\|u_p\|_\infty = 1$ then we can write

$$k(x_p, y_p, s_i, t_j).u_p(s_i, t_j) = k(x_p, y_p, s_i, t_j), \quad i, \; j = 1, \ldots, n+2. \tag{29}$$

On the other hand, we have

$$
\begin{aligned}
\|K_{mn}\|_\infty \geq& \|K_{mn} u_p\|_\infty \geq |(K_{mn} u_p)(x_p, y_p)| \\
&= h_x h_y \Bigg(\max \sum_{j=1}^{m'} \sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} |k(x, y, s_i, t_j)|
\end{aligned}
$$

$$
\begin{aligned}
&+\max\sum_{j=1}^{m'}\sum_{i=1}^{m}v_j^{(m',2)}v_i^{(m,2)}|k(x,y,s_{n+3-i},t_j)|\\
&+\max\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m}v_j^{(m',2)}|k(x,y,s_i,t_j)|\\
&+\max\sum_{j=1}^{m'}\sum_{i=1}^{m}v_j^{(m',2)}v_i^{(m,2)}|k(x,y,s_i,t_{n'+3-j})|\\
&+\max\sum_{j=1}^{m'}\sum_{i=1}^{m}v_j^{(m',2)}v_i^{(m,2)}|k(x,y,s_{n+3-i},t_{n'+3-j})|\\
&+\max\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m}v_j^{(m',2)}|k(x,y,s_i,t_{n'+3-j})|\\
&+\max\sum_{j=m'+1}^{n'+2-m'}\sum_{i=1}^{m}v_i^{(m,2)}|k(x,y,s_i,t_j)|\\
&+\max\sum_{j=m'+1}^{n'+2-m'}\sum_{i=1}^{m}v_i^{(m,2)}|k(x,y,s_{n+3-i},t_j)|\\
&+\max\sum_{j=m'+1}^{n'+2-m'}\sum_{i=m+1}^{n+2-m}|k(x,y,s_{n+3-i},t_j)|\Bigg),
\end{aligned}
\tag{30}
$$

and the proof is completed. □

Now we can prove the following theorem, which shows that the quadrature method convergence at the rate of $O(h_x^{m+1})+O(h_y^{m'+1})$.

Theorem 2 *Let $k(.,.,.,.)\in C([a,b]\times[c,d]\times[a,b]\times[c,d])$ and the nonlinear function F satisfy the generalized Lipschitz condition with constant L, i.e.*

$$|F(.,.,z(.,.))-F(.,.,v(.,.))|\le L|z(.,.)-v(.,.)|.$$

Then quadrature operator $(K_{mn}(Fu))(x,y)$ is a uniform convergence sequence and there exist constant c_1, c_2 and M such that

$$\max|u(x,y)-u^{m,m'}(x,y)|\le\frac{c_1h_x^{m+1}+c_2h_y^{m'+1}}{|1-9LM'h_xh_y|}. \tag{31}$$

Proof For $(x,y)\in[a,b]\times[c,d]$, we have

$$
\begin{aligned}
&|(K(Fu))(x,y)-(K_{mn}(Fu^{m,m'}))(x,y)|=\bigg|\int_c^d\int_a^b k(x,y,s,t)F(s,t,u(s,t))\mathrm{d}s\mathrm{d}t\\
&\quad-h_xh_y(\sum_{j=1}^{m'}\sum_{i=1}^{m}v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_i,t_j)F(s_i,t_j,u^{m,m'}(s_i,t_j))\\
&\quad+\sum_{j=1}^{m'}\sum_{i=1}^{m}v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_{n+3-i},t_j)F(s_{n+3-i},t_j,u^{m,m'}(s_{n+3-i},t_j))
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m} v_j^{(m',2)} k(x,y,s_i,t_j) F(s_i,t_j,u^{m,m'}(s_i,t_j))\\
&+\sum_{j=1}^{m'}\sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} k(x,y,s_i,t_{n'+3-j}) F(s_i,t_{n'+3-j},u^{m,m'}(s_i,t_{n'+3-j}))\\
&+\sum_{j=1}^{m'}\sum_{i=1}^{m} v_j^{(m',2)} v_i^{(m,2)} k(x,y,s_{n+3-i},t_{n'+3-j}) F(s_{n+3-i},t_{n'+3-j},u^{m,m'}(s_{n+3-i},t_{n'+3-j}))\\
&+\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m} v_j^{(m',2)} k(x,y,s_i,t_{n'+3-j}) F(s_i,t_{n'+3-j},u^{m,m'}(s_i,t_{n'+3-j}))\\
&+\sum_{j=m'+1}^{n'+2-m'}\sum_{i=1}^{m} v_i^{(m,2)} k(x,y,s_i,t_j) F(s_i,t_j,u^{m,m'}(s_i,t_j))\\
&+\sum_{j=m'+1}^{n'+2-m'}\sum_{i=1}^{m} v_i^{(m,2)} k(x,y,s_{n+3-i},t_j) F(s_{n+3-i},t_j,u^{m,m'}(s_{n+3-i},t_j))\\
&+\sum_{j=m'+1}^{n'+2-m'}\sum_{i=m+1}^{n+2-m} k(x,y,s_{n+3-i},t_j) F(s_{n+3-i},t_j,u^{m,m'}(s_{n+3-i},t_j)))|\\
&\le h_x h_y (\sum_{j=1}^{m'}\sum_{i=1}^{m} |v_j^{(m',2)} v_i^{(m,2)} k(x,y,s_i,t_j)| L |u(s_i,t_j)-u^{m,m'}(s_i,t_j)\Big|\\
&+\sum_{j=1}^{m'}\sum_{i=1}^{m} |v_j^{(m',2)} v_i^{(m,2)} k(x,y,s_{n+3-i},t_j)| L |u(s_{n+3-i},t_j)-u^{m,m'}(s_{n+3-i},t_j)|\\
&+\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m} |v_j^{(m',2)} k(x,y,s_i,t_j)| L |u(s_i,t_j)-u^{m,m'}(s_i,t_j)|\\
&+\sum_{j=1}^{m'}\sum_{i=1}^{m} |v_j^{(m',2)} v_i^{(m,2)} k(x,y,s_i,t_{n'+3-j})| L |u(s_i,t_{n'+3-j})-u^{m,m'}(s_i,t_{n'+3-j})|\\
&+\sum_{j=1}^{m'}\sum_{i=1}^{m} |v_j^{(m',2)} v_i^{(m,2)} k(x,y,s_{n+3-i},t_{n'+3-j})| L |u(s_{n+3-i},t_{n'+3-j})-u^{m,m'}(s_{n+3-i},t_{n'+3-j})|\\
&+\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m} |v_j^{(m',2)} k(x,y,s_i,t_{n'+3-j})| L |u(s_i,t_{n'+3-j})-u^{m,m'}(s_i,t_{n'+3-j})|\\
&+\sum_{j=m'+1}^{n'+2-m'}\sum_{i=1}^{m} |v_i^{(m,2)} k(x,y,s_i,t_j)| L |u(s_i,t_j)-u^{m,m'}(s_i,t_j)|\\
&+\sum_{j=m'+1}^{n'+2-m'}\sum_{i=1}^{m} |v_i^{(m,2)} k(x,y,s_{n+3-i},t_j)| L |u(s_{n+3-i},t_j)-u^{m,m'}(s_{n+3-i},t_j)|\\
&+\sum_{j=m'+1}^{n'+2-m'}\sum_{i=m+1}^{n+2-m} |k(x,y,s_{n+3-i},t_j)| L |u(s_{n+3-i},t_j)-u^{m,m'}(s_{n+3-i},t_j)|)\\
&+O(h_x^{m+1})+O(h_y^{m'+1}).
\end{aligned}
$$

Having considered the assumption on the kernel of theorem, we can write

$$\sum_{j=1}^{m'}\sum_{i=1}^{m}|v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_i,t_j)|\le M_1, \sum_{j=1}^{m'}\sum_{i=1}^{m}|v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_{n+3-i},t_j)|\le M_2,$$

$$\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m}|v_j^{(m',2)}k(x,y,s_i,t_j)|\le M_3, \sum_{j=1}^{m'}\sum_{i=1}^{m}|v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_i,t_{n'+3-j})|\le M_4,$$

$$\sum_{j=1}^{m'}\sum_{i=1}^{m}|v_j^{(m',2)}v_i^{(m,2)}k(x,y,s_{n+3-i},t_{n'+3-j})|\le M_5,$$

$$\sum_{j=1}^{m'}\sum_{i=m+1}^{n+2-m}|v_j^{(m',2)}k(x,y,s_i,t_{n'+3-j})|\le M_6, \sum_{j=m'+1}^{n'+2-m'}\sum_{i=1}^{m}|v_i^{(m,2)}k(x,y,s_i,t_j)|\le M_7,$$

$$\sum_{j=m'+1}^{n'+2-m'}\sum_{i=1}^{m}|v_i^{(m,2)}k(x,y,s_{n+3-i},t_j)|\le M_8, \sum_{j=m'+1}^{n'+2-m'}\sum_{i=m+1}^{n+2-m}|k(x,y,s_{n+3-i},t_j)|\le M_9,$$

where $\max M_i = M^{'}, i = 1,\ldots,9$.
Then we can conclude that

$$\max|u(x,y)-u^{m,m'}(x,y)|\le\frac{c_1h_x^{m+1}+c_2h_y^{m'+1}}{|1-9LM'h_xh_y|}.$$

□

5 Numerical example

In this section, to illustrate the performance of the presented methods in solving two-dimensional Fredholm integral equations and justify the accuracy and efficiency of the method, we consider the following examples.

Example 1 Consider the following two-dimensional Fredholm integral equation

$$u(x,y)-\int_0^1\int_0^1\frac{x}{(8+y)(1+s+t)}u(s,t)\mathrm{d}s\mathrm{d}t=\frac{1}{(1+x+y)^2}-\frac{x}{6(8+y)},$$

where the exact solution is $u(x,y)=\frac{1}{(1+x+y)^2}$.
In Table 2, numerical results are presented for rule $\mathcal{I}_{Q_2}^{m,m'}(f)$, $m=m'=5,7,9$, which we use Eq. (26) for obtaining solution. Numerical results illustrate accuracy of the proposed quadrature rule. By increasing the values of n, n', the errors have been decreased.
In Tables 3 and 4, we compare the absolute errors of the spline quasi-interpolant method for $n=n'=16, 32$ and $m=m'=7$, with the two-dimensional triangular orthogonal functions method (Mirzaee and Piroozfar 2010) and a novel numerical method based on integral mean value theorem (Ma et al. 2015).

Example 2 We consider the two-dimensional Fredholm integral equation

$$u(x,y)-\int_0^1\int_0^1(s.\sin(t)+1)u(s,t)dsdt=x.\cos(y)-\frac{1}{6}(\sin(1)+3)\sin(1),$$

Table 2 Max. Abs. Err. for Example 1 ($\mathcal{I}_{Q_2}^{m,m'}$)

$n = n'$	$m = m' = 5$	$m = m' = 7$	$m = m' = 9$
20	8.57358×10^{-10}	2.46277×10^{-11}	1.00828×10^{-12}
25	2.48118×10^{-10}	5.01529×10^{-12}	1.49075×10^{-13}
30	8.88374×10^{-11}	1.33091×10^{-12}	2.9976×10^{-14}

Table 3 Numerical results for Example 1

(x, y)	$n = 16$	$n = 32$	$n = 16$ (Mirzaee and Piroozfar 2010)	$n = 32$ (Mirzaee and Piroozfar 2010)
(0.2, 0.2)	2.2619×10^{-11}	1.6153×10^{-13}	9.96×10^{-03}	9.54×10^{-03}
(0.4, 0.4)	4.4161×10^{-11}	3.1541×10^{-13}	9.49×10^{-03}	9.03×10^{-03}
(0.6, 0.6)	6.4701×10^{-11}	4.6215×10^{-13}	8.029×10^{-03}	4.3×10^{-04}
(0.8, 0.8)	8.4308×10^{-11}	6.0218×10^{-13}	6.78×10^{-03}	2.62×10^{-04}

Table 4 Numerical results for Example 1

(x, y)	$n = 16$	$n = 32$	$n = 16$ (Ma et al. 2015)	$n = 32$ (Ma et al. 2015)
(0.2, 0.2)	2.2619×10^{-11}	1.6153×10^{-13}	8.8102×10^{-06}	2.5841×10^{-06}
(0.4, 0.4)	4.4161×10^{-11}	3.1541×10^{-13}	1.7201×10^{-05}	5.0452×10^{-06}
(0.6, 0.6)	6.4701×10^{-11}	4.6215×10^{-13}	2.52×10^{-05}	7.392×10^{-06}
(0.8, 0.8)	8.4308×10^{-11}	6.0218×10^{-13}	3.284×10^{-05}	9.632×10^{-06}

Table 5 Maximum absolute errors for Example 2 ($\mathcal{I}_{Q_2}^{m,m'}$)

$n = n'$	$m = m' = 5$	$m = m' = 7$	$m = m' = 9$
20	4.84148×10^{-10}	1.43963×10^{-12}	4.32987×10^{-15}
25	1.30321×10^{-10}	2.53852×10^{-13}	2.77556×10^{-16}
30	4.43818×10^{-11}	6.11733×10^{-14}	2.22045×10^{-16}

where the exact solution is $u(x, y) = x.\cos(y)$. In Table 5, numerical results are presented for rule $\mathcal{I}_{Q_2}^{m,m'}(f)$, $m = m' = 5, 7, 9$. In table 6, we compare the absolute errors of the spline quasi-interpolant method for $n = n' = 16, 32$ and $m = m' = 7$, with the two-dimensional Haar wavelets method (Derili et al. 2012). The results show the efficiency and rate of convergence of the method.

Example 3 Consider the following two-dimensional Fredholm integral equation

$$u(x, y) - \int_0^1 \int_0^1 \frac{x}{(1+y)}(1+s+t)u^2(s,t)\mathrm{d}s\mathrm{d}t = \frac{1}{(1+x+y)^2} - \frac{x}{6(1+y)},$$

Table 6 Numerical results for Example 2

(x, y)	$n = 16$	$n = 32$	$n = 16$ (Derili et al. 2012)	$n = 32$ (Derili et al. 2012)
(0.5, 0.5)	7.99616×10^{-12}	3.68594×10^{-14}	1.1×10^{-02}	8.6×10^{-03}
(0.25, 0.25)	7.99613×10^{-12}	3.68316×10^{-14}	1.6×10^{-02}	1.2×10^{-02}
(0.125, 0.125)	7.99617×10^{-12}	3.68733×10^{-14}	8.0×10^{-03}	8.9×10^{-03}
(0.0625, 0625)	7.99614×10^{-12}	3.68386×10^{-14}	4.1×10^{-02}	2.0×10^{-02}
(0.03125, 0.03125)	7.9962×10^{-12}	3.68045×10^{-14}	2.1×10^{-04}	6.0×10^{-03}

Table 7 Maximum absolute error for Example 3 ($\mathcal{I}_{Q_2}^{m,m'}$)

$n = n'$	$m = m' = 5$	$m = m' = 7$	$m = m' = 9$
20	1.0096×10^{-8}	2.90008×10^{-10}	1.1873×10^{-11}
25	2.92176×10^{-9}	5.90583×10^{-11}	1.75523×10^{-12}
30	1.04612×10^{-9}	1.56726×10^{-11}	3.52884×10^{-13}

Table 8 Numerical results for Example 3

(x, y)	$n = 16$	$n = 32$	$n = 16$ (Babolian et al. 2011)	$n = 32$ (Babolian et al. 2011)
(0.5, 0.5)	4.55026×10^{-10}	3.02523×10^{-12}	1.5×10^{-02}	7.6×10^{-03}
(0.25, 0.25)	2.73016×10^{-10}	1.95011×10^{-12}	3.4×10^{-02}	1.8×10^{-02}
(0.125, 0.125)	1.51675×10^{-10}	1.08336×10^{-12}	5.9×10^{-02}	3.1×10^{-02}
(0.0625, 0.0625)	8.02988×10^{-11}	5.7343×10^{-13}	8.0×10^{-02}	4.2×10^{-02}
(0.03125, 0.03125)	4.1366×10^{-11}	2.9543×10^{-13}	1.5×10^{-03}	5.0×10^{-02}

Table 9 Numerical results for Example 3

n	$m = 7$	Han and Wang (2002)
16	1.36508×10^{-9}	3.869×10^{-4}
32	9.75067×10^{-12}	9.693×10^{-5}

where the exact solution is $u(x, y) = \frac{1}{(1+x+y)^2}$. In Table 7, numerical results are presented for rule $\mathcal{I}_{Q_2}^{m,m'}(f)$, $m = m' = 5, 7, 9$.

Numerical results illustrate accuracy the proposed quadrature rules. By increasing the values of n, n', the errors have been decreased. In Tables 8 and 9, we compare the absolute errors of the spline quasi-interpolant method for $n = n' = 16, 32$ and $m = m' = 7$, with rationalized Haar functions method (Babolian et al. 2011) and iterated discrete Galerkin method (Han and Wang 2002).

Example 4 Consider the following two-dimensional Fredholm integral equation

$$u(x, y) - \int_0^1 \int_0^1 (\cos(\pi s) + y)u^2(s, t)\mathrm{d}s\mathrm{d}t = y.\sin(\pi x) - \frac{y}{6},$$

Table 10 Maximum absolute errors for Example 4 ($\mathcal{I}_{Q_2}^{m,m'}$)

$n = n'$	$m = m' = 5$	$m = m' = 7$	$m = m' = 9$
20	4.0374×10^{-8}	2.04599×10^{-9}	9.2916×10^{-11}
25	8.60993×10^{-9}	2.86549×10^{-10}	8.67972×10^{-12}
30	2.42475×10^{-9}	5.6822×10^{-11}	1.22138×10^{-12}

Table 11 Numerical results for Example 4

(x, y)	$n = 20$	$n = 20$ (Khan and Fardi 2015)
(0.5, 0.5)	1.02299×10^{-9}	7.651×10^{-5}
(0.25, 0.25)	5.11496×10^{-10}	3.826×10^{-5}
(0.125, 0.125)	2.55748×10^{-10}	1.913×10^{-5}
(0.0625, 0.0625)	1.27874×10^{-10}	9.567×10^{-6}
(0.03125, 0.03125)	6.3937×10^{-11}	4.783×10^{-6}

Table 12 Maximum absolute errors for Example 5 ($\mathcal{I}_{Q_2}^{m,m'}$)

$n = n'$	$m = m' = 5$	$m = m' = 7$	$m = m' = 9$
20	2.06277×10^{-10}	1.2942×10^{-11}	6.46927×10^{-13}
25	4.36942×10^{-11}	1.49969×10^{-12}	7.33857×10^{-14}
30	1.37256×10^{-11}	3.03646×10^{-13}	1.16573×10^{-14}

where the exact solution is $u(x, y) = y.\sin(\pi x)$. In Table 10, numerical results are presented for rule $\mathcal{I}_{Q_2}^{m,m'}(f)$, $m = m' = 5, 7, 9$. In Table 11, we compare the absolute errors of the spline quasi-interpolant method for $n = n' = 20$ and $m = m' = 7$, with homotopy analysis method (Khan and Fardi 2015).

Example 5 Consider the following two-dimensional Fredholm integral equation:

$$u(x, y) - \int_0^1 \int_0^1 \frac{(1 - xs)(2 - \mathrm{e}^{u(s,t)})}{t^2 y + 3} \mathrm{d}s\mathrm{d}t = \ln\left(2 + \frac{x}{1 + y^2}\right) - \frac{(2x - 3)\left(-3\pi + 4\sqrt{3y}\arctan\sqrt{\frac{y}{3}}\right)}{72(y - 3)}$$

where the exact solution is $u(x, y) = \ln(2 + \frac{x}{1+y^2})$. In Table 12, numerical results are presented for rule $\mathcal{I}_{Q_2}^{m,m'}(f)$, $m = m' = 5, 7, 9$. In Table 13, the absolute errors of the spline quasi-interpolant method for $n = n' = 30$ and $m = m' = 7$, $m = m' = 9$ are displayed.

Example 6 Consider the following two-dimensional Fredholm integral equation:

$$u(x, y) - \int_0^2 \int_0^2 k(x, y, s, t) u(s, t) \mathrm{d}s\mathrm{d}t = g(x, y),$$

Table 13 Numerical results for Example 5

(x, y)	$n = 30, m = 7$	$n = 30, m = 9$
$(\frac{1}{2}, \frac{1}{2})$	1.35447×10^{-13}	4.8849×10^{-15}
$(\frac{1}{4}, \frac{1}{4})$	1.57545×10^{-13}	5.5511×10^{-15}
$(\frac{1}{8}, \frac{1}{8})$	1.5743×10^{-13}	5.7731×10^{-15}
$(\frac{1}{16}, \frac{1}{16})$	1.59872×10^{-13}	5.8841×10^{-15}
$(\frac{1}{32}, \frac{1}{32})$	1.60982×10^{-13}	5.8841×10^{-15}

Table 14 Maximum absolute errors for Example 6 ($\mathcal{I}_{Q_2}^{m,m'}$)

$n = n'$	$m = m' = 5$	$m = m' = 7$	$m = m' = 9$
20	3.21558×10^{-6}	1.2075×10^{-6}	9.99654×10^{-7}
25	7.61194×10^{-7}	4.22466×10^{-7}	4.59151×10^{-7}
30	2.21196×10^{-7}	1.35096×10^{-7}	1.17357×10^{-7}

Table 15 Numerical results for Example 6

(x, y)	$n = 30, m = 7$	$n = 30, m = 9$
$(\frac{1}{4}, \frac{1}{4})$	2.54852×10^{-8}	2.92787×10^{-8}
$(\frac{1}{8}, \frac{1}{8})$	2.4186×10^{-8}	1.94871×10^{-8}
$(\frac{1}{16}, \frac{1}{16})$	1.73592×10^{-8}	1.00631×10^{-8}
$(\frac{1}{32}, \frac{1}{32})$	1.08312×10^{-9}	6.96115×10^{-9}
$(\frac{1}{64}, \frac{1}{64})$	2.16537×10^{-8}	2.99332×10^{-8}

where

$$k(x, y, s, t) = \begin{cases} \left[\cos\left(\frac{\pi(x-s)}{2}\right)\cos\left(\frac{\pi(y-t)}{2}\right)\right]^4, & -1 \leq x - s, y - t \leq 1, \\ 0, & \text{else}, \end{cases}$$

$g(x, y) = 1 - T(x)T(y)$ with

$$T(x) = \begin{cases} \frac{1}{16\pi}[\sin(2\pi x) + 8\sin(\pi x) + 6\pi x + 6\pi], & 0 \leq x \leq 1, \\ \frac{1}{16\pi}[-\sin(2\pi x) - 8\sin(\pi x) - 6\pi x + 18\pi], & 1 \leq x \leq 2, \end{cases}$$

and the exact solution is $u(x, y) = 1$. In Table 14, numerical results are presented for rule $\mathcal{I}_{Q_2}^{m,m'}(f)$, $m = m' = 5, 7, 9$. In Table 15, the absolute errors of the spline quasi-interpolant method for $n = n' = 30$ and $m = m' = 7$, $m = m' = 9$ are displayed.

Table 16 Quadrature weights $v_i^{(m,2)}$

$m = 5$	$m = 7$	$m = 9$	$m = 13$
0.1307936	0.1374149	0.1400901	0.1414888
0.8359375	0.8190165	0.8109525	0.8060753
1.0449652	1.0698175	1.0870467	1.1014595
0.9861458	0.9603402	0.9321887	0.8977344
1.0021577	1.0177210	1.0478285	1.1060041
–	0.9950634	0.9742957	0.9037215
–	1.0006252	1.0095197	1.0720481
–	–	0.9978581	0.9571099
–	–	1.0002198	1.0196830
–	–	–	0.9933074
–	–	–	1.0015852
–	–	–	0.9997667
–	–	–	1.00001605

6 Conclusion

The spline quasi interpolant quadrature rule $\mathcal{I}_{Q_2}^{m,m'}$ is used to solve the two-dimensional Fredholm integral equation. The convergence analysis of the presented method is discussed. Also the obtained results are compared with the exact solution and methods in Babolian et al. (2011), Derili et al. (2012), Han and Wang (2002), Khan and Fardi (2015), Ma et al. (2015) and Mirzaee and Piroozfar (2010). The method is computationally attractive and application is demonstrate through illustrative examples.

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