Computational and Applied Mathematics (2020) 39:104
https://doi.org/10.1007/s40314-020-1112-1

A modified Euler method for solving fuzzy differential equations under generalized differentiability

N. Ahmady[1] · T. Allahviranloo[2] · E. Ahmady[3]

Received: 15 June 2019 / Revised: 27 January 2020 / Accepted: 4 February 2020 /
Published online: 12 March 2020
© SBMAC - Sociedade Brasileira de Matemática Aplicada e Computacional 2020

Abstract
In this paper, we intend to introduce a modified approach for solving fuzzy differential equations (FDEs) under generalized differentiability. Modified Euler method estimated FDEs by using a two-stage predictor–corrector algorithm with local truncation error of order two. The consistency, convergence, and stability of the proposed method are also investigated in detail. The acceptable accuracy of the Modified Euler method is illustrated by some examples.

Keywords Fuzzy differential equations · Generalized Hukuhara differentiability · Numerical method

Mathematics Subject Classification 34A07

1 Introduction

Fuzzy differential equations (FDEs) are applied for modeling problems in science and engineering under uncertainly and has been rapidly developing in recent years Allahviranloo (2020).

The concept of fuzzy derivatives is a very important notation in FDEs. Fuzzy derivatives were first initiated by Chang and Zadeh (1972), and followed up by Dubois and Prade (1982), Puri and Ralescu (1986) and Goetschel and Voxman (1987).

The concept of FDEs was introduced by Kaleva (1987), Seikkala (1987), and other researches. Many works have been done to study the numerical solution of FDEs under Hukuhara differentiability like Euler method, Taylor method, Runge–Kutta method, Predictor–Corrector and Improved Predictor–Corrector method (Ma et al. 1999; Abbasbandy and Allahviranloo 2002, 2004; Allahviranloo et al. 2007, 2009).

Communicated by Leonardo Tomazeli Duarte.

✉ N. Ahmady
n.ahmadi@iauvaramin.ac.ir

1 Department of Mathematics, Varamin-Pishva Branch, Islamic Azad University, Varamin, Iran

2 Faculty of Engineering and Natural Sciences, Bahcesehir University, Istanbul, Turkey

3 Department of Mathematics, Shahr-e-Qods Branch, Islamic Azad University, Tehran, Iran

It is well-known that the solution of FDEs under Hukuhara derivatives has the property that the diameter is non-decreasing as t increase; to overcome this shortcoming, Bede and Gal introduced the strongly generalized differential of set value fuzzy function (Bede and Gal 2005) and studied more properties in Bede and Stefanini (2011), Bede and Gal (2006), Bede and Stefanini (2013).

Rapidly, the numerical methods for solving fuzzy differential equation and fuzzy integral equation under Hukuhara differentiability extended to fuzzy differential equation and fuzzy integral equation under strongly generalized differentiability (Rabiei et al. 2013; Hajighasemi et al. 2010; BaloochShahryari and Salashour 2012; Ahmadian et al. 2018, 2012).
Nieto et al. (2009) obtained solution of first-order fuzzy differential equations using generalized differentiability by interpreting the original fuzzy differential equations with two crisp ordinary differential equations.

Allahviranloo and Salahshour proposed a new approach for solving first order fuzzy differential equations under strongly generalized H-differentiability; the main part of the proposed technique was extending 1-cut solution of original FDEs by allocating some unknown spreads in Salahshour et al. (2018). Also Allahviranloo, Gouyandeh, and Armand by introducing the fuzzy Taylor expansion proposed the Euler method for solving FDEs under generalized differentiability; the local truncation error for the Euler method is $o(h)$ (Allahviranloo et al. 2015). Tapaswini and Chakraverty proposed improved Euler methods for solving FDEs under Hukuhara differentiability in Tapaswini and Chakraverty (2012). In the present paper, we intend to improve the Euler method under generalized differentiability by two-stage predictor–corrector method, to get better accuracy without losing the simplicity of the Euler method. The convergence and stability of the proposed method are proved. By some examples, the capability of the proposed method is shown.

The paper is organized as follows: In Sect. 2, some basic definitions are brought. In the proposed method, moreover consistency, convergence, and stability of the Modified Euler method are introduced in Sect. 3. The numerical examples are presented in Sect. 4 and finally, conclusion is drawn.

2 Preliminaries

In this section the most basic notations and definitions used are introduced:
The set of fuzzy numbers, that is, normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets which are defined over the real line and denoted by $\mathbb{R}_{\mathcal{F}}$. For $0 < \alpha \leq 1$, set $[u]^{\alpha} = \left\{t \in \mathbb{R} \middle| u(t) \geq \alpha\right\}$, and $[u]^{0} = cl\left\{t \in \mathbb{R} \middle| u(t) > 0\right\}$. We represent $[u]^{\alpha} = [u^{-}(\alpha),\ u^{+}(\alpha)]$, so if $u \in \mathbb{R}_{\mathcal{F}}$, the α-level set $[u]^{\alpha}$ is a closed interval for all $\alpha \in [0,\ 1]$. For arbitrary $u,\ v \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$, the addition and scalar multiplication are defined by $[u+v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}\ ,\ [ku]^{\alpha} = k[u]^{\alpha}$, respectively.

Definition 2.1 (*see* Kaleva 1987) The Hausdorff distance between fuzzy numbers is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^{+} \cup \{0\}$ as

$$D(u,\ v) = \sup_{\alpha \in [0,\ 1]} \max \left\{|u^{-}(\alpha) - v^{-}(\alpha)|,\ |u^{+}(\alpha) - v^{+}(\alpha)|\right\}.$$

Consider $u, v, w, z \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$; then the following properties are well-known for metric D:(see Negoita and Ralescu 1975):

1. $D(u \oplus w,\ v \oplus w) = D(u,\ v)$;
2. $D(\lambda u,\ \lambda v) = |\lambda| D(u,\ v)$;
3. $D(u \oplus v,\ w \oplus z) \le D(u,\ w) + D(v,\ z)$;
4. $D(u \ominus v,\ w \ominus z) \le D(u,\ w) + D(v,\ z)$, as long as $u \ominus v$ and $w \ominus z$ exist, where $u,\ v,\ w,\ z \in \mathbb{R}_{\mathcal{F}}$,

where $\ominus$ is the Hukuhara difference(H-difference); it means that $w \ominus v = u$ if and only if $u \oplus v = w$.

Definition 2.2 (*see* Stefanini 2008) The generalized Hukuhara difference of two fuzzy numbers $u,\ v \in \mathbb{R}_{\mathcal{F}}$ is defined as follows:

$$u \ominus v = w \Longleftrightarrow \begin{cases} (a).\ u = v + w; \\ or\ (b).\ v = u + (-1)w. \end{cases}$$

Proposition 2.1 (see Stefanini 2008) *Let $A, B \in K_n^C$ be two compact convex set; then*

1. *if the gH-difference exists, it is unique and it is a generalization of the usual Hukuhara difference since $A \ominus B = A \sim_h B$, whenever $A \sim_h B$ exists,*
2. $A \ominus A = 0$,
3. *if $A \ominus B$ exists in the sence (a), then $B \ominus A$ exists in sense (b) and vice versa,*
4. $(A \oplus B) \ominus B = A$,
5. $\{0\} \ominus (A \ominus B) = (-B) \ominus (-A)$,
6. *we have $(A \ominus B) = (B \ominus A) = C$ if and only if $C = \{0\}$ and $A = B$.*

Definition 2.3 (*see* Nieto et al. 2009) A fuzzy valued function $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is said to be continuous at $t_0 \in [a, b]$ if for each $\epsilon > 0$ there is $\delta > 0$ such that $D(f(t), f(t_0)) < \epsilon$, whenever $t \in [a, b]$ and $|t - t_0| < \delta$. We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $t_0 \in [a,\ b]$.

Definition 2.4 (*see* Bede and Gal 2005) Let $F : I \to \mathbb{R}$. Fix $t_0 \in I$. We say F is strongly generalized differentiable at t_0, if there exists an element $F'(t_0) \in \mathbb{R}$ such that either

1. for all $h > 0$ sufficiently closed to 0, the H-differences $F(t_0+h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0-h)$ exist and limits (in the metric D)
$$\lim_{h \to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0) \tag{2.1}$$
or
2. for all $h > 0$ sufficiently closed to 0, the H-differences $F(t_0) \ominus F(t_0+h)$, $F(t_0-h) \ominus F(t_0)$ exist and limits (in the metric D)
$$\lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = \lim_{h \to 0^+} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t_0) \tag{2.2}$$
or
3. for all $h > 0$ sufficiently closed to 0, the H-differences $F(t_0+h) \ominus F(t_0)$, $F(t_0-h) \ominus F(t_0)$ exist and limits (in the metric D)
$$\lim_{h \to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \to 0^+} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t_0) \tag{2.3}$$
or

4. for all $h > 0$ sufficiently closed to 0, the H-differences $F(t_0) \ominus F(t_0+h)$, $F(t_0) \ominus F(t_0-h)$ exist and limits (in the metric D)

$$\lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0+h)}{-h} = \lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0-h)}{h} = F'(t_0) \tag{2.4}$$

Definition 2.5 Let $f : I \to \mathbb{R}_F$. We say f is $[(i) - gH]$-differentiable on I if f is differentiable in the sense (1) of Definition 2.4 and its derivative is denoted $f_{i.gH}$ and similarly for $[(ii) - gH]-$ differentiable we have $f_{ii.gH}$ if f is differentiable in the sense (2) of Definition 2.4.

Theorem 2.1 (see Chalco-Cano and Roman-Flores 2008) *Let $f : I \to \mathbb{R}_F$ and put $[f(t)]^\alpha = [(f^+)(t;\alpha)\,,\ (f^-)(t;\alpha)]$ for each $\alpha \in [0, 1]$.*

1. *If f is $[(i) - gH]$ differentiable the f^+ and f^- are differentiable functions and*
$$f'_{i.gH}(t_0;\alpha) = [(f^-)'(t;\alpha),\ (f^+)'(t;\alpha)], \tag{2.5}$$
2. *If f is $[(ii) - gH]$ differentiable the f^+ and f^- are differentiable functions and*
$$f'_{ii.gH}(t_0;\alpha) = [(f^+)'(t;\alpha),\ (f^-)'(t;\alpha)], \tag{2.6}$$

Definition 2.6 (*see* Chalco-Cano and Roman-Flores 2008) We say that a point $t_0 \in (a, b)$ is a switching point for the differentiability of f if in any neighborhood V of t_0 there exist points $t_1 < t_0 < t_2$ such that

type (I) at t_1 (2.5) holds while (2.6) does not hold and at t_2 (2.6) holds and (2.5) does not hold, or

type (II) at t_1 (2.6) holds while (2.5) does not hold and at t_2 (2.5) holds and (2.6) does not hold.

The following condition and notations are used in the reminder of paper:
We assume that in the whole paper, the generalized Hukuhara difference of two fuzzy numbers exists.
$\mathcal{C}_\mathcal{F}([a, b], \mathbb{R}_\mathcal{F})$ is the set of fuzzy valued functions f which are defined on $[a, b]$ and is fuzzy continuous from the interior points of $[a, b]$ such that the continuity is one-sided at endpoints a, b.
$\mathcal{C}^k_{gH}([a, b],\ \mathbb{R}_\mathcal{F})$, is the space of functions f such that f and its first k, gH-derivtives are all in $\mathcal{C}_\mathcal{F}([a, b], \mathbb{R}_\mathcal{F})$.

Theorem 2.2 (see Allahviranloo et al. 2015) *Consider $f : [a, b] \longrightarrow \mathbb{R}_\mathcal{F}$ is gH-differentiable such that type of differentiability f in $[a, b]$ does not change. Then for $a \le s \le b$*

(i) *if $f(t)$ is $[(i) - gH]$-differentiable, then $f'_{i.gH}(t)$ is (FR)-integrable over $[a,\ b]$ and*
$$f(s) = f(a) \oplus \int_a^s f'_{i.gH}(t)dt. \tag{2.7}$$

(ii) *If $f(t)$ is $[(ii) - gH]$-differentiable, then $f'_{ii.gH}(t)$ is (FR)-integrable over $[a,\ b]$ and*
$$f(a) = f(s) \oplus (-1)\int_a^s f'_{ii.gH}(t)dt. \tag{2.8}$$

Theorem 2.3 (see Allahviranloo et al. 2015) *Let $f^{(i)} : [a,\ b] \to \mathbb{R}_\mathcal{F}$ and $f \in \mathcal{C}^4_{gH}([a, b],\ \mathbb{R}_\mathcal{F})$. For all $s \in [a,\ b]$*

(i) *Consider* $f_{gH}^{(i)},\ \ i=1,\ldots,n$ *are* $[(i)-gH]$*-differentiable and type of differentiability does not change in interval* $[a,\ b]$*; then*

$$f_{i.gH}^{(i-1)}(s)=f_{i.gH}^{(i-1)}(a)\oplus\int_a^s f_{i.gH}^{(i)}(t)dt. \tag{2.9}$$

(ii) *If* $f_{gH}^{(i)},\ \ i=1,\ldots,n$ *are* $[(ii)-gH]$*-differentiable and type of differentiability does not change in interval* $[a,\ b]$*, then*

$$f_{ii.gH}^{(i-1)}(s)=f_{ii.gH}^{(i-1)}(a)\oplus\int_a^s f_{ii.gH}^{(i)}(t)dt. \tag{2.10}$$

(iii) *Suppose that* $f^{(i)},\ i=2k-1,\ k\in\mathbb{N}$ *are [(i)-gH]-differentiable and* $f(t),\ f^{(i)},\ i=2k,\ k\in\mathbb{N}$ *are [(ii)-gH]-differentiable, so*

$$f_{i.gH}^{(i-1)}(s)=f_{i.gH}^{(i-1)}(a)\ominus(-1)\int_a^s f_{ii.gH}^{(i)}(t)dt. \tag{2.11}$$

(iv) *Consider for* $i=2k-1, k\in\mathbb{N},\ f^{(i)}$ *are [(ii)-gH]-differentiable and* $f(t),\ f^{(i)}$ *are [(i)-gH]-differentiable for* $i=2k,\ k\in\mathbb{N}$*, then*

$$f_{ii.gH}^{(i-1)}(s)=f_{ii.gH}^{(i-1)}(a)\ominus(-1)\int_a^s f_{i.gH}^{(i)}(t)dt. \tag{2.12}$$

3 Proposed method

Now, let us consider the following fuzzy differential equation (FDE):

$$\begin{cases} y'_{gH}(t)=f(t,y(t)),\ t\in[0,T]; \\ y(0)=y_0\in\mathbb{R}_{\mathcal{F}}; \end{cases} \tag{3.1}$$

here $y(t)$ is an unknown fuzzy function of crisp variable t and $f:[0,\ T]\times\mathbb{R}_{\mathcal{F}}\to\mathbb{R}_{\mathcal{F}}$ is continuous; also $y'_{gH}(t)$ is the generalized Hukuhara derivative of $y(t)$ such that the set of switching points is finite.

First, we are going to approximate $y''_{gH}(t)$ in four different cases by using concept of generalized differentiability.

Theorem 3.1 *Let* $f\in\mathcal{C}_{gH}^4([a,b],\ \mathbb{R}_{\mathcal{F}})$*. Then*

(i) *if* $f_{gH}^{(i)},\ \ i=1,\ldots,4$ *are* $[(i)-gH]$*-differentiable and type of differentiability does not change in interval* $[a,\ b]$*, then*

$$f''_{i.gH}(a)\approx\frac{f'_{i.gH}(s)\ominus f'_{i.gH}(a)}{s-a}, \tag{3.2}$$

(ii) *let* $f_{gH}^{(i)},\ \ i=1,\ldots,4$ *are* $[(ii)-gH]$*-differentiable and type of differentiability does not change in interval* $[a,\ b]$*, then*

$$f''_{ii.gH}(a)\approx\frac{f'_{ii.gH}(s)\ominus f'_{ii.gH}(a)}{s-a}, \tag{3.3}$$

(iii) *suppose that* $f^{(i)}$, $i = 1, 3$ *are* $[(i) - gH]$*-differentiable and* $f(t)$, $f^{(i)}$, $i = 2, 4$ *are* $[(ii) - gH]$*-differentiable, so*

$$(-1)f''_{i.gH}(a) \approx \frac{f'_{ii.gH}(s) \ominus f'_{ii.gH}(a)}{s-a}, \tag{3.4}$$

(iv) *consider for* $i = 1, 3$, $f^{(i)}$ *are* $[(ii) - gH]$*-differentiable and* $f(t)$, $f^{(i)}$ *are* $[(i) - gH]$*-differentiable for* $i = 2, 4$, *then*

$$(-1)f''_{ii.gH}(a) \approx \frac{f'_{i.gH}(s) \ominus f'_{i.gH}(a)}{s-a}. \tag{3.5}$$

Proof Since $f \in \mathcal{C}^4_{gH}([a, b], \mathbb{R}_{\mathcal{F}})$, so $f^{(i)}_{gH}$, $i = 0, 1, 2, 3, 4$, are (FR)-integrable on T.

(i) Let $f^{(i)}$ be $[(i) - gH]$-differentiable; by Theorem 2.3, we can write

$$f'_{i.gH}(s) = f'_{i.gH}(a) \oplus \int_a^s f''_{i.gH}(s_1)\mathrm{d}s_1,$$

where

$$f''_{i.gH}(s_1) = f''_{i.gH}(a) \oplus \int_a^{s_1} f'''_{i.gH}(s_2)\mathrm{d}s_2. \tag{3.6}$$

by integration of Eq. (3.6), we get

$$\begin{aligned}\int_a^s f''_{i.gH}(s_1)\mathrm{d}s_1 &= \int_a^s f''_{i.gH}(a)\mathrm{d}s_1 \oplus \int_a^s \left(\int_a^{s_1} f'''_{i.gH}(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1,\\ &= f''_{i.gH}(a) \odot (s-a) \oplus \int_a^s \left(\int_a^{s_1} f'''_{i.gH}(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1,\end{aligned}$$

where the last double (FR)-integral belongs to $\mathbb{R}_{\mathcal{F}}$. So

$$\begin{aligned}f'_{i.gH}(s) = f'_{i.gH}(a) \oplus f''_{i.gH}(a) \odot (s-a)\\ \oplus \int_a^s \left(\int_a^{s_1} f'''_{i.gH}(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1.\end{aligned}$$

Consequently,

$$f'''_{i.gH}(s_2) = f'''_{i.gH}(a) \oplus \int_a^{s_2} f^{(4)}_{i.gH}(s_3)\mathrm{d}s_3;$$

applying the (FR)-integral operator to $f'''_{i.gH}(s_2)$, gives

$$\begin{aligned}&\int_a^{s_1} f'''_{i.gH}(s_2)\mathrm{d}s_2\\ &= f'''_{i.gH}(a) \odot (s_1 - a) \oplus \int_a^{s_1} \left(\int_a^{s_2} f^{(4)}_{i.gH}(s_3)\mathrm{d}s_3\right)\mathrm{d}s_2;\end{aligned}$$

furthermore,

$$\int_a^s \left(\int_a^{s_1} f'''_{i.gH}(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1 = f'''_{i.gH}(a) \odot \int_a^s (s_1 - a)\mathrm{d}s_1 \oplus \int_a^s \left(\int_a^{s_1} \left(\int_a^{s_2} f^{(4)}_{i.gH}(s_3)\mathrm{d}s_3\right)\mathrm{d}s_2\right)\mathrm{d}s_1,$$

where the last triple integral belongs to $\mathbb{R}_{\mathcal{F}}$. Hence

$$f'_{i.gH}(s) = f'_{i.gH}(a) \oplus f''_{i.gH}(a) \odot (s-a) \oplus f'''_{i.gH}(a) \odot \frac{(s-a)^2}{2!} \oplus \int_a^s \left(\int_a^{s_1} \left(\int_a^{s_2} f^{(4)}_{i.gH}(s_3)\mathrm{d}s_3\right)\mathrm{d}s_2\right)\mathrm{d}s_1.$$

If s is very closed to a and $(s-a) \to 0$,

$$D(f'_{i.gH}(s), f'_{i.gH}(a) \oplus f''_{i.gH}(a) \odot (s-a)) \to 0,$$

and

$$D(f'_{i.gH}(s) \ominus f'_{i.gH}(a), f''_{i.gH}(a) \odot (s-a)) \to 0,$$

this means where $(s-a) \to 0$, we obtain:

$$f''_{i.gH}(a) \approx \frac{f'_{i.gH}(s) \ominus f'_{i.gH}(a)}{s-a}.$$

(ii) Suppose that $f^{(i)}$ is $[(ii) - gH]$-differentiable for $i = 1, 2, 3, 4$; by Theorem 2.3, we have

$$f'_{ii.gH}(s) = f'_{ii.gH}(a) \oplus \int_a^s f''_{ii.gH}(s_1)\mathrm{d}s_1,$$

where $f''_{ii.gH}$ is gained by

$$f''_{ii.gH}(s_1) = f''_{ii.gH}(a) \oplus \int_a^{s_1} f'''_{ii.gH}(s_2)\mathrm{d}s_2.$$

It is easy to see that

$$\int_a^s f''_{ii.gH}(s_1)\mathrm{d}s_1 = f''_{ii.gH}(a) \odot (s-a) \oplus \int_a^s \left(\int_a^{s_1} f'''_{ii.gH}(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1,$$

where the last double (FR)-integral belongs to $\mathbb{R}_{\mathcal{F}}$. Thus we conclude

$$f'_{ii.gH}(s) = f'_{ii.gH}(a) \oplus f''_{ii.gH}(a) \odot (s-a) \oplus \int_a^s \left(\int_a^{s_1} f'''_{ii.gH}(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1.$$

Consequently $f'''_{ii.gH}$ can be written as

$$f'''_{ii.gH}(s_2) = f'''_{ii.gH}(a) \oplus \int_a^{s_2} f^{(4)}_{ii.gH}(s_3)\mathrm{d}s_3,$$

applying the (FR)-integral operator to $f'''_{ii.gH}(s_2)$, gives

$$\int_a^{s_1} f'''_{ii.gH}(s_2)\mathrm{d}s_2 = f'''_{ii.gH}(a) \odot (s_1 - a) \oplus \int_a^{s_1} \left(\int_a^{s_2} f^{(4)}_{ii.gH}(s_3)\mathrm{d}s_3\right)\mathrm{d}s_2;$$

furthermore,

$$\int_a^s \left(\int_a^{s_1} f'''_{ii.gH}(s_2)\mathrm{d}s_2 \right)\mathrm{d}s_1 = f'''_{ii.gH}(a) \odot \int_a^s (s_1 - a)\mathrm{d}s_1 \oplus \int_a^s \left(\int_a^{s_1} \left(\int_a^{s_2} f^{(4)}_{ii.gH}(s_3)\mathrm{d}s_3 \right)\mathrm{d}s_2 \right)\mathrm{d}s_1,$$

where the last triple integral belongs to $\mathbb{R}_{\mathcal{F}}$. Hence

$$f'_{ii.gH}(s) = f'_{ii.gH}(a) \oplus f''_{ii.gH}(a) \odot (s-a) \oplus f'''_{ii.gH}(a) \odot \frac{(s-a)^2}{2!} \oplus \int_a^s \left(\int_a^{s_1} \left(\int_a^{s_2} f^{(4)}_{ii.gH}(s_3)\mathrm{d}s_3 \right)\mathrm{d}s_2 \right)\mathrm{d}s_1.$$

If s is very closed to a and $(s-a) \to 0$,

$$D(f'_{ii.gH}(s) \ominus f'_{ii.gH}(a), f''_{i.gH}(a) \odot (s-a)) \to 0,$$

this means where $(s-a) \to 0$ we have

$$f''_{ii.gH}(a) \approx \frac{f'_{ii.gH}(s) \ominus f'_{ii.gH}(a)}{s-a}.$$

(iii) Suppose that $f^{(i)}$, $i = 1, 3$, are $[(i) - gH]$-differentiable and $f^{(i)}$, $i = 2, 4$, are $[(ii) - gH]$-differentiable, so

$$f'_{ii.gH}(s) = f'_{ii.gH}(a) \ominus (-1) \odot \int_a^s f''_{i.gH}(s_1)\mathrm{d}s_1,$$

As it is known, $f''_{i.gH}$ is obtained from

$$f''_{i.gH}(s_1) = f''_{i.gH}(a) \ominus (-1) \odot \int_a^{s_1} f'''_{ii.gH}(s_2)\mathrm{d}s_2;$$

by integration we obtain

$$\int_a^s f''_{i.gH}(s_1)\mathrm{d}s_1 = f''_{i.gH}(a) \odot (s-a) \ominus (-1) \odot \int_a^s \left(\int_a^{s_1} f'''_{ii.gH}(s_2)\mathrm{d}s_2 \right) ds_1;$$

therefore,

$$f'_{ii.gH}(s) = f'_{ii.gH}(a) \oplus (-1) f''_{i.gH}(a) \odot (s-a) \ominus \int_a^s \left(\int_a^{s_1} f'''_{ii.gH}(s_2)\mathrm{d}s_2 \right);$$

similarly, in order to find $f'''_{ii.gH}$ by Theorem 2.3, we have

$$f'''_{ii.gH}(s_2) = f'''_{ii.gH}(a) \ominus (-1) \int_a^{s_2} f^{(4)}_{i.gH}(s_3)\mathrm{d}s_3,$$

and

$$\int_a^s \left(\int_a^{s_1} f'''_{ii.gH}(s_2)\mathrm{d}s_2 \right)\mathrm{d}s_1 = \frac{(s-a)^2}{2} \odot f'''_{ii.gH}(a) \ominus (-1) \int_a^s \left(\int_a^{s_1} \left(\int_a^{s_2} f^{(4)}_{ii.gH}(s_3)\mathrm{d}s_3)\mathrm{d}s_2 \right) \right)\mathrm{d}s_1,$$

thus, we conclude:

$$f'_{ii.gH}(s) = f'_{ii.gH}(a) \oplus (-1) f''_{i.gH}(a) \odot (s-a) \ominus (-1) f'''_{ii.gH}(a) \odot \frac{(s-a)^2}{2} \oplus \int_a^s \left(\int_a^{s_1} \left(\int_a^{s_2} f^{(4)}_{i.gH}(s_3) \mathrm{d}s_3 \right) \mathrm{d}s_2 \right) \mathrm{d}s_1;$$

now, if $(s-a) \to 0$, we get

$$D(f'_{ii.gH}(s) \ominus f'_{ii.gH}(a), (-1) f''_{i.gH}(a) \odot (s-a)) \to 0;$$

this means

$$(-1) f''_{i.gH}(a) \approx \frac{f'_{ii.gH}(s) \ominus f'_{ii.gH}(a)}{s-a}.$$

(iv) Suppose that $f^{(i)}$, $i = 1, 3$, are $[(ii) - gH]$-differentiable and $f^{(i)}$, $i = 2, 4$, are $[(i) - gH]$-differentiable, so base on the above procedures,

$$f'_{i.gH}(s) = f'_{i.gH}(a) \ominus (-1) \odot \int_a^s f''_{ii.gH}(s_1) \mathrm{d}s_1,$$

where

$$f''_{ii.gH}(s_1) = f''_{ii.gH}(a) \ominus (-1) \odot \int_a^{s_1} f'''_{i.gH}(s_2) \mathrm{d}s_2.$$

also

$$\int_a^s f''_{ii.gH}(s_1) \mathrm{d}s_1 = f''_{ii.gH}(a) \odot (s-a) \ominus (-1) \odot \int_a^s \left(\int_a^{s_1} f'''_{i.gH}(s_2) \mathrm{d}s_2 \right) ds_1.$$

By a similar way, it is easy to see that

$$f'_{i.gH}(s) = f'_{i.gH}(a) \oplus (-1) f''_{ii.gH}(a) \odot (s-a) \ominus (-1) f'''_{i.gH}(a) \odot \frac{(s-a)^2}{2} \oplus \int_a^s \left(\int_a^{s_1} \left(\int_a^{s_2} f^{(4)}_{ii.gH}(s_3) \mathrm{d}s_3 \right) \mathrm{d}s_2 \right) \mathrm{d}s_1,$$

and finally if $(s-a) \to 0$, we can conclude

$$f''_{i.gH}(a) \approx \frac{f'_{i.gH}(s) \ominus f'_{i.gH}(a)}{s-a}.$$

Now, based on the above theorem, we introduce the Modified Euler method for solving FDE (3.1). First to integrate the system given Eq. (3.1), we replace the interval $[0, T]$ by a set of discrete equally spaced grid points, $0 = t_0 < t_1 < \cdots < t_N = T$, where $t_n = nh$, $h = \frac{T}{N}$. □

Now in order to obtain the proposed method, we consider three cases:

Case 1

Let $y(t) \in \mathcal{C}^4_{gH}([0,T],\ \mathbb{R}_{\mathcal{F}})$ is $[(i)-gH]$-differentiable solution of problem (3.1) and $y^{(i)}(t),\ i=1,\ldots,4$ are $[(i)-gH]$-differentiable. Now by using the Taylor series expansion of $y(t_{k+1})$ at the point t_k , for each $k=0,1,\ldots,N$, we get

$$\begin{aligned} y(t_{k+1}) &= y(t_k) \oplus (t_{k+1}-t_k) \odot y'_{i.gH}(t_k) \\ &\oplus \frac{(t_{k+1}-t_k)^2}{2} \odot y''_{i.gH}(t_k) \oplus \frac{(t_{k+1}-t_k)^3}{3!} \odot y'''_{i.gH}(\eta_k); \end{aligned}$$

for some points η_k lie between t_k and t_{k+1}. Since $h=t_{k+1}-t_k$, gives

$$y(t_{k+1}) = y(t_k) \oplus h \odot y'_{i.gH}(t_k) \oplus \frac{h^2}{2} \odot y''_{i.gH}(t_k) \oplus \frac{h^3}{3!} \odot y'''_{i.gH}(\eta_k). \tag{3.7}$$

Consequently, by taking Eq. (3.2) into Eq. (3.7) we obtain:

$$y(t_{k+1}) = y(t_k) \oplus \frac{h}{2} \odot \left(\frac{y'_{i.gH}(t_{k+1}) \ominus y'_{i.gH}(t_k)}{h}\right) \oplus \frac{h^3}{3!} \odot y'''_{i.gH}(\eta_k).$$

Fuzzy differential equation (3.1) implies that $y'_{gH}(t)=f(t,y(t))$; so we have

$$y(t_{k+1}) = y(t_k) \oplus \frac{h}{2} \odot \left(\frac{f(t_{k+1},y(t_{k+1})) \oplus f(t_k,y(t_k))}{h}\right) \oplus \frac{h^3}{3!} \odot y'''_{i.gH}(\eta_k). \tag{3.8}$$

In order to obtain a numerical method, the value of $y(t_{k+1})$ appearing on the RHS is not known. To handel this, the value of $y(t_{k+1})$ is first predicted by Euler method (Allahviranloo et al. 2015), and then the predicted value is used in Eq. (3.8).

Thus, the Modified Euler method can be written as follows:

$$\begin{cases} y^*_{k+1} = y_k \oplus h \odot f(t_k,y_k), \\ y_{k+1} = y_k \oplus \frac{h}{2} \odot \Big(f(t_{k+1},y^*(t_{k+1})) \oplus f(t_k,y(t_k))\Big), \quad k=0,1,\ldots,N-1. \end{cases} \tag{3.9}$$

Case 2

Now, consider $y(t)$ is $[(ii)-gH]$-differentiable and belongs to $\mathcal{C}^3_{gH}([0,T],\mathbb{R}_{\mathcal{F}})$ and $y^{(i)}(t),\quad i=1,\ldots,4$ are $[(ii)-gH]$-differentiable. So the Taylor's series expansion of $y(t)$ about the point t_k at t_{k+1} is

$$\begin{aligned} y(t_{k+1}) &= y(t_k) \ominus (-1)h \odot y'_{ii.gH}(t_k) \\ &\ominus (-1)\frac{h^2}{2} \odot y''_{ii.gH} \\ &\ominus (-1)\frac{h^3}{3!} \odot y'''_{ii.gH}(\eta_k), \end{aligned} \tag{3.10}$$

Now, by taking Eq. (3.3) into Eq. (3.10) we obtain

$$y(t_{k+1}) = y(t_k) \ominus (-1)\frac{h}{2} \odot \Big(y'_{ii.gH}(t_k) \oplus y'_{ii.gH}(t_{k+1})\Big),$$

As described in Case 1, the Modified Euler method takes the following form:

$$\begin{cases} y^*_{k+1} = y_k \ominus (-1)h \odot f(t_k, y_k), \\ y_{k+1} = y_k \ominus (-1)\frac{h}{2} \odot \Big(f(t_{k+1}, y^*(t_{k+1})) \oplus f(t_k, y(t_k)) \Big), k = 0, 1, \ldots, N-1. \end{cases} \tag{3.11}$$

Case 3
We Consider partition of $[0, T]$ as follows:

$$t_0 = 0, t_1, \ldots, t_j, \gamma, t_{j+1}, \ldots, t_N = T. \tag{3.12}$$

If the assumptions of case 1 are true for $y(t),\ t \in [0, t_j]$ and case 2 are true for $y(t),\ t \in [t_{j+1}, T]$ ($\gamma \in [0, T]$ is a switching point type I), the Modified Euler method can be written as follows:

$$\begin{cases} y^*_{k+1} = y_k \oplus h \odot f(t_k, y_k), \\ y_{k+1} = y_k \oplus \frac{h}{2} \odot \Big(f(t_{k+1}, y^*(t_{k+1})) \oplus f(t_k, y(t_k)) \Big), \quad k = 0, 1, \ldots, j. \\ \\ y^*_{k+1} = y_k \ominus (-1)h \odot f(t_k, y_k), \\ y_{k+1} = y_k \ominus (-1)\frac{h}{2} \odot \Big(f(t_{k+1}, y^*(t_{k+1})) \oplus f(t_k, y(t_k)) \Big), \quad k = j+1, 1, \ldots, N-1. \end{cases} \tag{3.13}$$

Case 4
If the assumptions of case 2 are true for $y(t),\ t \in [0, t_j]$ and case 1 are true for $y(t),\ t \in [t_{j+1}, T]$ ($\gamma \in [0, T]$ is a switching point type II); hence the Modified Euler method can be written as follows:

$$\begin{cases} y^*_{k+1} = y_k \ominus (-1)h \odot f(t_k, y_k), \\ y_{k+1} = y_k \ominus (-1)\frac{h}{2} \odot \Big(f(t_{k+1}, y^*(t_{k+1})) \oplus f(t_k, y(t_k)) \Big), \quad k = 0, 1, \ldots, j, \\ \\ y^*_{k+1} = y_k \oplus h \odot f(t_k, y_k), \\ y_{k+1} = y_k \oplus \frac{h}{2} \odot \Big(f(t_{k+1}, y^*(t_{k+1})) \oplus f(t_k, y(t_k)) \Big), \quad k = j+1, 1, \ldots, N-1. \end{cases} \tag{3.14}$$

3.1 Consistency, convergence and stability

In this section, consistency, convergence and stability of the Modified Euler Method are discussed in detail.

Theorem 3.2 *The Modified Euler method is consistent.*

Proof First, let $y(t)$ is $[(i) - gH]$-differentiable; by Eq. (3.9) we can write

$$\phi_k = y_{k+1} \ominus \left(y_k \oplus \frac{h}{2} \odot \Big(f(t_{k+1}, y_k \oplus f(t_k, y_k)) \oplus f(t_k, y_k) \Big) \right).$$

The local truncation error is defined as

$$\tau_k = \frac{1}{h} \odot \phi_k;$$

now, it is sufficient to show

$$\lim_{h\to 0} \max D(\tau_k, 0) = 0,$$

where $\tau_k = \frac{h^2}{3!} y'''_{i.gH}(\xi_k)$ and $D(y'''_{i.gH}, 0) \le M_1$.
Consequently,

$$\lim_{h\to 0} \max D(\tau_k, 0) = \lim_{h\to 0} \frac{h^2}{3!} \odot \max D(y'''_{i.gH}(\xi_k), 0) \le \frac{h^2}{3!} M_1 = 0.$$

If $y(t)$ is $[(ii) - gH]$-differentiable, then

$$\tau_k = \ominus(-1)\frac{h^2}{3!} \odot y'''_{ii.gH}(\xi_k),$$

and $D(y'''_{ii.gH}, 0) \le M_2$;
therefore,

$$\lim_{h\to 0} \max D(\tau_k, 0) = \lim_{h\to 0} \left|\frac{-h^2}{3!}\right| \odot \max D(\ominus y'''_{i.gH}(\xi_k), 0) \le \frac{h^2}{3!} M_2 = 0;$$

so, the proof of the theorem is complete. □

Lemma 3.1 (see Nieto et al. 2009) *For all real z,*

$$1 + z \le e^z. \tag{3.15}$$

Theorem 3.3 *Let $y'''_{gH}(t)$ exist and $f(t, y(t))$ satisfy in Lipschitz condition on the $\{(t, y(t))|t \in [0, p],\ y \in \overline{B}(y_0, q),\ p, q > 0\}$; then the Modified Euler Method converges to the solution of fuzzy differential Eq.* (3.1).

Proof Let $y(t)$ is $[(i) - gH]$-differentiable; then the Modified Euler method may be written in the following form:

$$y_{k+1} = y_k \oplus h \odot \phi(t_n, y_n; h), \tag{3.16}$$

where

$$\phi(t_n, y_n; h) = \frac{1}{2}[f(t_n, y_n) \oplus f(t_n + h, y_n \oplus h \odot f(t_n, y_n))],$$

and $\phi(., .; .)$ is continuous function of its variables. First we want to verify the Lipschitz condition of the function ϕ for the Modified Euler method.
From Lipschitz condition, we obtain

$$\begin{aligned} D(\phi(t, u; h), \phi(t, v; h)) &\le \frac{1}{2} D(f(t, u), f(t, v)) \\ &\quad + \frac{1}{2} D(f(t + h, u \oplus h \odot f(t, u)), f(t + h, v \oplus h \odot f(t, v))), \\ &\le \frac{L}{2} D(u, v) + \frac{L}{2} D(u \oplus h \odot f(t, u), v \oplus h \odot f(t, v)), \\ &\le L\left(1 + \frac{hL}{2}\right) D(u, v) = L_\phi D(u, v), \end{aligned}$$

i.e. ϕ satisfies a Lipschitz condition with constant $(L + \frac{1}{2} h L^2)$.

Now the exact solution of Eq. (3.1) is satisfied:

$$y(t_{k+1}) = y(t_k) \oplus h \odot \phi(t_k, y(t_k); h) \oplus R_k, \tag{3.17}$$

where $R_k = \frac{h^3}{3!} \odot y'''_{i.gH}(t_k)$.

Subtracting Eq. (3.17) from Eq. (3.16) and by using Lipschitz condition, the following equation is obtained:

$$D(y(t_{k+1}), y_{k+1}) \leq (1 + hL_{\phi_k})D(y(t_k), y_k) \oplus D(R_k, 0); \tag{3.18}$$

we put

$$L_\phi = \max_{0\leq k\leq N} \{L_{\phi_k}\}, \quad R = \max_{0\leq k\leq N} D(R_k, 0),$$

and by substituting in Eq. (3.18), get

$$D(y(t_{k+1}), y_{k+1}) \leq (1 + hL_\phi)D(y(t_k), y_k) \oplus R,$$

it is easy to see that

$$D(y(t_{k+1}), y_{k+1}) \leq (1 + hL_\phi)^{k+1} D(y(t_0), y_0) \oplus [1 + (1 + hL_\phi) + \cdots + (1 + hL_\phi)^k] \odot R.$$

On the other hand,

$$\sum_{i=0}^{k} (1 + hL_\phi)^i = \frac{(1 + hL_\phi)^{k+1} - 1}{hL_\phi},$$

Now for $0 \leq (k + 1)h \leq T$, $(k + 1) \leq (N - 1)$, and by using Eq. (3.15), we obtain

$$D(y(t_{k+1}), y_{k+1}) \leq e^{TL_{\phi_k}} D(y(t_0), y_0) + \frac{R}{hL_\phi}[e^{TL_\phi} - 1],$$

where $R = \max_{0\leq k\leq N} D(\frac{h^3}{3!} \odot y'''_{i.gH}(\xi_k), 0)$.

It is clear that $D(y(t_0), y_0) = 0$; so

$$D(y(t_{k+1}), y_{k+1}) \leq \frac{h^2}{6L_\phi}[e^{TL_\phi} - 1] \max_{0\leq t\leq T} D(y'''_{i.gH}(t), 0).$$

Thus, $\lim_{h\to 0} D(y(t_{k+1}), y_{k+1}) \to 0$ and in this case, Modified Euler method converges.

Consider $y(t)$ is $[(ii) - gH]$-differentiable, the Modified Euler method is written as follows:

$$y_{k+1} = y_k \ominus (-1)h \odot \varphi(t_n, y_n; h), \tag{3.19}$$

where

$$\varphi(t_n, y_n; h) = \frac{1}{2}[f(t_n, y_n) \oplus f(t_n + h, y_n \ominus (-1)h \odot f(t_n, y_n))],$$

and $\varphi(., .; .)$ is continuous function of its variables. The Lipschitz condition of the function φ for the Modified Euler method is as follows:

$$D(\varphi(t, u; h), \varphi(t, v; h)) \leq \frac{1}{2}D(f(t, u), f(t, v)) + \frac{1}{2}D(f(t + h, u \ominus (-1)hf(t, u)), f(t + h, v \ominus (-1)hf(t, v))),$$

$$\leq \frac{L}{2}D(u,v) + \frac{L}{2}D(u \ominus (-1)h \odot f(t,u), v \ominus (-1)h \odot f(t,v)),$$
$$\leq L\left(1 - \frac{hL}{2}\right)D(u,v) = L_\varphi D(u,v),$$

i.e. φ satisfies a Lipschitz condition with constant $(L - \frac{1}{2}hL^2)$.

Now the exact solution of Eq. (3.1) is satisfied:

$$y(t_{k+1}) = y(t_k) \ominus (-1)h \odot \varphi(t_k, y(t_k); h) \oplus J_k, \tag{3.20}$$

where $J_k = -\frac{h^3}{3!} \odot y'''_{ii.gH}(t_k)$.

Subtracting Eq. (3.20) from Eq. (3.19) and by using Lipschitz condition, the following equation is obtained:

$$D(y(t_{k+1}), y_{k+1}) \leq (1 - hL_{\phi_k})D(y(t_k), y_k) \oplus D(J_k, 0); \tag{3.21}$$

now by putting

$$L_\varphi = \max_{0 \leq k \leq N}\{L_{\varphi_k}\}, \quad J = \max_{0 \leq k \leq N} D(J_k, 0),$$

and by similar procedure, we obtain

$$D(y(t_{k+1}), y_{k+1}) \leq \frac{-h^2}{6L_\phi}\left[\frac{1}{e^{TL_\phi}} - 1\right] \max_{0 \leq t \leq T} D(y'''_{ii.gH}(t), 0).$$

Thus, $\lim_{h\to 0} D(y(t_{k+1}), y_{k+1}) \to 0$. □

Theorem 3.4 *The Modified Euler method is stable.*

Proof Let $y_{k+1}, k+1 \geq 0$ be the solution of the Modified Euler method with initial condition $y_0 \in \mathbb{R}_F$ and let τ_{k+1} be the solution of the Modified Euler method with perturbed fuzzy initial condition $\tau_0 = y_0 \oplus \epsilon \in \mathbb{R}_F$; therefore, if $y(t)$ is $[(i) - gH]$-differentiable

$$\tau_{k+1} = \tau_k \oplus \frac{h}{2} \odot \Big(f(t_{k+1}, \tau_k \oplus h \odot f(t_k, \tau_k)) \oplus f(t_k, \tau_k)\Big), \quad \tau_0 = y_0 \oplus \epsilon,$$

then by using Lipschitz condition and properties of Hausdorff distance, we have

$$D(y_{k+1}, \tau_{k+1}) \leq D(y_k, \tau_k) + \frac{h}{2}L\Big(D(y_k, \tau_k) + hLD(y_k, \tau_k)\Big) + \frac{h}{2}LD(y_k, \tau_k),$$

Thus

$$D(y_{k+1}, \tau_{k+1}) \leq \left(1 + hL + \frac{(hL)^2}{2}\right)D(y_k, \tau_k).$$

Now by iterating the above inequality, we can write

$$D(y_{k+1}, \tau_{k+1}) \leq \left(1 + hL + \frac{(hL)^2}{2}\right)^k D(y_0, \tau_0) \leq (e^{hL})^k D(y_0, \tau_0)$$
$$\leq e^{LT} D(y_0, \tau_0) \leq \kappa D(y_0, \tau_0),$$

where $\kappa = e^{TL}$, and $kh \leq (k+1)h \leq T$. So, if $y(t)$ is $[(i) - gH]$-differentiable, this method is stable. □

Now, we suppose that $y(t)$ is $[(ii)-gH]$-differentiable; then it is easy to see that

$$D(y_{k+1},\tau_{k+1}) \le D(y_k,\tau_k) - \frac{h}{2}L\bigg(D(y_k,\tau_k) - hLD(y_k,\tau_k)\bigg) - \frac{h}{2}LD(y_k,\tau_k);$$

therefore

$$D(y_{k+1},\tau_{k+1}) \le \left(1 - hL + \frac{(hL)^2}{2}\right) D(y_k,\tau_k);$$

by repeating the earlier procedure we obtain

$$\begin{aligned} D(y_{k+1},\tau_{k+1}) &\le \left(1 - hL + \frac{(hL)^2}{2}\right)^k D(y_0,\tau_0) \le (e^{-hL})^k D(y_0,\tau_0) \\ &\le e^{-LT} D(y_0,\tau_0) \le \kappa D(y_0,\tau_0), \end{aligned}$$

where $\kappa = e^{-TL}$, and $kh \le (k+1)h \le T$; therefore, the Modified Euler method is stable.

4 Numerical examples

In this section we will provide three examples to emphasize acceptable accuracy of Modified Euler Method. All numerical computations were performed using Maple 13 software package.

Example 4.1 (see. Allahviranloo et al. 2015) Consider the first-order fuzzy initial value problem as follows:

$$\begin{cases} y'_{i.gH}(t) = y(t) \oplus (1.3, 2, 2.1), \quad 0 \le t \le 1; \\ y(0) = (0.82, 1, 1.2), \end{cases}$$

such that the exact $[(i)-gH]$-differentiable solution is obtained by solving the following system:

$$\begin{cases} (y_-)'(t;\alpha) = y_-(t;\alpha) + 1.3 + 0.7\alpha, \quad 0 \le t \le 1; \\ (y^+)'(t;\alpha) = y^+(t;\alpha) + 2.1 - 0.1\alpha, \quad 0 \le t \le 1; \\ y(0;\alpha) = [0.82 + 0.18\alpha, 1.2 - 0.2\alpha], \ . \end{cases}$$

Now we use the Modified Euler method to obtain approximate solution and we compare our solution with Euler method (Allahviranloo et al. 2015). The global truncation errors of the Modified Euler and Euler method have been reported for $h = 0.025$ and 0.005 in Table 1, and the approximated solution by the Modified Euler method is shown in Fig. 1.

Example 4.2 (*see* Allahviranloo et al. 2015) Let us consider the following initial value problem:

$$\begin{cases} y'_{ii.gH}(t) = -y(t) \oplus t \odot (0.7, 1, 1.8), \quad 0 \le t \le 1, \\ y(0) = (0, 1, 2.2) \end{cases} \tag{4.1}$$

Since $y(t)$ has $[(ii)-gH]$-derivative, the $[(ii)-gH]$-differentiable solution is obtained by solving

$$\begin{cases} (y_-)'(t;\alpha) = -y_-(t;\alpha) + t\,[1.8 - 0.8\alpha], \quad 0 \le t \le 1, \\ (y^+)'(t;\alpha) = -y^+(t;\alpha) + t\,[0.7 + 0.3\alpha], \quad 0 \le t \le 1, \\ y(0;\alpha) = [\alpha, 2.2 - 1.2\alpha] \end{cases}$$

Table 1 The global truncation errors for Example 4.1

t	$h = 0.025$		$h = 0.005$	
	Modified Euler	Euler	Modified Euler	Euler
0	0	0	0	0
0.1	2.3952×10^{-5}	4.48149×10^{-3}	1.514×10^{-7}	4.44089×10^{-4}
0.2	5.2943×10^{-5}	9.89954×10^{-3}	3.346×10^{-6}	2.00812×10^{-3}
0.3	1.36619×10^{-4}	1.64009×10^{-2}	5.547×10^{-6}	3.32856×10^{-3}
0.4	2.01315×10^{-4}	2.41529×10^{-2}	8.174×10^{-6}	4.90422×10^{-3}
0.5	2.78108×10^{-4}	3.33459×10^{-2}	1.1291×10^{-5}	6.77416×10^{-3}
0.6	3.68824×10^{-4}	4.41964×10^{-2}	1.4973×10^{-5}	8.98281×10^{-3}
0.7	4.75546×10^{-4}	5.69503×10^{-2}	1.9305×10^{-5}	1.15806×10^{-2}
0.8	6.00637×10^{-4}	7.18871×10^{-2}	2.4382×10^{-5}	1.46252×10^{-2}
0.9	7.46780×10^{-4}	8.93237×10^{-2}	3.0317×10^{-5}	1.81815×10^{-2}
1.0	9.17016×10^{-4}	1.09619×10^{-1}	3.7230×10^{-5}	2.23235×10^{-2}

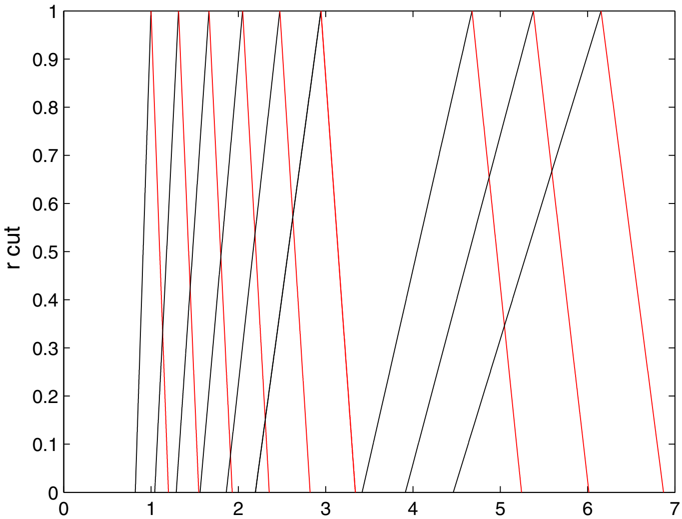

Fig. 1 Approximated solution for Example 4.1. Red lines: $y^{+}(t, r)$; black lines: $y^{-}(t, r)$

The global truncation errors of Modified Euler and Euler method have been reported for $h = 0.025$ and 0.005 in Table 2, and the approximated solution by the Modified Euler method is shown in Fig. 2.

Example 4.3 (*see* Allahviranloo et al. 2015) Consider the initial value problem

$$\begin{cases} y'_{gH}(t) = (1-t)\, y(t), & 0 \le t \le 2, \\ y(0) = (0, 1, 2) \end{cases}$$

Obviously that initial value problem on [0, 1] is $[(i) - gH]$-differentiable and at $t = 1$ the problem is switched to $[(ii) - gH]$-differentiable. So, the point $t = 1$ is a switching point and

Table 2 The global truncation errors for Example 4.2

t	$h = 0.025$		$h = 0.005$	
	Modified Euler	Euler	Modified Euler	Euler
0	0	0	0	0
0.1	2.7852×10^{-5}	3.33362×10^{-3}	1.098×10^{-6}	6.58119×10^{-4}
0.2	5.0403×10^{-5}	6.02895×10^{-3}	1.984×10^{-6}	1.19083×10^{-3}
0.3	6.8410×10^{-5}	8.17763×10^{-3}	2.695×10^{-6}	1.61606×10^{-3}
0.4	8.2535×10^{-5}	9.85964×10^{-3}	3.252×10^{-6}	1.94945×10^{-3}
0.5	9.3351×10^{-5}	1.11446×10^{-2}	3.679×10^{-6}	2.20464×10^{-3}
0.6	1.01361×10^{-4}	1.20932×10^{-2}	3.994×10^{-6}	2.39351×10^{-3}
0.7	1.07002×10^{-4}	1.27580×10^{-2}	4.217×10^{-6}	2.52638×10^{-3}
0.8	1.10650×10^{-4}	1.31847×10^{-2}	4.360×10^{-6}	2.61220×10^{-3}
0.9	1.12636×10^{-4}	1.34127×10^{-2}	4.438×10^{-6}	2.65874×10^{-3}
1.0	1.13242×10^{-4}	1.34763×10^{-2}	4.461×10^{-6}	2.67269×10^{-3}

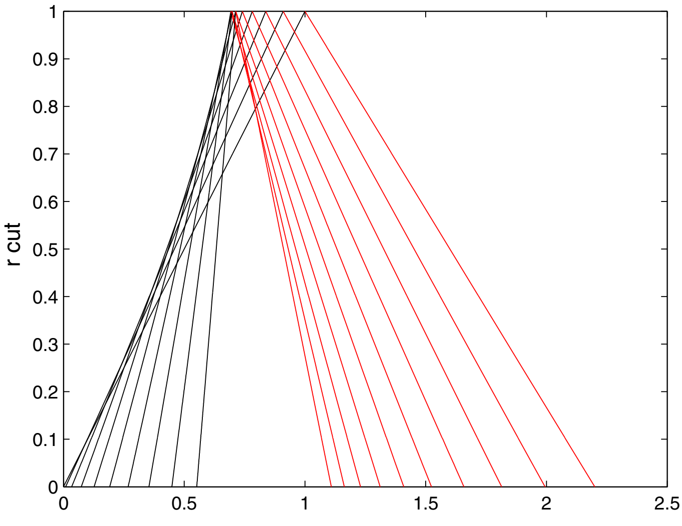

Fig. 2 Approximated solution for Example 4.2. Red lines: $y^+(t, r)$; black lines: $y^-(t, r)$

the obtained solution on $[0, \ 1]$ is $[(i) - gH]$-differentiable and $[(ii) - gH]$-differentiable on $(1, \ 2]$. The $[(i) - gH]$-differentiable solution can be obtained by solving

$$\begin{cases} (y_-)'(t;\alpha) = \begin{cases} (1-t)y_-(t;\alpha), & 0 \le t \le 1, \\ (1-t)y^+(t;\alpha), & 1 < t \le 2, \end{cases} \\ (y^+)'(t;\alpha) = \begin{cases} (1-t)y^+(t;\alpha), & 0 \le t \le 1, \\ (1-t)y_-(t;\alpha), & 1 < t \le 2, \end{cases} \\ y(0;\alpha) = [\alpha, 2-\alpha] \end{cases}$$

Table 3 The global truncation errors for Example 4.3

t	$h = 0.025$		$h = 0.005$	
	Modified Euler	Euler	Modified Euler	Euler
0	0	0		0
0.2	3.7098×10^{-5}	1.05772×10^{-3}	1.476×10^{-6}	2.21057×10^{-4}
0.4	6.3540×10^{-5}	4.62954×10^{-3}	2.507×10^{-6}	9.48966×10^{-4}
0.6	7.9558×10^{-5}	1.07346×10^{-2}	3.109×10^{-6}	2.18262×10^{-3}
0.8	8.7732×10^{-5}	1.87015×10^{-2}	3.39×10^{-6}	3.78194×10^{-3}
1.0	9.0927×10^{-5}	2.72488×10^{-2}	3.471×10^{-6}	5.48657×10^{-3}
1.2	9.22345×10^{-5}	1.84424×10^{-2}	3.4815×10^{-6}	3.73717×10^{-3}
1.4	9.39650×10^{-5}	8.95003×10^{-3}	3.5120×10^{-6}	1.85160×10^{-3}
1.6	9.82381×10^{-5}	6.46455×10^{-4}	3.6421×10^{-6}	5.51592×10^{-5}
1.8	1.120221×10^{-4}	9.62457×10^{-3}	4.1481×10^{-6}	1.84070×10^{-3}
2.0	1.53036×10^{-4}	1.72255×10^{-2}	5.752×10^{-6}	3.35558×10^{-3}

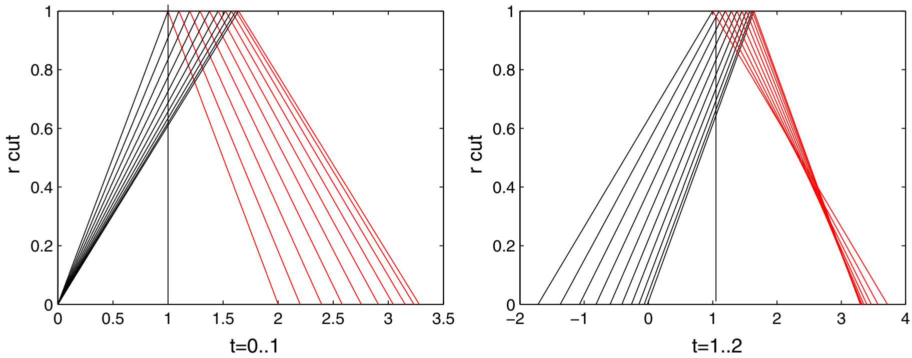

Fig. 3 Approximated solution for Example 4.3. Red lines: $y^+(t, r)$; black lines:$y^-(t, r)$

The global truncation errors of Modified Euler and Euler method have been reported for $h = 0.025$ and 0.005 in Table 3, and the approximated solution by the Modified Euler method is plotted in Fig. 3.

Example 4.4 Jafari and Razvarz (2018) A tank with a heating system is displayed in Fig. 4, where $\tilde{R} = 0.5$, the thermal capacitance is $\tilde{C} = 2$, and the temperature is ψ. The model is formulated as follows:

$$\begin{cases} \phi'(t) = -\frac{1}{\tilde{R}\tilde{C}}\phi(t), & 0 \le t \le T, \\ \phi(0) = (\phi^-(0, \alpha), & \phi^+(0, \alpha)), \end{cases} \tag{4.2}$$

where the initial condition is a symmetric triangular fuzzy number as $\phi(0) = (-a(1 - \alpha), a(1 - \alpha))$.

The solution of FDE (4.2), by using Modified Euler method, for $a = 2, h = 0.005$ is shown in Fig. 5.

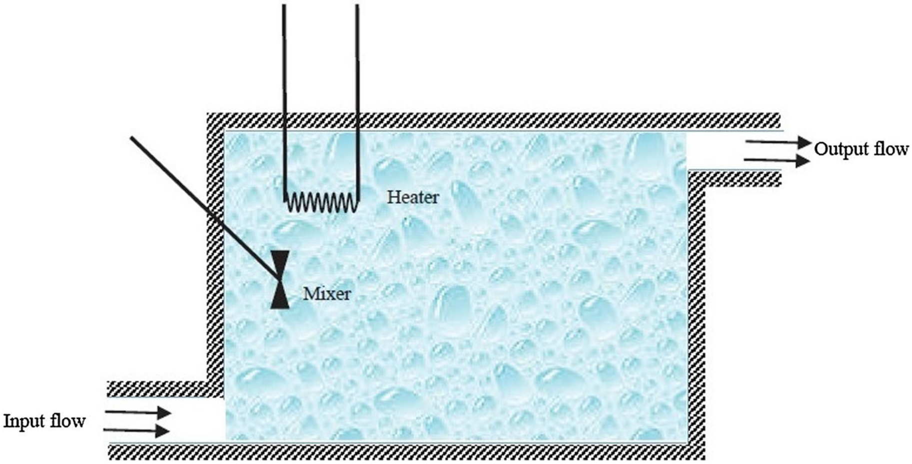

Fig. 4 Thermal system

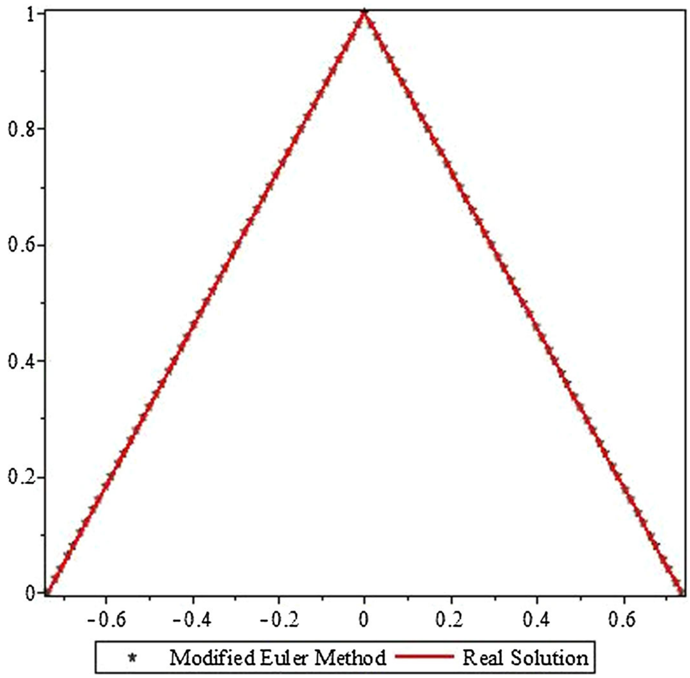

Fig. 5 Approximated and Real solution for Example 4.4

5 Conclusion

In this paper, we provided the numerical method for solving fuzzy differential equations under generalized differentiability. By improving Euler method (Allahviranloo et al. 2015), the Modified Euler method without losing its simplicity was introduced, and it was shown that the global error for the Modified Euler method was $o(h^2)$, while the global error of Euler method was $o(h)$. Finally, some examples were provided to emphasize acceptable accuracy of proposed method.

References

Abbasbandy S, Allahviranloo T (2002) Numerical solution of fuzzy differential equation by Taylor method. J. Comput Method Appl Math 2:113–124

Abbasbandy S, Allahviranloo T (2004) Numerical solution of fuzzy differential equation by Runge–Kutta method. Nonlinear Stud 11(1):117–129

Ahmadian A, Salahshour S, Chan C, Baleanu D (2018) Numerical solutions of fuzzy differential equations by an efficient Runge–Kutta method with generalized differentiability. Fuzzy Sets Syst 331:47–67

Ahmadian A, Suleiman M, Ismail F (2012) An improved Runge–Kutta method for solving fuzzy differential equations under generalized differentiability. AIP Conf Proc 1482:325–330

Allahviranloo T (2020) Uncertain information and linear systems. In: Studies in systems, decision and control, vol 254. Springer, pp 109–119. ISBN 978-3-030-31323-4

Allahviranloo T, Gouyandeh Z, Armand A (2015) A full fuzzy method for solvingdifferential equation based on Taylor expansion. Journal of Inteligent and Fuzzy Systems 29:1039–1055

Allahviranloo T, Ahmady N, Ahmady E (2007) Numerical solution of fuzzy differential equations by predictor–corrector method. Inform Sci 177(7):1633–1647

Allahviranloo T, Abbasbandy S, Ahmady N, Ahmady E (2009) Improved predictor-corrector method for solving fuzzy initial value problems. Inf Sci 179:945–955

BaloochShahryari MR, Salashour S (2012) Improved predictor–corrector method for solving fuzzy differential equations under generalized differentiability. J Fuzzy Set Val Anal 2012:1–16

Bede B, Gal SG (2005) Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. Fuzzy Set Syst 151:581–599

Bede B, Gal SG (2006) Remark on the new solutions of fuzzy differential equations. Chaos Solitons Fractals

Bede B, Stefanini L (2011) Solution of Fuzzy Differential Equations with generalized differentiability using LU-parametric representation. EUSFLAT 1:785–790

Bede B, Stefanini L (2013) Generalized differentiability of fuzzy-valued functions. Fuzzy Sets Syst 230:119–141

Chang S, Zadeh L (1972) On fuzzy mapping and control. IEEE Trans Syst Cybern 2:30–34

Chalco-Cano Y, Roman-Flores H (2008) On new solutions of fuzzy differential equations. Chaos Solitons Fractals 38:112–119

Dubois D, Prade H (1982) Toward fuzzy differential calculus: Part 3. Differ, Fuzzy Sets and Systems, pp 225–233

Epperson JF (2007) An introduction to numerical methods and analysis. Wiley, Hoboken

Goetschel R, Voxman W (1987) Elementary fuzzy calculus. Fuzzy Sets Syst 24:31–43

Hajighasemi S, Allahviranloo T, Khezerloo M, Khorasany M, Salahshour S (2010) Existence and uniqueness of solutions of fuzzy Volterra integro-differential equations. Inf Process Manag Uncert Knowl Based Syst 81:491–500

Jafari R, Razvarz S (2018) Solution of fuzzy differential equations using fuzzy Sumudu transforms. Math Comput Appl . https://doi.org/10.3390/mca23010005

Kaleva O (1987) Fuzzy differential equations. Fuzzy Sets Syst 24:301–317

Ma M, Friedman M, Kandel A (1999) Numerical solutions of fuzzy differential equations. Fuzzy Sets Syst 105:133–138

Negoita CV, Ralescu D (1975) Applications of fuzzy sets to systems analysis. Wiley, New York

Nieto JJ, Khastan A, Ivaz K (2009) Numerical solution of fuzzy differential equation under generalized differentiability. Nonlinear Anal Hybrid Syst 3:700–707

Puri ML, Ralescu DA (1986) Differentials of fuzzy functions. J Math Anal Appl 114:409–422

Rabiei F, Ismail F, Ahmadian A, Salahshour S (2013) Numerical solution of second-order fuzzy differential equation using improved Runge–Kutta Nystrom method. Math Probl Eng 2013:1–10

Salahshour S, Ahmadian A, Abbasbandy S, Baleanud D (2018) M-fractional derivative under interval uncertainty: theory, properties and applications. Chaos Solitons Fractals 117:84–93

Stefanini L (2008) A generalization of Hukuhara difference for interval and fuzzy arithmetic. In: Series on advances in soft computing, vol. 48, Springer. An extended version is available online at the RePEc service. https://econpapers.repec.org/paper/urbwpaper/08-5f01.htm

Seikkala S (1987) On the fuzzy initial value problem. Fuzzy Sets Syst 24:319–330

Stefanini L, Bede B (2009) Eneralized Hukuhara differentiability of interval-valued functions and interval differential equations. Nonlinear Anal 71:1311–1328

Tapaswini S, Chakraverty S (2012) A new approach to fuzzy initial value problem by improved Euler method. Fuzzy Inf Eng 3:293–312

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.